VECTORS, MATRICES and COMPLEX NUMBERS

by

Jean-Paul GINESTIER and John EGSGARD

with
International Baccalaureate
questions

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A Search for Meaning

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Originally French, Jean-Paul Ginestier was brought up in England and took his first degree there. He then taught in Cameroon, then the UK, before settling with his wife and two children in Canada, where he gained teaching and administrative experience in both the private and public systems, in English and in French. He also took his B.Ed, then his M.Sc. at the University of Toronto. In 1984-85, he was on exchange at the International School of Geneva, Switzerland, where John Egsgard had taught in the same capacity two years earlier.

Jean-Paul Ginestier also has a long experience, both as a teacher and an examiner, with the International Baccalaureate. From 1988 to 1995, he was Deputy Chief Examiner for IB Higher Level Mathematics. He has also made presentations at a many mathematics conferences in a number of countries.

Since 1995 he has been Head of Mathematics at two United World Colleges - the Nordic College in Norway, then the UWC of the Adriatic in Italy from 2004.

John Christian Egsgard, M.A.

John Egsgard retired from teaching in 1998 after 47 years in secondary school classrooms. Besides being an author of 15 secondary mathematics texts and editor of the Ontario Mathematics Gazette, he has been active in mathematics associations around the world. He has been president of both the National Council of Teachers of Mathematics of the USA, and the Ontario Association for Mathematics Education in Canada. John has been on many committees preparing for the first 9 International Congresses of Mathematics Education throughout the world. He has delivered papers or major speeches at most of the ICMEs.

John's experience with the IB comes from his year of teaching at the International School of Geneva, and in the marking of exams and preparing questions for them.

INTRODUCTION

The International Baccalaureate is a high quality programme of study that tries to bring together the best features inspired from various educational systems around the world; the IB Diploma is recognized for university entrance in most countries. In this book, past IB examination questions can be found in each of the Review Exercises and the Problem Supplement. The questions have been selected from papers for Standard Level Mathematical Studies (SMS), Standard Level Mathematics (S) and Higher Level Mathematics (H).

The original edition of this book, published by Gage (Canada Publishing Corporation) in 1989 was designed primarily to cover the Ontario Academic Credit called Algebra and Geometry. It so happens that this course was almost entirely a subset of IB Higher Level Mathematics at that time. The coverage of Vectors and Complex Numbers is thorough, and clearly explained for students in their last years of secondary education.

Teachers will find that the most relevant chapters for the IB HL Mathematics course are 1, 3, 5 and 6, although some of the notions in chapter 2 are necessary. All Matrix and Transformation work is found in chapter 7, and all Complex Numbers in chapter 10. Chapter 9 on Induction also covers a section of the IB HL Mathematics syllabus.

One characteristic of this book is that it tries to pay tribute, both by taking a historical approach where appropriate (as in the case of Complex Numbers), and by mentioning their names, to the many mathematicians who have led the evolution of these ideas for the last 500 years or so.

The book was reprinted with modifications in 1994. Although it has become the norm to publish entirely new textbooks as often as possible, then to discard them as quickly as possible, the authors believe that the best textbooks are those that are continuously revised, corrected, and updated.

The new text-searchable version of the book is now available on the web at
http://ginestier.hostned.ws

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Text Organization

- The text is divided into 10 chapters.
- *In Search of* sections within each chapter encourage students to investigate and explore challenging topics individually, or in small groups.
- *Making Connections* pages relate mathematics to other fields and disciplines, and encourage a greater understanding of the nature and purpose of mathematics.
- The *problem supplement* at the end of the text provides an opportunity for students to synthesize the skills and ideas acquired throughout the course.
- The *answer key* provides answers for all exercises, reviews, and inventories, as well as for the problem supplement.
- The *glossary* provides definitions of relevant mathematical terms.
- The *index* lists topics and main concepts for easy reference.

Chapter Organization

- Each chapter begins with a discussion of a problem which can be solved using the mathematics developed throughout the chapter.
- Teaching material is clearly separated from exercise material.
- Worked examples enhance the understanding of each topic.
- Colour is used to highlight generalizations, rules, and formulas.
- Mathematical terms appear in **boldface** type when they are first introduced. All relevant terms are defined in the glossary.
- *Italic* type is used for emphasis.
- Exercise material is carefully sequenced from questions that utilize and apply knowledge to those that develop critical thinking skills.
- The main concepts covered in each chapter are listed concisely in the *Chapter Summary*.
- The *Chapter Inventory* provides students with an opportunity to test their skills and understanding of the mathematical concepts in the chapter.
- The *Chapter Review* provides additional opportunities for students to apply their problem-solving skills.
- IB questions (numbered in colour) are to be found at the end of chapters, and in the problem supplement.

CONTENTS

VECTORS, MATRICES
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CHAPTER ONE

INTRODUCTION TO VECTORS

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CHAPTER ONE

Introduction to Vectors

In the first seven chapters of this book, you will be studying the algebra of mathematical objects called **vectors**.

Vectors are among the most recent inventions in mathematics that you will encounter in high school. Indeed, vector analysis was only fully developed at the turn of the 20th century. Unfortunately, very few properties and theorems in vector analysis are named after their originators. To compensate for this, a short history of the development of vectors is presented here.

In 1843, the famous Irish mathematician Sir William Hamilton invented an algebraic system that he thought could model any physical situation. He called the elements of his algebraic system *quaternions*. (Complex numbers, which you will study in chapter 10, form a subset of quaternions.) Hamilton, with his followers, attempted to apply the theory of quaternions to many areas in mathematics, and to use these to describe physical phenomena. A movement was created, which kept quaternion theory in the forefront of mathematical research until the end of the century.

Around the same time, the German mathematician Hermann Grassmann published a treatise, called *Die Ausdehnungslehre*, discussing much more general extensions of the number system than Hamilton's quaternions. (These extensions include complex numbers, quaternions, vectors, and matrices. The general name for these objects is *hypernumbers* or *holors*.) However, Grassmann's work was considered incomprehensible at the time of its publication, in 1844.

In 1881, the American mathematician Josiah Willard Gibbs, who was the first professor of mathematical physics at Yale, used ideas based on the works of Hamilton and Grassmann to publish a pamphlet called *Vector Analysis*. At first, Gibbs' ideas were rejected strongly by the supporters of quaternions, who maintained that their theory was 'complete', and that vectors were unpalatable 'hermaphrodites'.

The self-taught English scientist, Oliver Heaviside, like Gibbs, found quaternions unsatisfactory for the description of many physical phenomena in mechanics and electromagnetism. Despite his lack of a formal education beyond the age of 16, he started producing original and entertaining scientific papers in 1872, at the age of 22. The works that he published between 1893 and 1912 firmly established the superiority of vectors over quaternions to explain electromagnetic theory.

Today, vector analysis is an important part of mathematics. Furthermore, any physicist or engineer must have a firm grasp of the methods and symbolism of vector analysis.

HAMILTON was portrayed by a contemporary artist with his mace of office as President of the Royal Academy of Ireland. He was Royal Astronomer of Ireland from 1826 to 1865.

1.1 What is a Vector?

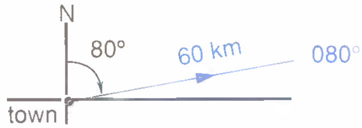
In mathematics, you often deal with numbers. However, many of the concepts used are not just numbers, or numbers alone. For example, lines, points, sets, matrices are not numbers.

Vectors are objects that generally need more than one number, to be defined. In this way, they are similar to points. However, vectors have other qualities that make them radically different from points. The nature of vectors should become apparent through the examples of statements below, as you shall see presently.
Note: Each statement makes use of a single number.

1. She is 60 km out of town.
2. The wind is blowing from the northeast (that is, from bearing 045°).
3. The elevator is stuck on the 12th floor.
4. The temperature in the office is 25°C.
5. The truck drove 5 blocks away from the Post Office.
6. The rocket moved in a straight line away from the earth with a speed of 5000 km/h.

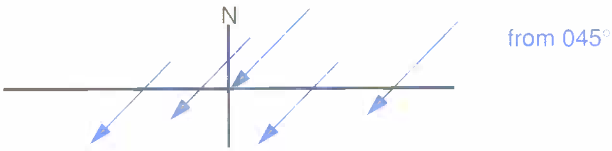
Some of these statements require more information to be complete, whereas others do tell you everything you want to know with the single number contained. You will see the idea of a vector emerging from those statements which appear incomplete.

In 1, you need to know in *which direction* the 60 km is; thus you need to know *another number*, the *bearing*. (The bearing of a object is its direction measured clockwise from north in degrees, and expressed as three digits.)



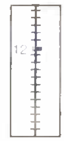
The diagram indicates that she is 60 km out of town, *on a bearing of 080°*. You need the two numbers, 60 and 080. The 60 km describes her distance from the town. The *two* numbers 60 and 080 describe her **displacement** from the town.

In 2, you know where the wind is coming from, but you do not know its *speed*. The statement is incomplete. You need another *number*.

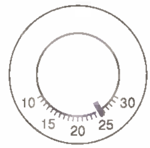


The diagram indicates a speed of *15 km/h* as well as the bearing 045°.
1 cm represents 10 km/h.

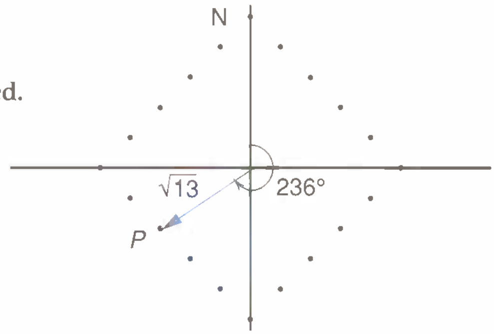
In 3, you *do* have complete information. The single number 12 tells you exactly where the elevator is.



In 4, you again have enough information. The single number is 25.

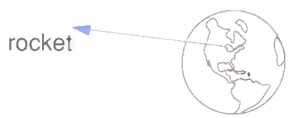


In 5, direction must be considered. After travelling 5 blocks, the truck may be in many different places.



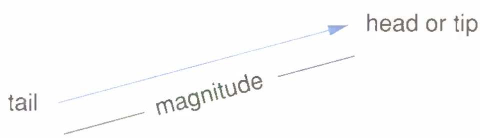
If it is at point P as the diagram indicates, it would be more informative to have, for example, the two numbers 3 and 2. These numbers tell you respectively how far west and south it is. Or, the two numbers could be the length of $OP = \sqrt{3^2 + 2^2} = \sqrt{13}$, and the bearing of OP, namely 236°.

In 6, besides the number 5000 representing the speed, one or two other numbers indicating the direction of travel of the rocket would be needed.



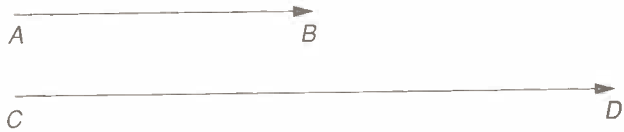
Observe the following about the preceding examples: Whenever *more than one number* was needed, a specific direction was involved, and it was convenient to draw a line segment with an *arrow* on it.

These 'directed line segments' represent *vectors*.



- The *length of the directed line segment* represents the *magnitude of the vector*.

In the diagram, the directed line segment *CD* is twice as long as the directed line segment *AB*. If *AB* represents a magnitude of 5, then *CD* will represent a magnitude of $2 \times 5 = 10$.



The *direction of the line segment*, as indicated by the arrow, represents the *direction of the vector*.

(Any two *parallel lines* with arrows pointing the same way are said to define the same *direction*.)

Thus,

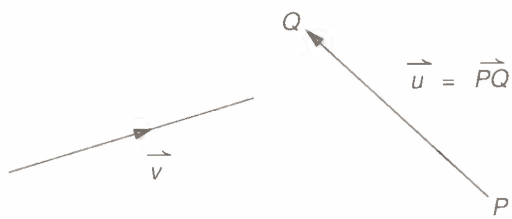
the *magnitude* and *direction* of a vector can be represented by a directed line segment.

A directed line segment gives you a 'picture' of a vector. The vector can also be represented by two (or more) numbers, as you saw in the examples. You will be investigating that type of representation further in section 1.3.

NOTATION

A vector can be represented by a single letter like this: $\vec{v}$. Note that an arrow $\rightharpoonup$ is put over the "v"; this is to indicate that $\vec{v}$ represents a vector.

A similar notation is also used for the vector $\vec{u}$ represented by the directed line segment joining a point *P* to a point *Q*: you can write $\vec{u} = \overrightarrow{PQ}$.



Note: In some texts, vectors are represented in bold print, without an arrow, thus: **v**.

The *length* or *magnitude* of a vector is a real number. (Any real number is called a **scalar**, to distinguish it from a vector.) The length of the vector $\vec{v}$ is denoted by $|\vec{v}|$. The length of the vector $\overrightarrow{PQ}$ is denoted by $|\overrightarrow{PQ}|$.

Note: In some texts, the length of the vector $\vec{v}$ is represented by v, and the length of the vector $\overrightarrow{PQ}$ by PQ.

An Important Property of Vectors

Take another look at the directed line segments representing the vectors in the examples 1, 2, 5, and 6 above. In statement 2 about the wind, many directed line segments were drawn, although there is only one 'wind'! This indicated that the wind does not blow on only one point. The directed line segments all represent the *same vector*.

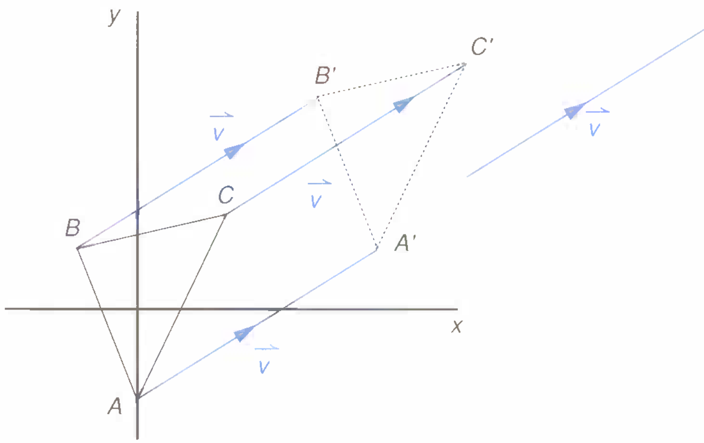
Indeed, one of the most important attributes of a vector is that it has magnitude and direction *only*. It does *not* have a particular *place*. Because a vector can be represented by any one of a family of directed line segments having the same length and direction, the following is true.

A vector is *everywhere*. A directed line segment representing a vector can be *drawn where you want.*

Translations and Vectors

You have studied translations before, and you will be seeing them again in more detail in chapter 8.

A translation is a transformation in which a figure or an object is moved to any other position, without altering its shape or size and *without turning*. (Instead of saying that an object is *translated*, you can say that it is *shifted* or *displaced*.)



For example, the triangle ABC is translated to the triangle $A'B'C'$ in the xy-plane above. The translation is depicted by the vector $\vec{v}$.

Indeed, there is a one-to-one correspondence between translations and vectors. It may help you to understand better that a vector is everywhere if you imagine translating the entire plane with the vector $\vec{v}$, then drawing the infinite number of equal directed line segments showing the translation of every point in the plane.

Here, $\vec{v} = \overrightarrow{AA'} = \overrightarrow{BB'} = \overrightarrow{CC'}$, etc.

Thus, any directed line segment with the appropriate magnitude and direction will represent the vector correctly. This leads to the following definition of the equality of vectors.

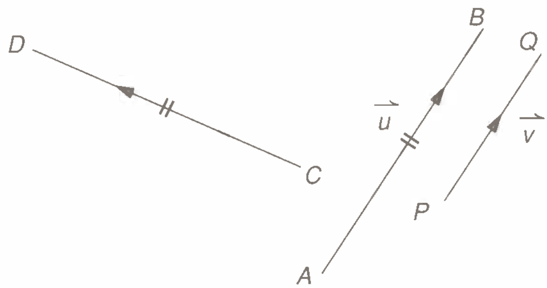
DEFINITION

Two vectors are equal if and only if they have the same magnitude and the same direction.

Alternatively,

two directed line segments that have the same length and the same direction represent the same vector.

Example Given $\overrightarrow{AB} = \vec{u}$, $|\overrightarrow{CD}| = |\vec{u}|$, $\overrightarrow{PQ} = \vec{v}$, with line segment PQ parallel to line segment AB, as shown.



Justify the following: a) $\overrightarrow{AB} \neq \overrightarrow{BA}$

b) $\overrightarrow{CD} \neq \vec{u}$

c) $\vec{u} \neq \vec{v}$

Solution

a) Although $|\overrightarrow{AB}| = |\overrightarrow{BA}|$, the direction of $\overrightarrow{AB}$ is opposite to the direction of $\overrightarrow{BA}$. Therefore the directions are not the same. Thus, $\overrightarrow{AB} \neq \overrightarrow{BA}$.

b) $\overrightarrow{CD}$ and $\vec{u}$ have different directions, thus are not equal.

c) $|\vec{u}| \neq |\vec{v}|$, that is, $\vec{u}$ and $\vec{v}$ have different lengths. Thus $\vec{u}$ and $\vec{v}$ are not equal. ■

SUMMARY

Any number of parallel lines with arrows pointing the same way define a particular direction.

A vector is everywhere; it can be represented by any directed line segment which has the correct magnitude and direction.

Equal vectors have the same magnitude and direction.

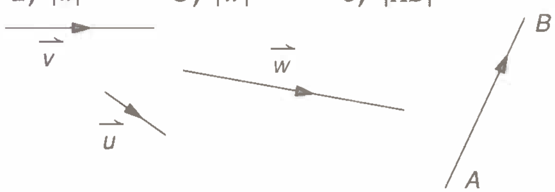
1.1 Exercises

1. Which of the following should be represented by vectors?
   a) weight
   b) frequency
   c) velocity
   d) volume (of a solid)
   e) volume (of a sound)
   f) age
   g) temperature change
   h) distance
   i) displacement
   j) capacity
   k) force
   l) acceleration

2. Most calculations done on a small scale, compared to the size of the earth, assume that the earth is flat.
   With that assumption, which of the following can be considered *directions*?
   a) east c) downtown
   b) up d) next door

   In reality, knowing the earth to be approximately spherical, which of the above are directions?

3. If $|\vec{v}| = 2$, state the value of the following.
   a) $|\vec{u}|$ b) $|\vec{w}|$ c) $|\overrightarrow{AB}|$

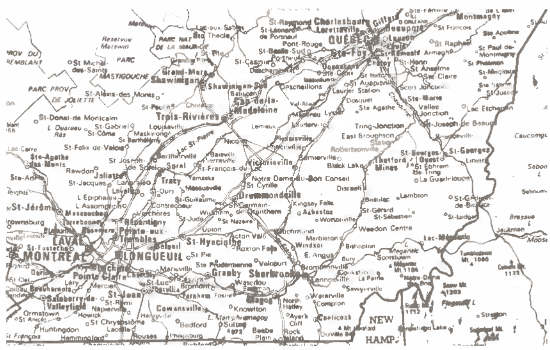


4. A directed line segment of length 0.5 cm pointing directly downwards represents a weight of 10 newtons (N). Use the same scale to draw directed line segments to represent the following.
   a) Juan is pushing upward with a force of 100 N.
   b) Helen is pulling a sled westward with a force of 80 N.
   c) The briefcase weighs 25 N.

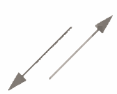
5. Given that $\overrightarrow{AB} = \vec{v}$ and $\overrightarrow{PQ} = \vec{v}$, what can you say about the line segments AB and PQ?

6. a) If $\vec{u} = \vec{v}$, does it follow that $|\vec{u}| = |\vec{v}|$? Explain.
   b) If $|\vec{u}| = |\vec{v}|$, does it follow that $\vec{u} = \vec{v}$? Explain.

7. The vector representing the displacement from Montréal to Québec can be described by its magnitude, namely 245 km, and its bearing, namely 041°. Describe in a similar manner
   a) the displacement from Québec to Montréal,
   b) the displacement from Québec to St Célestin, given that St Célestin is exactly halfway between Montréal and Québec.



8. State, with reasons, whether or not the following pairs of directed line segments represent the same vector.

a)

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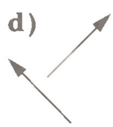
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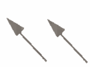
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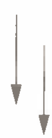
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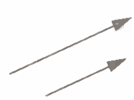
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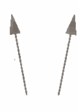
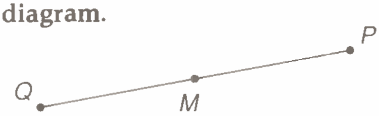
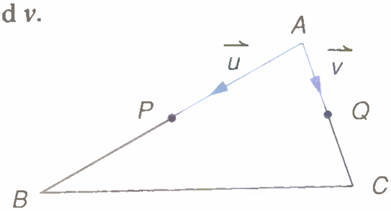


9. $ABCD$ is a square. State, with reasons, whether or not the following statements are true.
   a) $\overrightarrow{AC} = \overrightarrow{BD}$
   b) $\overrightarrow{AC} = \overrightarrow{DB}$
   c) $|\overrightarrow{AC}| = |\overrightarrow{BD}|$

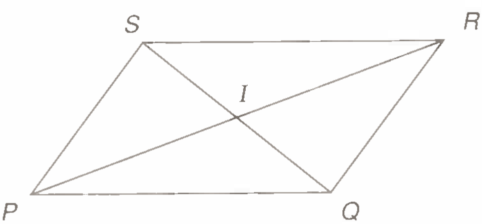
10. Given that M is the midpoint of segment PQ,
   a) give reasons why $\overrightarrow{QM} = \overrightarrow{MP}$
   b) state any other vector equality from the diagram.



11. In the triangle ABC, P is the midpoint of AB, and Q is the midpoint of AC. If $\overrightarrow{AP} = \vec{u}$ and $\overrightarrow{AQ} = \vec{v}$, express $\overrightarrow{PB}$ and $\overrightarrow{QC}$ in terms of $\vec{u}$ and $\vec{v}$.

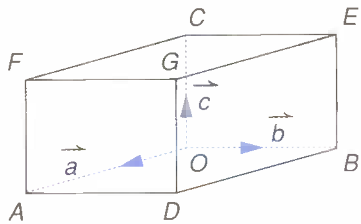


12. $PQRS$ is a parallelogram whose diagonals intersect at I. Assuming all the properties of a parallelogram, state, where possible, another vector equal to
   a) $\overrightarrow{PQ}$
   b) $\overrightarrow{PR}$
   c) $\overrightarrow{PI}$
   d) $\overrightarrow{RQ}$
   e) $\overrightarrow{IQ}$
   f) $\overrightarrow{SQ}$

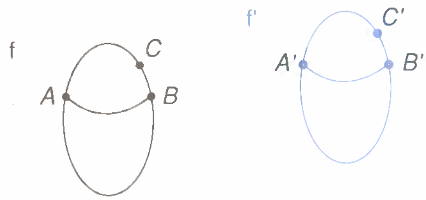


13. In question 12, if $|\overrightarrow{PQ}| = 5$, $|\overrightarrow{PS}| = 4$, and $|\overrightarrow{PI}| = 3$, state the value of each of the following.
   a) $|\overrightarrow{QP}|$
   b) $|\overrightarrow{QR}|$
   c) $|\overrightarrow{RI}|$

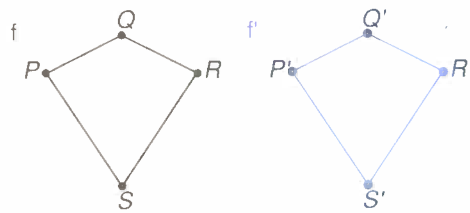
14. $OADBECFG$ is a rectangular solid where $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, and $\overrightarrow{OC} = \vec{c}$. Name all the other vectors equal to $\vec{a}$, $\vec{b}$, and $\vec{c}$.



15. For each of the following, name and draw three representatives of the vector depicting the translation from figure f to figure f′, where $A \rightarrow A'$, $B \rightarrow B'$, etc.
   a)



   b)

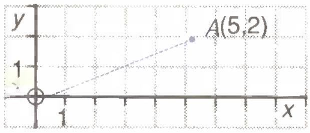


16. In question 15 a), what would the diagram look like if *all* the possible representatives of the translation vector where drawn?

1.2 Three-dimensional Space

Vectors can be represented by directed line segments. In two dimensions, a directed line segment drawn on a page represents a vector. In three dimensions, a vector could be modelled by a pencil held above your desk, with the point indicating the direction of the vector. To understand vectors better, you need familiarity with three-dimensional space, and with diagrams on a plane surface that represent three-dimensional objects. You must learn how to represent three-dimensional objects on paper, so as to produce a visual impression of the third dimension.

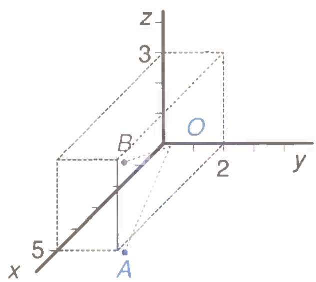
In two-dimensional coordinate geometry, or 2-space, you are accustomed to the *xy*-plane. The figure shows how the point $A = (5,2)$ is represented.



The *x* and *y* axes are both real number lines. Thus, the set of points in this plane can be called $\mathbb{R} \times \mathbb{R}$ or $\mathbb{R}^2$, that is, the set of all ordered pairs of real numbers.

In three-dimensional coordinate geometry, or 3-space, you use three mutually perpendicular axes *x*, *y*, and *z*. The *xy*-plane becomes part of the *xyz*-space. By convention, you put the *y* and *z* axes on the paper, and you try to give the impression of the *x*-axis rising out of the page at right angles to the paper.

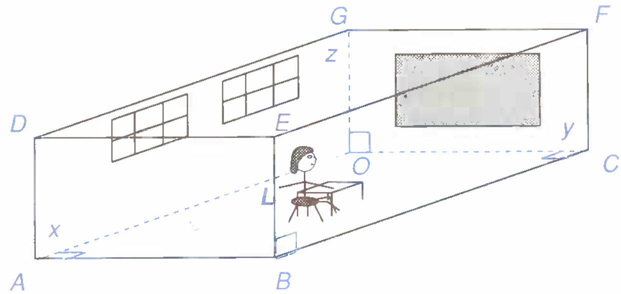
The coordinate system using the triple (x,y,z) is said to form a **right-handed system**. Use your right hand as a model. Stretch out the thumb and the first two fingers. Let the positive *x*-axis be represented by your thumb, and the positive *y*-axis by your first finger. Then the direction of the positive *z*-axis will be represented by your second finger.



If you followed the same instructions, using your left hand instead of the right, you would find the *z*-axis pointing in the *opposite* direction. That is why it is important to define the triple (x,y,z) in a foolproof way. If you use an (x,y,z) system, instead of saying that the point *A* of the *xy*-plane has coordinates $(x,y) = (5,2)$, you say that its coordinates are $(x,y,z) = (5,2,0)$. The point *B*, shown on the same diagram, has coordinates $(5,2,3)$.

3-space is not difficult to imagine, since it is all around you. In fact, any rectangular room, such as your classroom, gives you a good framework for imagining a 3-space coordinate system.

In the drawing shown, Maria is sitting at her desk, facing the blackboard. There are windows on the wall to her left. The corners are represented by the letters as shown. If you choose the point *O* as **origin**, then you could represent the *x*-axis by *OA*, the *y*-axis by *OC*, and the *z*-axis by *OG*.



In this figure, the floor *OABC* is the *xy*-plane; here, every point has coordinates $(x,y,0)$.

The blackboard wall *OCFG* is the *yz*-plane; points here have coordinates $(0,y,z)$.

The window wall *OADG* is the *zx*-plane; points here have coordinates $(x,0,z)$.

Once you have specified a unit of measurement along the axes, any point in the room can now be described by its three coordinates.

For example, suppose you choose the unit as 1 metre. Maria's left hand, *L*, is 3 m from the blackboard, 2 m from the window wall, and 1 m above the floor. Hence the coordinates of *L* are $(x,y,z) = (3,2,1)$.

The drawing of the classroom above illustrates the three essential rules of mathematical 3-dimensional drawing.

For a 3-dimensional mathematical drawing:

1. do not overlap distinct lines
2. keep verticals vertical
3. keep parallels parallel

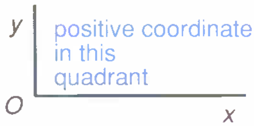
It is especially important to mark right-angles according to these rules. Observe how some of the right angles at *O*, *A*, *B* and *C* are marked in the figure. Incorrectly drawn right-angle markers can lead to very confusing 3-D drawings. Producing a good drawing is more of a challenge in 3-D than in 2-D. It requires practice, and sometimes more than one attempt.

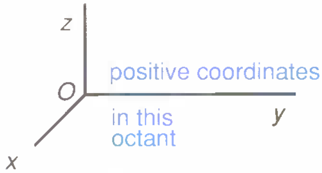
So far you have only looked at *positive values* along the axes of your 3-space. In the diagram above of a classroom, any point in the room could be specified by its three positive number coordinates.

How can you represent the points *outside* the room?

If you use negative numbers too, a 3-space coordinate system will allow you to specify *any* point in space, in the same way that a 2-space coordinate system allows you to specify *any* point in its plane.

The two-dimensional plane is divided by the axes into *four quadrants*. Points whose coordinates are all positive are found in only *one* of these quadrants.



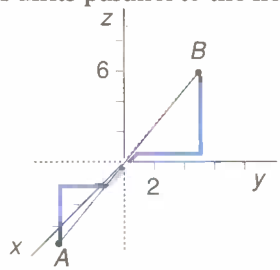


Three-dimensional space is divided into *eight octants* (from the Greek "octo", meaning eight). Points whose coordinates are all positive are found in only *one* of these octants.

The set of all points in 3-space, regardless of the signs of their coordinates, is called $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ or $\mathbb{R}^3$.

Example Plot the points $A = (2,-3,-4)$ and $B = (-1,5,6)$ in a 3-space coordinate system, and draw the segment AB.

Solution To locate A, proceed 2 units along the positive x-axis, then 3 units parallel to the negative y-axis, then 4 units parallel to the negative z-axis.



B is located in a similar fashion.
The points A and B can be joined to obtain the segment AB. ■

Note: From the diagram, it looks as if AB passes through O. However, this is *not* the case; but this does show that you can sometimes have difficulty in interpreting a 3-dimensional drawing. You must beware of the pitfalls of a drawing in perspective.

You cannot yet prove that AB does not pass through O. However, the methods shown in chapter 5 will allow you to do that.

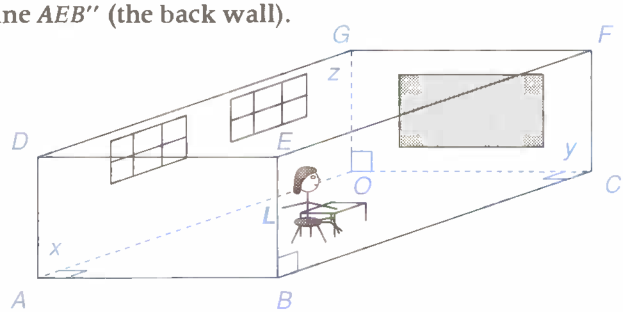
General 3-space Concepts

Look again at your classroom. Without making any specific reference to the 'origin' and the 'axes', you can discover other important facts about lines and planes in 3-space.

Points

You know that any two distinct points determine a straight line. Similarly, any *three distinct points*, not all on a single straight line, determine a *plane*.

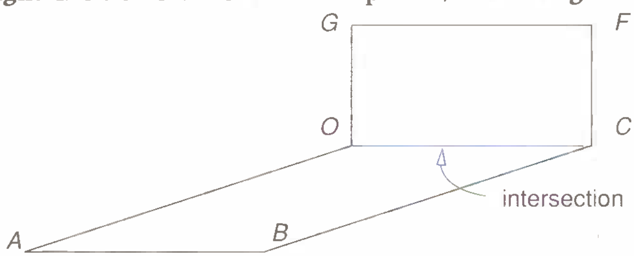
In the classroom drawn below, whenever you name any two points, you are identifying a *unique line* containing those two points:
"the line *AE*" (a diagonal across the back wall).
Whenever you mention three points, you are identifying a *unique plane* containing those three points:
"the plane *AEB*" (the back wall).



Intersections of Planes

You can see that the plane of the ceiling (*EFG* or *DEFG*) and the plane of the floor (*OABC*) *never meet*. These are *parallel planes*.

However, the floor and the front wall (*OCFG*) do intersect. The points of the straight line *OC* lie in both of these planes, as the diagram shows.



Two planes are either parallel, or they intersect in a straight line.

Since any straight line can be the intersection of two planes, the following converse is true.

Any straight line is contained in an infinite number of planes.

Intersections of Lines

In the drawing of the classroom, the lines *BC* and *CF* meet at the point *C*. Note that they are in the same plane *BCFE*.

The lines *BC* and *EF* are parallel—they never meet.

Although the lines *BC* and *OG* never meet, they do not seem to fit the notion of 'parallelism'. Indeed, they are not parallel, and they do not intersect. Such lines in space are called **skew lines**. You can draw skew lines as follows.

Two distinct *parallel* lines in 3-space *never meet, but are in the same plane*. This is not the case for *BC* and *OG*, so they are skew. On the other hand, the lines *BC* and *EF*, which also never meet, *are* in the same plane *BCFE*, so they are parallel.
Given two distinct lines, L_1 and L_2, in 3-space, there are three possibilities.

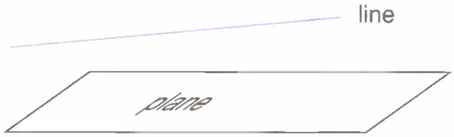
1. L_1, L_2 meet; therefore they define a plane. They are coplanar.
2. L_1, L_2 are parallel; therefore they do not meet, but they are coplanar.
3. L_1, L_2 are skew; they do not meet, and they are not coplanar.

Intersections of Lines and Planes

In the drawing of the classroom, the line *CF* meets the floor *OABC* at the point *C*.

In general, a line intersects a plane in a single point. The diagram shows the line *L* crossing the plane Π at the point *A*. (The dotted line indicates that part of *L* which is *behind* Π.) A line perpendicular to a plane is perpendicular to all the lines in that plane.

Some lines and planes never meet. For example, the lines *EF*, *ED*, *DF* never meet the plane of the floor. In these cases, the line is said to be *parallel* to the plane.

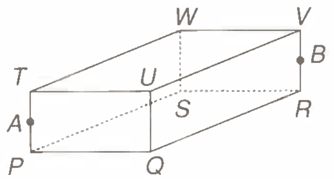


SUMMARY

Three distinct points, not on the same straight line, determine a plane.
Two planes are either parallel, or intersect in a straight line.
Two lines which are neither parallel nor intersecting are skew.
A line is either parallel to a plane, contained in the plane, or intersects the plane in a single point.

1.2 Exercises

(The diagram is to be used to answer questions 1-4.)



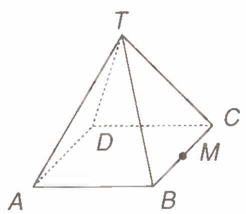
1. Above is a drawing of a rectangular box *PQRSWTUV*. *A* is the midpoint of segment *PT*, and *B* is the midpoint of segment *RV*. State two lines parallel to each of the following.
   a) *TW* b) *WS* c) *PR*

2. State two lines perpendicular to each of the following.
   a) *TW* b) *WS* c) *AB*

3. State whether the following pairs of lines are parallel, intersecting, or skew.
   a) *TQ* and *WR*
   b) *TQ* and *SV*
   c) *PW* and *QW*
   d) *PV* and *QS*
   e) *PV* and *QW*
   f) *AB* and *SQ*

4. Copy the above drawing and join the line *AB*. Does the line *AB* really pass through *S*? What could you do to enhance the three-dimensional drawing, indicating precisely whether or not *S* is on the line *AB*?

5. *ABCDT* is a right square pyramid. It is called "square" because it has a square base *ABCD*, and "right" because its apex, *T*, is vertically above the centre of the base.



   a) Draw the pyramid, and locate the centre *O* of its base (the intersection of the diagonals *AC* and *DB*).
   b) Join *TO*, and mark the right angles *TOA* and *TOD*.
   c) Given that *M* is the midpoint of *BC*, join *OM*, and *MT*. Mark the right angle in the triangle *TOM*.

6. In question 5, if *AB* = 6 cm, and *TO* = 4 cm, use the theorem of Pythagoras to calculate the exact value of the lengths of *TM* and *TB*.

7. In question 5, name the following.
   a) two skew lines
   b) the three planes intersecting at point *B*

8. Plot the following points in $\mathbb{R}^3$.

   $A(1,1,1)$
   $B(2,0,0)$
   $C(0,3,0)$
   $D(0,0,-1)$
   $E(2,0,-3)$
   $F(0,-2,5)$
   $G(-3,-3,-3)$
   $H(1,2,-5)$
   $J(-1,2,5)$

9. $\mathbb{R}^3$ is divided into eight octants. State the signs of the coordinates (x,y,z) of a point in each octant.

10. State the condition that must be satisfied by the coordinates (x,y,z) of a point positioned as follows.
    a) on the x-axis
    b) on the y-axis
    c) on the z-axis

11. State the condition that must be satisfied by the coordinates (x,y,z) of a point positioned as follows.
    a) in the xy-plane
    b) in the yz-plane
    c) in the zx-plane

12. How would you describe the set of points $P(x,y,z)$, given that $x = y$? Draw this set in $\mathbb{R}^3$.

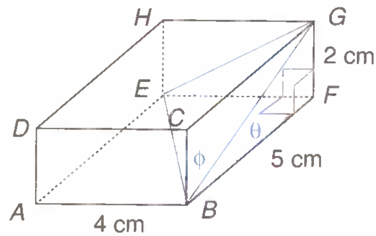
In Search of Trigonometry as an Aid To Visualization of 3-Space

The use of trigonometry to solve triangles that occur in three-dimensional situations should help you to visualize and familiarize yourself with 3-space concepts. To solve triangles that are not right-angled, you can use the sine law and the cosine law. You will find these described on page 542.

Examine the following examples and their diagrams very carefully. The examples should help you to grasp some intuitive notions about angles in 3-space.

Example 1 Given the rectangular box shown, find, correct to the nearest degree, the angle between

a) the lines *BG* and *BF*
b) the lines *BG* and *BE*
c) the line *BH* and the plane *ABFE*
d) the planes *ABGH* and *ABFE*.



Solution

a) Since *F* is a right angle, you can apply trigonometry to the right triangle *BFG*. The angle required is θ where $\tan\theta = \frac{GF}{BF} = \frac{2}{5} = 0.4$ hence $\theta \doteq 22°$.

b) The angle ϕ between *BG* and *BE* is in triangle *BGE* which is not a right triangle. You can use the cosine law to find ϕ, if you can determine the sides *BG*, *BE* and *GE*.

BG is the hypotenuse of the triangle you used in part a), that is, triangle *BFG*, thus $BG^2 = BF^2 + FG^2 = 5^2 + 2^2 = 29$, so $BG = \sqrt{29}$.

Also, the triangle *BFE* (on the 'floor') is right-angled at *F*, as is the triangle *EFG* (on the 'front wall'). You can find the lengths *BE* and *EG* in the same way.

$BE^2 = BF^2 + FE^2 = 5^2 + 4^2 = 41$, so $BE = \sqrt{41}$

$GE^2 = GF^2 + FE^2 = 2^2 + 4^2 = 20$, so $GE = \sqrt{20}$.

Finally, you can apply the cosine law to the triangle *BGE* to find the angle ϕ.

$$GE^2 = BE^2 + BG^2 - (2)(BE)(BG)\cos\phi$$

$$20 = 41 + 29 - 2\sqrt{41}\sqrt{29}\cos\phi$$

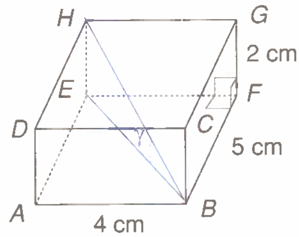
hence $$\cos\phi = \frac{(41 + 29 - 20)}{2\sqrt{41}\sqrt{29}} = 0.725\ 018\ldots,$$

giving $$\phi \doteq 44°.$$

c) If you hold your pencil against the paper you are writing on, you will notice that there are many different angles between the pencil and the paper. Similarly, there are many different possible angles between *BH* and the 'floor' *ABFE*. It depends on which line from *B* you choose in the floor!

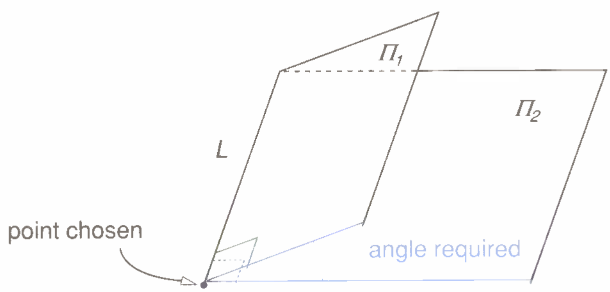
The *angle between a line and a plane* means *the smallest possible angle* between the line and the plane. This angle will be found between the line and its perpendicular projection on the plane. In our example, it is the angle γ between *BH* and *BE*. Note that *BEH* is a right triangle. Thus

$\tan\gamma = \frac{EH}{BE} = \frac{2}{\sqrt{41}} = 0.3123\ldots$, so $\gamma \doteq 17°$.



(In the activities you will be calculating the angle between *BH* and *BF*, and the angle between *BH* and *BA*. You will find that both of these are greater than 17°.)

d) The angle between two planes Π_1 and Π_2 can be found as follows.
1 Find the line of intersection, *L*, of the two planes.
2 Choose a point on *L*, that you will use to find
a line perpendicular to *L*, in Π_1, and a line perpendicular to *L*, in Π_2.
3 The angle between those two lines is the required angle.

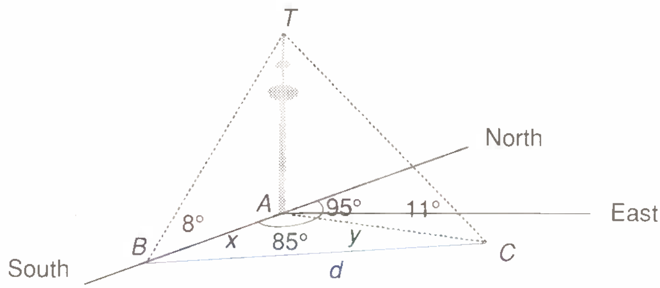


In Example 1, the line of intersection of the planes is *AB*. If you choose *B* as the point on this line, *BF* and *BG* are lines which obey the above criteria. Thus the angle required is θ, calculated in a) as 22°. ■

Example 2 The CN tower in Toronto is 553 m high. A person in a boat on Lake Ontario, at a point B due south of the tower, observes the top T of the tower at an angle of elevation of 8°. At the same time, another person, at a point C on a bearing 095° from the tower, observes the angle of elevation of T to be 11°.

Calculate the distance BC between the two people, correct to 3 significant figures.

Solution Your diagram must indicate bearings at ground level, as well as a vertical tower. Look carefully at the features of the diagram.



Denote the distance of the first person from the base of the tower by x, and the distance of the second person from the base of the tower by y.

From right triangles BAT and CAT respectively,

$\tan 8° = \dfrac{553}{x}$ and $\tan 11° = \dfrac{553}{y}$

thus $x = \dfrac{553}{0.1405\ldots} \doteq 3935$ m, and $y = \dfrac{553}{0.1943\ldots} \doteq 2945$ m.

Note: The distances x and y are written here to 4 significant digits for clarity of reading. You should retain the actual values using the full accuracy of your calculator for subsequent use of x and y.

You can now find the distance d between the people by using the cosine law in the surface triangle BAC. Note that the angle opposite d is $180° - 95° = 85°$.

$d^2 = x^2 + y^2 - (2)(x)(y) \cos 85°$

$= 21\,625\,032.67$ using full calculator accuracy

so $d \doteq 4650$ correct to 3 significant digits

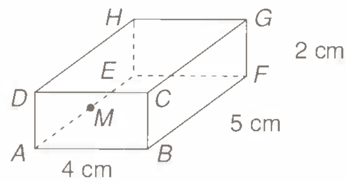
Thus the distance between the two people is about 4650 m. ■

SUMMARY

The angle between a line and a plane is the angle between the line and its perpendicular projection on the plane.

The angle between two planes Π_1 and Π_2 is the angle between a line in Π_1 and a line in Π_2, each line chosen to be perpendicular to the line of intersection of the planes.

Activities



Give all angles correct to nearest degree, and all lengths correct to 3 significant digits.

1. Given the rectangular box shown, where *M* is the midpoint of *AE*, find the following.
   a) the angle α between the lines *BH* and *BF*
   b) the angle β between the lines *BH* and *BA*
   c) the angle γ between the lines *BH* and *BM*

Note: All your answers should be greater than 17°, which is the value of the angle between *BH* and *BE*, that is, the angle between *BH* and the plane *ABFE*.

2. Given a right pyramid *ABCDT*, on a square base *ABCD*, with $AB = 6$ cm, and height $TO = 4$ cm, calculate the following.
   a) the angle *TAB*
   b) the angle θ between a slant edge (such as *TB*) and the base *ABCD*
   c) the angle ϕ between a slant face (such as *TBC*) and the base *ABCD*.

3. A plane sheet of plywood measuring 1.2 m by 2.4 m is inclined with its shorter edge at an angle of 30° to the horizontal.
   a) How high is the top edge of the plywood?
   b) Calculate the angle between a diagonal of the sheet of plywood and the horizontal.

4. A woman on a frozen lake observes the top of a radio tower, due north of her, at an angle of elevation of 21°. She then skis for 500 m on a bearing 065°, and finds herself due east of the tower. Calculate the following.
   a) the height of the radio tower
   b) the angle of elevation of the top of the tower from the second point of observation

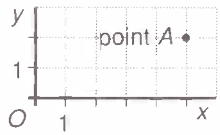
5. A flagpole is placed at one corner of a courtyard 25 m long and 20 m wide. The angle of elevation of the top of the flagpole from the opposite corner of the courtyard is 18°. Calculate the height of the flagpole, and the angles of elevation of the top of the flagpole from the other two corners of the courtyard.

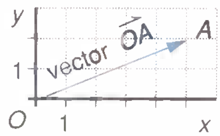
6. The angles of elevation of the top *T* of a vertical post *TO* are observed to be α and β from points *A* and *B* due west and due north of the post. If the distance $AB = d$, show that the height of the post is

$$\frac{d}{\sqrt{(\cot^2 \alpha + \cot^2 \beta)}}$$

1.3 Vectors as Ordered Pairs or Triples

A point in a plane coordinate system can be represented by an ordered pair of numbers. In the diagram the point A has coordinates (5,2).





Draw the vector $\overrightarrow{OA}$.
This vector, represented by a directed line segment joining the origin O to a point A, is called the **position vector** of point A. Recall from section 1.1 that vectors can also be represented by two (or more) numbers; here, you could represent vector $\overrightarrow{OA}$ by the same ordered pair, (5,2), as point **A**. *But beware*: it is very important to distinguish between *vectors* and *points*. When writing vectors as ordered pairs, an **arrow notation** will therefore be used, as follows: vector $\overrightarrow{OA} = \overrightarrow{(5,2)}$, as distinct from the point $A = (5,2)$.
The entries for the vector $\overrightarrow{OA}$ are called *components*.
The entries for the point A are called *coordinates*.

In some texts, vectors as ordered pairs are written as columns, in order to be distinguished from points, as follows: $\overrightarrow{OA} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

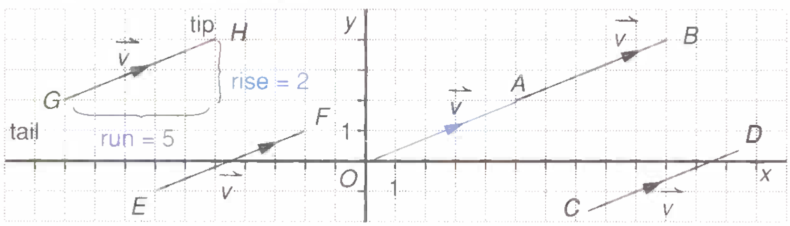
Indeed, you will be using this notation for vectors in chapters 7 and 8.

Other texts use square brackets for vectors as ordered pairs. Yet others make no distinction between representing points or vectors. In these texts, the reader must be vigilant as to which is which, noting the distinction from the context.

Unfortunately, there is no standard notation. As a student of mathematics, you should familiarize yourself with the different representations in use.

The vector $\vec{v}$, equal to the position vector $\overrightarrow{OA}$, can be drawn wherever you like. In other words, it can be represented by any directed line segment parallel to OA, pointing the same way as OA, and congruent to OA.

$\vec{v} = \overrightarrow{OA} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{GH} = \overrightarrow{AB} = \overrightarrow{(5,2)}$



In each of these cases, the run from the tail of the vector to the tip is 5, and the corresponding rise is 2.

Because vectors can be drawn anywhere, you can draw a vector expressed as an ordered pair using grid lines only, without an x-axis, a y-axis, or an origin. The diagram shows the following vectors:

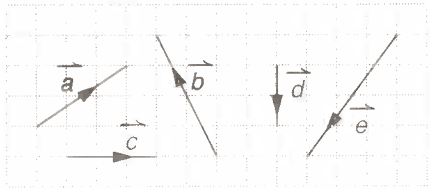
$\vec{a} = \overrightarrow{(3,2)}$

$\vec{b} = \overrightarrow{(-2,4)}$

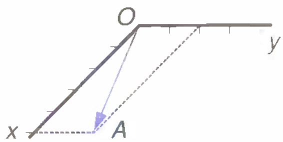
$\vec{c} = \overrightarrow{(3,0)}$

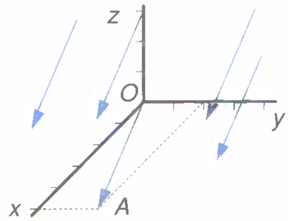
$\vec{d} = \overrightarrow{(0,-2)}$

$\vec{e} = \overrightarrow{(-3,-4)}$



A corresponding result holds in 3-space. Draw the xy-plane in perspective, with the x-axis coming out of the page, at right angles to it. Again, draw the vector $\vec{v} = \overrightarrow{OA}$.





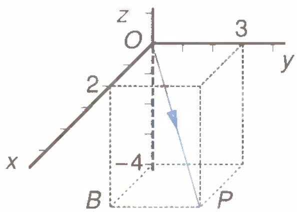
Because a vector can be drawn where you want, as long as it has the correct magnitude and direction, the vector $\vec{v}$ can be represented by directed line segments hovering *above* or *below* the plane, as you can see from the second diagram.

If you now think of A as a point in 3-space, you would call A by its *three* coordinates (x,y,z), thus: $A = (5,2,0)$.

The vector $\overrightarrow{OA}$, which is the position vector of the point A, would be represented by an *arrowed ordered triple*, thus: $\overrightarrow{OA} = \overrightarrow{(5,2,0)}$

The coordinates of point $P(2,3,-4)$ in the figure are the numbers 2, 3, and -4.

The components of vector $\overrightarrow{OP} = \overrightarrow{(2,3,-4)}$ are also the numbers 2, 3 and -4.



In general, the point $P(a,b,c)$, defined in a 3-space coordinate system with origin O, has position vector $\overrightarrow{OP} = \overrightarrow{(a,b,c,)}$.

Similarly, the point $P(a,b)$, defined in a 2-space coordinate system with origin O, has position vector $\overrightarrow{OP} = \overrightarrow{(a,b)}$.

The abbreviations $\overrightarrow{OP} = \vec{p}$, $\overrightarrow{OQ} = \vec{q}$ etc., are often used in problems that involve the position vectors of different points.

The set of two-dimensional vectors, or vectors in 2-space, will be designated by $\mathbb{V}_2$.

The set of three-dimensional vectors, or vectors in 3-space, will be designated by $\mathbb{V}_3$.

Equality of Vectors

You saw in section 1.1 that vectors are equal if and only if they have the same magnitude and direction. Thus, two equal vectors can be represented by a directed line segment from the origin to the same point, and hence by the same ordered pair. This leads to the following.

In $\mathbb{V}_2$, $\overrightarrow{(a,b)} = \overrightarrow{(r,s)}$ if and only if $a = r$ and $b = s$

Similarly, in $\mathbb{V}_3$,

$$\overrightarrow{(a,b,c)} = \overrightarrow{(r,s,t)} \text{ if and only if } a = r \text{ and } b = s \text{ and } c = t$$

Length of a Vector

The length of a vector is defined as the length of a directed line segment which represents the vector.

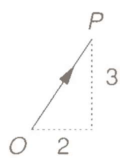
Note: The length of a vector is sometimes called its "magnitude" or its "norm."

First, look at an example in 2-space.

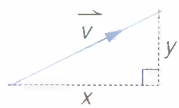
Example 1 Find the length of the vector $\vec{v} = \overrightarrow{(2,3)}$.

Solution $\vec{v}$ can be represented by $\overrightarrow{OP}$, the position vector of the point $P(2,3)$.

You need to find $|\overrightarrow{OP}|$ = the length of line segment OP.
By using the theorem of Pythagoras,
$|\vec{v}|^2 = |\overrightarrow{OP}|^2 = 2^2 + 3^2$, so
$|\vec{v}| = |\overrightarrow{OP}| = \sqrt{2^2 + 3^2} = \sqrt{13}$.

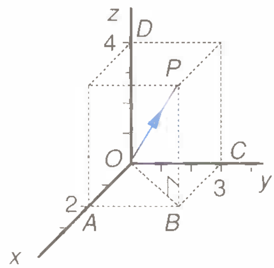


In general, in $\mathbb{V}_2$,
if $\vec{v} = \overrightarrow{(x,y)}$, then $|\vec{v}| = \sqrt{x^2 + y^2}$



A similar result is true for the length of a vector in 3-space.

Example 2 Find the length of the vector $\vec{p} = \overrightarrow{(2,3,4)}$.



Solution $\vec{p}$ can be represented by $\overrightarrow{OP}$, the position vector of the point $P(2,3,4)$.

In the diagram, $OA = CB = 2$, $OC = AB = 3$, and $OD = BP = 4$. You need to find $|\vec{p}| = |\overrightarrow{OP}| =$ length of segment OP.

From the right triangle OBP, $\quad OP^2 = OB^2 + BP^2$

but from the right triangle OAB, $OB^2 = OA^2 + AB^2$

$\Rightarrow \quad OP^2 = OA^2 + AB^2 + BP^2$

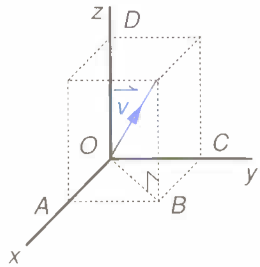
and since $BP = OD$, $AB = OC$, then $\quad OP^2 = OA^2 + OC^2 + OD^2$

or $\quad |\overrightarrow{OP}|^2 = 2^2 + 3^2 + 4^2 = 29$

thus $\quad |\vec{p}| = |\overrightarrow{OP}| = \sqrt{29}$. ■

In general, if $\overrightarrow{OP} = \overrightarrow{(x,y,z)}$, then $OA = x$, $OB = y$, and $OC = z$; you have the following result in $\mathbb{V}_3$.

If $\vec{v} = \overrightarrow{(x,y,z)}$, then $|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$



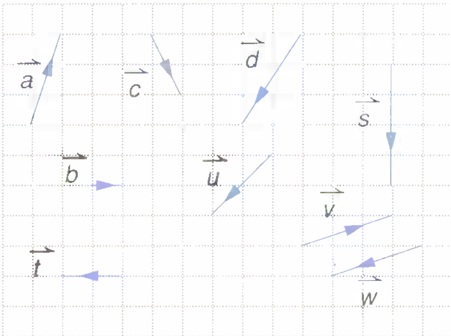
SUMMARY

If P is a point in a coordinate system of origin O, then $\overrightarrow{OP}$ is called the *position vector* of P.

in 2-space	*in 3-space*
If $P = (a,b)$, then $\overrightarrow{OP} = \overrightarrow{(a,b)}$	If $P = (a,b,c,)$, then $\overrightarrow{OP} = \overrightarrow{(a,b,c)}$
Vectors $\overrightarrow{(a,b)} = \overrightarrow{(r,s)}$ if and only if the numbers $a = r$ and $b = s$	Vectors $\overrightarrow{(a,b,c,)} = \overrightarrow{(r,s,t)}$ if and only if the numbers $a = r$, $b = s$ and $c = t$
The length of $\vec{v} = \overrightarrow{(x,y)}$ is $|\vec{v}| = \sqrt{x^2 + y^2}$	The length of $\vec{v} = \overrightarrow{(x,y,z)}$ is $|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$

1.3 Exercises

1. Write the vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{s}, \vec{t}, \vec{u}, \vec{v}, \vec{w}$ represented below, as ordered pairs.



2. On a grid, draw six directed line segments representing the vector $\vec{u} = \overrightarrow{(6,2)}$.

3. Repeat question 2 for the vector $\vec{v} = \overrightarrow{(-3,-5)}$.

4. Use a grid to draw the vectors $\vec{a} = \overrightarrow{(3,4)}$, $\vec{b} = \overrightarrow{(4,-1)}$, $\vec{c} = \overrightarrow{(-2,5)}$, and $\vec{d} = \overrightarrow{(-1,-1)}$.

5. Calculate the lengths of the vectors $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$ of question 4.

6. a) Given the point $P(4,-3)$, draw the position vector $\overrightarrow{OP}$.

b) If $\overrightarrow{OQ} = \overrightarrow{(-1,7)}$, state the coordinates of the point Q.

c) If $\overrightarrow{OR} = \overrightarrow{(0,2,-2)}$, state the coordinates of the point R.

7. In a 3-space coordinate system, draw the position vector $\vec{p}$ of the point $P(2,3,5)$.

8. a) On a grid, draw points A, B, and C such that $\overrightarrow{AB} = \overrightarrow{(4,3)}$ and $\overrightarrow{BC} = \overrightarrow{(1,-5)}$.

b) Use your drawing to express $\overrightarrow{AC}$ as an ordered pair.

c) Conjecture a rule by which the components of $\overrightarrow{AC}$ can be obtained from the components of $\overrightarrow{AB}$ and of $\overrightarrow{BC}$.

9. A point P, whose position vector is $\overrightarrow{OP} = \overrightarrow{(1,2)}$, is translated to position P' according to the vector $\vec{v} = \overrightarrow{(4,1)}$. What are the coordinates of P'?

10. Show that the length of $\vec{v} = \overrightarrow{\left(\frac{3}{5}, -\frac{4}{5}\right)}$ is 1 unit. ($\vec{v}$ is known as a **unit vector**.)

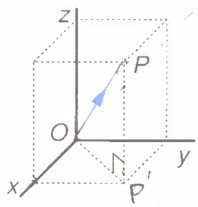
11. Given that $\overrightarrow{OP} = \overrightarrow{(2,x)}$ and that $|\overrightarrow{OP}| = 5$, calculate x.

12. Given that $\vec{u} = \overrightarrow{(2,3)}$ and $\vec{v} = \overrightarrow{(n,-1)}$, calculate the following.

a) $|\vec{u}|$

b) n, given that $|\vec{u}| = |\vec{v}|$

13. The position vector of P is $\overrightarrow{OP} = \overrightarrow{(x,y,z)}$. Show that $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$.



14. In a certain city the blocks are 100 m square. You walk, from a point O, 4 blocks east, then 3 blocks north, to arrive at a point P.

a) How far have you walked?

b) What is the (direct) distance from O to P?

15. The vectors $\vec{p} = \overrightarrow{(3,x)}$ and $\vec{q} = \overrightarrow{(w,-6)}$ are equal. State the values of x and w.

16. The vectors $\vec{a} = \overrightarrow{(2,-1,k)}$ and $\vec{b} = \overrightarrow{(m,n,7)}$ are equal. State the values of k, m and n.

17. $\vec{u} = \overrightarrow{(2h+k,-3)}$, and $\vec{v} = \overrightarrow{(4,h+k)}$. Given that $\vec{u} = \vec{v}$, calculate the values of h and k.

18. Use a 2-space coordinate system to find the vector $\overrightarrow{PQ}$ given the following points.

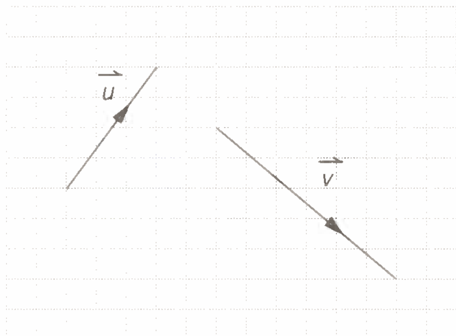
a) $P(0,0)$, $Q(3,2)$,
b) $P(3,2)$, $Q(0,0)$,
c) $P(3,2)$, $Q(4,7)$
d) $P(3,2)$, $Q(-1,2)$
e) $P(5,-2)$, $Q(-1,-3)$
f) $P(a,b)$, $Q(c,d)$

1.4 Vector Addition

You know that vectors can be represented by directed line segments, or by ordered pairs or ordered triples of numbers. Vectors will be more useful to you once you can combine them according to certain operations.

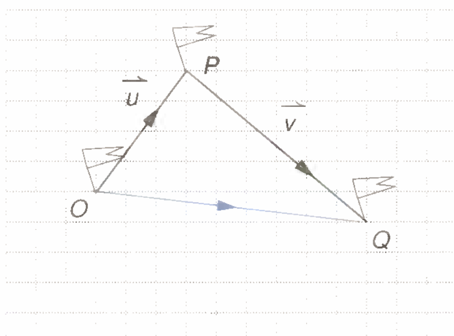
The first operation you will learn will be vector addition. Recall that there is a one-to-one correspondence between translations and vectors.

To find a definition for the addition of two vectors, consider two translations, represented by vectors $\vec{u}$ and $\vec{v}$, performed in succession. (This is sometimes called the **composition** of two translations.)



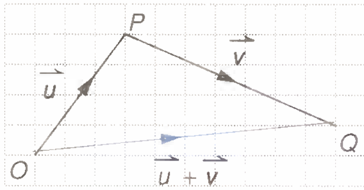
Recall that a vector can be drawn anywhere. To find a vector that represents the result of performing these translations in succession, let $\vec{u} = \overrightarrow{OP}$ and $\vec{v} = \overrightarrow{PQ}$. Then

the translation from O to P (vector $\vec{u} = \overrightarrow{OP}$)
followed by
the translation from P to Q (vector $\vec{v} = \overrightarrow{PQ}$)
gives
a "resultant" translation from O to Q (vector $\overrightarrow{OQ}$).



This idea suggests the following definition of **vector addition** for vectors represented by directed line segments.

DEFINITION Given any three points O, P, and Q: $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$



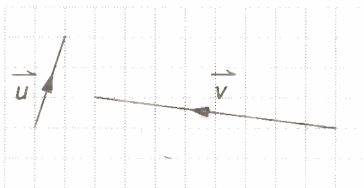
This definition is also called the **triangle law of vector addition.**

This law states that, if two vectors are represented by directed line segments such that the tail of the second is the same point as the tip of the first, then the directed line segment from the tail of the first to the tip of the second represents their vector sum.

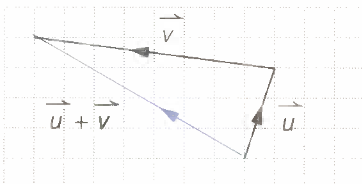
Note 1 Because a triangle is always in a *plane*, the triangle law works in $\mathbb{V}_3$ as well.

2 If you consider the vectors as displacements, then the resultant or sum vector is the *short cut* from the initial point to the final destination.

Example 1 Given the vectors $\vec{u}$ and $\vec{v}$ shown, draw a directed line segment representing the vector $\vec{u} + \vec{v}$.



Solution You must draw at least one of the vectors again, so that the tip of $\vec{u}$ is coincident with the tail of $\vec{v}$. (It *does not matter where* you draw them).



■

The following example will suggest a rule for vector addition for vectors represented by ordered pairs.

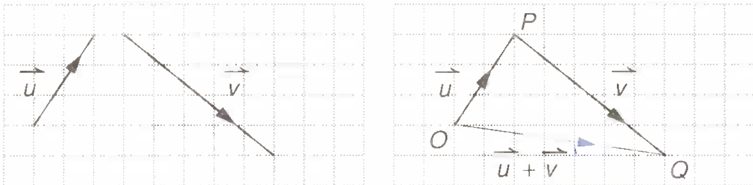
Example 2 Given the vectors $\vec{u} = \overrightarrow{(2,3)}$ and $\vec{v} = \overrightarrow{(5,-4)}$.

a) Use a grid to draw $\vec{u}$ and $\vec{u}$ wherever you wish; then draw them again so that $\overrightarrow{OP} = \vec{u}$ and $\overrightarrow{PQ} = \vec{v}$.

b) Express $\overrightarrow{OQ} = \vec{u} + \vec{v}$ as an ordered pair.

c) State an algebraic relationship among $\vec{u}$, $\vec{v}$, and $\vec{u} + \vec{v}$.

Solution a)



b) $\overrightarrow{OQ} = \vec{u} + \vec{v} = \overrightarrow{(7,-1)}$

c) $\vec{u} + \vec{v} = \overrightarrow{(2,3)} + \overrightarrow{(5,-4)} = \overrightarrow{(2+5,3+(-4))} = \overrightarrow{(7,-1)}$. ■

It appears that the resultant vector can be obtained in component form by adding the first components of $\vec{u}$ and $\vec{v}$, then the second components of $\vec{u}$ and $\vec{v}$.

This leads to the following definition of vector addition in $\mathbb{V}_2$, for vectors represented by ordered pairs.

DEFINITION $\overrightarrow{(a,b)} + \overrightarrow{(p,q)} = \overrightarrow{(a+p,b+q)}$

Similarly, the definition of vector addition in $\mathbb{V}_3$ for vectors in component form is as follows.

DEFINITION $\overrightarrow{(a,b,c)} + \overrightarrow{(p,q,r)} = \overrightarrow{(a+p,b+q,c+r)}$

Example 3 Given $\vec{p} = \overrightarrow{(-5,6)}$, $\vec{q} = \overrightarrow{(4,3)}$, and $\vec{u} = \overrightarrow{(2,3,4)}$, $\vec{w} = \overrightarrow{(-1,1,5)}$ find

a) $\vec{p} + \vec{q}$,

b) $\vec{u} + \vec{w}$.

Solution

a) $\vec{p} + \vec{q} = \overrightarrow{(-5,6)} + \overrightarrow{(4,3)}$
$= \overrightarrow{(-5+4,6+3)}$
$= \overrightarrow{(-1,9)}$

b) $\vec{u} + \vec{w} = \overrightarrow{(2,3,4)} + \overrightarrow{(-1,1,5)}$
$= \overrightarrow{(2+(-1),3+1,4+5)}$
$= \overrightarrow{(1,4,9)}$ ■

Example 4 An object is displaced according to the vector $\vec{u} = \overrightarrow{(4,7)}$, then according to the vector $\vec{v} = \overrightarrow{(1,-6)}$. What is the resultant displacement of the object?

Solution The resultant displacement is given by $\vec{u} + \vec{v}$

$\vec{u} + \vec{v} = \overrightarrow{(4,7)} + \overrightarrow{(1,-6)} = \overrightarrow{(4+1,7-6)} = \overrightarrow{(5,1)}$. ■

Example 5 Given $\vec{u} = \overrightarrow{(2,-1,-6)}$, $\vec{v} = \overrightarrow{(2,5,x)}$, and $\vec{u} + \vec{v} = \overrightarrow{(y,4,-3)}$, calculate the values of x and y.

Solution $\vec{u} + \vec{v} = \overrightarrow{(2,-1,-6)} + \overrightarrow{(2,5,x)} = \overrightarrow{(2+2,-1+5,-6+x)} = \overrightarrow{(4,4,-6+x)}$
but $\vec{u} + \vec{v} = \overrightarrow{(y,4,-3)}$,
so $\overrightarrow{(4,4,-6+x)} = \overrightarrow{(y,4,-3)}$.

Thus, from the definition of equality of vectors,
$y = 4$, and $-6 + x = -3$ so that $x = 3$. ■

Note: The calculation of the middle component, 4, confirms that the arithmetic is correct.

A mathematical object that can be represented by directed line segments, or by ordered pairs or ordered triples of numbers, is a vector provided that it obeys the laws of addition defined above.

Such a vector is the ordered triple used to describe the monthly sales of a realtor, for example,
$\vec{v} = \overrightarrow{(h,c,b)}$, where h = number of houses sold
c = number of condominiums sold
b = number of business locations sold.

Example 6 Suppose a real estate agent, Jessie LaRue, made the following sales in the winter of 1989:
in January, $\vec{j} = \overrightarrow{(2,1,1,)}$,
in February, $\vec{f} = \overrightarrow{(1,3,0)}$.

She then left town for her annual holiday for the next three months, so she made no further sales. Indicate, by means of a vector, how many properties of each type she sold during the winter period.

Solution $\vec{w} = \overrightarrow{(2+1,1+3,1+0)} = \overrightarrow{(3,4,1)}$
Thus she sold 3 houses, 4 condominiums and 1 business location. ■

SUMMARY

Component laws of vector addition:

in $\mathbb{V}_2$: $\overrightarrow{(a,b)} + \overrightarrow{(p,q)} = \overrightarrow{(a+p,b+q)}$

in $\mathbb{V}_3$: $\overrightarrow{(a,b,c)} + \overrightarrow{(p,q,r)} = \overrightarrow{(a+p,b+q,c+r)}$

Geometric law of vector addition (the triangle law):
$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$

Vectors are mathematical objects that may be represented either

by directed line segments that combine according to the triangle law of addition, or

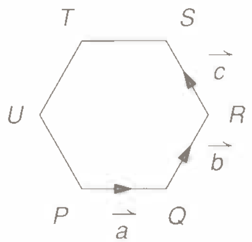
by ordered pairs or ordered triples of numbers, that combine by the addition of components.

1.4 Exercises

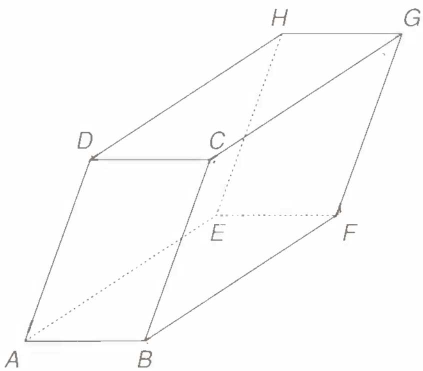
1. Given the parallelogram $PQRS$, use the triangle law of vector addition to simplify
   a) $\vec{PQ} + \vec{QR}$, b) $\vec{PS} + \vec{SR}$.

2. In the parallelogram $PQRS$, $\vec{PQ} = \vec{SR} = \vec{u}$, and $\vec{PS} = \vec{QR} = \vec{v}$. What can you conclude about the sums $\vec{u} + \vec{v}$ and $\vec{v} + \vec{u}$?

3. Given the regular hexagon $PQRSTU$ shown, where $\vec{PQ} = \vec{a}$, $\vec{QR} = \vec{b}$, and $\vec{RS} = \vec{c}$, find the following vectors in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$.
   a) $\vec{TS}$
   b) $\vec{UT}$
   c) $\vec{PR}$
   d) $\vec{US}$
   e) $\vec{PT}$
   f) $\vec{QS}$



4. A parallelepiped is a prism whose opposite sides are congruent parallelograms. Given the parallelepiped shown, find a single vector equal to each of the following.
   a) $\vec{AB} + \vec{BC}$ d) $\vec{AB} + (\vec{CG} + \vec{FG})$
   b) $\vec{AB} + \vec{FG}$ e) $\vec{DH} + \vec{CB}$
   c) $\vec{CG} + \vec{FG}$ f) $\vec{HC} + \vec{BF}$



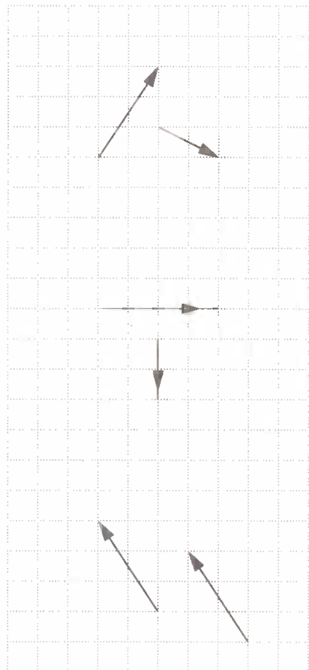
5. In question 4, if $\vec{AB} = \vec{u}$, $\vec{AE} = \vec{v}$, and $\vec{AD} = \vec{w}$, express each vector sum of parts a)–d) in terms of $\vec{u}$, $\vec{v}$, $\vec{w}$.

6. Redraw the following vectors appropriately to find a directed line segment representing their sum.

a)

b)

c)



7. If $\vec{a} = \overline{(3,5)}$, $\vec{b} = \overline{(2,-7)}$, $\vec{c} = \overline{(5,-2)}$, show that $\vec{a} + \vec{b} = \vec{c}$ by drawing the three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ on a grid.

8. An object is displaced in a plane according to the vector $\vec{u} = \overline{(-1,4)}$, then displaced again according to the vector $\vec{v} = \overline{(-1,-4)}$.
   a) Calculate the resultant displacement, $\vec{w}$.
   b) Draw the three displacements $\vec{u}$, $\vec{v}$, $\vec{w}$, on a grid.

9. Given $\vec{p} = \overrightarrow{(2,3)}, \vec{q} = \overrightarrow{(-1,5)}, \vec{r} = \overrightarrow{(3,-4)}$, find the following vector sums.
   a) $\vec{p} + \vec{q}$ c) $\vec{q} + \vec{r}$
   b) $\vec{p} + \vec{r}$ d) $\vec{r} + \vec{q}$

10. Given $\vec{u} = \overrightarrow{(0,1,-2)}, \vec{v} = \overrightarrow{(3,-3,7)}, \vec{w} = \overrightarrow{(-4,-5,1)}$, find the following vector sums.
   a) $\vec{u} + \vec{v}$ c) $\vec{v} + \vec{w}$
   b) $\vec{u} + \vec{w}$ d) $\vec{w} + \vec{v}$

11. What conclusion can you draw from parts c) and d) of questions 9 and 10?

12. Given that $\vec{u} = \overrightarrow{(2,5)}$ and $\vec{v} = \overrightarrow{(4,3)}$,
   a) calculate $\vec{u} + \vec{v}$ as an ordered pair
   b) draw $\vec{u}$, $\vec{v}$, and $\vec{u} + \vec{v}$ on a grid
   c) calculate $|\vec{u}|$, $|\vec{v}|$, and $|\vec{u} + \vec{v}|$.
   d) Does $|\vec{u}| + |\vec{v}| = |\vec{u} + \vec{v}|$? Explain.

13. Repeat question 12 for $\vec{u} = \overrightarrow{(-1,3)}, \vec{v} = \overrightarrow{(-2,6)}$. Do you get the same answer for part d)? Explain.

14. Given $\vec{u} = \overrightarrow{(2,5)}, \vec{v} = \overrightarrow{(4,3)}, \vec{w} = \overrightarrow{(1,-2)}$, $\vec{p} = \vec{u} + \vec{v}$, and $\vec{q} = \vec{v} + \vec{w}$
   a) calculate $\vec{p} + \vec{w}$ as an ordered pair
   b) calculate $\vec{u} + \vec{q}$ as an ordered pair.

   What conclusion can you draw from your results?

15. Given $\vec{u} = \overrightarrow{(3,p,q)}, \vec{v} = \overrightarrow{(-1,p,7)}$, and $\vec{u} + \vec{v} = \overrightarrow{(2,8,-2)}$, find the values of p and q.

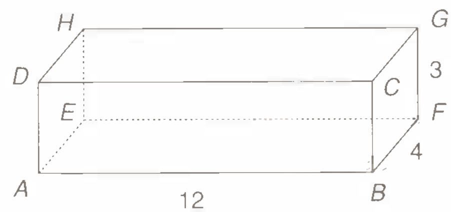
16. The realtor Jessie LaRue, back in town for July and August, notes $\vec{j} = \overrightarrow{(1,2,1)}$ according to the number of houses, condominiums, and business locations she sold during July. Her summer sales, for the two months July and August, are represented by the vector $\vec{s} = \overrightarrow{(2,2,1)}$. Write the vector $\vec{a}$, representing her sales in August, as an ordered triple.

17. A skater at a point A on a frozen lake goes 240 m towards the north, then 100 m towards the east to arrive at point B. Draw a vector diagram to represent the resultant displacement $\overrightarrow{AB}$. What is the magnitude of this displacement? What is the bearing of B from A?

18. A particle is displaced 5 cm along bearing 295°, then 8 cm along bearing 190°. Find the magnitude, correct to 2 decimal places, and direction of the resultant displacement.

19. Two vectors $\vec{p}$ and $\vec{q}$ are drawn so that they have a common tail and form an angle of 110°. If $|\vec{p}| = 7$, $|\vec{q}| = 3$, calculate the following.
   a) $|\vec{p} + \vec{q}|$, correct to 3 significant digits
   b) the angle θ between $\vec{p}$ and $(\vec{p} + \vec{q})$, to the nearest degree

20. Given the rectangular box shown, where $|\overrightarrow{AB}| = 12$, $|\overrightarrow{BF}| = 4$, and $|\overrightarrow{FG}| = 3$, calculate the following.
   a) $|\overrightarrow{AB} + \overrightarrow{BF}|$
   b) $|(\overrightarrow{AB} + \overrightarrow{BF}) + \overrightarrow{FG}|$
   c) Show that $|\overrightarrow{AB} + \overrightarrow{BF}| \leqslant |\overrightarrow{AB}| + |\overrightarrow{BF}|$



21. The relationship in question 20 part c) is known as the *triangle inequality*. Why? There is a special case of three points A, B and F which make the *equality* hold true. How then are the points A, B, and F positioned?

Between Pigeons and Problem Solving

If six people are in a room, prove that at least three of them are mutual acquaintances or at least three are mutual strangers.

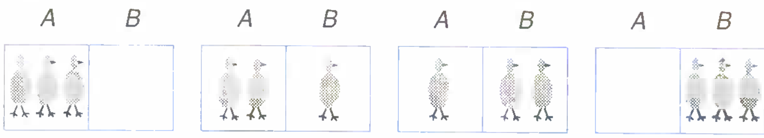
What does this problem have to do with pigeons? There is a very simple idea in discrete mathematics called "the pigeonhole principle" that will enable you to solve this problem.

In its simplest form the pigeonhole principle is stated as follows.

If m pigeons are placed in k pigeonholes and $m > k$, then at least one of the holes must contain at least two pigeons.

To understand the principle, consider the case of $m = 3$ and $k = 2$, that is, three pigeons and two pigeon holes.

The diagrams show all possible situations.



At least two pigeons are in one of the holes in each of the four cases.

You can also see the truth of the principle for three pigeons and two pigeonholes in another way.

Suppose you try to distribute the pigeons as evenly as possible in an attempt to avoid two pigeons in one hole. Then you would put one pigeon in A and a second pigeon in B. This still leaves one pigeon to be pigeonholed, so that one of A or B must contain two pigeons.

The following problems can be solved using the pigeonhole principle.

You have 10 identical red socks and 12 identical yellow socks in a drawer. It is so dark that you cannot see. How many socks must you remove from the drawer so that you will have at least two socks of the same colour?

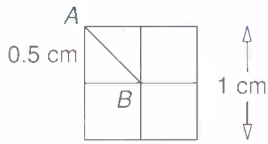
Think of the colours as pigeonholes and the socks as pigeons. With only two pigeonholes (colours), if you select three pigeons (socks) you must have at least two pigeons in the same hole. This means you must have two socks of the same colour.

You select, at random, 5 points in the xy-plane whose coordinates are integers. The points can be joined by a total of 10 line segments. Prove that at least one of the line segments contains a third point whose coordinates are integers.

The coordinates of the 5 points can be separated into 4 classes as follows.
(even, even) (even, odd) (odd, even) (odd, odd)

Since there are 5 points (pigeons) and 4 classes (pigeonholes), at least two points must be in the same class. If these points are (a,b) and (c,d), then their midpoint $\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$ must have coordinates that are integers. This is true because a and b will be both even or both odd. Also c and d will be both even or both odd. Since the sum of two odd numbers and the sum of two even numbers are each divisible evenly by two, both $\frac{a+b}{2}$ and $\frac{c+d}{2}$ will be integers.

Five points are randomly selected in a square whose sides are 1 cm. Prove that at least two of the points are at most a distance $\sqrt{0.5}$ cm apart.



Divide the square into four congruent squares (pigeonholes). Because of the pigeonhole principle at least two of the points (pigeons) must be located in the same small square, say in the top left one. Now the furthest apart these two points can be is at the corners of a diagonal, say at A and B. But by the Pythagorean theorem, $AB = \sqrt{0.5^2 + 0.5^2} = \sqrt{0.25 + 0.25} = \sqrt{0.5}$ Hence there must be at least two points that are at most $\sqrt{0.5}$ cm apart.

You may wish to try these problems.

1. Five points are randomly selected in an equilateral triangle whose sides are 2 cm. Prove that at least two of the points are at most a distance 1 cm apart.

2. 26 distinct numbers are selected from among the first 50 natural numbers. Prove that at least two of these numbers must be consecutive.

3. How many playing cards must you draw from a deck of 52 playing cards to be certain that at least two cards are from the same suit?

4. How many students must be in your school to be certain that at least two of them have the same birthday?

The following problems, the second of which is the problem at the beginning, need the generalized pigeonhole principle for their solution. This is stated as follows.

If more than sn pigeons are placed in n pigeonholes, then at least one hole must contain at least $s + 1$ pigeons.

5. How many students must be in your school to be certain that at least four of them have the same birthday?

6. If six people are in a room, prove that at least three of them are mutual acquaintances or at least three are mutual strangers.

1.5 Properties of Vector Addition

Sulu and Mary met downtown and decided to shop separately before having lunch together. Mary walked 3 blocks south, then 4 blocks east. Sulu walked 4 blocks east, then 3 blocks south. Did they then arrive at the same spot to meet for lunch?

Vector addition has two important properties which you will discover in the examples that follow.

Example 1 Given that $\vec{u} = \overrightarrow{(2,3)}$ and $\vec{v} = \overrightarrow{(5,-1)}$, find the following.

a) $\vec{u} + \vec{v}$, geometrically as a directed line segment, and algebraically as an ordered pair

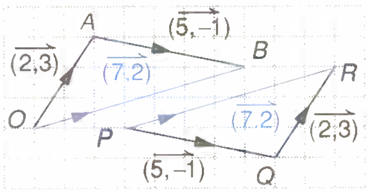
b) $\vec{v} + \vec{u}$, also geometrically and algebraically.

c) Draw conclusions from your results.

Solution

geometric | algebraic

a)
b)



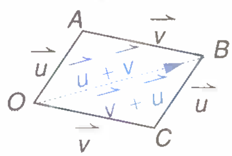
$$\vec{u} + \vec{v} = \overrightarrow{(2,3)} + \overrightarrow{(5,-1)} = \overrightarrow{(2+5,3-1)} = \overrightarrow{(7,2)}$$

$$\vec{v} + \vec{u} = \overrightarrow{(5,-1)} + \overrightarrow{(2,3)} = \overrightarrow{(5+2,-1+3)} = \overrightarrow{(7,2)}$$

c) $\overrightarrow{OB} = \overrightarrow{PR} = \overrightarrow{(7,2)}$, $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ $\quad$ $\vec{u} + \vec{v} = \vec{v} + \vec{u} = \overrightarrow{(7,2)}$ ■

Thus, you can add two vectors in either order and obtain the same resultant vector.

This holds true for any vectors. By virtue of the diagram below (diagrams of Example 1 combined) the **triangle law of vector addition** is sometimes known as the **parallelogram law of vector addition**.



Given any vectors $\vec{u}$ and $\vec{v}$, the following property holds.

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$

Vector addition is thus said to be commutative.

Note: The triangle law also shows that commutativity holds, as follows.

$$\vec{u} + \vec{v} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

and

$$\vec{v} + \vec{u} = \overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OB}$$

(Observe that this can be checked on the above diagram.)

Example 2 Given $\vec{u} = \overrightarrow{(3,2)}$, $\vec{v} = \overrightarrow{(5,-1)}$, and $\vec{w} = \overrightarrow{(1,-1)}$, find the following.

a) $[\vec{u} + \vec{v}] + \vec{w}$, geometrically as a directed line segment, and algebraically as an ordered pair

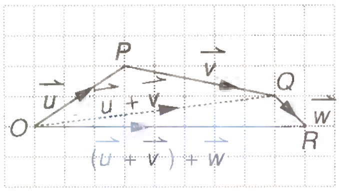
b) $\vec{u} + [\vec{v} + \vec{w}]$, also geometrically and algebraically.

c) Draw conclusions from your results.

Solution

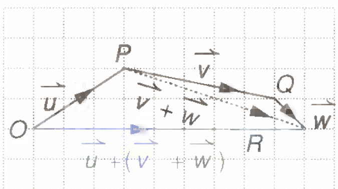
geometric | algebraic

a) $[\vec{u} + \vec{v}] + \vec{w} = \overrightarrow{OR}$



$$[\vec{u} + \vec{v}] + \vec{w}$$
$$= [\overrightarrow{(3 + 5, 2 - 1)}] + \overrightarrow{(1,-1)}$$
$$= \overrightarrow{(8,1)} + \overrightarrow{(1,-1)}$$
$$= \overrightarrow{(9,0)}$$

b) $\vec{u} + [\vec{v} + \vec{w}] = \overrightarrow{OR}$



$$\vec{u} + [\vec{v} + \vec{w}]$$
$$= \overrightarrow{(3,2)} + [\overrightarrow{(5 + 1, -1 - 1)}]$$
$$= \overrightarrow{(3,2)} + \overrightarrow{(6,-2)}$$
$$= \overrightarrow{(9,0)}$$

c) $[\vec{u} + \vec{v}] + \vec{w} = \vec{u} + [\vec{v} + \vec{w}]$ | $[\vec{u} + \vec{v}] + \vec{w} = \vec{u} + [\vec{v} + \vec{w}]$ ■

The following property holds true for any vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$.

$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

Vector addition is thus said to be associative.

The Associative Property of Vector Addition

Given any three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, draw the vectors so that $\vec{u} = \overrightarrow{OP}$, $\vec{v} = \overrightarrow{PQ}$, and $\vec{w} = \overrightarrow{QR}$. The triangle law shows that associativity holds as follows.

$$(\overrightarrow{OP} + \overrightarrow{PQ}) + \overrightarrow{QR} = \overrightarrow{OQ} + \overrightarrow{QR} = \overrightarrow{OR}$$

and

$$\overrightarrow{OP} + (\overrightarrow{PQ} + \overrightarrow{QR}) = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}$$

(Observe that this can be checked on the above diagram.)

Since the operation of vector addition is associative, you do not *need* the brackets. You have a perfectly clear meaning for $\vec{u} + \vec{v} + \vec{w}$,

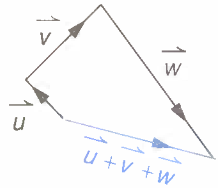
namely, $\vec{u} + \vec{v} + \vec{w} = (\vec{u} + \vec{v}) + \vec{w}$.

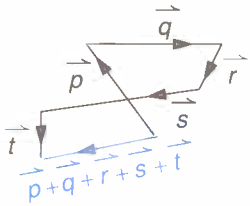
This means the additions are performed in order one after the other.

Similarly, $\vec{p} + \vec{q} + \vec{r} + \vec{s} + \ldots$ also indicates that the additions are to be performed in order, one after the other.

Thus, you can add any number of vectors (as long as they have the same dimensions), as follows.

Geometrically, use the triangle law repeatedly, with each vector's tip joined to the tail of the next one, as shown in the diagrams below. This law, for adding more than two vectors represented by directed line segments, is known as the **polygon law of vector addition.**





Algebraically, add the respective components, for example,
$\overrightarrow{(3,5)} + \overrightarrow{(4,-6)} + \overrightarrow{(-2,8)} = \overrightarrow{(3+4-2, 5-6+8)} = \overrightarrow{(5,7)}$

The polygon law of vector addition can also be expressed as follows.

For any points A, B, C, D, E: $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{AE}$

Note: The polygon is not necessarily in one plane; a polygon that does not lie in a plane is known as a **skew polygon.**

Example 3

Given the vectors $\vec{u} = \overrightarrow{(5,2,0)}$, $\vec{v} = \overrightarrow{(0,6,0)}$, $\vec{w} = \overrightarrow{(0,0,-4)}$,

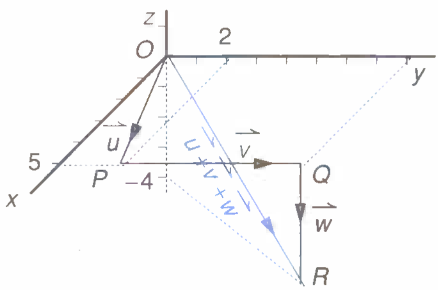
a) find $\vec{u} + \vec{v} + \vec{w}$

b) show a geometrical interpretation of this sum by drawing the tail of the vector $\vec{u}$ at the origin O of a 3-space coordinate system.

Solution

a) $\vec{u} + \vec{v} + \vec{w}$
$= \overrightarrow{(5,2,0)} + \overrightarrow{(0,6,0)} + \overrightarrow{(0,0,-4)}$
$= \overrightarrow{(5+0+0, 2+6+0, 0+0-4)}$
$= \overrightarrow{(5,8,-4)}$

b) Using the polygon law,
$\overrightarrow{OP} + \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{OR}$



This is a good example of a skew polygon ($OPQR$ is a skew quadrilateral). ■

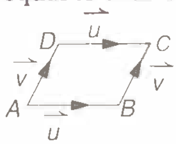
SUMMARY

Vector addition is commutative: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

Vector addition is associative: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
(thus brackets are not required for multiple additions.)

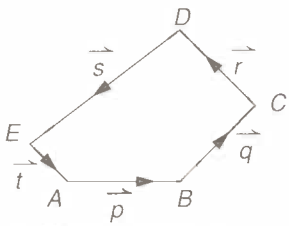
1.5 Exercises

1. Name a single vector equal to each of the following sums.



   a) $\vec{u} + \vec{v}$
   b) $\vec{v} + \vec{u}$

2. Repeat question 1 for the following sums.



   a) $\vec{p} + \vec{q} + \vec{r} + \vec{s} + \vec{t}$
   b) $\vec{p} + \vec{q}$
   c) $\vec{p} + \vec{q} + \vec{r}$
   d) $\vec{p} + \vec{q} + \vec{r} + \vec{s}$
   e) $\vec{q} + \vec{p}$

3. a) State the commutative law for vector addition.
   b) Use any two vectors $\vec{u}$ and $\vec{v}$ to illustrate geometrically that vector addition is commutative.
   c) Use the vectors $\vec{u} = \overrightarrow{(-3,5)}$ and $\vec{v} = \overrightarrow{(4,1)}$ to illustrate algebraically that vector addition is commutative.

4. a) State the associative law for vector addition.
   b) Use any three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ to illustrate geometrically that vector addition is associative.
   c) Use the vectors $\vec{u} = \overrightarrow{(-3,5)}$, $\vec{v} = \overrightarrow{(4,1)}$, and $\vec{w} = \overrightarrow{(2,-7)}$ to illustrate algebraically that vector addition is associative.

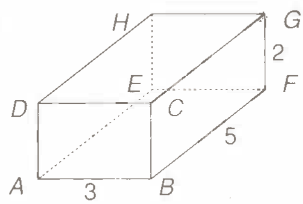
5. State which of the four usual operations in $\mathbb{R}$, namely addition, multiplication, subtraction and division, are commutative and associative.
   If an operation does *not* have either the commutative property or the associative property, give an example to demonstrate this. (Such examples, used to prove that a property does not hold true, are known as counterexamples. *One* counterexample is sufficient for disproof.)

6. a) State whether or not the operation of exponentiation is associative in $\mathbb{R}$. That is, state whether or not it is true that, for any three real numbers a, b, c, $[a^b]^c = a^{[b^c]}$
   b) If exponentiation is associative, prove it. If it is not, disprove it by using a counterexample.

7. Simplify the following.
   a) $\overrightarrow{AP} + \overrightarrow{PC}$
   b) $\overrightarrow{YZ} + \overrightarrow{XY}$
   c) $\overrightarrow{AD} + \overrightarrow{DC} + \overrightarrow{CB}$
   d) $\overrightarrow{QR} + \overrightarrow{RQ}$

8. Given any five points in space P, Q, R, S, T,
   a) explain why $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{ST} = \overrightarrow{PT}$
   b) simplify $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{ST} + \overrightarrow{TP}$.
   c) What is the magnitude of your answer to b)?
   d) Are there any special cases in which any of the answers to a) or b) do not hold true?

9. Given the rectangular box shown, calculate the following
   a) $|\overrightarrow{AB} + \overrightarrow{BF} + \overrightarrow{FG}|$
   b) $|\overrightarrow{BF} + \overrightarrow{FE} + \overrightarrow{EH}|$
   c) $|\overrightarrow{BF} + \overrightarrow{FG} + \overrightarrow{GH}|$
   d) $|\overrightarrow{AB} + \overrightarrow{BF} + \overrightarrow{FE}|$



10. Find $\vec{u} + \vec{v} + \vec{w}$ in the following cases.
    a) $\vec{u} = \overrightarrow{(2,-5)}$, $\vec{v} = \overrightarrow{(1,3)}$, $\vec{w} = \overrightarrow{(-6,1)}$,
    b) $\vec{u} = \overrightarrow{(0,0,3)}$, $\vec{v} = \overrightarrow{(4,4,-1)}$, $\vec{w} = \overrightarrow{(1,1,-2)}$,

11. a) Show a geometric interpretation of each of the sums in question 10.
    b) Which of the above cases, if any, give an example of a skew polygon?

12. Given the skew quadrilateral $OPQR$ where $\overrightarrow{OP} = \overrightarrow{(5,2,0)}$, $\overrightarrow{PQ} = \overrightarrow{(0,6,0)}$, and $\overrightarrow{QR} = \overrightarrow{(0,0,-4)}$, calculate
    a) $|\overrightarrow{OQ}|$,
    b) $|\overrightarrow{OR}|$,
    c) the angle QOR.

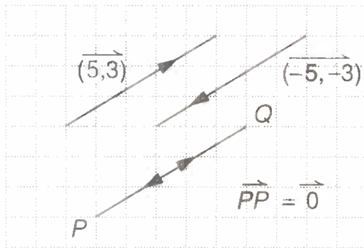
1.6 Vector Subtraction

The Zero Vector

In $\mathbb{V}_2$, suppose $\vec{u} = \overrightarrow{(5,3)}$ and $\vec{v} = \overrightarrow{(-5,-3)}$,
then $\vec{u} + \vec{v} = \overrightarrow{(5,3)} + \overrightarrow{(-5,-3)} = \overrightarrow{(5-5,3-3)} = \overrightarrow{(0,0)}$.
The vector $\overrightarrow{(0,0)}$ is called the **zero vector** of $\mathbb{V}_2$: $\overrightarrow{(0,0)} = \vec{0}$.

The vector $\vec{v} = \overrightarrow{(-5,-3)}$ is written $-\vec{u}$ and is called the **opposite** of $\vec{u}$.

Geometrically, if the triangle law is used:
$\overrightarrow{PQ} + \overrightarrow{QP} = \overrightarrow{PP}$,
thus $\overrightarrow{PP} = \vec{0} = \overrightarrow{(0,0)}$



In general, $\vec{v} = \overrightarrow{(a,b)}$ and $-\vec{v} = \overrightarrow{(-a,-b)}$ are called **opposite vectors**, and $\overrightarrow{(a,b)} + \overrightarrow{(-a,-b)} = \vec{0}$.

These concepts hold in a similar fashion in $\mathbb{V}_3$: $-\vec{v} = \overrightarrow{(-a,-b,-c)}$ is the opposite of vector $\vec{v} = \overrightarrow{(a,b,c)}$, and the zero vector is $\vec{0} = \overrightarrow{(0,0,0)}$.

The vectors $\vec{v}$ and $-\vec{v}$ are said to have **opposite directions**.

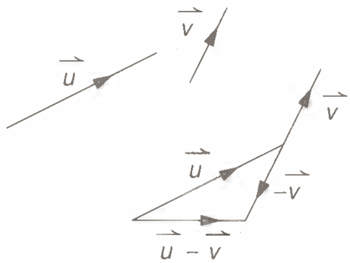
Subtraction of Vectors

The subtraction of vectors is defined by 'adding the opposite' vector.

Thus

$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$$

The geometric interpretation of this is shown in the diagram.



Now the zero vector can be defined by

$$\vec{u} - \vec{u} = \vec{0}$$

As an example of vector subtraction, if $\vec{u} = \overrightarrow{(8,6)}$ and $\vec{v} = \overrightarrow{(3,2)}$,
$\vec{u} - \vec{v} = \vec{u} + (-\vec{v}) = \overrightarrow{(8,6)} + \overrightarrow{(-3,-2)} = \overrightarrow{(8-3,6-2)} = \overrightarrow{(5,4)}$.

In general, $\overrightarrow{(a,b)} - \overrightarrow{(p,q)} = \overrightarrow{(a,b)} + \overrightarrow{(-p,-q)} = \overrightarrow{(a-p,b-q)}$.

Thus vector subtraction in component form is carried out as follows.

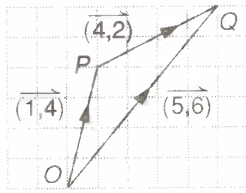
in $\mathbb{V}_2$,
$$\overrightarrow{(a,b)} - \overrightarrow{(p,q)} = \overrightarrow{(a-p,b-q)}$$

in $\mathbb{V}_3$,
$$\overrightarrow{(a,b,c)} - \overrightarrow{(p,q,r)} = \overrightarrow{(a-p,b-q,c-r)}$$

Example 1 a) Calculate the vector $\overrightarrow{(5,6)} - \overrightarrow{(1,4)}$.
b) Demonstrate this subtraction by using a diagram.

Solution a) $\overrightarrow{(5,6)} - \overrightarrow{(1,4)}$
$= \overrightarrow{(5-1, 6-4)}$
$= \overrightarrow{(4,2)}$

b)



Notice that the diagram indicates that $\overrightarrow{OQ} - \overrightarrow{OP} = \overrightarrow{PQ}$. ■

This is a universal result, and is known as the **subtraction form of the triangle law**.

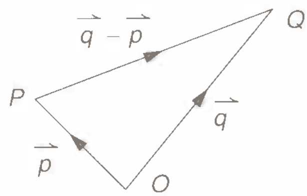
(Watch the order of the letters carefully.)

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

Note: When the law is written this way, any other single letter could be substituted for O. This very important property allows you to *choose* an origin, and this can help you through some tricky vector problems which you will encounter later.

The subtraction form of the triangle law can also be written as follows, using the abbreviations $\overrightarrow{OQ} = \vec{q}$, $\overrightarrow{OP} = \vec{p}$.

$\overrightarrow{PQ} = \vec{q} - \vec{p}$



Example 2 Use the subtraction form of the triangle law to simplify the following.
a) $\overrightarrow{ED} - \overrightarrow{EF}$
b) $\overrightarrow{CD} + \overrightarrow{BC} + \overrightarrow{AB}$

Solution a) Using E as origin, and applying the law directly,
$\overrightarrow{ED} - \overrightarrow{EF} = \overrightarrow{FD}$

b) Use any origin O, and the abbreviations $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$.

Then $\overrightarrow{CD} + \overrightarrow{BC} + \overrightarrow{AB}$
$= \vec{d} - \vec{c} + \vec{c} - \vec{b} + \vec{b} - \vec{a}$
$= \vec{d} - \vec{a}$
$= \overrightarrow{AD}$ ■

Example 3

a) Given the points in 2-space $A = (2,3)$ and $B = (7,4)$, find the vector $\overrightarrow{AB}$.

b) Given the points in 3-space $P = (2,0,-1)$ and $Q(-3,4,1)$, find the vector $\overrightarrow{PQ}$.

Solution

a) You can use the points to determine the following position vectors.

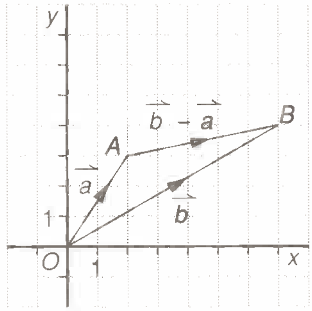
Since $A = (2,3)$, $\overrightarrow{OA} = \vec{a} = \overrightarrow{(2,3)}$

Since $B = (7,4)$, $\overrightarrow{OB} = \vec{b} = \overrightarrow{(7,4)}$

Now $\overrightarrow{AB} = \vec{b} - \vec{a}$

$= \overrightarrow{(7,4)} - \overrightarrow{(2,3)}$

$= \overrightarrow{(5,1)}$



b) In a similar manner, the position vectors of P and Q are respectively $\vec{p} = \overrightarrow{(2,0,1)}$ and $\vec{q} = \overrightarrow{(-3,4,1)}$.

Thus $\overrightarrow{PQ} = \vec{q} - \vec{p} = \overrightarrow{(-3,4,1)} - \overrightarrow{(2,0,1)} = \overrightarrow{(-5,4,0)}$. ■

In this example, you have used vector subtraction. Be careful not to attempt to 'subtract points'. Such an operation has *not* been defined.

The example leads to the following rules.

RULES

In $\mathbb{V}_2$, given points $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$, the vector $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \overrightarrow{(x_2 - x_1, y_2 - y_1)}$.

Thus $|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

In $\mathbb{V}_3$, given points $P_1 = (x_1,y_1,z_1)$ and $P_2 = (x_2,y_2,z_2)$, the vector $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \overrightarrow{(x_2 - x_1, y_2 - y_1, z_2 - z_1)}$.

Thus $|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Example 4

If vector $\overrightarrow{RS} = \overrightarrow{(-1,4,6)}$, and the point $S = (1,2,6)$, find the coordinates of R.

Solution

Let $R = (x,y,z)$, so that $\vec{r} = \overrightarrow{(x,y,z)}$.

$$\begin{aligned}
\text{Since} \quad \overrightarrow{RS} &= \vec{s} - \vec{r} \\
\overrightarrow{(-1,4,6)} &= \overrightarrow{(1,2,6)} - \overrightarrow{(x,y,z)} \\
\text{thus} \quad \overrightarrow{(x,y,z)} &= \overrightarrow{(1,2,6)} - \overrightarrow{(-1,4,6)} \\
&= \overrightarrow{(2,-2,0)}
\end{aligned}$$

Hence the coordinates of R are $(2,-2,0)$. ■

You might note that examples involving vector subtraction can be resolved using vector addition. For instance, in Example 3a), you could state that $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{(-2,-3)} + \overrightarrow{(7,4)} = \overrightarrow{(5,1)}$, as before.

However, you will find that handling subtraction and addition equally well gives you a lot more flexibility.

Further Properties of Vector Addition

Observe that, given any vector $\vec{v}$,

$$\vec{v} + \vec{0} = \vec{v} \text{ and } \vec{0} + \vec{v} = \vec{v}.$$

Because of this property, the zero vector is called the **neutral element**, or the **identity element** for vector addition.

Observe also that, given any vector $\vec{v}$,

$$\vec{v} + (-\vec{v}) = \vec{0} \text{ and } (-\vec{v}) + \vec{v} = \vec{0}.$$

Because of this property, $(-\vec{v})$ and $\vec{v}$ are called **inverses** of each other, for vector addition.

SUMMARY

$\vec{u} = \overrightarrow{(a,b)}$ and $-\vec{u} = \overrightarrow{(-a,-b)}$ are opposite vectors in $\mathbb{V}_2$

$\vec{v} = \overrightarrow{(a,b,c)}$ and $-\vec{v} = \overrightarrow{(-a,-b,-c)}$ are opposite vectors in $\mathbb{V}_3$

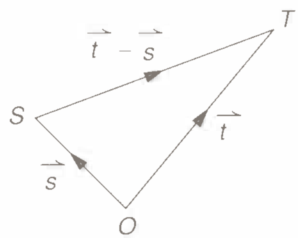
The sum of two opposite vectors is the zero vector.

Subtraction of Vectors

$\overrightarrow{ST} = \overrightarrow{OT} - \overrightarrow{OS} = \vec{t} - \vec{s}$

$\overrightarrow{(a,b)} - \overrightarrow{(p,q)} = \overrightarrow{(a-p,b-q)}$ in $\mathbb{V}_2$

$\overrightarrow{(a,b,c)} - \overrightarrow{(p,q,r)} = \overrightarrow{(a-p,b-q,c-r)}$ in $\mathbb{V}_3$



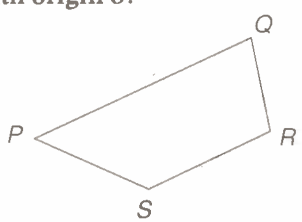
1.6 Exercises

1. Determine whether the following are true or false.
   a) $\overrightarrow{AX} - \overrightarrow{XC} = \overrightarrow{AC}$
   b) $\overrightarrow{AX} - \overrightarrow{AC} = \overrightarrow{CX}$
   c) $\overrightarrow{CA} - \overrightarrow{XA} = \overrightarrow{CX}$
   d) $-\overrightarrow{AX} - \overrightarrow{XC} = \overrightarrow{AC}$

2. If Q, R and S are any points, express the vector $\overrightarrow{RS}$ in terms of $\overrightarrow{QR}$ and $\overrightarrow{QS}$.

3. Simplify by using the subtraction form of the triangle law.
   a) $\overrightarrow{OP} - \overrightarrow{OR}$
   b) $\overrightarrow{QZ} - \overrightarrow{QX}$
   c) $\overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{DC}$
   d) $\overrightarrow{AB} + \overrightarrow{CA} + \overrightarrow{BC}$
   e) $\overrightarrow{OB} - \overrightarrow{OA} + \overrightarrow{BC}$

4. $PQRS$ is a plane quadrilateral, and O is *any* other point in space. Express the following vectors in subtraction form, using position vectors with origin O.
   a) $\overrightarrow{PQ}$
   b) $\overrightarrow{QR}$
   c) $\overrightarrow{RS}$
   d) $\overrightarrow{RP}$



5. $ABCD$ is a square and O is *any* point. If $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, and $\overrightarrow{OC} = \vec{c}$, express the vector $\overrightarrow{OD} = \vec{d}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$. (Hint: Recall that in a square, $\overrightarrow{AB} = \overrightarrow{DC}$; write these vectors as subtractions.)

6. Use the properties of vector addition to prove that $(-\vec{q}) + (\vec{p} + \vec{q}) = \vec{p}$

7. Given $\vec{p} = \overrightarrow{(2,3)}$, $\vec{q} = \overrightarrow{(-1,5)}$, $\vec{r} = \overrightarrow{(3,-4)}$, find the following.
   a) $\vec{p} - \vec{q}$
   b) $\vec{q} - \vec{r}$
   c) $\vec{r} - \vec{p}$
   d) $\vec{p} - \vec{r}$

8. Given $\vec{u} = \overrightarrow{(0,1,-2)}$, $\vec{v} = \overrightarrow{(3,-3,7)}$, $\vec{w} = \overrightarrow{(-4,-5,1)}$, find the following.
   a) $\vec{u} - \vec{v}$
   b) $\vec{v} - \vec{w}$
   c) $\vec{w} - \vec{u}$
   d) $\vec{u} - \vec{w}$

9. What conclusion can you draw from parts c) and d) of questions 7 and 8?

10. Given the vectors of questions 7 and 8, simplify the following.
    a) $\vec{p} + \vec{q} - \vec{r}$
    b) $\vec{p} - \vec{q} + \vec{r}$
    c) $\vec{w} - \vec{v} + \vec{u}$
    d) $-\vec{u} - \vec{v} - \vec{w}$

11. Given $\vec{a} = \overrightarrow{(11,-2,k)}$, $\vec{b} = \overrightarrow{(m,n,8)}$, and $\vec{a} - \vec{b} = \overrightarrow{(5,0,1)}$, find the values of the real numbers k, m, and n.

12. Given the points $P(2,3)$, $Q(7,4)$, and $R(-1,1)$, use vector subtraction to determine the following vectors in component form.
    a) $\overrightarrow{PQ}$
    b) $\overrightarrow{QR}$
    c) $\overrightarrow{RP}$

13. Use the answers of question 12 to calculate the sum $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP}$. Explain your result.

14. Given the points $L(1,0,-7)$, $M(2,2,9)$ and $N(-3,-4,6)$, use vector subtraction to determine the following vectors in component form.
    a) $\overrightarrow{LM}$
    b) $\overrightarrow{MN}$
    c) $\overrightarrow{NL}$

15. Given that R is at $(12,10)$, S is at $(15,11)$, and $\overrightarrow{OP} = \overrightarrow{RS}$, find the coordinates of the point P.

16. Repeat question 15 for $R(6,-1,3)$ and $S(2,-5,-1)$.

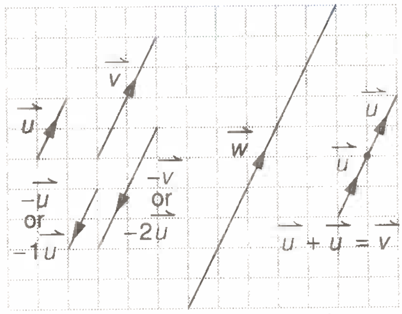
17. Given the points $A(2,5)$, $B(3,-1)$, $C(-4,6)$, $D(0,2)$, find the coordinates of the point M such that $\overrightarrow{OM} = \overrightarrow{AB} - \overrightarrow{CD}$.

18. Given two perpendicular vectors $\vec{u}$ and $\vec{v}$, whose magnitudes are not necessarily the same,
    a) find $|\vec{u} + \vec{v}|$ in terms of $|\vec{u}|$ and $|\vec{v}|$,
    b) show that $|\vec{u} + \vec{v}| = |\vec{u} - \vec{v}|$.

1.7 Multiplication of a Vector by a Scalar

Consider the vectors $\vec{u} = \overrightarrow{(1,2)}$, $\vec{v} = \overrightarrow{(2,4)}$, $\vec{w} = \overrightarrow{(5,10)}$.

Is there any relationship among them?



By vector addition, you can see that

$$\vec{v} = \vec{u} + \vec{u}$$

and $\vec{w} = \vec{u} + \vec{u} + \vec{u} + \vec{u} + \vec{u}$.

It seems natural to write

$$\vec{v} = 2\vec{u}$$

and $\vec{w} = 5\vec{u}$

Note: $\vec{u}$, $\vec{v}$, and $\vec{w}$ are all in the same direction (that is, parallel, and pointing the same way).

In the last section, you saw that the vector $-\vec{u}$ had the same length as $\vec{u}$, but the opposite direction. You can think of $-\vec{u}$ as being the same as $-1\vec{u}$.

Similarly, $-\vec{v}$ has the same length as $\vec{v}$, but the opposite direction. Thus $-\vec{v} = -2\vec{u}$.

Although these give rise to an unusual combination of symbols

$$\vec{w} = 5\vec{u}$$

vector — real number — vector

this operation is accepted. It is called multiplication of a vector by a scalar or **scalar multiplication of a vector.**

If k is a *positive* scalar and $\vec{u}$ is a vector, then $k\vec{u}$ is a vector in the *same direction* as $\vec{u}$, with length k times the length of $\vec{u}$.

If k is *negative*, then the direction of $k\vec{u}$ is reversed (that is, the direction of $k\vec{u}$ is *opposite to the direction* of $\vec{u}$), and the *length* of $k\vec{u}$ is $|k|$ times *the length* of $\vec{u}$.

In general,

the vectors $\vec{u}$ and $k\vec{u}$ are parallel;

the length of vector $k\vec{u}$ is $|k||\vec{u}|$.

Note: $|k|$ means the absolute value of the real number k, whereas $|\vec{u}|$ means the length of the vector $\vec{u}$.

This should not lead to any confusion, because both length and absolute value are always numbers (scalars) greater than or equal to zero.

In some texts, a distinction is made by writing the length of the vector $\vec{u}$ as $\|\vec{u}\|$.

Scalar multiplication can also be defined for vectors in component form, as shown in the following example.

Example 1 Calculate the following by using vector addition, given $\vec{u} = \overrightarrow{(1,3)}$.

a) $\vec{v} = 2\vec{u}$ b) $\vec{w} = 5\vec{u}$

Solution a) $\vec{v} = \vec{u} + \vec{u} = \overrightarrow{(1,3)} + \overrightarrow{(1,3)}$
$= \overrightarrow{(2,6)} = \overrightarrow{(2 \times 1, 2 \times 3)}$ or
$2\overrightarrow{(1,3)} = \overrightarrow{(2 \times 1, 2 \times 3)}$

b) $\vec{w} = \vec{u} + \vec{u} + \vec{u} + \vec{u} + \vec{u} = \overrightarrow{(1,3)} + \overrightarrow{(1,3)} + \overrightarrow{(1,3)} + \overrightarrow{(1,3)} + \overrightarrow{(1,3)}$
$= \overrightarrow{(1 + 1 + 1 + 1 + 1, 3 + 3 + 3 + 3 + 3)}$
$= \overrightarrow{(5,15)} = \overrightarrow{(5 \times 1, 5 \times 3)}$ or
$5\overrightarrow{(1,3)} = \overrightarrow{(5 \times 1, 5 \times 3)}$. ■

This leads to the following definition for the multiplication of a vector in component form by a scalar.

DEFINITION $k\overrightarrow{(x,y)} = \overrightarrow{(kx,ky)}$ in $\mathbb{V}_2$ or $k\overrightarrow{(x,y,z)} = \overrightarrow{(kx,ky,kz)}$ in $\mathbb{V}_3$

Example 2 Find the lengths of vectors $\vec{u} = \overrightarrow{(1,3)}$, $\vec{v} = \overrightarrow{(2,6)}$ and $\vec{w} = \overrightarrow{(5,15)}$ of Example 1, to verify that the length of a vector $k\vec{a}$ equals $|k|$ times the length of $\vec{a}$.

Solution $|\vec{u}| = \sqrt{1^2 + 3^2} = \sqrt{10}$
$|\vec{v}| = \sqrt{2^2 + 6^2} = \sqrt{40} = 2\sqrt{10} = 2|\vec{u}|$, as expected.
$|\vec{w}| = \sqrt{5^2 + 15^2} = \sqrt{250} = 5\sqrt{10} = 5|\vec{u}|$, as expected.
An alternate way of calculating $|\vec{w}|$ is to use some factoring:
$|\vec{w}| = \sqrt{5^2 + (3 \times 5)^2} = \sqrt{5^2(1^2 + 3^2)} = 5\sqrt{10}$, as before. ■

You can now combine the operations of vector addition and multiplication by a scalar.

Example 3 Express as a single vector $3\overrightarrow{(1,-6,4)} + 2\overrightarrow{(5,0,2)} - 4\overrightarrow{(1,1,1)}$.

Solution $3\overrightarrow{(1,-6,4)} + 2\overrightarrow{(5,0,2)} - 4\overrightarrow{(1,1,1)}$
$= \overrightarrow{(3,-18,12)} + \overrightarrow{(10,0,4)} - \overrightarrow{(4,4,4)}$
$= \overrightarrow{(3 + 10 - 4, -18 + 0 - 4, 12 + 4 - 4)}$
$= \overrightarrow{(9,-22,12)}$ ■

At this point, you will look at an example which will illustrate the various properties of the operation of multiplication of a vector by a scalar.

Example 4 Given the vectors $\vec{u} = \overrightarrow{(2,0,-3)}$ and $\vec{v} = \overrightarrow{(1,-6,-2)}$, calculate the following.

a) $5(4\vec{u})$ b) $5(\vec{u} + \vec{v})$ c) $(2 + 3)\vec{u}$

Solution

a) $5(4\vec{u}) = 5\overrightarrow{(4 \times 2, 4 \times 0, 4 \times [-3])} = 5\overrightarrow{(8,0,-12)} = \overrightarrow{(40,0,-60)}$.
Note that this result is the same as
$(5 \times 4)\vec{u} = 20\overrightarrow{(2,0,-3)} = \overrightarrow{(20 \times 2, 20 \times 0, 20 \times [-3])} = \overrightarrow{(40,0,-60)}$.

b) $5(\vec{u} + \vec{v}) = 5\overrightarrow{(2 + 1, 0 - 6, -3 - 2)} = 5\overrightarrow{(3,-6,-5)} = \overrightarrow{(15,-30,-25)}$.
Note that this result is the same as
$$5\vec{u} + 5\vec{v} = 5\overrightarrow{(2,0,-3)} + 5\overrightarrow{(1,-6,-2)}$$
$$= \overrightarrow{(10,0,-15)} + \overrightarrow{(5,-30,-10)} = \overrightarrow{(15,-30,-25)}.$$

c) $(2 + 3)\vec{u} = 5\vec{u}$,
but also $2\vec{u} + 3\vec{u} = \vec{u} + \vec{u} + \vec{u} + \vec{u} + \vec{u} = 5\vec{u}$,
because vector addition is associative.
$5\vec{u} = 5\overrightarrow{(2,0,-3)} = \overrightarrow{(10,0,-15)}$ ■

The above example suggests the following properties of multiplication of a vector by a scalar.

PROPERTIES

For any vectors $\vec{u}$, $\vec{v}$, and scalars k, m:

1. $k(m\vec{u}) = (km)\vec{u}$
2. $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$
3. $(k + m)\vec{u} = k\vec{u} + m\vec{u}$

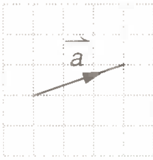
Example 5 Simplify $\overrightarrow{BD} - \overrightarrow{AD} + \overrightarrow{AB} - \overrightarrow{CB}$.

Solution Using vector subtraction from a common origin O, and the abbreviations $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$, $\overrightarrow{OD} = \vec{d}$,

$$\begin{aligned}\overrightarrow{BD} - \overrightarrow{AD} + \overrightarrow{AB} - \overrightarrow{CB} &= \vec{d} - \vec{b} - (\vec{d} - \vec{a}) + \vec{b} - \vec{a} - (\vec{b} - \vec{c}) \\ &= \vec{d} - \vec{b} - \vec{d} + \vec{a} + \vec{b} - \vec{a} - \vec{b} + \vec{c} \\ &= \vec{c} - \vec{b} \\ &= \overrightarrow{BC}\end{aligned}$$ ■

1.7 Exercises

1. Copy the vector $\vec{a}$ shown onto a grid.



On the same grid, draw representatives of the following vectors.

$2\vec{a}$
$3\vec{a}$
$5\vec{a}$
$-2\vec{a}$
$-\vec{a}$

2. Given $\vec{u} \parallel \vec{v}$, $\vec{p} \parallel \vec{q}$, but $\vec{u} \nparallel \vec{p}$, state which of the following are parallel to $\vec{u}$.
   a) $2\vec{u}$
   b) $-3\vec{v}$
   c) $4\vec{p}$
   d) $-5\vec{q}$

3. $ABCD$ is a parallelogram and M is the midpoint of BC. If $\overrightarrow{AB} = 2\vec{u}$ and $\overrightarrow{BM} = \vec{v}$, express the following vectors in terms of $\vec{u}$ and $\vec{v}$.
   a) $\overrightarrow{AM}$
   b) $\overrightarrow{BC}$
   c) $\overrightarrow{AD}$
   d) $\overrightarrow{DB}$
   e) $\overrightarrow{CA}$

4. Draw on a grid the vectors $\vec{p} = \overrightarrow{(2,-5)}$, $\vec{q} = \overrightarrow{(2,5)}$, $\vec{r} = \overrightarrow{(6,-15)}$, $\vec{s} = \overrightarrow{(-4,10)}$. Which of these vectors are parallel? Which are in the same direction?

5. The points A, B, C are such that $\overrightarrow{AB} = \overrightarrow{(2,-3)}$ and $\overrightarrow{BC} = \overrightarrow{(4,-6)}$.
   a) Express $\overrightarrow{BC}$ in terms of $\overrightarrow{AB}$.
   b) Express $\overrightarrow{AB}$ in terms of $\overrightarrow{BC}$.
   c) Show that the points A, B, and C must lie on the same straight line (that is, points A, B, C are collinear).

6. Using the points of question 5, calculate the following vectors in component form.
   a) $\overrightarrow{BA}$ b) $\overrightarrow{AC}$ c) $\overrightarrow{CA}$

7. Express each of the following as a single vector.
   a) $5\overrightarrow{(2,-1)} + 2\overrightarrow{(3,2)} + \overrightarrow{(-7,1)}$
   b) $\overrightarrow{(1,-4,-3)} - 4\overrightarrow{(2,3,-5)} + 2\overrightarrow{(2,2,2)}$
   c) $\frac{1}{2}\overrightarrow{(2,-7,1)} + \frac{5}{2}\overrightarrow{(3,2,1)}$

8. Given the vectors $\vec{u} = \overrightarrow{(2,1,-3)}$, $\vec{v} = \overrightarrow{(1,0,4)}$, $\vec{w} = \overrightarrow{(4,1,5)}$, express each of the following as a single vector.
   a) $3\vec{u} - 4\vec{v} + 2\vec{w}$,
   b) $\vec{u} + 2\vec{v} - \vec{w}$.
   c) What does your result in b) indicate about the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$?

9. Given the vectors of question 8, calculate the following.
   a) $|3\vec{u} - 4\vec{v} + 2\vec{w}|$
   b) $|\vec{u} + 2\vec{v} - \vec{w}|$

10. Given the vectors $\vec{u} = \overrightarrow{(-1,3)}$ and $\vec{v} = \overrightarrow{(2,1)}$, illustrate, both algebraically and geometrically, that
    a) $3(2\vec{u}) = 6\vec{u}$,
    b) $4(\vec{u} + \vec{v}) = 4\vec{u} + 4\vec{v}$
    c) $(3 + 1)\vec{u} = 3\vec{u} + \vec{u}$

11. Simplify each of the following.
    a) $3\vec{v} + 4\vec{v} - 6\vec{v}$
    b) $2(\vec{u} - 2\vec{v}) + 4\vec{v}$
    c) $-5(\vec{u} + 2\vec{v}) + 5\vec{u} + 9\vec{v}$
    d) $3\vec{u} - \vec{v} + 2\vec{w} - (\vec{u} - \vec{v}) - 2(\vec{u} + \vec{w})$

12. Simplify the following, by using a common origin O and vector subtraction.

a) $\overrightarrow{AC} - \overrightarrow{DC} + \overrightarrow{DB}$

b) $\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{BC} - \overrightarrow{AD}$

c) $\overrightarrow{AD} - \overrightarrow{BC} + \overrightarrow{DC} - 2\overrightarrow{AB}$

13. Given that $\vec{b} = k\vec{a}$, where $k \in \mathbb{R}$ is not zero, use the properties of scalar multiplication to prove that $\vec{a} = \frac{1}{k}\vec{b}$.

14. Prove the properties

$k(m\vec{u}) = (km)\vec{u}$

$k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

$(k + m)\vec{u} = k\vec{u} + m\vec{u}$,

for any scalars k, m and vectors $\vec{u}$, $\vec{v}$.

15. Given any non-zero vector $\vec{v}$, calculate the length of $\frac{1}{|\vec{v}|}\vec{v}$.

16. a) If A is the point (2,3) and B is the point (8,1), calculate in component form the vector $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}$.

b) Plot the points O, A, B, M in a 2-space coordinate system of origin O.

c) Show that M is the midpoint of AB.

17. Given any three points O, A, and B, a point M is positioned such that $\overrightarrow{MA} + \overrightarrow{MB} = \vec{0}$.

a) Express $\overrightarrow{MA}$ in terms of $\overrightarrow{MB}$.

b) How are the points A, B, and M related geometrically?

c) Use vector subtraction, with origin O, to find $\overrightarrow{OM}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

18. Given the two vectors $\vec{u}$ and $\vec{v}$, and $|\vec{u}| = 4$. Find the value of $|\vec{u} + \vec{v}|$ in each of the following cases.

a) $\vec{v} = 3\vec{u}$

b) $\vec{v} = -5\vec{u}$

c) $\vec{u}$ and $\vec{v}$ perpendicular, $|\vec{v}| = 3$

d) $\vec{u}$ perpendicular to $(\vec{u} + \vec{v})$, $|\vec{v}| = 9.6$

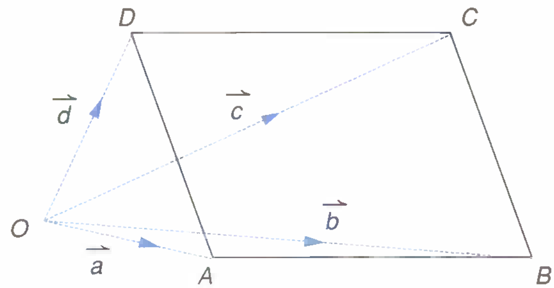
1.8 Applications of Vector Subtraction and Multiplication by a Scalar

You can now apply the properties you have learned, to solve certain geometric problems by using vectors.

Most problems can be solved by using vector addition. However, the technique of using the subtraction form of the triangle law, together with a common origin, unravels many geometric problems quite neatly. For clarity, the abbreviations $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}, \ldots$ are used for the position vectors of $A, B, \ldots$

Example 1 $ABCD$ is a quadrilateral in which AB is parallel to and congruent to DC. Prove that sides AD and BC are also parallel and congruent.

Solution Let O be any point. Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$, and $\overrightarrow{OD} = \vec{d}$.



Write a vector equality from what is given in the problem.

$$\overrightarrow{AB} = \overrightarrow{DC}$$

Writing as subtractions, $\vec{b} - \vec{a} = \vec{c} - \vec{d}$

rearranging, $\vec{d} - \vec{a} = \vec{c} - \vec{b}$

that is, $\overrightarrow{AD} = \overrightarrow{BC}$

hence AD is parallel to and congruent to BC, as required.

Alternative Solution

Using vector addition, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ ①

and $\overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC}$ ②

so from ① and ②, $\overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AB} + \overrightarrow{BC}$ ③

but $\overrightarrow{DC} = \overrightarrow{AB}$ ④ [given]

so ③ – ④ gives $\overrightarrow{AD} = \overrightarrow{BC}$, as required. ■

Note: This example proves the following property of a parallelogram.

PROPERTY

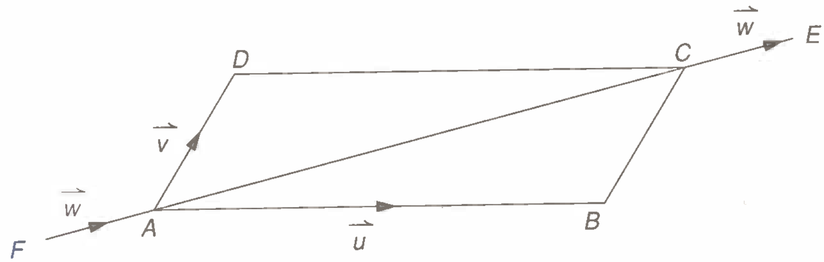
If a quadrilateral $ABCD$ is such that $AB = DC$ and $AB \| DC$, then that quadrilateral is a parallelogram.

The next example uses vector addition, and the above property.

Example 2 A parallelogram $ABCD$ is such that $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{AD} = \vec{v}$. AC is produced to E, and CA is produced to F such that $\overrightarrow{FA} = \overrightarrow{CE} = \vec{w}$.

a) Express the vectors $\overrightarrow{FB}$ and $\overrightarrow{DE}$ in terms of $\vec{u}$, $\vec{v}$, and $\vec{w}$.
b) Hence show that $FBED$ is a parallelogram.

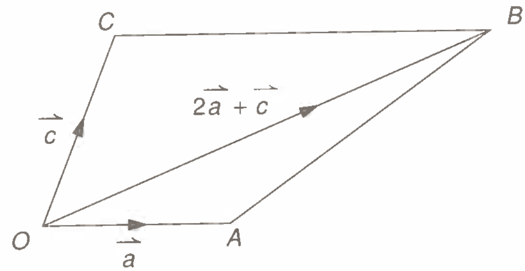
Solution



a) $\overrightarrow{FB} = \overrightarrow{FA} + \overrightarrow{AB} = \vec{w} + \vec{u}$
$\overrightarrow{DE} = \overrightarrow{DC} + \overrightarrow{CE} = \vec{u} + \vec{w}$

b) $\overrightarrow{FB} = \overrightarrow{DE}$ from a)
Hence, $FB = DE$ and $FB \| DE$.
Thus, by the above property, $FBED$ is a parallelogram. ■

Example 3 $OABC$ is a quadrilateral with $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OC} = \vec{c}$, and $\overrightarrow{OB} = 2\vec{a} + \vec{c}$.



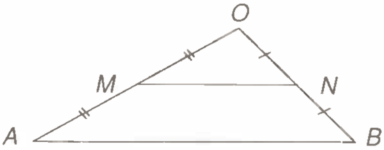
a) Express the vectors $\overrightarrow{AB}$ and $\overrightarrow{CB}$ in terms of $\vec{a}$ and $\vec{c}$.
b) Draw geometric conclusions about the quadrilateral.

Solution a) Using the subtraction form of the triangle law, with O as origin,
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2\vec{a} + \vec{c}) - \vec{a} = \vec{a} + \vec{c}$ and
$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC} = (2\vec{a} + \vec{c}) - \vec{c} = 2\vec{a}$

b) Thus $\overrightarrow{CB}$ is in the *same direction* as $\vec{a}$ and *twice the length* of $\vec{a}$. In geometrical terms, that means that CB is parallel to OA, thus the quadrilateral is a *trapezoid*. ■

Example 4 In triangle OAB, M is the midpoint of side OA and N is the midpoint of side OB. Prove that the side AB is parallel to segment MN, and that the length of AB is twice the length of MN.

Solution Choose O as origin, and use the abbreviations $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OM} = \vec{m}$, $\overrightarrow{ON} = \vec{n}$.



Write vector statements from what is given in the problem.

① $\vec{m} = \frac{1}{2}\vec{a}$ and ② $\vec{n} = \frac{1}{2}\vec{b}$

Subtracting the equations ② − ①,

$$\begin{aligned}\vec{n} - \vec{m} &= \frac{1}{2}\vec{b} - \frac{1}{2}\vec{a}\\ &= \frac{1}{2}(\vec{b} - \vec{a})\end{aligned}$$

thus $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AB}$

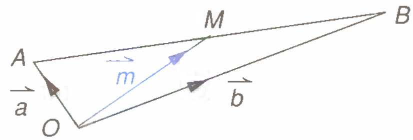
which proves that AB is parallel to MN, and is twice the length of MN, as required. ■

Note: In this example, you have proved the midpoint theorem.

THEOREM In any triangle, the line segment joining the midpoints of any two sides is parallel to, and half the length of, the third side.

Example 5 If M is the midpoint of AB, and O is any point, find an expression for $\overrightarrow{OM}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Solution



Use the abbreviations $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, and $\overrightarrow{OM} = \vec{m}$. Since M is the midpoint of AB, the lengths $AM = MB$, and $AM \| MB$,

thus

$$\begin{aligned}\overrightarrow{AM} &= \overrightarrow{MB}\\ \vec{m} - \vec{a} &= \vec{b} - \vec{m}\\ 2\vec{m} &= \vec{a} + \vec{b}\\ \vec{m} &= \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}\end{aligned}$$ ■

FORMULA Note: This example gives the following mid-point formula. If M is the midpoint of AB, and O is any point, then $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}$

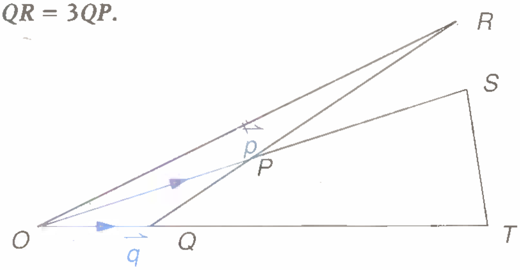
1.8 Exercises

1. *OABC* is a quadrilateral in which $\vec{OA} = 2\vec{v} - \vec{u}$, $\vec{OC} = \vec{u}$, and $\vec{OB} = 2\vec{v}$. Express $\vec{AB}$ and $\vec{CB}$ in terms of $\vec{u}$ and $\vec{v}$, and thus describe the quadrilateral.

2. Describe the quadrilateral of question 1, given that $|2\vec{v} - \vec{u}| = |\vec{u}|$.

3. *OABC* is a quadrilateral in which $\vec{OA} = \vec{a}$, $\vec{OC} = \vec{c}$, and $\vec{OB} = \vec{a} + \frac{1}{2}\vec{c}$. Express $\vec{AB}$ and $\vec{CB}$ in terms of $\vec{a}$ and $\vec{c}$, and thus describe the quadrilateral.

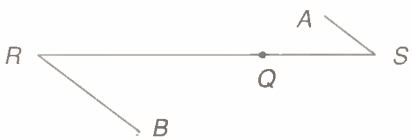
4. In the diagram, $\vec{OP} = \vec{p}$ and $\vec{OQ} = \vec{q}$. It is also known that $\vec{OS} = 2\vec{OP}$, $\vec{OT} = 4\vec{OQ}$, and $\vec{QR} = 3\vec{QP}$.



a) Express the following vectors in terms of $\vec{p}$ and $\vec{q}$: $\vec{OS}$, $\vec{OT}$, $\vec{QP}$, $\vec{TS}$, $\vec{QR}$, $\vec{OR}$, $\vec{TR}$.

b) Hence show that the points *T*, *S* and *R* are on straight line.

5. Given that $\vec{RQ} = 2\vec{QS}$, and $\vec{RB} = 2\vec{AS}$, use vectors to show that *AQB* is a straight line.



6. Assuming that the opposite sides of a parallelogram are parallel and congruent, prove that vector addition has the commutative property. That is, prove that for any vectors $\vec{u}$ and $\vec{v}$, $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

7. If $k\vec{a} + m\vec{b} - (k + m)\vec{c} = \vec{0}$, prove that $(\vec{a} - \vec{c})$ and $(\vec{b} - \vec{c})$ are parallel vectors.

8. Points *A* and *B* are the midpoints of sides *PQ* and *SR* respectively of parallelogram *PQRS*. Prove that *PBRA* is a parallelogram.

9. In parallelogram *PQRS*, prove that $\vec{QS} + \vec{RP} = 2\vec{RS}$.

10. *D*, *E* and *F* are the midpoints of sides *AB*, *BC* and *CA* respectively of a triangle *ABC*. If *O* is any point, prove that $\vec{OA} + \vec{OB} + \vec{OC} = \vec{OD} + \vec{OE} + \vec{OF}$.

11. Given any quadrilateral *ABCD* (which may be skew), let *K*, *L*, *M*, and *N* be the midpoints of sides *AB*, *BC*, *CD*, and *DA* respectively. Prove that *KLMN* is a parallelogram.

12. Given a quadrilateral *ABCD* (which may be skew), let *M* be the midpoint of *AC* and *N* be the midpoint of *BD*.

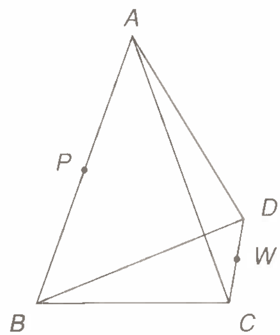
a) Show that $\vec{AB} + \vec{AD} + \vec{CB} + \vec{CD} = 4\vec{MN}$.

b) If *M* and *N* coincide, show that $\vec{AB} = \vec{DC}$.

c) What kind of a quadrilateral is *ABCD* if *M* and *N* coincide?

13. Given a tetrahedron (a triangular pyramid) *ABCD*, let *P* be the midpoint of *AB* and *W* be the midpoint of *CD*. *M* is the midpoint of *PW*. The position vectors of *A*, *B*, *C*, *D*, *M* from some origin *O* are $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{m}$ respectively.

Show that $\vec{m} = \frac{1}{4}(\vec{a} + \vec{b} + \vec{c} + \vec{d})$.



1.9 Unit Vectors— Standard Basis of a Vector Space

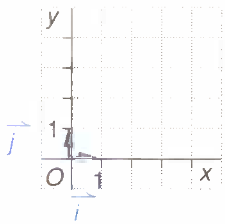
In this section, you will learn how to express vectors of $\mathbb{V}_2$ and $\mathbb{V}_3$ in terms of certain vectors whose length is 1 unit.

Any vector whose length is 1 is called a **unit vector.**

Example 1 Find the lengths of $\vec{i} = \overrightarrow{(1,0)}$ and $\vec{j} = \overrightarrow{(0,1)}$.

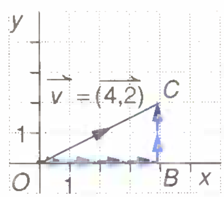
Solution $|\vec{i}| = \sqrt{1^2 + 0^2} = 1$ and similarly,
$|\vec{j}| = 1$.
$\vec{i}$ and $\vec{j}$ are thus unit vectors. ■

If you draw these vectors with their tails at the origin, O, you can see that they are unit vectors along the x and y axes respectively.



Example 2 Write the vector $\vec{v} = \overrightarrow{(4,2)}$ in terms of $\vec{i}$ and $\vec{j}$, using the operations of vector addition, and multiplication by a scalar.

Solution $\vec{v} = \overrightarrow{(4,2)}$ is the position vector of point $C(4,2)$.
$OB = 4$, so $\overrightarrow{OB} = 4\vec{i}$, and
$BC = 2$, so $\overrightarrow{BC} = 2\vec{j}$.
But $\vec{v} = \overrightarrow{OB} + \overrightarrow{BC}$
so $\overrightarrow{(4,2)} = 4\vec{i} + 2\vec{j}$. ■



A similar property holds true for any vector.

PROPERTY

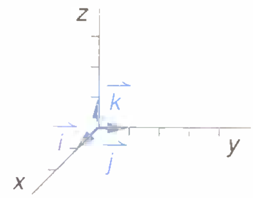
If $P = (x,y)$ is any point in 2-space, then
$\overrightarrow{OP} = x\vec{i} + y\vec{j}$ is the position vector of P or
$\overrightarrow{(x,y)} = x\vec{i} + y\vec{j}$

Every vector of $\mathbb{V}_2$ is equal to a position vector. Hence, every vector of $\mathbb{V}_2$ can be expressed in terms of the vectors $\vec{i}$ and $\vec{j}$. For this reason, $\vec{i}$ and $\vec{j}$ are called the **standard basis vectors of** $\mathbb{V}_2$.

In $\mathbb{V}_3$, the unit vectors along the x-axis, and y-axis, and z-axis are called $\vec{i}$, $\vec{j}$ and $\vec{k}$ respectively, where
$\vec{i} = \overrightarrow{(1,0,0)}$, $\vec{j} = \overrightarrow{(0,1,0)}$, and $\vec{k} = \overrightarrow{(0,0,1)}$.
Similarly, every vector of $\mathbb{V}_3$ can be expressed in terms of these three vectors according to the following property.

PROPERTY

If $P = (x,y,z)$ is any point in 3-space, then
$\overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}$ is the position vector of P or
$\overrightarrow{(x,y,z)} = x\vec{i} + y\vec{j} + z\vec{k}$



For this reason, $\vec{i}$, $\vec{j}$ and $\vec{k}$ are called the **standard basis vectors of $\mathbb{V}_3$.**

These basis vectors, whether in $\mathbb{V}_2$ or in $\mathbb{V}_3$, are all perpendicular or **orthogonal** to each other.
Furthermore, because they are all unit vectors, they are called **normed** vectors.

These words are combined and used to describe the set of vectors $\{\vec{i},\vec{j}\}$ as an **orthonormal basis of $\mathbb{V}_2$.**
Similarly, the set of vectors $\{\vec{i},\vec{j},\vec{k}\}$ is an **orthonormal basis of $\mathbb{V}_3$.**

You will study other bases of $\mathbb{V}_2$ and $\mathbb{V}_3$ further in chapter 2.

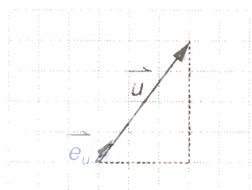
Other Unit Vectors

Example 3

a) Find a unit vector in the direction of $\vec{u} = \overrightarrow{(3,4)}$.
b) Verify that your solution is a unit vector.

Solution

a) The length $|\vec{u}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$. You must find a vector that has the same direction as $\vec{u}$, but whose length is only $\frac{1}{5}$ of $|\vec{u}|$. Call this vector $\vec{e}_u$: then $\vec{e}_u = \frac{1}{5}\vec{u} = \frac{1}{5}\overrightarrow{(3,4)} = \overrightarrow{\left(\frac{3}{5},\frac{4}{5}\right)}$.



b) $|\vec{e}_u| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{(9 + 16)}{25}} = \sqrt{1} = 1$, as required. ■

In general, given a vector $\vec{v}$,

the unit vector in the direction of $\vec{v}$ is $\vec{e}_v = \frac{1}{|\vec{v}|}\vec{v}$

Replacing $\vec{v}$ by its unit counterpart $\vec{e}_v$ is called **normalizing** $\vec{v}$.

Example 4 Normalize the following.

a) $\vec{v} = \overrightarrow{(-2,5)}$

b) $\vec{w} = \overrightarrow{(0.2,0.2,0.1)}$

Solution a) Let $\vec{e}_v$ be a unit vector in the direction of $\vec{v}$. Because $|\vec{v}| = \sqrt{(-2)^2 + 5^2} = \sqrt{29}$,

$$\vec{e}_v = \frac{1}{\sqrt{29}}\vec{v} = \frac{1}{\sqrt{29}}\overrightarrow{(-2,5)} = \overrightarrow{\left(-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)}.$$

b) Let $\vec{e}_w$ be a unit vector in the direction of $\vec{w}$. Because $|\vec{w}| = \sqrt{0.2^2 + 0.2^2 + 0.1^2} = \sqrt{0.09} = 0.3$,

$$\vec{e}_w = \frac{1}{0.3}\vec{w} = \frac{10}{3}\overrightarrow{(0.2,0.2,0.1)} = \overrightarrow{\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)}. \blacksquare$$

Example 5 Normalize $4\vec{v}$, where $\vec{v}$ is any vector.

Solution Let the unit vector in the direction of $4\vec{v}$ be $\vec{e}_{4v}$

$$\begin{aligned}\vec{e}_{4v} &= \frac{1}{|4\vec{v}|}(4\vec{v})\\ &= \frac{1}{|4\vec{v}|}4\vec{v}\\ &= \frac{\vec{v}}{|\vec{v}|}\\ &= \vec{e}_v. \blacksquare\end{aligned}$$

This last example shows that there is only *one* unit vector in the direction of $\vec{v}$. In other words, the unit vector in the direction of $\vec{v}$ does *not* depend on the length of $\vec{v}$.

Vector Spaces

The properties of vectors that you have learned so far will now allow you to define the following.

DEFINITION

A vector space $\mathbb{V}$ is a set of mathematical objects called vectors, together with two operations, called vector addition and multiplication by a scalar, having the following properties.

PROPERTIES

Vector Addition

A1. $\mathbb{V}$ is closed under addition: $\vec{u}, \vec{v} \in \mathbb{V}$ implies $\vec{u} + \vec{v} \in \mathbb{V}$

A2. Addition is associative: $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

A3. There is a $\vec{0} \in \mathbb{V}$ such that for all $\vec{u} \in \mathbb{V}$, $\vec{u} + \vec{0} = \vec{u}$

A4. If $\vec{u} \in \mathbb{V}$, then there exists $-\vec{u} \in \mathbb{V}$ such that $\vec{u} + (-\vec{u}) = \vec{0}$

A5. Addition is commutative: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(These properties mean that $\mathbb{V}$ is a **commutative group** with respect to addition.)

Multiplication of a Vector by a Scalar

M1. If $\vec{u} \in \mathbb{V}$, $k \in \mathbb{R}$, then $k\vec{u} \in \mathbb{V}$

M2. $(km)\vec{u} = k(m\vec{u})$, $k, m \in \mathbb{R}$

M3. $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

M4. $(k + m)\vec{u} = k\vec{u} + m\vec{u}$

M5. There exists $1 \in \mathbb{R}$ such that $1\vec{u} = \vec{u}$

Whenever you work with vectors, you are working in a "vector space". You have been doing just that in this entire chapter. What you have above is a summary of all the properties of vectors, valid for $\mathbb{V}_2$, and $\mathbb{V}_3$, and any other vector space. You may need to refer back to these properties in the chapters to come.

Note: Just as a vector is different from a point, a vector space is *not* a set of points such as $\mathbb{R}^2$ or $\mathbb{R}^3$. (There are no *points* in a vector space.)

SUMMARY

A vector whose length is 1 is a unit vector (or a normed vector).

$\{\vec{i},\vec{j}\}$, where $\vec{i} = \overrightarrow{(1,0)}$ and $\vec{j} = \overrightarrow{(0,1)}$,
is the standard basis for the vector space $\mathbb{V}_2$
$\{\vec{i},\vec{j},\vec{k}\}$, where $\vec{i} = \overrightarrow{(1,0,0)}$, $\vec{j} = \overrightarrow{(0,1,0)}$, and $\vec{k} = \overrightarrow{(0,0,1)}$,
is the standard basis for the vector space $\mathbb{V}_3$
These sets of vectors are called orthonormal bases.

In $\mathbb{V}_2$, $\overrightarrow{(p,q)} = p\vec{i} + q\vec{j}$
In $\mathbb{V}_3$, $\overrightarrow{(p,q,r)} = p\vec{i} + q\vec{j} + r\vec{k}$

To normalize $\vec{v}$ is to find the unit vector $\vec{e}_v$ in the direction of $\vec{v}$:

$$\vec{e}_v = \frac{1}{|\vec{v}|}\vec{v}$$

1.9 Exercises

1. Express the following vectors as ordered pairs.
   a) $\vec{i} + \vec{j}$
   b) $-5\vec{i} + \vec{j}$
   c) $-2\vec{j}$

2. Express the following vectors of $\mathbb{V}_2$ in terms of $\vec{i}$ and $\vec{j}$.
   a) $\vec{u} = \overrightarrow{(2,-7)}$
   b) $\vec{v} = \overrightarrow{(6,1)}$
   c) $\vec{w} = \overrightarrow{(-3,0)}$

3. Express the following vectors as ordered triples.
   a) $\vec{i} + 2\vec{j} + 3\vec{k}$
   b) $4\vec{i} - \vec{k}$
   c) $-\vec{j} - \vec{k}$

4. Express the following vectors of $\mathbb{V}_3$ in terms of $\vec{i}$, $\vec{j}$ and $\vec{k}$.
   a) $\vec{u} = \overrightarrow{(2,-4,6)}$
   b) $\vec{v} = \overrightarrow{(0,-1,-1)}$
   c) $\vec{w} = \overrightarrow{(0,10,0)}$

5. Simplify the following.
   a) $6(3\vec{i} - \vec{j})$
   b) $-2(-\vec{i} + \vec{j} - \vec{k})$
   c) $5(3\vec{i} - \vec{k})$
   d) $\sqrt{2}(\sqrt{2}\,\vec{i} + \sqrt{8}\,\vec{j})$

6. Simplify the following.
   a) $5(\vec{i} + \vec{j}) - 3(2\vec{i} - \vec{j})$
   b) $-(4\vec{i} - 2\vec{j} + \vec{k}) + 2(\vec{i} - 5\vec{k}) - 2\vec{j}$
   c) $\frac{1}{2}(3\vec{i} + 5\vec{j} - \vec{k}) + \frac{5}{2}(-\vec{i} - \vec{j} - 3\vec{k}) + \vec{i}$

7. Which of the following are unit vectors?
   a) $\vec{a} = \overrightarrow{\left(\frac{1}{2},\frac{1}{2}\right)}$
   b) $\vec{b} = \overrightarrow{\left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right)}$
   c) $\vec{c} = \overrightarrow{\left(\frac{1}{\sqrt{5}},0,-\frac{2}{\sqrt{5}}\right)}$
   d) $\vec{d} = \overrightarrow{\left(\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right)}$
   e) $\vec{e} = \overrightarrow{(0.2,0.4,0.4)}$
   f) $\vec{f} = \overrightarrow{\left(\frac{3}{13},-\frac{4}{13},\frac{12}{13}\right)}$

8. Normalize the following vectors.
   a) $\vec{u} = \overrightarrow{(3,-4)}$
   b) $\vec{v} = \overrightarrow{(0.1,0,0)}$
   c) $\vec{w} = \overrightarrow{(2,6,-1)}$
   d) $\vec{z} = \overrightarrow{(1,1,1)}$
   e) $\vec{p} = \overrightarrow{(\sqrt{7},3)}$
   f) $\vec{q} = \overrightarrow{(5,-4)}$

9. Given the non-zero vector $\vec{r} = \overrightarrow{(x,y,z)}$, prove that the unit vector in the direction of $\vec{r}$ is $\vec{e}_r = \overrightarrow{\left(\frac{x}{|\vec{r}|},\frac{y}{|\vec{r}|},\frac{z}{|\vec{r}|}\right)}$

10. Find the unit vector in the direction of $\overrightarrow{PQ}$ in each of the following cases.
    a) P is the point $(-3,6)$ and Q is the point $(4,1)$
    b) P is the point $(2,-3,5)$ and Q is the point $(1,-1,0)$

11. Given that $\vec{u} = \overrightarrow{(-5,12)}$, $\vec{v} = \overrightarrow{\left(\frac{3}{2},2,-6\right)}$ and $\vec{w} = \overrightarrow{(2,0,7)}$, find the unit vectors $\vec{e}_u$, $\vec{e}_v$, $\vec{e}_w$ in the directions of $\vec{u}$, $\vec{v}$, $\vec{w}$ respectively.

12. $\vec{e}_1$ and $\vec{e}_2$ are two perpendicular unit vectors. Calculate the following lengths.
    a) $|\vec{e}_1 + \vec{e}_2|$
    b) $|\vec{e}_1 + 2\vec{e}_2|$.

13. If $\vec{u} = \overrightarrow{(4,1,-2)}$ and $\vec{v} = \overrightarrow{(0,3,3)}$, find the unit vector in the direction of each of the following.
    a) $\vec{u} + \vec{v}$
    b) $3\vec{u} - \vec{v}$
    c) $\vec{u} - 4\vec{i} - \vec{j}$

14. For any $\vec{v} \in \mathbb{V}_3$ and any $k \in \mathbb{R}$, prove that the unit vector in the direction of $\vec{v}$ is equal to the unit vector in the direction of $k\vec{v}$.

15. If $\overrightarrow{OA} = 2\vec{i} - 3\vec{j} - \vec{k}$ and $\overrightarrow{OB} = \vec{i} + \vec{j} - \vec{k}$, show that the vector $\overrightarrow{AB}$ is parallel to the xy-plane.

16. If $\vec{u} = \overrightarrow{\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)}$ and $\vec{v} = \overrightarrow{\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)}$, prove that the vectors $\vec{u}$ and $\vec{v}$ form an orthonormal set.

In Search of Vectors as Classes of an Equivalence Relation

This section attempts to show how vectors as particular mathematical objects can be manufactured from elementary mathematical building blocks.

Relations and their Graphs

When you link certain elements between two sets, you say that you are setting up a **relation** between the two sets.

A relation is often described by a sentence.

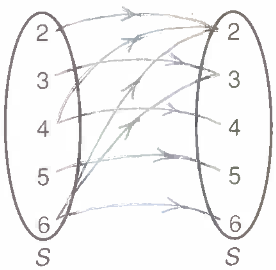
Example

"....is exactly divisible by...." between the set $S = \{2,3,4,5,6\}$ and itself creates the following relation.
$2 \rightarrow 2$, $3 \rightarrow 3$, $4 \rightarrow 2$, etc.,
where the arrow replaces the words "is exactly divisible by".

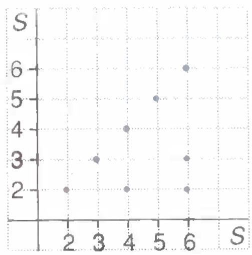
The set of all ordered pairs satisfying the sentence defines the relation, as follows.

$R = \{(2,2), (3,3), (4,2), (4,4), (5,5), (6,2), (6,3), (6,6)\}$

The relation can be graphed, or represented pictorially, in different ways. Here are two examples.

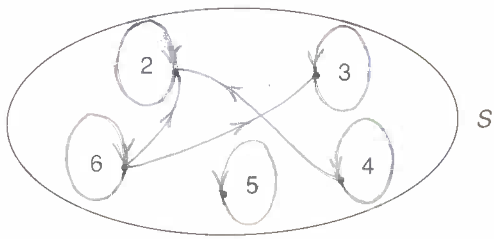


arrowgraph



Cartesian graph

However, a relation "from S to S" can also be called a relation "in S". The above relation in S can be graphed with the following special type of arrowgraph.



Equivalence Relation in a Set

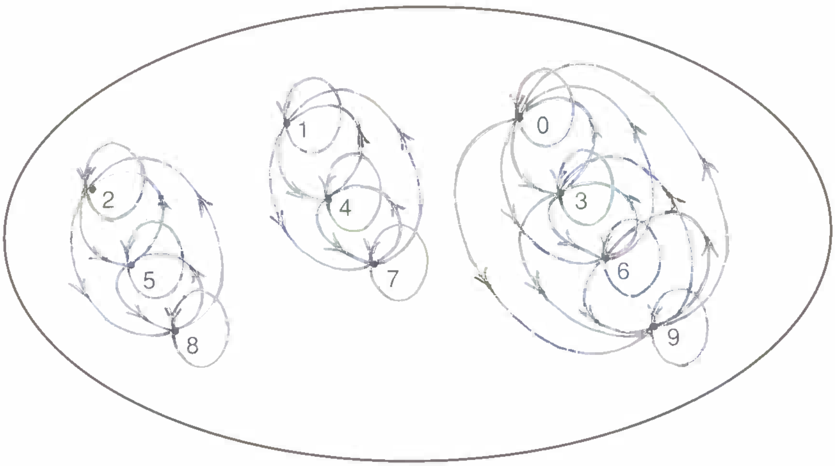
An equivalence relation in a set S has the following properties. It is

1. reflexive (for all $x \in S$, $x \rightarrow x$)
2. symmetric ($x \rightarrow y \Rightarrow y \rightarrow x$)
3. transitive ($x \rightarrow y$ and $y \rightarrow z \Rightarrow x \rightarrow z$)

Example

Consider the relation in the set of whole numbers, $\mathbb{W}$, defined by "...has the same remainder on division by 3 as...".

Observe what happens as the arrows are placed in a diagram.



- A set of subsets called a **partition** naturally forms.
- Each subset (or element of the partition) is called an **equivalence class** for the relation.

Good names for the equivalence classes here would be

{0,3,6,9,...} = "the class of 0" = *0*,

{1,4,7,...} = "the class of 1" = *1*, and

{2,5,8,...} = "the class of 2" = *2*.

You could call these classes "remainders".

Notice that *0* = *3*, *1* = *4*, etc., and that these classes are infinite sets.

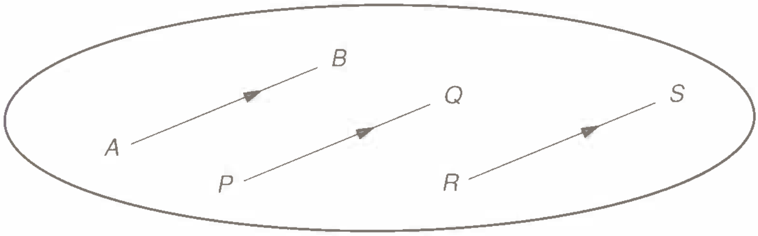
- The partition is the set of classes $\mathbb{W}_3 = \{0, 1, 2\}$.

(The classes in this example are sometimes called the "integers modulo 3")

Vectors

Consider the set D of all directed line segments in a plane. (Denote the directed line segment from A to B by $\underline{AB}$.)
In this set, consider the relation defined by
"$\underline{AB} \to \underline{PQ}$ if and only if $ABQP$ forms a parallelogram."

You will have the opportunity to verify that this is an equivalence relation.



The diagram shows that the directed line segments $\underline{AB}$, $\underline{PQ}$, $\underline{RS}$ satisfy the relation. There are many others! The "class of $\underline{AB}$" = $\{\underline{AB}, \underline{PQ}, \underline{RS}, \ldots\}$
Another name for this class is the vector $\overrightarrow{AB}$.

From this precise definition of a vector in a plane, you can see that 'drawing' a vector would actually cover the entire plane. Thus the vector is effectively *everywhere*. That is what allows you to draw a directed line segment representing the vector wherever you choose.

Activities

1. Verify that the relation defined by "$\underline{AB} \to \underline{PQ}$ if and only if $ABQP$ is a parallelogram", in the set of all directed line segments of a plane, is an equivalence relation.
   That is, verify that this relation is
   reflexive ($\underline{AB} \to \underline{AB}$),
   symmetric ($\underline{AB} \to \underline{PQ} \Rightarrow \underline{PQ} \to \underline{AB}$), and
   transitive ($\underline{AB} \to \underline{PQ}$ and $\underline{PQ} \to \underline{RS} \Rightarrow \underline{AB} \to \underline{RS}$)

2. Describe the partition created by each of the following equivalence relations.
   a) "is similar to" in the set of all triangles
   b) "has the same mother as" in the set of all Canadians
   c) "is in the same class as" in the set of children attending a public school
   d) "is parallel to" in the set of all straight lines
   e) "is congruent to" in the set of all line segments in a plane
   f) "$(p,q) \to (p',q')$ if and only if $p + q' = q + p'$" in the set of ordered pairs of whole numbers $\mathbb{W} \times \mathbb{W}$
   g) "$(p,q) \to (p',q')$ if and only if $pq' = qp'$" in the set of ordered pairs of integers $\mathbb{Z} \times [\mathbb{Z} - \{0\}]$ (that is, the ordered pairs (p,q) where p, q are integers and $q \neq 0$).

Summary

General Concepts

- Any number of parallel lines with arrows pointing the same way define a particular direction.
- Any real number is called a scalar, to distinguish it from a vector.
- A vector is everywhere: it can be represented by any directed line segment that has the correct magnitude and direction.

Equality and Length

- Equal vectors have the same magnitude and the same direction.
- If P is a point in a coordinate system of origin O, then $\overrightarrow{OP}$ is called the position vector of P.

in 2-space

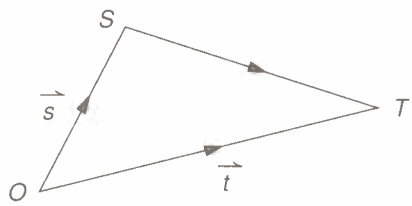
- If $P = (a,b)$, then $\overrightarrow{OP} = \overrightarrow{(a,b)}$
- The length of $\vec{v} = \overrightarrow{(x,y)}$ is $|\vec{v}| = \sqrt{x^2 + y^2}$

in 3-space

- If $P = (a,b,c)$, then $\overrightarrow{OP} = \overrightarrow{(a,b,c)}$
- The length of $\vec{v} = \overrightarrow{(x,y,z)}$ is $|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$

Vector Addition and Subtraction

- geometric law of vector addition (the triangle law)
  $\overrightarrow{OS} + \overrightarrow{ST} = \overrightarrow{OT}$



- geometric law of vector subtraction (the triangle law)
  $\overrightarrow{ST} = \overrightarrow{OT} - \overrightarrow{OS} = \vec{t} - \vec{s}$
- component laws of vector addition and vector subtraction

in $\mathbb{V}_2$

$\overrightarrow{(a,b)} + \overrightarrow{(p,q)} = \overrightarrow{(a+p,b+q)}$

$\overrightarrow{(a,b)} - \overrightarrow{(p,q)} = \overrightarrow{(a-p,b-q)}$

in $\mathbb{V}_3$

$\overrightarrow{(a,b,c)} + \overrightarrow{(p,q,r)} = \overrightarrow{(a+p,b+q,c+r)}$

$\overrightarrow{(a,b,c)} - \overrightarrow{(p,q,r)} = \overrightarrow{(a-p,b-q,c-r)}$

Multiplication of a Vector by a Scalar

- If k is a scalar and $\vec{u}$ is a vector, then $k\vec{u}$ is a vector parallel to $\vec{u}$, whose length is $|k||\vec{u}|$
  ($\vec{u}$ and $k\vec{u}$ have the same direction if $k > 0$, but have opposite directions if $k < 0$.)
- In component form, $k\overrightarrow{(x,y)} = \overrightarrow{(kx,ky)}$ in $\mathbb{V}_2$ or $k\overrightarrow{(x,y,z)} = \overrightarrow{(kx,ky,kz)}$ in $\mathbb{V}_3$.

Unit Vectors

- Any vector whose length is 1 is a unit vector (or a normed vector).
- To normalize $\vec{v}$ is to find the unit vector $\vec{e}_v$ in the direction of $\vec{v}$:
  $\vec{e}_v = \frac{1}{|\vec{v}|}\vec{v}$
- Given the standard basis vectors
  $\vec{i} = \overrightarrow{(1,0)}$ and $\vec{j} = \overrightarrow{(0,1)}$ of $\mathbb{V}_2$, then $\overrightarrow{(p,q)} = p\vec{i} + q\vec{j}$

  $\vec{i} = \overrightarrow{(1,0,0)}$, $\vec{j} = \overrightarrow{(0,1,0)}$ and $\vec{k} = \overrightarrow{(0,0,1)}$ of $\mathbb{V}_3$, then $\overrightarrow{(p,q,r)} = p\vec{i} + q\vec{j} + r\vec{k}$

Vector Spaces

A vector space $\mathbb{V}$ is a set of mathematical objects called vectors, together with two operations, called vector addition and multiplication by a scalar, having the following properties.

Vector Addition

A1. $\mathbb{V}$ is closed under addition: $\vec{u}, \vec{v} \in \mathbb{V}$ implies $\vec{u} + \vec{v} \in \mathbb{V}$

A2. Addition is associative: $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

A3. There is a $\vec{0} \in \mathbb{V}$ such that for all $\vec{u} \in \mathbb{V}$, $\vec{u} + \vec{0} = \vec{u}$

A4. If $\vec{u} \in \mathbb{V}$, then there exists $-\vec{u} \in \mathbb{V}$ such that $\vec{u} + (-\vec{u}) = \vec{0}$

A5. Addition is commutative: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(These properties mean that $\mathbb{V}$ is a commutative group with respect to addition.)

Multiplication of a Vector by a Scalar

M1. If $\vec{u} \in \mathbb{V}$, $k \in \mathbb{R}$, then $k\vec{u} \in \mathbb{V}$

M2. $(km)\vec{u} = k(m\vec{u})$, $k, m \in \mathbb{R}$

M3. $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

M4. $(k + m)\vec{u} = k\vec{u} + m\vec{u}$

M5. There exists $1 \in \mathbb{R}$ such that $1\vec{u} = \vec{u}$

Inventory

Complete each of the following statements.

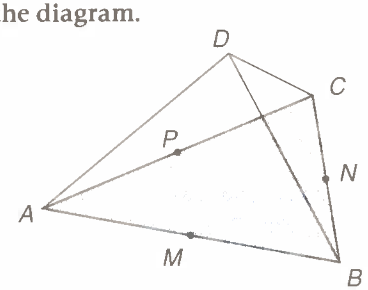
1. A number of arrowed parallel lines with the arrows pointing the same way is said to define a ________.
2. A ________ has both length and direction.
3. A real number is called a ________, to distinguish it from a vector.
4. A vector can be represented by a ________ line segment. A vector can also be represented by an ________ or an ________ of numbers.
5. Two directed line segments represent the same vector if they have the same ________ and the same ________.
6. If $\overrightarrow{(x,2)} = \overrightarrow{(-3,y)}$, then $x =$ ________ and $y =$ ________.
7. A plane can be determined by three distinct non-collinear ________, or by two distinct ________ lines.
8. In a three-dimensional mathematical drawing, you must keep ________ vertical, and ________ parallel.
9. Skew lines are lines that are neither ________ nor ________.
10. Given $P(2,-3,4)$, then the position vector $\overrightarrow{OP} =$ ________.
11. If $\vec{v} = \overrightarrow{(2,2,-1)}$, then the length of the vector $\vec{v}$, written ________, is equal to ________.
12. The vector sum $\overrightarrow{(1,-7)} + \overrightarrow{(-1,9)} =$ ________.
13. The vector sum $\overrightarrow{FG} + \overrightarrow{GH} =$ ________.
14. The vector difference $\overrightarrow{NK} - \overrightarrow{NL} =$ ________.
15. The product of the scalar 5 and the vector $\overrightarrow{(2,1,4)}$ is ________.
16. If $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{CD} = k\vec{u}$, where k is a scalar, then the lines AB and CD are ________. The length of the vector $\overrightarrow{ku}$ is ________.
17. Given the points $P(3,8)$ and $Q(1,6)$, the vector $\overrightarrow{PQ}$ in component form is ________.
18. A vector of length one is called a ________ vector.
19. The vector $\vec{v} = \overrightarrow{(4,-3)}$ can be expressed in terms of the standard basis vectors $\vec{i}$ and $\vec{j}$ as follows. $\vec{v} =$ ________.
20. Normalizing $\vec{v}$ means finding the ________ vector $\vec{e}_v$ in the direction of $\vec{v}$. If $\vec{v} = \overrightarrow{(4,-3)}$, then $\vec{e}_v =$ ________.

Review Exercises

1. M is the midpoint of segment AB, P is the midpoint of AM, Q is the midpoint of MB.
   a) Give reasons why $\overrightarrow{AP} = \overrightarrow{PM}$.
   b) State all other vectors in the diagram equal to $\overrightarrow{AP}$.
   c) State all vectors in the diagram equal to $\overrightarrow{BM}$.

2. In the square $ABCD$, P, Q, R and S are the midpoints of AB, BC, CD, and DA respectively. If $\overrightarrow{AP} = \vec{u}$ and $\overrightarrow{BQ} = \vec{v}$, express the following in terms of $\vec{u}$ and $\vec{v}$.
   a) $\overrightarrow{PB}$ b) $\overrightarrow{RC}$ c) $\overrightarrow{DR}$ d) $\overrightarrow{QC}$ e) $\overrightarrow{AS}$

3. $ABCD$ is a **regular tetrahedron**, that is, a solid made up of four congruent equilateral triangular faces. Its apex, D, is vertically above the centre of the base. The midpoints of AB, BC, and CA are M, N, and P respectively.
   a) Draw the tetrahedron, and locate the centroid O of its base (the intersection of the medians AN, BP, and CM of the triangular base).
   b) Join DO, and mark the right angles DOA, DOB, and DOC.
   c) List the other right angles defined in the diagram.



4. In the tetrahedron of question 3, calculate the following.
   a) the angle between two edges (such as angle DAB)
   b) the angle between an edge and a face (such as angle DAN)

5. In the tetrahedron of question 3, name the following.
   a) a pair of skew lines
   b) the three planes intersecting at point D

6. In a 3-space coordinate system, draw the position vector $\vec{p}$ of the point $P(-1,2,3)$.

7. A point P, whose position vector is $\overrightarrow{OP} = \overline{(5,-2,-4)}$, is translated to position P' by the vector $\vec{v} = \overline{(1,2,3)}$. What are the coordinates of P'?

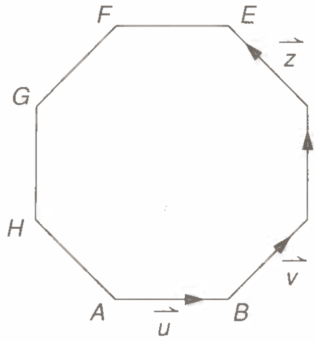
8. Find k so that the length of the vector $\vec{v} = \overline{\left(\frac{1}{2}, -\frac{2}{3}, k\right)}$ is one unit.

9. Given that $\vec{u} = \overline{(-2,2,5)}$ and $\vec{v} = \overline{(k,-1,-3)}$, calculate k, where $2|\vec{u}| = |\vec{v}|$.

10. An object is displaced in a plane according to the vector $\vec{u} = \overline{(2,-3)}$, then displaced again according to the vector $\vec{v} = \overline{(6,1)}$.
    a) Calculate the resultant displacement, $\vec{w}$.
    b) Draw the three displacements $\vec{u}$, $\vec{v}$, $\vec{w}$ on a grid.
    c) Calculate $|\vec{w}|$.

11. Given the regular octogon *ABCDEFGH* shown, where $\vec{AB} = \vec{u}$, $\vec{BC} = \vec{v}$, $\vec{CD} = \vec{w}$, and $\vec{DE} = \vec{z}$, find the following vectors in terms of $\vec{u}$, $\vec{v}$, $\vec{w}$, and $\vec{z}$.

a) $\vec{FE}$
b) $\vec{GF}$
c) $\vec{AF}$
d) $\vec{GE}$
e) $\vec{AE}$



12. Given $\vec{u} = \vec{(k,4,m)}$, $\vec{v} = \vec{(1,n,-8)}$, and $\vec{u} + \vec{v} = \vec{(-3,4,-2)}$, find the values of k, m and n.

13. A boat starts at a point *A* and sails 600 m on bearing 045°, then 400 m towards the east to arrive at point *B*. Draw a vector diagram to represent the resultant displacement $\vec{AB}$. What is the magnitude of this displacement? What is the bearing of *B* from *A*?

14. Simplify the following by using the subtraction form of the triangle law.

a) $\vec{QP} - \vec{QR}$
b) $\vec{XA} - \vec{XY} + \vec{AY}$

15. Given the points $L(5,6,-3)$ and $M(-8,11,4)$, use vector subtraction to determine the vector $\vec{LM}$ in component form.

16. Given any four points *A*, *B*, *C*, and *D* in space, express the following vectors in subtraction form, using position vectors with origin *A*.

a) $\vec{BC}$ b) $\vec{CD}$ c) $\vec{BA}$

17. *PQRS* is a parallelogram and *O* is any point. If $\vec{OP} = \vec{p}$, $\vec{OR} = \vec{r}$, and $\vec{OS} = \vec{s}$, express the vector $\vec{QO}$ in terms of $\vec{p}$, $\vec{r}$, and $\vec{s}$.

18. Given $\vec{u} = \vec{(5,-6,-3)}$, $\vec{v} = \vec{(2,0,4)}$, $\vec{w} = \vec{(-1,2,-5)}$, find the following.

a) $\vec{u} - \vec{v}$
b) $\vec{v} - \vec{w}$
c) $\vec{w} - \vec{u}$
d) $2\vec{u} - 3\vec{v}$
e) $\vec{u} + 2\vec{v} + 3\vec{w}$
f) $\frac{1}{2}\vec{u} - \vec{v} - 2\vec{w}$

19. The points *P*, *Q*, *R* are such that $\vec{PQ} = \vec{(4,-1,2)}$ and $\vec{PR} = \vec{(12,-3,6)}$.

a) Express $\vec{QR}$ in terms of $\vec{PQ}$.
b) What can you say about the points *P*, *Q* and *R*?

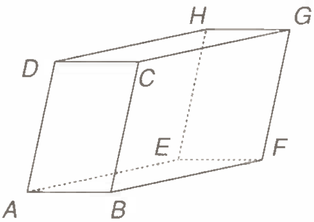
20. Find the coordinates of the points *P*, *Q*, and *R*, of question 19, given that $\vec{OP} = \vec{(3,0,-1)}$.

21. Given the two vectors $\vec{u}$ and $\vec{v}$, and $|\vec{v}| = 5$, find the value of $|\vec{u} + \vec{v}|$ in each of the following cases.

a) $3\vec{v} = 2\vec{u}$
b) $\vec{v} = -\vec{u}$
c) $\vec{u}$ and $\vec{v}$ are perpendicular, and $|\vec{u}| = 2$

22. Given the parallelepiped shown, state a single vector equal to each of the following.

a) $\vec{AC} - \vec{AD}$
b) $\vec{AB} - \vec{AD}$
c) $\vec{BF} - \vec{CG}$
d) $\vec{AC} - \vec{EH}$
e) $\vec{HD} - \vec{GB}$



23. The diagonals of a quadrilateral bisect each other. Prove that the quadrilateral is parallelogram.

24. *ABCD* is a rectangle and *PQCD* is a parallelogram. Prove that *ABQP* is a parallelogram.

25. Simplify the following.
 a) $6(-\vec{i} + 2\vec{j}) - 3(3\vec{i} - 4\vec{j})$
 b) $10(4\vec{i} + 2\vec{j} - 5\vec{k})$
 $-\frac{1}{2}(16\vec{i} - 4\vec{j})$
 $-6\vec{j} + 3\vec{k}$

26. Find the unit vector in the direction of $\overrightarrow{PQ}$ in each of the following cases.
 a) P is the point $(-1,-2)$ and Q is the point $(1,3)$
 b) P is the point $(4,3,-3)$ and Q is the point $(6,-5,-1)$
 c) P is the point (a,b,c) and Q is the point (d,e,f)

27. $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ form an orthonormal set. Calculate the following lengths.
 a) $|\vec{e}_1 + \vec{e}_3|$
 b) $|\vec{e}_1 + \vec{e}_2 + \vec{e}_3|$
 c) $|\vec{e}_2 + 3\vec{e}_3|$
 d) $|\vec{e}_1 - 2\vec{e}_2 - 3\vec{e}_3|$

28. In triangle OAB, points P and Q divide the side AB into three equal segments with P closer to vertex A than to B.
 a) Show that $\overrightarrow{AQ} = \overrightarrow{PB}$.
 b) Prove that $\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{OB}$.

29. A vector having the same magnitude as $-6\vec{i} + 8\vec{k}$ is

 A. $3\vec{i} + 4\vec{j} - 5\vec{k}$

 B. $-3\vec{i} + 4\vec{k}$

 C. $-6\vec{i} + 4\vec{j} + 4\vec{k}$

 D. $10\vec{i} + 10\vec{j} + 10\vec{k}$

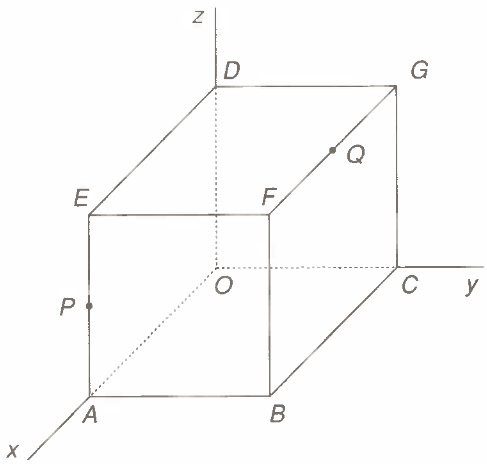
 E. $-10\vec{j}$

 *(83 H)

*For information about these questions, see the introductory pages of this book.

30. In a three dimensional rectangular Cartesian co-ordinate system, the points O, A, C and D have co-ordinates $(0,0,0)$, $(6,0,0)$, $(0,6,0)$ and $(0,0,6)$ respectively. $OABCGFED$ is a cube, as shown in the figure.

The points P and Q are the mid-points of $[AE]$ and $[FG]$ respectively.



 a) Write, in column vector form, each of the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.
 b) i. Find the square of the length of each of the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.
 ii. Show that $PQ = 3\sqrt{6}$ units.
 iii. Examine whether OPQ is a right angled triangle, giving a reason for your result.
 c) Given that the points R and S are the midpoints of $[BP]$ and $[CQ]$ respectively,
 i. prove that $\overrightarrow{OR} = \frac{1}{2}\overrightarrow{OP} + \frac{1}{2}\overrightarrow{OB}$ and
 ii. find the length of RS.
 d) An ant walks from P to Q along the surface of the cube. By considering the net of the cube, or otherwise, find how far the ant walks, given that the distance travelled must be as small as possible.

(87 SMS)

VECTORS, MATRICES and COMPLEX NUMBERS

with
International Baccalaureate
questions

Jean-Paul GINESTIER
and
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CHAPTER TWO

LINEAR DEPENDENCE

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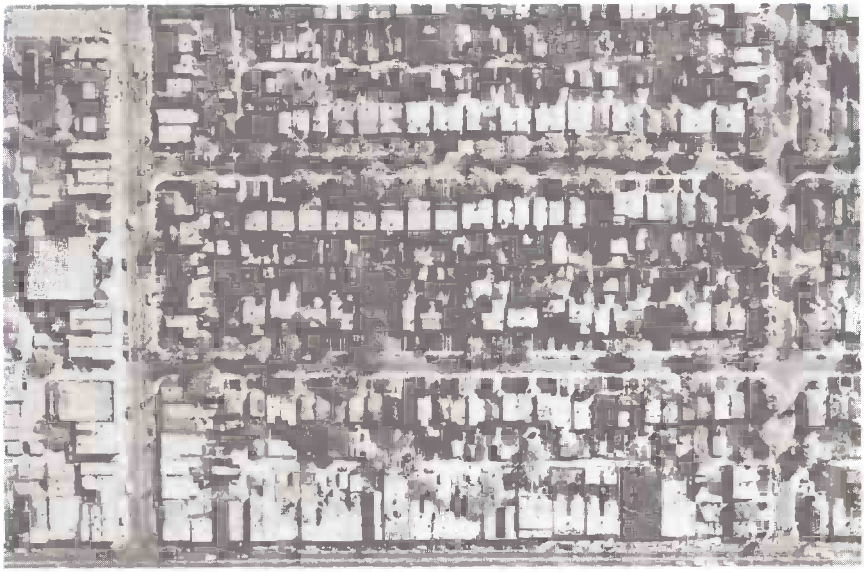
CHAPTER TWO

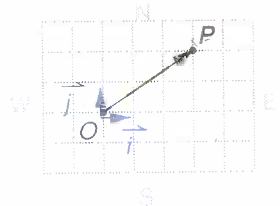
Linear Dependence



The photographs show two cities A and B. The streets of city A run east to west and south to north. The streets of city B go east to west and southwest to northeast.

If you wish to locate an intersection P in city A you can use origin O and position vector $\overrightarrow{OP}$. The diagram shows two perpendicular unit vectors $\vec{i}$ and $\vec{j}$ where $|\vec{i}|$ equals one city block east to west, and $|\vec{j}|$ equals one city block south to north.

City A



Since intersection P is three blocks east and two blocks north of O, $\overrightarrow{OP} = 3\vec{i} + 2\vec{j}$.

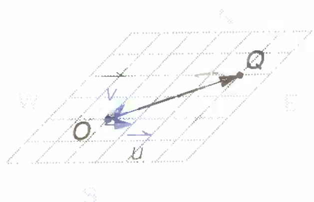
Every intersection point in city A can be expressed as a *combination* of a scalar times $\vec{i}$, plus a scalar times $\vec{j}$. Such a combination is called a **linear combination** of vectors $\vec{i}$ and $\vec{j}$.

Observe that the vector $3\vec{i}$ depends on vector $\vec{i}$. Vector $3\vec{i}$ is described as being **linearly dependent** with vector $\vec{i}$.

Recall that $3x + 2y$ is a *linear* expression, and that $3x + 2y = k$ defines a *linear* relation.

To locate an intersection Q in city B you will need origin O and two unit vectors $\vec{u}$ and $\vec{v}$. These vectors are *not* perpendicular to each other. Observe that $|\vec{u}|$ is one city block east to west while $|\vec{v}|$ is one city block southwest to northeast.

City B



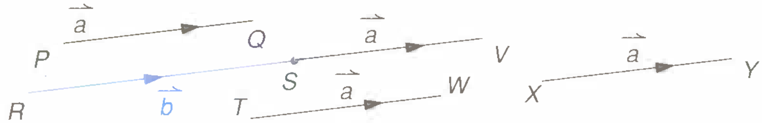
Since the point Q is three blocks east and two blocks northeast of point O, $\overrightarrow{OQ} = 3\vec{u} + 2\vec{v}$. Again vector $\overrightarrow{OQ}$ is called a linear combination of vectors $\vec{u}$ and $\vec{v}$, and the vector $3\vec{u}$ is described as being linearly dependent with vector $\vec{u}$.

In this chapter you will learn more about the linear combinations of vectors and about the dependence of vectors on each other.

2.1 Linear Dependence of Two Vectors

Vectors are not directed line segments but vectors can be represented by directed line segments. A vector in $\mathbb{V}_2$ can also be represented by an ordered pair; a vector in $\mathbb{V}_3$ by an ordered triple. There is a relationship that is derived from the directed line segment representation of a vector. This relationship is called *linear dependence*.

Suppose $\vec{a}$ and $\vec{b}$ can be represented by the parallel directed line segments $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ respectively.



In this situation you can write $\vec{a} \| \vec{b}$. But every directed line segment which is congruent to, parallel to, and in the same direction as $\overrightarrow{PQ}$ represents the vector $\vec{a}$. In particular, the directed line segment $\overrightarrow{SV} = \vec{a}$. Because the three points R, S, and V are collinear, $\vec{a}$ and $\vec{b}$ are called *collinear vectors*. Note that $\vec{a}$ and $\vec{b}$ can also be described as being parallel. Thus, for vectors *there is no distinction made between vectors that are parallel and vectors that are collinear*. The symbol $\|$ when used with vectors can be read either "parallel to" or "collinear with".

Two vectors that are collinear or parallel are *linearly dependent vectors*. *The zero vector $\vec{0}$ is parallel and linearly dependent with every other vector.*

DEFINITION

Two vectors $\vec{a}$ and $\vec{b}$ are linearly dependent if and only if $\vec{a} \| \vec{b}$.

What is the algebraic condition for two vectors to be linearly dependent? In section 1.5 you learned that every scalar multiple of $\vec{a}$ is parallel to $\vec{a}$. Hence $\vec{a}$ and $s\vec{a}$, $s \in \mathbb{R}$, are linearly dependent. Also, any two vectors that are linearly dependent are parallel, and so one vector must be a scalar multiple of the other vector.

An Algebraic Condition for Two Vectors to be Linearly Dependent

Two non-zero vectors $\vec{a}$ and $\vec{b}$ are linearly dependent if and only if $\vec{b} = s\vec{a}$, for some $s \in \mathbb{R}$.

Example 1 Given $\vec{c} = 3\vec{a}$ and $\vec{d} = 2\vec{a}$, prove that $\vec{c}$ and $\vec{d}$ are linearly dependent.

Solution

Geometric Proof

You must prove that $\vec{c} \| \vec{d}$.

Since $\vec{c}$ and $\vec{d}$ are each scalar multiples of $\vec{a}$, each vector is parallel to $\vec{a}$.

But $\vec{c} \| \vec{a}$ and $\vec{d} \| \vec{a}$ implies that $\vec{c} \| \vec{d}$.

Thus, $\vec{c}$ and $\vec{d}$ are linearly dependent.

Algebraic Proof

You must prove that $\vec{c}$ is a scalar multiple of $\vec{d}$ or that $\vec{d}$ is a scalar multiple of $\vec{c}$.

Since $\vec{c} = 3\vec{a}$, therefore $\vec{a} = \frac{1}{3}\vec{c}$.

Thus, $\vec{d} = 2\vec{a} = 2\left(\frac{1}{3}\vec{c}\right) = \frac{2}{3}\vec{c}$.

Therefore, $\vec{c}$ and $\vec{d}$ are linearly dependent. ■

In Example 1, you proved that

$\vec{d} = \frac{2}{3}\vec{c}$ or

$3\vec{d} = 2\vec{c}$ or

$3\vec{d} + (-2)\vec{c} = \vec{0}$.

Thus, real numbers $m = 3$, and $k = -2$ exist such that $m\vec{d} + k\vec{c} = \vec{0}$. This leads to the following alternate form of the first algebraic condition.

Another Algebraic Condition for Two Vectors to be Linearly Dependent.

Two vectors $\vec{a}$ and $\vec{b}$ are linearly dependent if and only if $m, k \in \mathbb{R}$ exist, not both equal to 0, such that $m\vec{a} + k\vec{b} = \vec{0}$.

Proof of this algebraic condition

Part 1 Given $\vec{a}$ and $\vec{b}$ are linearly dependent,
prove $m, k \in \mathbb{R}$ exist, not both equal to 0, such that $m\vec{a} + k\vec{b} = \vec{0}$.

Proof: $\vec{a}$ and $\vec{b}$ linearly dependent $\Rightarrow \vec{b} = p\vec{a}$, for some $p \in \mathbb{R}$

$\Rightarrow \vec{b} - p\vec{a} = \vec{0}$

$\Rightarrow (-p)\vec{a} + (1)\vec{b} = \vec{0}$

Therefore, $m = -p$, $k = 1$ exist, not both 0, such that $m\vec{a} + k\vec{b} = \vec{0}$

Part 2 Given $m, k \in \mathbb{R}$ exist, not both equal to 0, such that $m\vec{a} + k\vec{b} = \vec{0}$, prove $\vec{a}$ and $\vec{b}$ are linearly dependent.

Proof: At least one of k and m cannot be 0.

Suppose $k \neq 0$, then $m\vec{a} + k\vec{b} = \vec{0}$ can be written

$k\vec{b} = -m\vec{a}$ or

$\vec{b} = -\frac{m}{k}\vec{a}$

Thus, $\vec{a}$ and $\vec{b}$ are linearly dependent.

Note: If m and k are both zero, the statement $m\vec{a} + k\vec{b} = \vec{0}$ would imply that $0\vec{a} + 0\vec{b} = \vec{0}$. This statement is true for all vectors $\vec{a}$ and $\vec{b}$, whether or not they are linearly dependent.

Example 2 Which of the following vectors are linearly dependent with $\vec{a} = \overrightarrow{(1,2,-3)}$?

$\vec{b} = \overrightarrow{(4,8,-12)}$, $\vec{c} = \overrightarrow{\left(-\frac{1}{2},-1,\frac{3}{2}\right)}$, $\vec{d} = \overrightarrow{(2,4,-5)}$

Solution You must check to see which vectors are scalar multiples of vector $\vec{a}$.

$\vec{b} = \overrightarrow{(4,8,-12)} = 4\overrightarrow{(1,2,-3)} = 4\vec{a}$

Therefore, $\vec{b}$ and $\vec{a}$ are linearly dependent.

$\vec{c} = \overrightarrow{\left(-\frac{1}{2},-1,\frac{3}{2}\right)} = -\frac{1}{2}\overrightarrow{(1,2,-3)} = -\frac{1}{2}\vec{a}$

Therefore, $\vec{c}$ and $\vec{a}$ are linearly dependent.

If $\vec{d} = k\vec{a}$, $k \in \mathbb{R}$

then $\overrightarrow{(2,4,-5)} = k\overrightarrow{(1,2,-3)} = \overrightarrow{(k,2k,-3k)}$

and $2 = k$, $4 = 2k$, $-5 = -3k$

This forces $k = 2$ and $k = \frac{5}{3}$ at the same time, which is impossible.

Therefore, $\vec{d}$ and $\vec{a}$ are not linearly dependent. ■

DEFINITION Two vectors that are not linearly dependent are *linearly independent*.

Important Facts about Two Linearly Independent Vectors

Geometric

If $\vec{a}$ and $\vec{b}$ are linearly independent, then $\vec{a} \nparallel \vec{b}$.

Algebraic

1. No scalar p exists such that $\vec{a} = p\vec{b}$.
2. If $\vec{a}$ and $\vec{b}$ are linearly independent *and* $m\vec{a} + k\vec{b} = \vec{0}$ then $m = k = 0$.

Intuitively, this last statement says that the only way that you can add two non-parallel vectors to obtain the vector $\vec{0}$ is to multiply each vector by the scalar 0.

Example 3 The two vectors $\vec{a}$ and $\vec{b}$ are linearly independent. If $x\vec{a} + (y-3)\vec{b} = \vec{0}$, then find the values of x and y.

Solution Since $\vec{a}$ and $\vec{b}$ are linearly independent, and

$x\vec{a} + (y-3)\vec{b} = \vec{0}$, then

$x = 0$, and $y - 3 = 0$.

Thus $x = 0$, and $y = 3$. ■

2.1 Exercises

1. Vectors $\vec{a}$ and $\vec{b}$ are linearly dependent.
   a) What is the geometric relationship between $\vec{a}$ and $\vec{b}$?
   b) State two algebraic equations that are true relating $\vec{a}$ and $\vec{b}$.
   c) What conditions, if any, are imposed on the scalars in the equations in b)?

2. You are given two vectors $\vec{x}$ and $\vec{y}$ such that $\vec{x} \| \vec{y}$. What is the algebraic relationship between $\vec{x}$ and $\vec{y}$?

3. Use the fact that a vector as a directed line segment can be drawn anywhere to explain how two collinear vectors can be represented by two line segments that are not collinear.

4. If $\vec{z} = w\vec{d}$ where $w \in \mathbb{R}$, then how are vectors $\vec{z}$ and $\vec{d}$ related?

5. Scalars s and t exist, not both 0, such that $s\vec{m} + t\vec{k} = \vec{0}$. How are vectors $\vec{m}$ and $\vec{k}$ related geometrically?

6. Points P and Q are such that $\overrightarrow{PQ} = \vec{a} \neq \vec{0}$.
   a) If R is any point on line PQ explain why $\vec{a}$ and $\overrightarrow{PR}$ are linearly dependent.
   b) If T is any point not on line PQ explain why $\vec{a}$ and $\overrightarrow{PT}$ are linearly independent.

7. Given that $\vec{a} \| \vec{c}$, $\vec{c} \nparallel \vec{b}$, and $\vec{d} = 5\vec{c}$, which of the following vectors are linearly dependent with $\vec{a}$?
   a) $\vec{u} = 3\vec{a}$
   b) $\vec{v} = -2\vec{b}$
   c) $\vec{w} = 7\vec{c}$
   d) $\vec{r} = \pi\vec{c}$
   e) $\vec{t} = -6\vec{d}$?

8. List all sets of parallel vectors from among the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{u}$, $\vec{v}$, $\vec{w}$, $\vec{r}$, and $\vec{t}$ of question 7.

9. Given $\vec{a} = \overrightarrow{(2,3)}$,
   a) write three vectors collinear with $\vec{a}$
   b) write three vectors linearly dependent with $\vec{a}$
   c) write one vector linearly independent with $\vec{a}$.

10. Given $\vec{b} = \overrightarrow{(4,1,3)}$,
   a) write three vectors collinear with $\vec{b}$
   b) write three vectors linearly dependent with $\vec{b}$
   c) write one vector linearly independent with $\vec{b}$.

11. $\vec{a}$ and $\vec{b}$ are linearly independent. Use the following equations to name vectors that are linearly dependent with $\vec{a}$.
   $7\vec{c} + 4\vec{a} = \vec{0}$
   $5\vec{b} + 2\vec{d} = \vec{0}$
   $m\vec{a} - k\vec{e} = \vec{0}$, $m, k \in \mathbb{R}, m \neq 0$.

12. a) $\vec{a}$ and $\vec{b}$ are linearly independent. What is the geometric relationship between $\vec{a}$ and $\vec{b}$?
   b) $\vec{x} \nparallel \vec{y}$. Are $\vec{x}$ and $\vec{y}$ necessarily linearly independent? Explain.

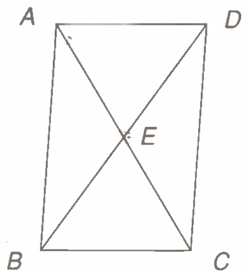
13. a) $\vec{p}$ and $\vec{q}$ are linearly independent vectors and $r\vec{p} + v\vec{q} = \vec{0}$. What conclusion can you draw about the scalars r and v?
   b) Scalars m and k exist, both equal to 0, such that $m\vec{a} + k\vec{b} = \vec{0}$. Are $\vec{a}$ and $\vec{b}$ necessarily linearly independent?

14. Which of the following pairs of vectors are linearly dependent? Justify your answer.
   a) $\overrightarrow{(1,2)}$, $\overrightarrow{(4,8)}$
   b) $\overrightarrow{(-3,2)}$, $\overrightarrow{(-6,3)}$
   c) $\overrightarrow{(8,-2)}$, $\overrightarrow{\left(-1,\frac{1}{4}\right)}$
   d) $\overrightarrow{(3,6,2)}$, $\overrightarrow{(6,12,4)}$
   e) $\overrightarrow{(-1,-1,6)}$, $\overrightarrow{(6,6,-36)}$
   f) $\overrightarrow{(4,0,1)}$, $\overrightarrow{(4,1,1)}$

15. $ABCD$ is a parallelogram. Name vectors linearly dependent with each of the following.

a) $\overrightarrow{AB}$ c) $\overrightarrow{AE}$

b) $\overrightarrow{AC}$ d) $\overrightarrow{BD}$



16. Explain why each of the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{BD}$ from question 15 is linearly independent with each of the other vectors.

17. Given the points $P(-2,4)$, $Q(-3,7)$, and $R(-4,10)$,

a) write $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ in component form

b) prove that $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are linearly dependent

c) use part b) to draw a geometric conclusion about the points P, Q, and R.

18. Given the points $A(1,3,-2)$, $B(5,5,4)$, and $C(-1,2,-5)$,

a) write $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in component form

b) prove that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly dependent

c) use part b) to draw a geometric conclusion about the points A, B, and C.

19. a) $\vec{a} \nparallel \vec{b}$. For what real numbers m and k is $m\vec{a} + k\vec{b} = \vec{0}$? Are $\vec{a}$ and $\vec{b}$ linearly dependent?

b) $\vec{a} \parallel \vec{b}$, and $|\vec{a}| = |\vec{b}|$. For what real numbers m and k is $m\vec{a} + k\vec{b} = \vec{0}$? Are $\vec{a}$ and $\vec{b}$ linearly dependent?

20. Given that $\vec{a}$ and $\vec{b}$ are linearly independent, find the values of the scalars in each of the following.

a) $s\vec{a} + t\vec{b} = \vec{0}$

b) $r\vec{a} + (3 - m)\,\vec{b} = \vec{0}$

c) $(x - 1)\vec{a} + (y + 2)\,\vec{b} = \vec{0}$

d) $(2z - 6)\,\vec{a} + (7 + 3k)\,\vec{b} = \vec{0}$

21. $\vec{a}$ and $\vec{b}$ are linearly independent non-zero vectors such that $3\vec{a} + k\vec{b} = m\vec{a} - 5\vec{b}$. Find the values of the scalars k and m.

22. $\vec{p}$ and $\vec{q}$ are linearly independent non-zero vectors where $5c\vec{p} + d\vec{r} - 6\vec{q} = \vec{0}$ and $3\vec{q} + 2\vec{p} + \vec{r} = \vec{0}$. Find the values of the real numbers c and d.

23. Given that
$\vec{a} = 3\vec{b} - 2\vec{c} + 4\vec{d}$,
$\vec{e} = 2\vec{b} + 6\vec{c} - 2\vec{d}$,
$\vec{f} = 4\vec{b} - 10\vec{c} + 10\vec{d} + \vec{e}$,
prove that $\vec{a}$ and $\vec{f}$ are linearly dependent.

24. Given $\vec{a} = m\vec{c} \neq \vec{0}$, and $\vec{b} = k\vec{d} \neq \vec{0}$, where $\vec{c}$ and $\vec{d}$ are linearly independent,

a) use a geometric argument to prove that $\vec{a}$ and $\vec{b}$ are linearly independent

b) use an algebraic argument to prove that $\vec{a}$ and $\vec{b}$ are linearly independent.

25. $\vec{a}$ and $\vec{b}$ are linearly dependent and P, Q, R are points such that $\overrightarrow{PQ} = \vec{a}$ and $\overrightarrow{PR} = \vec{b}$. D is any point on the line containing points P and Q. Prove that scalars m and k exist such that $\overrightarrow{QD} = m\vec{a}$, and $\overrightarrow{QD} = k\vec{b}$.

26. $\vec{a}$ and $\vec{b}$ are linearly independent. O, A, B, and C are points such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, and $\overrightarrow{OC} = 4\vec{a} - 3\vec{b}$. Prove that points A, B, and C are collinear.

In Search of a Solution for a System with Three Variables: Elimination

Problem

Solve the linear system

$3x - y + 2z = -1$ ①
$5x + 3y + z = 17$ ②
$x - 2y - 3z = 11$ ③

Solution

One method you can employ to find the solution of a system of 3 equations in 3 variables is *elimination*. Another method using matrices will be found on page 95.

The first step in elimination is to *obtain two equations in two variables by eliminating one variable* from each of two different pairs of the original equations.

Eliminate z from ① and ②.

2 × ②	$10x + 6y + 2z = 34$	④
	$3x - y + 2z = -1$	①
④ − ①	$7x + 7y = 35$	⑤

Eliminate z from ② and ③.

3 × ②	$15x + 9y + 3z = 51$	⑥
	$x - 2y - 3z = 11$	③
⑥ + ③	$16x + 7y = 62$	⑦

Now eliminate one of the variables, say y, from the two equations in two variables, ⑤ and ⑦.

⑤ − ⑦ $-9x = -27$
$x = 3$

By back-substitution you can find $y = 2$ and $z = -4$, giving the solution $(x,y,z) = (3,2,-4)$

2.2 Linear Dependence of Three Vectors

In the last section you studied the linear dependence of *two* vectors. What could it mean to say that *three* vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent?

If two vectors $\vec{a}$ and $\vec{b}$ are linearly dependent, recall that the algebraic condition between the vectors can be expressed in two equivalent ways.

1. m and k exist, not both 0, such that $m\vec{a} + k\vec{b} = \vec{0}$.
2. Some $s \in \mathbb{R}$ exists such that $\vec{b} = s\vec{a}$

The definition chosen for the meaning of the linear dependence of three vectors will be an extension of the first algebraic condition for two vectors to be linearly dependent.

DEFINITION

Three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent if and only if m, k, and p exist, not all equal to 0, such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$ $\quad m, k, p \in \mathbb{R}$.

Example 1 will demonstrate the geometric significance of this equation.

Example 1 Suppose $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent vectors in $\mathbb{V}_3$ such that $3\vec{a} + 2\vec{b} - 4\vec{c} = \vec{0}$. Find a geometric relationship among $\vec{a}$, $\vec{b}$, and $\vec{c}$.

Solution Three vectors in 3-space are not usually coplanar. You will show that the above algebraic condition forces the three vectors to be parallel to the same plane.

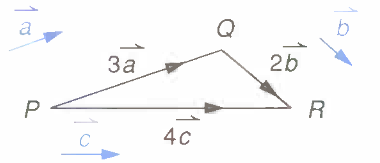
Points P, Q, and R can be selected such that $\overrightarrow{PQ} = 3\vec{a}$, and $\overrightarrow{QR} = 2\vec{b}$.

Then $\overrightarrow{PQ} + \overrightarrow{QR} - 4\vec{c} = \vec{0}$.

Thus, $\quad 4\vec{c} = \overrightarrow{PQ} + \overrightarrow{QR}$.

But, $\quad \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

thus, $\quad 4\vec{c} = \overrightarrow{PR}$.



Now any three points lie in one plane, that is, are coplanar. Thus, points P, Q, and R are coplanar, and segments PQ, QR, and PR lie in this same plane. So, $3\vec{a}$, $2\vec{b}$, and $4\vec{c}$ all lie in this plane. Since $\vec{a} \| 3\vec{a}$ and $\vec{b} \| 2\vec{b}$ and $\vec{c} \| -4\vec{c}$, then $\vec{a}$, $\vec{b}$, and $\vec{c}$ are parallel to this plane.

Hence, $\vec{a}$, $\vec{b}$, and $\vec{c}$ are parallel to the same plane. ■

Note: *Two* vectors are *collinear* if directed line segments that represent them can be translated so that each segment lies along the same line.

DEFINITION

Three vectors are said to be *coplanar* if directed line segments that represent them can be translated so that each segment lies in the same plane Π. These vectors can be coplanar in any one of a family of planes parallel to plane Π.

You should visualize this definition using the top of your desk and three pencils as $\vec{a}$, $\vec{b}$, and $\vec{c}$ to realize that three vectors being coplanar is the exception rather than the rule.

Two non-collinear vectors always lie in the same plane. (Place two pencils, $\vec{a}$ and $\vec{b}$, on your desk. If you introduce any third vector (a third pencil, $\vec{c}$), it does not have to lie in this plane. (The pencil can make a non-zero angle with the desk.)

PROPERTY

Three vectors are linearly dependent if and only if the three vectors are coplanar.

Proof of coplanar property

Part 1 Given three linearly dependent vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ in $\mathbb{V}_3$, prove $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar.

Proof:

You will need to show that three line segments representing $\vec{a}$, $\vec{b}$, and $\vec{c}$, can be drawn in such a way that the line segments form a triangle. The sides of a triangle must lie in the same plane.

Since $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly dependent, then $m, k, p \in \mathbb{R}$ exist such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$, where not all of m, k, and p are equal to 0.

Suppose $m \neq 0$. Then distinct points P and Q exist such that $\overrightarrow{PQ} = m\vec{a}$. Let R be the point such that $\overrightarrow{QR} = k\vec{b}$.

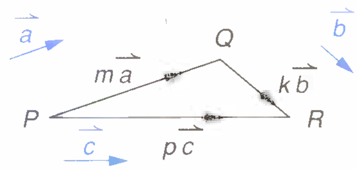
Then $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$ becomes

$\overrightarrow{PQ} + \overrightarrow{QR} + p\vec{c} = \vec{0}$.

Thus, $\quad p\vec{c} = -(\overrightarrow{PQ} + \overrightarrow{QR})$.

But $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

thus, $\quad p\vec{c} = -\overrightarrow{PR}$.



Now any three points lie in one plane, that is, are coplanar. Thus, points P, Q, and R are coplanar, and segments PQ, QR, and PR lie in this same plane. So vectors $m\vec{a}$, $k\vec{b}$ and $p\vec{c}$ all lie in this plane. Since $\vec{a} \| m\vec{a}$, $\vec{b} \| k\vec{b}$ and $\vec{c} \| p\vec{c}$, then $\vec{a}$, $\vec{b}$, $\vec{c}$ are parallel to this plane.

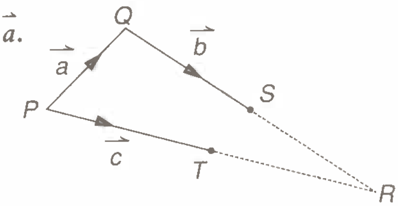
Hence, $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar.

Part 2 Given $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar, prove $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent, that is, prove that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$, where not all of m, k, and p are equal to 0, $m, k, p \in \mathbb{R}$.

Proof:

Case 1: No two vectors are linearly dependent.

Select points P and Q such that $\overrightarrow{PQ} = \vec{a}$.



Since $\vec{b} \nparallel \vec{a}$, a point S, not on line PQ, exists such that $\overrightarrow{QS} = \vec{b}$.

Since $\vec{c} \nparallel \vec{a}$, and $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar, then a point T exists such that $\overrightarrow{PT} = \vec{c}$, where T is in the plane PQS, but T is not in line PQ.

Since $\vec{b} \nparallel \vec{c}$, the lines containing segments PT and QS must intersect at some point, say R.

Thus, for some real numbers k and t,
$\overrightarrow{QR} = k\vec{b}$, and $\overrightarrow{PR} = t\vec{c}$

but $\quad \overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP} = \vec{0}$

thus $\quad \vec{a} + k\vec{b} - t\vec{c} = \vec{0}$

thus $\quad (1)\vec{a} + k\vec{b} + (-t)\vec{c} = \vec{0}$.

Therefore, $m = 1$, $k = k$, and $p = -t$ exist, not all 0, such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$.

Therefore, $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent.

Case 2: Two vectors are linearly dependent.

Suppose $\vec{a}$ and $\vec{b}$ are linearly dependent. Then $w \in \mathbb{R}$ exists such that $\vec{b} = w\vec{a}$.

Thus, $w\vec{a} - \vec{b} = \vec{0}$.

Thus, $w\vec{a} + (-1)\vec{b} + (0)\vec{c} = \vec{0}$.

Therefore, $m = w$, $k = -1$, and $p = 0$ exist, not all 0, such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$.

Therefore, $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent.

A relationship similar to the second algebraic condition for two dependent vectors is true for three linearly dependent vectors.

You know that $\vec{a}$, $\vec{b}$, and $\vec{c}$ being linearly dependent implies that m, k, and p exist, not all 0, such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$.

Suppose $m \neq 0$. Then $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$ can be written

$$m\vec{a} = -k\vec{b} - p\vec{c}, \text{ or}$$

$$\vec{a} = -\frac{k}{m}\vec{b} + -\frac{p}{m}\vec{c}$$

This relationship is described by saying that vector $\vec{a}$ is a *linear combination* of vectors $\vec{b}$ and $\vec{c}$.

Thus the previous property can now be written in terms of linear combinations of vectors.

PROPERTY

$\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent if and only if *at least one* vector can be expressed as a linear combination of the other two vectors.

Example 2

a) Prove the vectors $\vec{d} = \overrightarrow{(2,3)}$, $\vec{e} = \overrightarrow{(6,1)}$, $\vec{f} = \overrightarrow{(4,2)}$, are linearly dependent.

b) Express one of the vectors as a linear combination of the other two vectors.

Solution

a) A geometric proof is simpler here. You must show that $\vec{d}$, $\vec{e}$, and $\vec{f}$ are coplanar.

The three vectors $\vec{d}$, $\vec{e}$, and $\vec{f}$ are 2-space vectors (in $\mathbb{V}_2$), so they are coplanar in the plane of this 2-space. But three coplanar vectors are linearly dependent. Thus, the three vectors $\vec{d}$, $\vec{e}$, and $\vec{f}$ are linearly dependent.

b) If $\vec{d}$ is a linear combination of $\vec{e}$ and $\vec{f}$, then

$$\vec{d} = s\vec{e} + t\vec{f},$$

or $\overrightarrow{(2,3)} = s\overrightarrow{(6,1)} + t\overrightarrow{(4,2)}$

or $\overrightarrow{(2,3)} = \overrightarrow{(6s + 4t, s + 2t)}$

Equating components gives

$$2 = 6s + 4t$$
$$3 = s + 2t$$

Solving these equations gives $s = -1$ and $t = 2$.

Thus, $\vec{d}$ can be expressed as a linear combination of $\vec{e}$ and $\vec{f}$, namely $\vec{d} = -\vec{e} + 2\vec{f}$. ■

Note: Part b) of Example 2 provides an algebraic proof for part a) but the algebraic argument is more complex than the geometric one.

Example 3 Prove the following vectors are linearly dependent. Express one of the vectors as a linear combination of the other two vectors.
$\vec{a} = \overrightarrow{(2,1,3)}$, $\vec{b} = \overrightarrow{(3,-5,4)}$, $\vec{c} = \overrightarrow{(12,-7,17)}$

Solution You can use either of the two equivalent conditions for linear dependence to prove that $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent. You will see a proof using both conditions.

When employing the first condition as in method 1, additional algebra must be performed to obtain a linear combination. If the second condition is used as in method 2, the linear combination appears as part of the proof.

Method 1 Prove m, k, and p exist, not all 0, such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$.

You must find m, k, and p such that

$$m\overrightarrow{(2,1,3)} + k\overrightarrow{(3,-5,4)} + p\overrightarrow{(12,-7,17)} = \overrightarrow{(0,0,0)}$$

or, $$\overrightarrow{(2m,m,3m)} + \overrightarrow{(3k,-5k,4k)} + \overrightarrow{(12p,-7p,17p)} = \overrightarrow{(0,0,0)}$$

or, $$\overrightarrow{(2m + 3k + 12p, m - 5k - 7p, 3m + 4k + 17p)} = \overrightarrow{(0,0,0)}$$

Hence, $2m + 3k + 12p = 0$ ①
$m - 5k - 7p = 0$ ②
$3m + 4k + 17p = 0$ ③

You can solve this system by the method of elimination or by the use of matrices as described on page 73 and page 95 respectively. Elimination will be used here.

Eliminate k from ① and ②

5 × ① + 3 × ② $13m + 39p = 0$ ④

Eliminate k from ② and ③

4 × ② + 5 × ③ $19m + 57p = 0$ ⑤

Eliminate m from ④ and ⑤

④ ÷ 13 − ⑤ ÷ 19 $0 + 0p = 0$, which is true for all values of p.

Select $p = 1$. From ④, $m = -3$. Substituting into ① gives $k = -2$.

Hence, real numbers $m = -3$, $k = -2$, $p = 1$ exist, not all zero, such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$.

Hence the three vectors are linearly dependent.

Using these values, $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$ gives $-3\vec{a} - 2\vec{b} + \vec{c} = \vec{0}$

Solving for $\vec{a}$, $-3\vec{a} = 2\vec{b} - \vec{c}$ or $\vec{a} = -\frac{2}{3}\vec{b} + \frac{1}{3}\vec{c}$.

This expresses $\vec{a}$ as a linear combination of $\vec{b}$ and $\vec{c}$.

Method 2 Prove that at least one of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is a linear combination of the other two vectors.

Suppose $\vec{a} = m\vec{b} + k\vec{c}$. Then,

$$\overrightarrow{(2,1,3)} = m\overrightarrow{(3,-5,4)} + k\overrightarrow{(12,-7,17)}$$

or $\overrightarrow{(2,1,3)} = \overrightarrow{(3m,-5m,4m)} + \overrightarrow{(12k,-7k,17k)}$

or $\overrightarrow{(2,1,3)} = \overrightarrow{(3m + 12k,-5m - 7k,4m + 17k)}$

Thus,
$$\begin{aligned} 2 &= 3m + 12k \quad ① \\ 1 &= -5m - 7k \quad ② \\ 3 &= 4m + 17k \quad ③ \end{aligned}$$

Eliminating m from ① and ②

$5 \times ① + 3 \times ②$ gives $13 = 39k$, or $k = \frac{1}{3}$.

Substituting in ② gives $1 = -5m - 7\left(\frac{1}{3}\right)$, or $m = -\frac{2}{3}$.

Because m and k must satisfy all three equations you *must* check these values in equation ③.

Substituting in ③, $L.S. = 3$, $R.S. = 4\left(-\frac{2}{3}\right) + 17\left(\frac{1}{3}\right) = \frac{9}{3} = 3 = L.S.$

Since $\vec{a} = -\frac{2}{3}\vec{b} + \frac{1}{3}\vec{c}$ is a linear combination of $\vec{b}$ and $\vec{c}$, then $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent. ■

Note: Method 2 will not work if $\vec{a}$ and $\vec{b}$ are multiples of each other. In this case, it will be necessary to show either $\vec{a}$ or $\vec{b}$ is a linear combination of the other two vectors. Method 1 will always work.

SUMMARY

Linear Dependence of Two or Three Vectors

	two vectors: $\vec{a}$, $\vec{b}$	three vectors: $\vec{a}$, $\vec{b}$, $\vec{c}$
geometric condition	$\vec{a} \parallel \vec{b}$	$\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar
algebraic conditions	1. m, k exist, not both 0, such that $m\vec{a} + k\vec{b} = \vec{0}$, or 2. $\vec{b} = s\vec{a}$, for some $s \in \mathbb{R}$	1. m, k, p exist, not all 0, such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$, or 2. at least one is a linear combination of the other two; for example $\vec{c} = s\vec{a} + t\vec{b}$ $s, t \in \mathbb{R}$

2.2 Exercises

1. Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent.
   a) What is the geometric relationship among $\vec{a}$, $\vec{b}$, and $\vec{c}$?
   b) State two algebraic equations that are true relating $\vec{a}$, $\vec{b}$, and $\vec{c}$.
   c) What conditions, if any, are imposed on the scalars in the equations in b)?

2. You are given three coplanar vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$. State two equations showing the algebraic relationship among $\vec{x}$, $\vec{y}$, and $\vec{z}$.

3. Explain why three coplanar vectors can be represented by three directed line segments that are not coplanar.

4. a) The vector $\vec{k}$ is a linear combination of the vectors $\vec{d}$ and $\vec{e}$. What is the geometric relationship among $\vec{k}$, $\vec{d}$, and $\vec{e}$?
   b) Scalars a, b, and c exist for the vectors in a) such that $a\vec{k} + b\vec{d} + c\vec{e} = \vec{0}$. What, if anything, must be true about the scalars a, b, and c?

5. Vectors $\vec{a}$ and $\vec{b}$ are linearly dependent. Vectors $\vec{c}$ and $\vec{d}$ are such that $\vec{c} = 3\vec{a} + 2\vec{b}$ and $\vec{d} = 4\vec{a} + 2\vec{c}$.
   a) Use a geometric argument to show that $\vec{c}$ and $\vec{d}$ lie in the plane of $\vec{a}$ and $\vec{b}$.
   b) Use an algebraic argument to show that $\vec{c}$ and $\vec{d}$ lie in the plane of $\vec{a}$ and $\vec{b}$.

6. a) Prove that $\vec{0} = \overrightarrow{(0,0)}$ is linearly dependent with every vector in $\mathbb{V}_2$.
   b) Prove that $\vec{0} = \overrightarrow{(0,0,0)}$ is linearly dependent with every pair of vectors in $\mathbb{V}_3$.

7. Prove that the vectors $\vec{a} = \overrightarrow{(4,1,2)}$, $\vec{b} = \overrightarrow{(-1,0,3)}$, and $\vec{c} = \overrightarrow{(2,1,8)}$ are linearly dependent.

8. Prove the vectors in each of the following are linearly dependent. In each case express one of the vectors as a linear combination of the other two vectors.
   a) $\vec{a} = \overrightarrow{(2,3,-4)}$, $\vec{b} = \overrightarrow{(-1,6,2)}$, $\vec{c} = \overrightarrow{(8,-3,-16)}$
   b) $\vec{d} = \overrightarrow{(-5,-6)}$, $\vec{e} = \overrightarrow{(2,3)}$, $\vec{f} = \overrightarrow{(-1,-3)}$
   c) $\vec{g} = \overrightarrow{(4,0,1)}$, $\vec{h} = \overrightarrow{(-8,15,-12)}$, $\vec{n} = \overrightarrow{(0,3,-2)}$
   d) $\vec{p} = \overrightarrow{(6,-4)}$, $\vec{q} = \overrightarrow{(-3,9)}$, $\vec{r} = \overrightarrow{(-3,2)}$

9. The vectors $\vec{a} = \overrightarrow{(0,-1,-4)}$, $\vec{b} = \overrightarrow{(1,5,-1)}$, and $\vec{c} = \overrightarrow{(3,k,5)}$ are linearly dependent. Find the value of k.

10. The vectors $\vec{x} = \overrightarrow{(8,-1,3)}$, $\vec{y} = \overrightarrow{(-4,2,m)}$, and $\vec{z} = \overrightarrow{(4,5,7)}$ are coplanar. Find the value of m.

11. Given the vectors $\vec{u} = \overrightarrow{(1,2)}$, $\vec{v} = \overrightarrow{(3,2)}$, and $\vec{w} = \overrightarrow{(-1,3)}$. If possible, write $\vec{a} = \overrightarrow{(3,6)}$ as a linear combination of each of the following.
    a) $\vec{u}$ only
    b) $\vec{v}$ only
    c) $\vec{u}$ and $\vec{v}$ only
    d) $\vec{u}$, $\vec{v}$, $\vec{w}$

12. Repeat question 11 for $\vec{a} = \overrightarrow{(-1,2)}$.

13. Given the vectors $\vec{u} = \overrightarrow{(2,0,0)}$, $\vec{v} = \overrightarrow{(0,-1,2)}$, $\vec{w} = \overrightarrow{(1,0,3)}$, and $\vec{t} = \overrightarrow{(0,0,1)}$. If possible, write $\vec{a} = \overrightarrow{(0,-3,6)}$ as a linear combination of each of the following.
    a) $\vec{u}$ and $\vec{v}$ only
    b) $\vec{u}$, $\vec{v}$, and $\vec{w}$ only
    c) $\vec{u}$, $\vec{v}$, $\vec{w}$, and $\vec{t}$

14. Repeat question 13 for $\vec{a} = \overrightarrow{(5,-1,3)}$.

15. If any two of three vectors are linearly dependent, then prove that the three vectors are linearly dependent.

16. Given $\vec{a} = \overrightarrow{(2,1,-3)}$, $\vec{b} = \overrightarrow{(0,2,5)}$ and $\vec{c} = \overrightarrow{(-4,-2,6)}$, prove that constants m, k, and p, not all zero, exist such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$

2.3 Linearly Independent Vectors and Basis Vectors

Two vectors can be linearly dependent or linearly independent but not both. You know that three vectors can be linearly dependent. You are now ready to define linear independence for three vectors. You will do this in such a way that three vectors (in general, *any* number of vectors) can be linearly dependent or linearly independent, but not both.

DEFINITION

Three vectors that are not linearly dependent are *linearly independent.*

Geometrically this definition suggests that three linearly independent vectors cannot be coplanar.

Also, three vectors being linearly dependent means that m, k, and p exist, not all equal to 0, such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$. This fact implies that if the only values that can be found for m, k, and p such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$ are $m = k = p = 0$, then $\vec{a}$, $\vec{b}$ and $\vec{c}$ are not linearly dependent. Hence, the three vectors are linearly independent.

SUMMARY

Linear Independence of Two or Three Vectors

	two vectors: $\vec{a}$, $\vec{b}$	three vectors: $\vec{a}$, $\vec{b}$, $\vec{c}$
geometric condition	$\vec{a} \nparallel \vec{b}$	$\vec{a}$, $\vec{b}$, $\vec{c}$ are not coplanar
algebraic conditions	1. If $m\vec{a} + k\vec{b} = \vec{0}$ then $m = k = 0$, or 2. no s exists such that $\vec{b} = s\vec{a}$, $s \in \mathbb{R}$	If $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$ then $m = k = p = 0$

Two linearly independent vectors you have met in $\mathbb{V}_2$ are the unit vectors $\vec{i}$ and $\vec{j}$. In chapter 1 you learned that any vector in $\mathbb{V}_2$ can be written $m\vec{i} + k\vec{j}$, where m and k are scalars. Thus every vector in the plane of $\vec{i}$ and $\vec{j}$ can be expressed as a linear combination of $\vec{i}$ and $\vec{j}$.

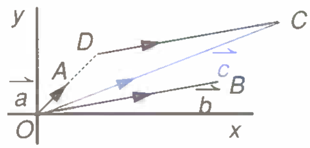
In Example 1 you will find that a similar fact is true for every pair $\vec{a}$, $\vec{b}$ of linearly independent vectors, that is, any vector in the plane of $\vec{a}$ and $\vec{b}$ can be expressed as a linear combination of $\vec{a}$ and $\vec{b}$.

Example 1 If $\vec{a}$ and $\vec{b}$ are two linearly independent vectors, and $\vec{c}$ is any other vector coplanar with $\vec{a}$ and $\vec{b}$, prove that $\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$.

Solution You will need to show that scalars s and t exist such that $\vec{c} = s\vec{a} + t\vec{b}$.

Since $\vec{a}$ and $\vec{b}$ are linearly independent, they are not parallel. But they are coplanar. Now the vectors $\vec{a}$ and $\vec{b}$ can be translated in their common plane so that each becomes a position vector with its tail at the origin of a 2-space coordinate system. Let $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$

Case 1: $\vec{c}$ is not parallel to either $\vec{a}$ or $\vec{b}$.



Let $\vec{c} = \overrightarrow{OC}$. From C draw a line parallel to OB intersecting OA or OA extended at point D. Then $\overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{DC}$.
Since D is in OA, a real number s exists such that $\overrightarrow{OD} = s\vec{a}$.
Since CD is parallel to OB, a real number t exists such that $\overrightarrow{DC} = t\vec{b}$.

Thus, $\overrightarrow{OC} = s\vec{a} + t\vec{b}$ or $\vec{c} = s\vec{a} + t\vec{b}$. Hence, $\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$.

Case 2: $\vec{c}$ is parallel to either $\vec{a}$ or $\vec{b}$.

Suppose $\vec{c}$ is parallel to $\vec{a}$. Then a real number p exists such that $\vec{c} = p\vec{a}$. Hence, $\vec{c} = p\vec{a} + 0\vec{b}$. Hence, $\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$. ■

The results of Example 1 are true for any two linearly independent vectors in $\mathbb{V}_2$. Two linearly independent vectors form a **basis** in the 2-space plane in which they lie.

DEFINITION

Any two linearly independent vectors $\vec{a}$ and $\vec{b}$ form a basis for $\mathbb{V}_2$.

If $\vec{c} = s\vec{a} + t\vec{b}$, then s and t are called the **components** of $\vec{c}$ in the basis $\{\vec{a},\vec{b}\}$.

Example 2 Prove that the components of a vector in the $\mathbb{V}_2$ basis $\{\vec{a},\vec{b}\}$ are unique.

Solution Suppose that a vector $\vec{c}$ has two pairs of components (s,t) and (p,q) in basis $\{\vec{a},\vec{b}\}$.

You will need to show that (s,t) and (p,q) are the same ordered pair.

Because both ordered pairs are components, you have two equations for $\vec{c}$, namely $\vec{c} = s\vec{a} + t\vec{b}$ and $\vec{c} = p\vec{a} + q\vec{b}$.

Thus, $s\vec{a} + t\vec{b} = p\vec{a} + q\vec{b}$ or, $(s - p)\vec{a} + (t - q)\vec{b} = \vec{0}$.
Since $\{\vec{a},\vec{b}\}$ is a basis in $\mathbb{V}_2$, $\vec{a}$ and $\vec{b}$ are linearly independent.

Hence, $s - p = 0$, and $t - q = 0$, or $s = p$ and $t = q$. Thus, the components of $\vec{c}$ in $\{\vec{a},\vec{b}\}$ are unique. ■

The most important basis in $\mathbb{V}_2$ consists of the unit vectors $\vec{i} = \overrightarrow{(1,0)}$ and $\vec{j} = \overrightarrow{(0,1)}$. You will prove $\vec{i}$ and $\vec{j}$ form a basis for $\mathbb{V}_2$ in Example 4.

You have now seen that if two vectors in 2-space or in 3-space are linearly independent then any third vector in the same plane can be written as a linear combination of the two vectors.

As you will see in the following example, every vector in 3-space can be written as a linear combination of three linearly independent vectors.

In other words, it is impossible for four 3-space vectors to be linearly independent.

Example 3

If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are three linearly independent vectors, and $\vec{d}$ is any other vector, then prove that $m, k, p \in \mathbb{R}$ exist such that $\vec{d} = m\vec{a} + k\vec{b} + p\vec{c}$.

Solution You will make use of the fact that the plane containing vectors $\vec{a}$ and $\vec{b}$ intersects with the plane containing $\vec{c}$ and $\vec{d}$.

$\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ can be translated so that each becomes a position vector with its tail at the origin of the 3-space coordinate system. Since the three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly independent, the three vectors are not coplanar.

Let Π_1 be the plane containing $\vec{a}$ and $\vec{b}$.

Let Π_2 be the plane containing $\vec{c}$ and $\vec{d}$.

Since Π_1 and Π_2 are distinct non-parallel planes, they intersect in a line L.

Let $\vec{v}$ be any vector along line L. Then $\vec{v}$ lies in the plane Π_1. Hence $\vec{v}$ is a linear combination of $\vec{a}$ and $\vec{b}$.

Thus, $\vec{v} = s\vec{a} + t\vec{b}$ ①

Also $\vec{v}$ lies in Π_2, so $\vec{v} = w\vec{d} + r\vec{c}$ ②

From ① and ②

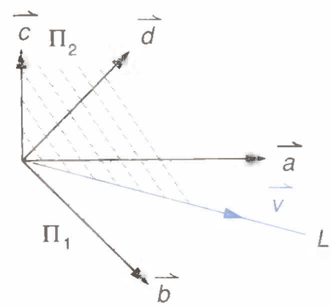
$s\vec{a} + t\vec{b} = w\vec{d} + r\vec{c}$

Thus, $w\vec{d} = s\vec{a} + t\vec{b} - r\vec{c}$

or, $\vec{d} = \frac{s}{w}\vec{a} + \frac{t}{w}\vec{b} - \frac{r}{w}\vec{c}$ ($w \neq 0$, see note).

Thus, $\vec{d} = m\vec{a} + k\vec{b} + p\vec{c}$,

where $m = \frac{s}{w}$, $k = \frac{t}{w}$, $p = -\frac{r}{w}$. ■



Note: The scalar w can not be 0, otherwise $\vec{v}$ and $\vec{c}$ would be scalar multiples and hence collinear. Thus, $\vec{c}$ would be coplanar with $\vec{b}$ and $\vec{a}$, which is not true.

If three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly independent then the vectors can not be coplanar but must exist in 3-space. Example 3 shows that any other vector in 3-space can be expressed as a linear combination of these vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

The vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are said to form a basis for $\mathbb{V}_3$.

DEFINITION

Any three linearly independent vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ form a basis for $\mathbb{V}_3$.

If $\vec{d} = m\vec{a} + k\vec{b} + p\vec{c}$, then the scalars m, k, and p are called the *components* of $\vec{d}$ in the basis $\{\vec{a},\vec{b},\vec{c}\}$.

The most important 3-space basis consists of the unit vectors $\vec{i} = \overrightarrow{(1,0,0)}$, $\vec{j} = \overrightarrow{(0,1,0)}$, and $\vec{k} = \overrightarrow{(0,0,1)}$ that you met in chapter 1. You will prove $\vec{i}$, $\vec{j}$ and $\vec{k}$ form a basis for $\mathbb{V}_3$ in Example 4.

Example 4

a) Prove that the vectors $\vec{i} = \overrightarrow{(1,0)}$ and $\vec{j} = \overrightarrow{(0,1)}$ form a basis for $\mathbb{V}_2$.

b) Prove that the vectors $\vec{i} = \overrightarrow{(1,0,0)}$, $\vec{j} = \overrightarrow{(0,1,0)}$, and $\vec{k} = \overrightarrow{(0,0,1)}$ form a basis for $\mathbb{V}_3$.

Solution

a) Vectors $\vec{i}$ and $\vec{j}$ form a basis for $\mathbb{V}_2$ if they are linearly independent. Hence, you must prove that $m\vec{i} + t\vec{j} = \vec{0}$ implies that $m = t = 0$.

Suppose $m\vec{i} + t\vec{j} = \vec{0}$

Thus $m\overrightarrow{(1,0)} + t\overrightarrow{(0,1)} = \overrightarrow{(0,0)}$

or $\overrightarrow{(m,0)} + \overrightarrow{(0,t)} = \overrightarrow{(0,0)}$

or $\overrightarrow{(m+0,0+t)} = \overrightarrow{(0,0)}$

Equating components you obtain $m = 0$ and $t = 0$. Hence, $\vec{i}$ and $\vec{j}$ form a basis for $\mathbb{V}_2$.

b) Vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ form a basis for $\mathbb{V}_3$ if they are linearly independent. Hence, you must prove that $m\vec{i} + t\vec{j} + p\vec{k} = \vec{0}$ implies that $m = t = p = 0$. Suppose $m\vec{i} + t\vec{j} + p\vec{k} = \vec{0}$

Thus $m\overrightarrow{(1,0,0)} + t\overrightarrow{(0,1,0)} + p\overrightarrow{(0,0,1)} = \overrightarrow{(0,0,0)}$

or $\overrightarrow{(m,0,0)} + \overrightarrow{(0,t,0)} + \overrightarrow{(0,0,p)} = \overrightarrow{(0,0,0)}$

or $\overrightarrow{(m+0+0,0+t+0,0+0+p)} = \overrightarrow{(0,0,0)}$

or $\overrightarrow{(m,t,p)} = \overrightarrow{(0,0,0)}$

Equating components you obtain $m = 0$, $t = 0$, and $p = 0$. Hence, $\vec{i}$, $\vec{j}$, and $\vec{k}$ form a basis for $\mathbb{V}_3$. ■

In Search of Vectors in Spaces with Dimension Higher than Three

Vectors in 2-space can be represented geometrically by directed line segments or algebraically by ordered pairs. The set of such vectors is called $\mathbb{V}_2$. Vectors in 3-space can be represented geometrically by directed line segments or algebraically by ordered 3-tuples. The set of such vectors is called $\mathbb{V}_3$.

Are there also sets of vectors $\mathbb{V}_4$, $\mathbb{V}_5$, $\mathbb{V}_6$, and so on?

In that part of *Finite Mathematics* called matrices you will find that ordered 4-tuples, ordered 5-tuples etc. are used to represent such things as the inventory of a factory or the wins of various teams.

City	Monitors	Printers	Disk Drives	Keyboards
Weston	10	12	25	15
Guelph	20	24	44	25
Kingston	12	15	28	10

City of Team	Games Played	Games Won	Games Lost	Games Tied	Points For	Points Against
Toronto	1	1	0	0	21	20
Ottawa	1	1	0	0	20	11
Winnipeg	1	0	1	0	11	20
Hamilton	1	0	1	0	20	21

Each set of ordered n-tuples, $n \in \mathbb{N}$, makes up the set of vectors $\mathbb{V}_n$ *provided that* an addition rule can be defined so that addition has all of the properties held by $\mathbb{V}_2$ and by $\mathbb{V}_3$. When a set of ordered n-tuples has these properties $\mathbb{V}_n$ is called an *n-dimensional vector space*. There is no readily available geometric model for vectors of more than three dimensions. (The corner of a room provides a model for three mutually perpendicular axes. Can you imagine *four* mutually perpendicular axes?) Nevertheless, ordered n-tuples can exhibit all the other properties of vectors—all you lose is the geometric model.

Properties of an n-dimensional vector space $\mathbb{V}_n$

Definition of addition

If $\vec{a} = \overrightarrow{(a_1,a_2,a_3,\ldots,a_n)}$ and $\vec{b} = \overrightarrow{(b_1,b_2,b_3,\ldots,b_n)}$ then
$\vec{a} + \vec{b} = \overrightarrow{(a_1 + b_1,a_2 + b_2,a_3 + b_3,\ldots,a_n + b_n)}$

Properties of addition

$\mathbb{V}_n$ is closed: $\vec{a}, \vec{b} \in \mathbb{V}_n$ implies $(\vec{a} + \vec{b}) \in \mathbb{V}_n$
Addition is associative: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
A neutral element $\vec{0} \in \mathbb{V}_n$ exists such that $\vec{a} + \vec{0} = \vec{0} + \vec{a}$ for all $\vec{a} \in \mathbb{V}_n$
Each $\vec{a} \in \mathbb{V}_n$ has an inverse $-\vec{a}$ such that $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$

Definition of scalar multiplication

If k is a scalar and $\vec{a} = \overrightarrow{(a_1,a_2,a_3,\ldots,a_n)}$ then $k\vec{a} = \overrightarrow{(ka_1,ka_2,ka_3,\ldots,ka_n)}$

Properties of scalar multiplication

$k\vec{a} \in \mathbb{V}_n$
$(km)\vec{a} = k(m\vec{a})$ [m is a scalar.]
$k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$
$(k + m)\vec{a} = k\vec{a} + m\vec{a}$

The linear dependence and linear independence of vectors in $\mathbb{V}_n$ can be defined in an manner similar to that for $\mathbb{V}_2$ and $\mathbb{V}_3$.

k vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3,\ldots,\vec{a}_k$ are linearly dependent if and only if k real numbers exist, $m_1, m_2, m_3,\ldots,m_k$, not all zero, such that
$\sum_{i=1}^{k} m_i\vec{a}_i = \vec{0}$

k vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3,\ldots,\vec{a}_k$ are linearly independent if they are not linearly dependent.

If the vectors are linearly independent, then $\sum_{i=1}^{k} m_i\vec{a}_i = \vec{0}$ implies

$m_1 = m_2 = m_3 = \ldots = m_k = 0.$

Any k linearly independent vectors in a space of k dimensions form a basis for $\mathbb{V}_k$.

Activity

Find applications of vector spaces of dimension higher than three.

2.3 Exercises

1. Vectors $\vec{s}$, $\vec{t}$, and $\vec{r}$ are not coplanar, and $w\vec{s} + x\vec{t} + z\vec{r} = \vec{0}$. What, if anything, is true about the scalars w, x, and z?

2. $\vec{x} \nparallel \vec{y}$, and $\vec{z}$ does not lie in the plane of $\vec{x}$ and $\vec{y}$. Which of $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ are linearly dependent with $\vec{x}$ and $\vec{y}$?
$\vec{a} = 3\vec{x} + 5\vec{y}$ $\quad \vec{b} = 4\vec{a} + 3\vec{x}$
$\vec{c} = 3\vec{x} + 5\vec{z}$ $\quad \vec{d} = \vec{a} - \vec{b}$

3. Vectors $\vec{a}$ and $\vec{b}$ are linearly independent. Scalars x, y, m, and k are such that $\vec{c} = x\vec{a} + y\vec{b}$ and $\vec{d} = m\vec{a} + k\vec{c}$. Prove that $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ are coplanar.

4. If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly independent, then prove that $\vec{u} = \vec{a} + 2\vec{b} + \vec{c}$, $\vec{v} = \vec{a} + 3\vec{b} - 2\vec{c}$, and $\vec{w} = \vec{a} + \vec{b} + 4\vec{c}$ are linearly dependent.

5. Prove that $\vec{a} = \overrightarrow{(4,1,2)}$, $\vec{b} = \overrightarrow{(-1,0,3)}$, and $\vec{d} = \overrightarrow{(3,1,4)}$ are linearly independent.

6. Determine whether or not the three vectors in each of the following are linearly dependent. In each case state the geometric significance of the result.
   a) $\overrightarrow{(0,1,3)}$, $\overrightarrow{(-3,5,2)}$, and $\overrightarrow{(-6,11,7)}$
   b) $\overrightarrow{(1,2,3)}$, $\overrightarrow{(-3,0,4)}$, and $\overrightarrow{(-1,4,6)}$
   c) $\overrightarrow{(4,1,8)}$, $\overrightarrow{(-2,1,0)}$, and $\overrightarrow{(0,3,16)}$
   d) $\overrightarrow{(1,2,4)}$, $\overrightarrow{(2,-3,-1)}$, and $\overrightarrow{(-1,-9,-13)}$
   e) $\overrightarrow{(3,5,1)}$, $\overrightarrow{(2,-2,-2)}$, and $\overrightarrow{(-4,-4,0)}$

7. Given the vectors $\vec{a} = \overrightarrow{(2,-5)}$ and $\vec{b} = \overrightarrow{(3,1)}$.
   a) Prove $\vec{a}$ and $\vec{b}$ form a basis for $\mathbb{V}_2$.
   b) Express each of the following vectors as a linear combination of $\vec{a}$ and $\vec{b}$.
   $\vec{c} = \overrightarrow{(7,-9)}$ $\vec{d} = \overrightarrow{(-2,-29)}$ $\vec{e} = \overrightarrow{(6,-15)}$

8. Which of the following pairs of vectors form a basis for $\mathbb{V}_2$?
   a) $\overrightarrow{(3,4)}$, $\overrightarrow{(2,3)}$
   b) $\overrightarrow{(1,-3)}$, $\overrightarrow{(4,2)}$
   c) $\overrightarrow{(4,-6)}$, $\overrightarrow{(6,-9)}$

9. Given the vectors $\vec{a} = \overrightarrow{(4,1,0)}$, $\vec{b} = \overrightarrow{(2,-3,4)}$, and $\vec{c} = \overrightarrow{(6,1,4)}$.
   a) Prove the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ form a basis for $\mathbb{V}_3$.
   b) Express each of the following vectors as a linear combination of $\vec{a}$, $\vec{b}$, and $\vec{c}$.
   $\vec{d} = \overrightarrow{(-8,-4,-8)}$ $\vec{e} = \overrightarrow{(18,13,4)}$
   $\vec{f} = \overrightarrow{(14,6,4)}$.

10. If $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$, then either $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar or $m = k = p = 0$. Prove.

11. Vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly independent. Vectors $\vec{d}$, $\vec{b}$ and $\vec{c}$ are not coplanar.
$\vec{d} = m\vec{a} + k\vec{b} + p\vec{c} \neq \vec{0}$.
$\vec{e} = r\vec{b} + t\vec{c} \neq \vec{0}$
Prove that $\vec{d}$ and $\vec{e}$ are linearly independent
   a) using a geometric argument
   b) using an algebraic argument.

12. Given vectors $\vec{a} = \overrightarrow{(w+1,3w+1)}$, and $\vec{b} = \overrightarrow{(2,w+2)}$.
   a) If $\{\vec{a},\vec{b}\}$ is *not* a basis for $\mathbb{V}_2$ find the value(s) for the scalar w.
   b) If $\{\vec{a},\vec{b}\}$ is a basis for $\mathbb{V}_2$ find the value(s) for the scalar w.

13. Given vectors $\vec{a} = \overrightarrow{(m,0,0)}$, $\vec{b} = \overrightarrow{(0,m,1)}$, and $\vec{c} = \overrightarrow{(0,1,m)}$.
   a) If $\{\vec{a},\vec{b},\vec{c}\}$ is *not* a basis for $\mathbb{V}_3$ find the value(s) for the scalar m.
   b) If $\{\vec{a},\vec{b},\vec{c}\}$ is a basis for $\mathbb{V}_3$ find the value(s) for the scalar m.

14. a) State an algebraic condition for four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ to be linearly dependent.
   b) The three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ form a basis for $\mathbb{V}_3$, and $\vec{d}$ is any other 3-space vector. Prove that $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ are linearly dependent.

The Prisoners' Dilemma—a Game

The game known as the *Prisoners' Dilemma* was introduced in 1950 by the Canadian-born mathematician Albert W. Tucker. The game involves the scenario of two suspects in a crime who are prevented from communicating with each other. Each is given one of two choices.

C: co-operation: maintain that both are innocent
N: nonco-operation: accuse the other of having committed the crime alone

It is usually in the individual's self-interest to accuse the other. Yet when both accuse, they reach a bad outcome. What is good for the prisoners *as a pair*, is to maintain that both are innocent.

This simple model can be used for a crucial international problem—the arms race between the USA and the USSR. Each of these two superpowers can independently select one of two policies.

C: co-operation: disarm, or at least agree to a partial ban on armaments
N: nonco-operation: heavily arm in preparation for any war contingency

Here, as in the original game of *Prisoners' Dilemma*, there are four possible outcomes, as indicated by the four ordered pairs (USA's choice, USSR's choice) that follow.

(C,C) Both the USA and the USSR co-operate by choosing to disarm. Most people would see this as the most preferred outcome, even though there are certain risks.

(N,N) Both the USA and the USSR refuse to co-operate by deciding to arm. From a global standpoint, most people would agree that this is the worst possible outcome.

(N,C) The USA decides not to co-operate and arm while the USSR elects to co-operate and disarm. This unilateral disarmament by the USSR would be the one preferred most of all by the USA and the one least preferred by the USSR.

(C,N) The USSR decides to arm while the USA elects to disarm. This would be the 'worst possible' outcome for the USA and the 'best' for the USSR.

The following matrix models this game.

To consider this game mathematically it is customary to assign a payoff from 0 to 5 for each event. Here (0,5) signifies a payoff of 0 for the USA and a payoff of 5 for the USSR.

		USSR C	USSR N
USA	C	both disarm (4,4)	favours USSR (0,5)
USA	N	favours USA (5,0)	arms race (2,2)

Should the USA select strategy C or strategy N? The USA can see what happens if the USSR selects C. The USA will receive a payoff of 4 for co-operating by disarming but a payoff of 5 for arming. Thus, the USA will get a better payoff by arming.

Now if the USSR selects strategy N, that is, the USSR decides to arm, the USA will receive 0 for disarming, and 2 for arming. Again the USA has the better payoff if it chooses to arm.

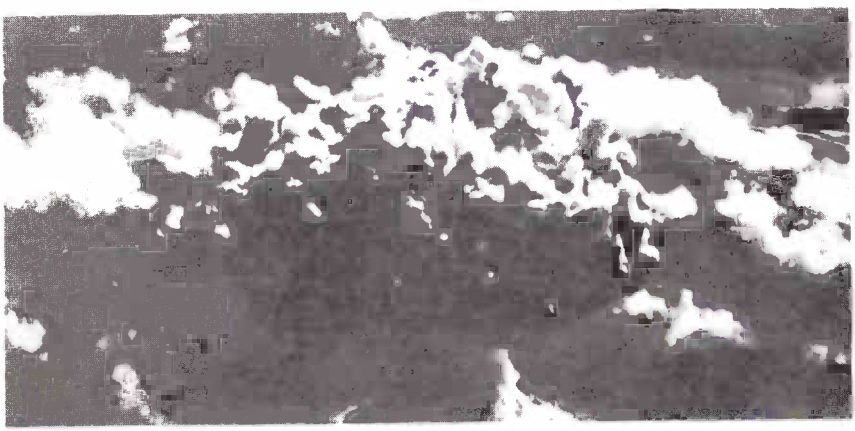
In either case, the USA gets a better payoff by arming than by disarming.

The same argument will lead the USSR to decide upon N, that is, no co-operation by arming. Thus, when each nation attempts to maximize its own payoff independently, the pair is driven into the outcome (N,N) with the payoff (2,2). The best payoff, (4,4), appears unattainable when the game is played in an atmosphere of nonco-operation.

Even if the USA and the USSR agree in advance to pursue together the globally optimal solution (C,C), this outcomes is fraught with problems. If either the USA or the USSR alone reneges on the agreement and secretly arms, it will benefit. Each country would be tempted to go back on its word and select N.

In real life, people, and sometimes nations, do manage to avoid the nonco-operative outcome in the game of *Prisoner's Dilemma*. The game is usually played within a larger context, where other incentives have their part to play. Also, the game is usually played on a repeating basis, so that elements such as reputation and trust also play a role. Players realize the mutual advantages in co-operation. Nations can also resort to other helpful measures, such as better communication, and more reliable inspection procedures.

The game of *Prisoners' Dilemma* pinpoints the dynamics behind a frequently occurring paradox. The nonco-operative outcome is not as satisfactory as a co-operative solution in which one is ready to allow one's own selfish interests to take second place.

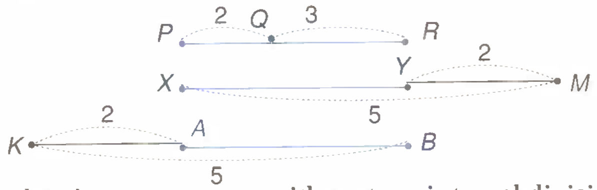


2.4 Points of Division of a Line Segment

When two directed line segments $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are linearly dependent, then the points P, Q, and R lie in a straight line. In this situation any one point divides the line segment joining the other two points into some ratio. Vectors can be used to find a special relationship among these points.

First recall the following about points of division of a line segment.

In the diagram, point Q divides PR *internally* in the ratio $2:3$.
Point M divides XY *externally* in the ratio $5:2$.
Point K divides AB *externally* in the ratio $2:5$.



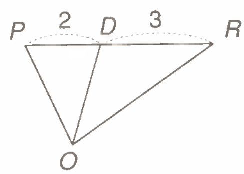
If direction is taken into account, as with vectors, internal division can be distinguished from external division by using negative signs.
Thus, if the direction from point X to point Y is taken to be positive, the point M divides XY externally in the ratio $5:(-2)$.
Similarily, if the direction from point A to point B is taken to be positive, the point K divides AB externally in the ratio $(-2):5$.

By convention the first term of the ratio is associated with the first point mentioned, that is, the division is from the first point mentioned to the dividing point, then from the dividing point to the second point mentioned. Observe that the point of division is closer to the point associated with the smaller term of the ratio.

Example 1 If point D divides the line segment PR internally in the ratio $2:3$, and O is any fixed point, then express $\overrightarrow{OD}$ in terms of $\overrightarrow{OP}$ and $\overrightarrow{OR}$.

Solution 1 The points are related as in the figure.

The key to solving this problem is to write an equation relating any two vectors along line PR and then to replace those vectors with position vectors *by subtraction*.



From the diagram $\overrightarrow{PD} = \frac{2}{5}\overrightarrow{PR}$ ①

But, $\overrightarrow{PD} = \overrightarrow{OD} - \overrightarrow{OP}$, and $\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP}$
Substituting in ① gives

$$\overrightarrow{OD} - \overrightarrow{OP} = \frac{2}{5}(\overrightarrow{OR} - \overrightarrow{OP})$$

$$\overrightarrow{OD} = \frac{2}{5}\overrightarrow{OR} - \frac{2}{5}\overrightarrow{OP} + \overrightarrow{OP}, \text{ hence, } \overrightarrow{OD} = \frac{2}{5}\overrightarrow{OR} + \frac{3}{5}\overrightarrow{OP}. \blacksquare$$

Solution 2 From the figure, replacing $\overrightarrow{OD}$ by the longer route $\overrightarrow{OP} + \overrightarrow{PD}$,

$\overrightarrow{OD} = \overrightarrow{OP} + \overrightarrow{PD}$

But $\overrightarrow{PD} = \frac{2}{5}\overrightarrow{PR}$, and $\overrightarrow{PR} = -\overrightarrow{OP} + \overrightarrow{OR}$

$$\begin{aligned}\text{Thus}\quad \overrightarrow{OD} &= \overrightarrow{OP} + \overrightarrow{PD}\\ &= \overrightarrow{OP} + \frac{2}{5}\overrightarrow{PR}\\ &= \overrightarrow{OP} + \frac{2}{5}(-\overrightarrow{OP} + \overrightarrow{OR})\\ &= \overrightarrow{OP} - \frac{2}{5}\overrightarrow{OP} + \frac{2}{5}\overrightarrow{OR}\end{aligned}$$

Thus $\overrightarrow{OD} = \frac{3}{5}\overrightarrow{OP} + \frac{2}{5}\overrightarrow{OR}$, as before. ■

Observe in Example 1,

1. the number 2 in the ratio 2 : 3 is connected to the point P in the diagram but multiplies $\overrightarrow{OR}$ in the equation
2. the number 3 in the ratio 2 : 3 is connected to the point R in the diagram but multiplies $\overrightarrow{OP}$ in the equation
3. the sum of the multipliers of $\overrightarrow{OP}$ and $\overrightarrow{OR}$, namely, $\frac{2}{5} + \frac{3}{5} = 1$
4. the denominator $5 = 2 + 3$.

Do a few more examples like Example 1 to see if these patterns continue.

Example 2 If point D divides the line segment PR externally in the ratio 5 : 3, and O is any fixed point, then express $\overrightarrow{OD}$ in terms of $\overrightarrow{OP}$ and $\overrightarrow{OR}$.

Solution The points are related as in the figure. Again, the key is to write an equation relating any two vectors along PR. From the figure,

$$\overrightarrow{DR} = \frac{3}{5}\overrightarrow{DP}$$

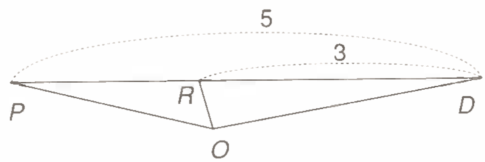
Using subtraction of position vectors for $\overrightarrow{DR}$ and $\overrightarrow{DP}$ gives

$$\overrightarrow{OR} - \overrightarrow{OD} = \frac{3}{5}(\overrightarrow{OP} - \overrightarrow{OD})$$

$$\overrightarrow{OR} - \overrightarrow{OD} = \frac{3}{5}\overrightarrow{OP} - \frac{3}{5}\overrightarrow{OD}$$

$$\overrightarrow{OR} - \frac{3}{5}\overrightarrow{OP} = \frac{2}{5}\overrightarrow{OD}$$

Hence, $\overrightarrow{OD} = \frac{5}{2}\overrightarrow{OR} - \frac{3}{2}\overrightarrow{OP}$. ■



Example 2 shows the same patterns as Example 1, if the ratio is considered as $5:(-3)$,

1. the number 5 in the ratio $5:(-3)$ is connected to the point P in the diagram but multiplies $\overrightarrow{OR}$ in the equation.
2. the number 3 in the ratio $5:(-3)$ is connected to the point R in the diagram but multiplies $\overrightarrow{OP}$ in the equation.
3. the sum of the multipliers of $\overrightarrow{OP}$ and $\overrightarrow{OR}$, namely, $-\frac{3}{2}+\frac{5}{2}=1$.
4. the denominator $2 = 5 + (-3)$.

It appears that the point D that divides segment PR in the ratio $m:k$ satisfies the following relationship. $\overrightarrow{OD}=\frac{k}{m+k}\overrightarrow{OP}+\frac{m}{m+k}\overrightarrow{OR}$

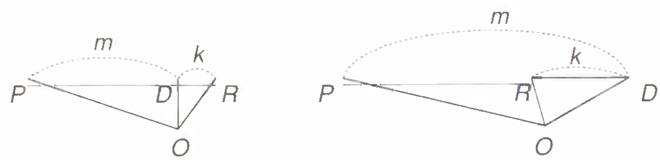
You will prove this in the following argument.

Proof of Internal/External Division Property

Given point D divides the line segment PR in the ratio $m:k$. For internal division m and k are positive. For external division the smaller of m and k is negative. If O is any fixed point, prove that

$$\overrightarrow{OD}=\frac{k}{(m+k)}\overrightarrow{OP}+\frac{m}{(m+k)}\overrightarrow{OR}$$

Proof:



The diagrams for internal and external division indicate that

$$\overrightarrow{PD}=\frac{m}{m+k}\overrightarrow{PR}$$

$$\overrightarrow{OD}-\overrightarrow{OP}=\frac{m}{m+k}(\overrightarrow{OR}-\overrightarrow{OP})$$

$$\overrightarrow{OD}-\overrightarrow{OP}=\frac{m}{m+k}\overrightarrow{OR}-\frac{m}{m+k}\overrightarrow{OP}$$

$$\overrightarrow{OD}=\frac{m+k-m}{m+k}\overrightarrow{OP}+\frac{m}{m+k}\overrightarrow{OR}$$

Thus, $\overrightarrow{OD}=\frac{k}{m+k}\overrightarrow{OP}+\frac{m}{m+k}\overrightarrow{OR}$

Note: Using the notation $\overrightarrow{OD}=\vec{d}$, $\overrightarrow{OP}=\vec{p}$, and $\overrightarrow{OR}=\vec{r}$, the property may be written

$$\vec{d}=\frac{k}{m+k}\vec{p}+\frac{m}{m+k}\vec{r}.$$

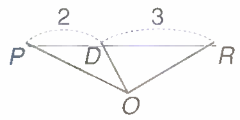
You can now use these results as a short cut for doing Examples 1 and 2.

Example 3 If point D divides the line segment PR as indicated, and O is any fixed point, then express $\overrightarrow{OD}$ in terms of $\overrightarrow{OP}$ and $\overrightarrow{OR}$.

a) internally in the ratio 2 : 3 b) externally in the ratio 5 : 3

Solution

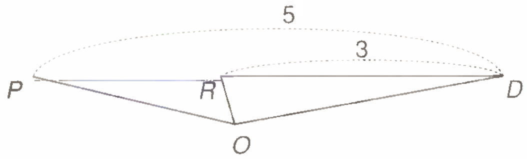
a)
D divides PR internally, 2 : 3



Here $m = 2$, $k = 3$, $m + k = 5$

Thus, $\overrightarrow{OD} = \frac{3}{5}\overrightarrow{OP} + \frac{2}{5}\overrightarrow{OR}$

b)
D divides PR externally, 5 : (−3)



Here $m = 5$, $k = -3$, $m + k = 2$

Thus $\overrightarrow{OD} = -\frac{3}{2}\overrightarrow{OP} + \frac{5}{2}\overrightarrow{OR}$ ■

Example 4 Find the coordinates of the point D that divides the line segment joining points $P(1,2,3)$ and $R(2,-4,3)$

a) internally in the ratio 5 : 7 b) externally in the ratio 3 : 2

Solution Let the fixed point O be $(0,0,0)$, and D have coordinates (x,y,z).

Thus, $\overrightarrow{OP} = \overrightarrow{(1,2,3)}$, $\overrightarrow{OR} = \overrightarrow{(2,-4,3)}$, and $\overrightarrow{OD} = \overrightarrow{(x,y,z)}$.

a) Here $m = 5$, $k = 7$, and $m + k = 12$.

Thus $\overrightarrow{OD} = \frac{7}{12}\overrightarrow{OP} + \frac{5}{12}\overrightarrow{OR}$, and

$$\overrightarrow{(x,y,z)} = \frac{7}{12}\overrightarrow{(1,2,3)} + \frac{5}{12}\overrightarrow{(2,-4,3)}$$

$$\overrightarrow{(x,y,z)} = \overrightarrow{\left(\frac{7}{12},\frac{14}{12},\frac{21}{12}\right)} + \overrightarrow{\left(\frac{10}{12},-\frac{20}{12},\frac{15}{12}\right)}$$

$$\overrightarrow{(x,y,z)} = \overrightarrow{\left(\frac{17}{12},-\frac{6}{12},\frac{36}{12}\right)}$$

Thus, the point of division D has coordinates $\left(\frac{17}{12},-\frac{1}{2},3\right)$.

b) Here $m = 3$, $k = -2$, and $m + k = 1$.

Thus $\overrightarrow{OD} = -2\overrightarrow{OP} + 3\overrightarrow{OR}$

$$\begin{aligned}\overrightarrow{(x,y,z)} &= -2\overrightarrow{(1,2,3)} + 3\overrightarrow{(2,-4,3)}\\ &= \overrightarrow{(-2,-4,-6)} + \overrightarrow{(6,-12,9)}\\ &= \overrightarrow{(4,-16,3)}\end{aligned}$$

The point dividing PR externally in the ratio 3 : 2 is $(4,-16,3)$. ■

2.4 Exercises

1. Draw a diagram showing a point D dividing the line segment PR as follows.
   a) internally in the ratio 2 : 5
   b) externally in the ratio 2 : 5

2. Point D divides the line segment PR internally in the ratio 4 : 3, and O is any fixed point. Express $\overrightarrow{OD}$ in terms of $\overrightarrow{OP}$ and $\overrightarrow{OR}$ as follows.
   a) using either method of Example 1
   b) using the point of division formula

3. In each of the following, find the vector $\overrightarrow{OD}$ such that point D divides the line segment PR internally in the indicated ratio, where O is any fixed point.
   a) 3 : 5 b) 4 : 1 c) 7 : 2 d) 6 : 11 e) 1 : 1

4. Point D divides the line segment PR *externally* in the ratio 4 : 3, and O is any fixed point. Express $\overrightarrow{OD}$ in terms of $\overrightarrow{OP}$ and $\overrightarrow{OR}$ as follows.
   a) using the method of Example 2
   b) using the point of division formula

5. In each of the following, find the vector $\overrightarrow{OD}$ such that point D divides the line segment PR externally in the indicated ratio, where O is any fixed point.
   a) 3 : 5 b) 4 : 1 c) 7 : 2 d) 6 : 11 e) 1 : 2

6. The point A divides the segment PQ internally in the ratio 2 : 3. The point B divides AQ externally in the ratio 3 : 4. If O is any fixed point, then express $\overrightarrow{OB}$ in terms of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.

7. Point D divides segment PR in the ratio $m : k$. Describe the position of the point D with respect to points P and R in each of the following cases.
   a) $m \geqslant 0, k \geqslant 0, m \geqslant k$
   b) $m \geqslant 0, k \leqslant 0, |m| \geqslant |k|$
   c) $k = 0$

8. Find the coordinates of the point D that divides the line segment joining $P(7,8)$ and $R(-4,5)$ internally in the ratio 3 : 7.

9. Repeat question 8 for the following ratios.
   a) 2 : 5, internally
   b) 7 : 6, internally
   c) 1 : 1
   d) 11 : 3, externally
   e) 3 : 11, externally
   f) 1 : 2, externally

10. Find the coordinates of the point D that divides the line segment joining $P(3,2,1)$ and $R(5,6,3)$ internally in the ratio 5 : 7.

11. Repeat question 10 for the following ratios.
   a) 5 : 2, internally
   b) 9 : 7, internally
   c) 1 : 1
   d) 3 : 13, externally
   e) 3 : 2, externally
   f) 2 : 1, externally

12. Point Q lies on the line PR. O is any point such that $\overrightarrow{OQ} = s\overrightarrow{OP} + \frac{5}{11}\overrightarrow{OR}$.
   a) Find the value of s.
   b) Into what ratio does the point Q divide the segment PR?

13. The point Q divides the segment PR internally in the ratio 2 : 1. The point A divides the segment PQ externally in the ratio 6 : 5. The point T divides PA internally in the ratio 2 : 3. If O is any point, then express $\overrightarrow{OT}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OR}$.

14. Given the triangle ABC with vertices $A(3,8)$, $B(-1,-6)$, and $C(7,4)$. D, E and F are the midpoints of sides BC, AC, and AB respectively.
   a) Express $\overrightarrow{OD}$, $\overrightarrow{OE}$, and $\overrightarrow{OF}$ as ordered pairs.
   b) Find the coordinates of the point K dividing the median AD internally in the ratio 2 : 1.
   c) Find the coordinates of the point M dividing the median BE internally in the ratio 2 : 1.
   d) Find the coordinates of the point N dividing the median CF internally in the ratio 2 : 1.
   e) Use your results of b), c), and d) to draw conclusions about the intersection of the medians AD, BE, and CF.

In Search of a Solution for a System with Three Variables: Matrices

In an *In Search of* on page 73 you learned the elimination method of solving a linear system. Here you will learn to solve the same system using equivalent matrices.

Problem

Solve the linear system

$$\begin{aligned} 3x - y + 2z &= -1 \quad ① \\ 5x + 3y + z &= 17 \quad ② \\ x - 2y - 3z &= 11 \quad ③ \end{aligned}$$

Solution

The method of matrices reduces the amount of writing that you must do by concentrating only on the coefficients in the three equations. Since the values of the variables do not change under the operations used in the method of elimination, only the coefficients are recorded in an array called a *matrix*. The position of each coefficient in the matrix corresponds to its position in the linear system.

Thus the above linear system is written in matrix form as

$$\begin{bmatrix} 3 & -1 & 2 & -1 \\ 5 & 3 & 1 & 17 \\ 1 & -2 & -3 & 11 \end{bmatrix}$$

This matrix is called the *augmented matrix* of the system. The name indicates that it *includes* the coefficient matrix $\begin{bmatrix} 3 & -1 & 2 \\ 5 & 3 & 1 \\ 1 & -2 & -3 \end{bmatrix}$

A matrix can be replaced by an equivalent matrix with zeros in certain positions by the multiplication of rows by numbers, to make elements equal, and then adding or subtracting rows.

First get 0's in the first position in row ② and in row ③.

$$\begin{array}{l} \\ 5 \times \text{row ①} - 3 \times \text{row ②} \\ \text{row ①} - 3 \times \text{row ③} \end{array} \begin{bmatrix} 3 & -1 & 2 & -1 \\ 0 & -14 & 7 & -56 \\ 0 & 5 & 11 & -34 \end{bmatrix}$$

Now get a 0 in the second position of row ③.

$$\begin{array}{l} \\ \\ 5 \times \text{row ②} + 14 \times \text{row ③} \end{array} \begin{bmatrix} 3 & -1 & 2 & -1 \\ 0 & -14 & 7 & -56 \\ 0 & 0 & 189 & -756 \end{bmatrix}$$

From row ③: $189z = -756$ thus $z = -4$.
From row ②: $-14y + 7z = -56$; using $z = -4$, $y = 2$.
From row ①: $3x - y + 2z = -1$; using $z = -4$ and $y = 2$, $x = 3$.

Therefore, the solution is $(x,y,z) = (3,2,-4)$.

Note 1 These matrices are known as *equivalent matrices* because the solution of each is the same as the solution of the original system.

2 The final matrix

$$\begin{bmatrix} 3 & -1 & 2 & -1 \\ 0 & -14 & 7 & -56 \\ 0 & 0 & 189 & -756 \end{bmatrix}$$

is called the *reduced matrix* for the system; that is, when a matrix corresponding to a linear system of three equations in three variables has the form

$$\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{bmatrix}$$

with the triangle of zeros on the left, the matrix is in *row-reduced form*. The values of a, b, c, d, e, f, g, h, and i are not unique because the row reduced form can be reached by different row operations. Nevertheless the solutions are the same.

Two special cases can occur.

Case 1 The last row is
0 0 0 i
where $i = 0$.
Then $0z = 0$ has an *infinity of solutions*. Hence an infinite number of (x,y,z) exist solving the system.

Here is an example of such a matrix in reduced form.

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Case 2 The last row is
0 0 0 i
where $i \neq 0$.
Then $0z = i$ has *no solution*. Hence no (x,y,z) exists solving the system.

Here is an example of such a matrix in reduced form.

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Activities

Solve the following linear systems using matrices.

a) $x + 2y - z = 2$
$2x - 3y + z = -1$
$4x + y + 2z = 12$

b) $2x - 3y + 4z = -8$
$3x + 4y + 2z = 13$
$5x + 2y - 3z = 25$

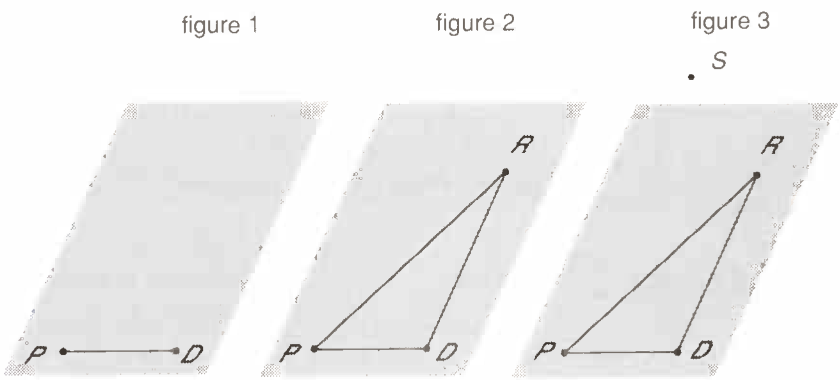
c) $2x + 3y - z = 12$
$3x - 2y + 3z = 1$
$x + 8y - 5z = 23$

d) $2x + 3y - z = 12$
$3x - 2y + 3z = 1$
$x + 8y - 5z = 1$

2.5 Collinear Points and Coplanar Points

In 3-space

1. two distinct points P and D are always collinear, (figure 1)
2. three distinct points P, D, and R are usually not collinear but will always lie in the same plane, (figure 2)
3. four distinct points P, D, R, and S will usually not be coplanar (figure 3).



In this section you will make use of linear dependence of vectors to determine whether or not three points are collinear and whether or not four points are coplanar.

Collinear Points

In the diagram, points P, D, and R lie in the same straight line. Thus, vectors $\overrightarrow{PD}$ and $\overrightarrow{DR}$ are collinear. Hence, $\overrightarrow{PD}$ and $\overrightarrow{DR}$ are linearly dependent.

Similarily, $\overrightarrow{PD}$ and $\overrightarrow{PR}$ are linearly dependent, and also $\overrightarrow{DR}$ and $\overrightarrow{PR}$ are linearly dependent.

Intuitively you should understand that line segment PD parallel to line segment DR with common point D implies that points P, D, and R are collinear points. This leads to the following property.

PROPERTY

If any two of the vectors, $\overrightarrow{PD}$, $\overrightarrow{PR}$, and $\overrightarrow{DR}$ are linearly dependent, then points P, D, and R are collinear.

Example 1 Prove that the points $P(3,2,-1)$, $D(5,4,1)$, and $R(-3,-4,-7)$ are collinear.

Solution

$\overrightarrow{PD} = \overrightarrow{OD} - \overrightarrow{OP} = \overrightarrow{(5,4,1)} - \overrightarrow{(3,2,-1)} = \overrightarrow{(2,2,2)}$.

$\overrightarrow{DR} = \overrightarrow{OR} - \overrightarrow{OD} = \overrightarrow{(-3,-4,-7)} - \overrightarrow{(5,4,1)} = \overrightarrow{(-8,-8,-8)}$.

But $\overrightarrow{DR} = -4\overrightarrow{(2,2,2)} = -4\overrightarrow{PD}$.

Therefore, $\overrightarrow{PD}$ and $\overrightarrow{DR}$ are linearly dependent.

Thus, points P, D, and R are collinear. ■

In section 2.4 you learned another fact about three collinear points P, D, and R. If the point D divides the segment PR in the ratio $m : k$ then

$$\overrightarrow{OD} = \frac{k}{m+k}\overrightarrow{OP} + \frac{m}{m+k}\overrightarrow{OR}$$

Note: The sum of the coefficients of $\overrightarrow{OP}$ and $\overrightarrow{OR}$, that is,

$$\frac{k}{m+k} + \frac{m}{m+k} = \frac{m+k}{m+k} = 1$$

The converse of this result is also true, as the following example shows.

Example 2

a) If $\overrightarrow{OD} = \frac{2}{5}\overrightarrow{OP} + \frac{3}{5}\overrightarrow{OR}$, prove that P, D, and R are collinear points.

b) Draw a diagram showing the relationship among the points P, D, and R.

Solution

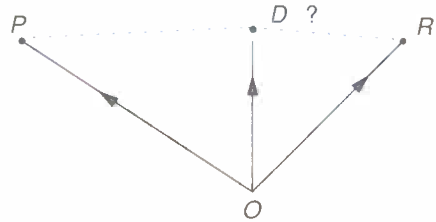
a) You will need to show that two of the vectors represented by the directed line segments $\overrightarrow{PD}$, $\overrightarrow{DR}$, or $\overrightarrow{PR}$ are linearly dependent.

Now $\overrightarrow{PD} = \overrightarrow{OD} - \overrightarrow{OP}$

$$= \frac{2}{5}\overrightarrow{OP} + \frac{3}{5}\overrightarrow{OR} - \overrightarrow{OP}$$

$$= \frac{3}{5}\overrightarrow{OR} - \frac{3}{5}\overrightarrow{OP}$$

$$= \frac{3}{5}(\overrightarrow{OR} - \overrightarrow{OP})$$



Thus, $\overrightarrow{PD} = \frac{3}{5}\overrightarrow{PR}$

Therefore, points P, D, and R are collinear.

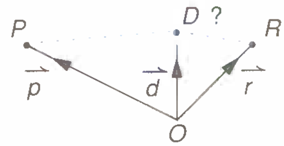
b) Since $\overrightarrow{PD} = \frac{3}{5}\overrightarrow{PR}$, the point D is positioned $\frac{3}{5}$ the distance from P to R as shown in the diagram.

■

The following example proves the result of Example 2 is true for any three collinear points.

Example 3 Given that points P, D, and R have position vectors, with respect to the origin O, of $\vec{p}$, $\vec{d}$, and $\vec{r}$ respectively, such that $\vec{d} = s\vec{p} + t\vec{r}$ where $s + t = 1$, prove that P, D, and R are collinear points.

Solution You will need to show that two of the directed line segments $\overrightarrow{PD}$, $\overrightarrow{DR}$, or $\overrightarrow{PR}$ are linearly dependent.



Now $\overrightarrow{PD} = \vec{d} - \vec{p}$

$= s\vec{p} + t\vec{r} - \vec{p}$

$= t\vec{r} + (s - 1)\vec{p}$ since $s + t = 1$, $s - 1 = -t$

$= t\vec{r} - t\vec{p}$

$= t(\vec{r} - \vec{p})$

Thus, $\overrightarrow{PD} = t\overrightarrow{PR}$

Therefore, points P, D, and R are collinear. ■

PROPERTY

The points P, D, and R are collinear if scalars s and t exist such that $\overrightarrow{OD} = s\overrightarrow{OP} + t\overrightarrow{OR}$, where $s + t = 1$.

This property can be described as the *condition for three vectors with tails at a fixed point O to have their tips in a line.*

Example 4

a) A, B, and C are collinear points such that $\overrightarrow{OB} = p\overrightarrow{OA} + (3 - 2p)\,\overrightarrow{OC}$. Find the value of p.

b) State the ratio into which B divides AC.

Solution

a) Since A, B, and C are collinear points and $\overrightarrow{OB} = p\overrightarrow{OA} + (3 - 2p)\,\overrightarrow{OC}$ then $p + (3 - 2p) = 1$
Therefore, $p = 2$.

b) Thus $\overrightarrow{OB} = 2\overrightarrow{OA} - 1\overrightarrow{OC}$
Hence B divides AC externally in the ratio $1 : 2$ ■

Coplanar Points

Suppose you have four points P, Q, R, and S such that $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ are linearly dependent. What can you say about the four points? Since $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ are linearly dependent, these three vectors must be coplanar. Hence the points P, Q, R, and S must lie in the same plane. Because the vectors $\overrightarrow{QR}$, $\overrightarrow{QS}$, and $\overrightarrow{RS}$ will also be in this plane, you can use the following property to prove that four points are coplanar.

PROPERTY

For the four points P, Q, R, S to be coplanar, three vectors (chosen with a common origin) must be linearly dependent.

Example 5 Prove that the four points $P(1,0,2)$, $D(3,2,0)$, $R(4,1,2)$, and $S(1,4,-4)$ are coplanar.

Solution You must select three of the vectors and show that they are linearly dependent. Suppose you choose the vectors $\overrightarrow{PD}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$, where

$\overrightarrow{PD} = \overrightarrow{OD} - \overrightarrow{OP} = \overrightarrow{(2,2,-2)}$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \overrightarrow{(3,1,0)}$

$\overrightarrow{PS} = \overrightarrow{OS} - \overrightarrow{OP} = \overrightarrow{(0,4,-6)}$.

You need to find m, k, and p, not all 0, such that

$m\overrightarrow{PD} + k\overrightarrow{PR} + p\overrightarrow{PS} = \vec{0}$

$m\overrightarrow{(2,2,-2)} + k\overrightarrow{(3,1,0)} + p\overrightarrow{(0,4,-6)} = \overrightarrow{(0,0,0)}$

$\overrightarrow{(2m,2m,-2m)} + \overrightarrow{(3k,k,0)} + \overrightarrow{(0,4p,-6p)} = \overrightarrow{(0,0,0)}$

$\overrightarrow{(2m + 3k, 2m + k + 4p, -2m - 6p)} = \overrightarrow{(0,0,0)}$

Thus

$$\begin{aligned} 2m + 3k \phantom{{}+4p} &= 0 \quad ① \\ 2m + k + 4p &= 0 \quad ② \\ -2m \phantom{{}+k} - 6p &= 0 \quad ③ \end{aligned}$$

You can solve this system by the method of elimination or by the use of matrices as described on page 73 and page 95 respectively. Matrices will be used here.

The augmented matrix for this system of three equations is

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 2 & 1 & 4 & 0 \\ -2 & 0 & -6 & 0 \end{bmatrix}$$

row ① − row ②
row ① + row ③

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 3 & -6 & 0 \end{bmatrix}$$

3 × row ② − 2 × row ③

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From row ③: $0p = 0$

Hence, p can be any real number.

From row ②: $2k - 4p = 0$, or

$k = 2p$

From row ①: $2m + 3k = 0$, or

$2m + 3(2p) = 0$, or

$m = -3p$

Thus, m, k, and p exist, for example, $p = 1$, $m = -3$, and $k = 2$ such that $m\overrightarrow{PD} + k\overrightarrow{PR} + p\overrightarrow{PS} = \vec{0}$. Thus, vectors $\overrightarrow{PD}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ are linearly dependent.

Hence, points P, D, R, and S are coplanar. ■

2.5 Exercises

1. a) State a vector condition for three points P, Q, and R to be collinear.
   b) State a vector condition for three points A, B, and C to be collinear.

2. Prove that the points $A(2,3)$, $B(-6,5)$, and $C(6,2)$ are collinear.

3. Prove that the points $A(2,3)$, $B(-6,5)$, and $C(-4,6)$ are not collinear.

4. Prove that the points $P(8,2,-4)$, $Q(7,0,-7)$, and $R(10,6,2)$ are collinear.

5. Prove that the points $P(8,2,-4)$, $Q(7,0,-7)$, and $R(10,6,0)$ are not collinear.

6. In each of the following decide whether or not the three points are collinear.
   a) $P(2,0,3)$, $Q(4,1,6)$, $R(6,2,9)$
   b) $X(4,5,6)$, $Y(12,1,-2)$, $Z(0,14,20)$
   c) $A(8,6)$, $B(-1,-7)$, $C(0,-25)$
   d) $D(7,11)$, $E(-3,8)$, $F(-23,2)$

7. For each of the following explain why the points P, D, and R are collinear, where $\overrightarrow{OP} = \vec{p}$, $\overrightarrow{OD} = \vec{d}$, and $\overrightarrow{OR} = \vec{r}$. In each case state the ratio into which point D divides segment PR. O is any point.
   a) $\vec{d} = \frac{1}{3}\vec{p} + \frac{2}{3}\vec{r}$
   b) $\vec{d} = \frac{4}{7}\vec{p} + \frac{3}{7}\vec{r}$
   c) $\vec{d} = -\frac{1}{3}\vec{p} + \frac{4}{3}\vec{r}$
   d) $\vec{d} = 8\vec{p} - 7\vec{r}$

8. Points P, D, and R are collinear and O is any point such that $\overrightarrow{OD} = \frac{2}{3}\overrightarrow{OP} + k\overrightarrow{OR}$. Find the value of k.

9. In each of the following, points P, D, and R are collinear. The position vectors of P, D, and R are $\vec{p}$, $\vec{d}$, and $\vec{r}$ respectively. Find the values of the scalars.
   a) $\vec{d} = m\vec{p} - 4\vec{r}$
   b) $\vec{d} = \frac{7}{9}\vec{p} + n\vec{r}$
   c) $\vec{d} = \frac{5}{3}\vec{p} + s\vec{r}$

10. a) State a vector condition for four points P, D, R, and S to be coplanar.
    b) State a vector condition for four points A, B, C, and D to be coplanar.

11. Prove that the points $P(4,0,3)$, $D(6,3,2)$, $R(3,2,7)$, and $S(5,12,13)$ are coplanar.

12. Prove that the points $P(4,0,3)$, $D(6,3,2)$, $R(3,2,7)$, and $S(5,7,14)$ are not coplanar.

13. In each of the following decide whether or not the four points are coplanar.
    a) $P(2,0,3)$, $Q(4,1,-6)$, $R(14,3,-3)$, $S(-16,-3,-12)$
    b) $W(3,1,2)$, $X(3,2,-1)$, $Y(0,6,4)$, $Z(-3,12,3)$
    c) $A(5,1,3)$, $B(4,3,0)$, $C(7,1,8)$, $D(5,2,6)$
    d) $K(1,6,3)$, $L(-2,-4,-1)$, $M(3,9,4)$, $N(-3,0,1)$

14. A, B, and C are points such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$. Scalars m, k, and p exist where $m \neq 0$ such that $m + k + p = 0$ and $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$. Prove that A, B, and C are collinear points.

15. O, A, B, C, and Z are five points in 3-space such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$, and $\overrightarrow{OZ} = -5\vec{a} + 2\vec{b} + 3\vec{c}$.
    a) Express $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AZ}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$.
    b) Prove that points A, B, C, and Z are coplanar.

16. You are given points O, P, D, R, and A such that $\overrightarrow{OP} = 2\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OR}$ and $\overrightarrow{OA} = k\overrightarrow{OD} + m\overrightarrow{OR}$. If points P, D, and R are collinear, then prove that $4k + 4m = 1$.

17. You are given points O, P, D, and R such that $m\overrightarrow{OD} - k\overrightarrow{OP} + (k - m)\,\overrightarrow{OR} = \vec{0}$. Prove that the points P, D, and R are collinear.

18. A, B, C, D, and E are five points in 3-space such that $\overrightarrow{AD} = \overrightarrow{AB} + \frac{2}{5}(\overrightarrow{EC} - \overrightarrow{EB})$. Prove that the three points B, C, and D are collinear.

2.6 Geometric Proofs Using Linear Independence of Vectors

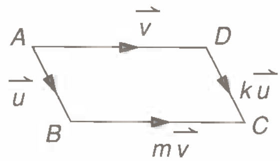
In your study of geometry you have proved geometric facts using the theorems of Euclidean geometry such as those on congruent triangles, angles in triangles, parallel lines and so on. In this section you will use the linear independence of vectors to prove some geometric facts involved with the division of segments internally or externally. The property of linear independence that you will use is the following.

If $\vec{u}$ and $\vec{v}$ are linearly independent, and $m\vec{u} + k\vec{v} = \vec{0}$, then $m = k = 0$.

In chapter 5 you will have the opportunity of doing the problems of this section by using vector equations of lines.

Example 1 Prove that the opposite sides of a parallelogram are congruent.

Solution Given: $AB \| DC$ and $AD \| BC$
Prove: $AB \cong DC$ and $AD \cong BC$



You must first translate the 'Given' and 'Prove' into vector information.

If $\overrightarrow{AB} = \vec{u}$, and $\overrightarrow{AD} = \vec{v}$, then the 'Given: $AB \| DC$ and $AD \| BC$' implies that k, m exist such that $\overrightarrow{DC} = k\vec{u}$, and $\overrightarrow{BC} = m\vec{v}$.

The 'Prove: $AB \cong DC$ and $AD \cong BC$' implies that you must show that $k = m = 1$.

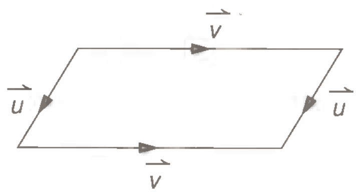
Starting at point A and moving around the parallelogram gives

$$\vec{u} + m\vec{v} - k\vec{u} - \vec{v} = \vec{0}$$

or $(1 - k)\,\vec{u} + (m - 1)\,\vec{v} = \vec{0}$

But $\vec{u}$ and $\vec{v}$ are not parallel, and so are linearly independent.
Thus, $1 - k = 0$, and $m - 1 = 0$.
Hence, $k = m = 1$, as required. ■

Example 1 implies that you may use the following or equivalent figure for problems involving a parallelogram.



Example 2 Prove that the diagonals of a parallelogram bisect each other.

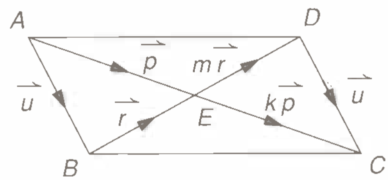
Solution

Given: $ABCD$ is a parallelogram with diagonals intersecting at E.
Prove: $AE \cong EC$, and $BE \cong ED$

Proof: As in the diagram, let $\overrightarrow{AE} = \vec{p}$, and $\overrightarrow{BE} = \vec{r}$, then $\overrightarrow{EC} = k\vec{p}$, and $\overrightarrow{ED} = m\vec{r}$.

(These statements are not needed if the facts are clearly shown on the diagram.)

You must show that $m = k = 1$.
In $\triangle AEB$, $\vec{u} = \vec{p} - \vec{r}$
In $\triangle CED$, $\vec{u} = -m\vec{r} + k\vec{p}$
Thus, $\vec{p} - \vec{r} = -m\vec{r} + k\vec{p}$
or $(1 - k)\,\vec{p} + (-1 + m)\,\vec{r} = \vec{0}$
Since $\vec{p}$ and $\vec{r}$ are not parallel
$1 - k = 0$, and $-1 + m = 0$
Thus, $k = m = 1$, as required. ■



Example 1 and Example 2 indicate a method of solving problems involving parallel or collinear segments where the ratio between the lengths of some segments is required to be found.

Step 1 Express parallel or collinear segments as $\vec{u}$ and $k\vec{u}$, $\vec{v}$ and $m\vec{v}$, etc. (If the ratio of the lengths of some segments is known, then use the terms of the ratio as scalar multipliers.)

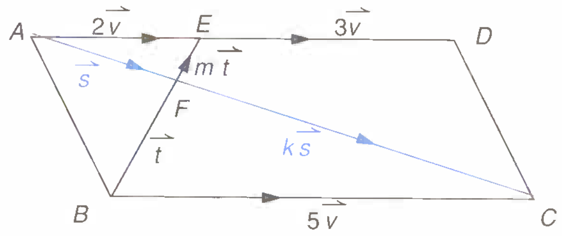
Step 2 a) Write a vector equation involving at least *two* linearly independent vectors and the unknown scalars of step 1.
b) If three linearly independent vectors appear in a), write one more equation involving the three vectors.
c) If four linearly independent vectors appear in a), write two more equations involving the four vectors.

Step 3 Use the vector equations from step 2 to eliminate all but two linearly independent vectors, say $\vec{x}$ and $\vec{y}$.

Step 4 Solve from the equation in step 3 for the required scalars k and m. First, write the equation in the form $A\vec{x} + B\vec{y} = \vec{0}$, then use the fact that $A = B = 0$.

Example 3 In parallelogram $ABCD$, E divides AD in the ratio $2:3$. BE and AC intersect at F. Find the ratio into which F divides AC.

Solution



Step 1

Let $\overrightarrow{AE} = 2\vec{v}$, and $\overrightarrow{ED} = 3\vec{v}$

$\overrightarrow{AF} = \vec{s}$ and $\overrightarrow{FC} = k\vec{s}$

$\overrightarrow{BF} = \vec{t}$ and $\overrightarrow{FE} = m\vec{t}$.

You must solve for the scalars k and m.

Step 2

In $\triangle AFE$: $2\vec{v} = \vec{s} + m\vec{t}$ ①

In $\triangle BFC$: $5\vec{v} = \vec{t} + k\vec{s}$ ②

Step 3

Eliminate vector $\vec{v}$ from ① and ②.

From ①: $10\vec{v} = 5\vec{s} + 5m\vec{t}$

From ②: $10\vec{v} = 2\vec{t} + 2k\vec{s}$

Thus $5\vec{s} + 5m\vec{t} = 2\vec{t} + 2k\vec{s}$

or $(5 - 2k)\vec{s} + (5m - 2)\vec{t} = \vec{0}$

Step 4

Since $\vec{s} \nparallel \vec{t}$

$5 - 2k = 0$ and $5m - 2 = 0$

Hence $k = \frac{5}{2}$, and $m = \frac{2}{5}$.

Using $k = \frac{5}{2}$ gives $\overrightarrow{FC} = \frac{5}{2}\vec{s}$

so $AF : FC = |\vec{s}| : \frac{5}{2}|\vec{s}| = 1 : \frac{5}{2}$ or $2:5$

Thus, F divides segment AC in the ratio $2:5$. ■

2.6 Exercises

Use vector methods to solve the following problems.

1. $OBCD$ is a parallelogram. E is the midpoint of side OD. Segments OC and BE intersect at point F. Find the ratio into which OC divides BE.

2. $OBCD$ is a parallelogram. E is the point that divides side OD in the ratio $2:5$. Segments OC and BE intersect at point F. Find the ratio into which OC divides BE.

3. a) In $\triangle OAB$, medians AD and BE intersect at point G. Find the ratios into which G divides AD and BE.
   b) Show the medians of a triangle trisect each other.

4. In $\triangle OBC$, E is the midpoint of side OB. Point F is on side OC such that segment EF is parallel to side BC. Into what ratio does F divide side OC?

5. In $\triangle OBC$, E is the point that divides side OB into the ratio $1:2$. Point F is on side OC such that segment EF is parallel to side BC. Into what ratio does F divide side OC?

6. In $\triangle OBC$, E is the point that divides side OB into the ratio $1:k$, $k \neq 0$. Point F is on side OC such that segment EF is parallel to side BC. Into what ratio does F divide side OC?

7. In $\triangle ABC$, D divides AB in the ratio $1:2$ and E divides AC in the ratio $1:4$. BE and CD intersect at point F. Find the ratios into which F divides each of BE and CD.

8. In parallelogram $PQRS$, A divides PQ in the ratio $2:5$, and B divides SR in the ratio $3:2$. Segments PR and AB intersect at C. Find the ratio into which C divides segment PR.

9. $PQRS$-is a trapezoid with PQ parallel to SR. PR and QS intersect at point A. If A divides segment QS in the ratio $2:3$, then find the ratio into which A divides PR.

10. $ABCD$ is a parallelogram. E is the point that divides side AD in the ratio $1:k$, where $k \geqslant 0$. Segments AC and BE intersect at point F. Find the ratio into which point F divides AC.

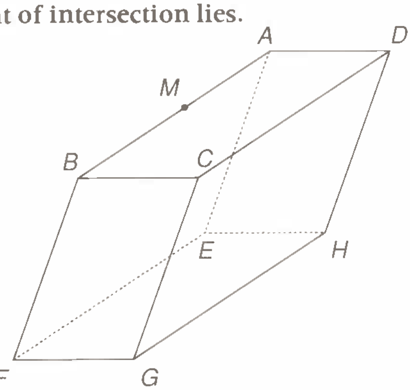
11. Let M be the midpoint of median AD of $\triangle ABC$. BM extended and AC intersect at K. Find the ratio into which K divides AC.

12. $ABCD$ is a trapezoid in which AD is parallel to BC. P and Y divide AB and DC respectively in the same ratio. Q is the point on diagonal AC such that PQ is parallel to BC. Prove that points P, Q, and Y are collinear.

13. In a tetrahedron, prove that the line segments joining a vertex to the centroid of the opposite face intersect at a point that divides the line segments in the ratio $1:3$. (The centroid of a triangle is the point of intersection of the medians. See also question 3.)

14. Show that the point found in question 13 is the same as the point of intersection of the line segments joining the midpoints of opposite edges of a tetrahedron.

15. The box shown, called a parallelepiped, is made up of three pairs of congruent parallelograms. Prove that the diagonals BH and EC intersect, and determine where the point of intersection lies.



16. In the box shown, let M be the midpoint of AB. Prove that MG and FD do not intersect.

Summary

- Two vectors $\vec{a}$ and $\vec{b}$ are *collinear or parallel* if they can be represented by parallel directed line segments. The relationship is written $\vec{a} \parallel \vec{b}$.
- *Two* non-zero vectors $\vec{a}$ and $\vec{b}$ are *linearly dependent* if and only if $\vec{a} \parallel \vec{b}$.
- The zero vector $\vec{0}$ is linearly dependent with every vector.
- *Two* vectors that are not linearly dependent are *linearly independent*.
- *Three* vectors are *coplanar* if they can be represented by directed line segments parallel to the same plane.
- *Three* vectors are *linearly dependent* if and only if they are coplanar.
- *Three* vectors that are not linearly dependent are *linearly independent*.

Linear Dependence of Two or Three Vectors

	two vectors: $\vec{a}, \vec{b}$	three vectors: $\vec{a}, \vec{b}, \vec{c}$
geometric condition	$\vec{a} \parallel \vec{b}$	$\vec{a}, \vec{b}, \vec{c}$ are coplanar
algebraic conditions	1. m, k exist, not both 0, such that $m\vec{a} + k\vec{b} = \vec{0}$, or 2. $\vec{b} = s\vec{a}$, for some $s \in \mathbb{R}$	1. m, k, p exist, not all 0, such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$, or 2. *at least one* is a linear combination of the other two; for example $\vec{c} = s\vec{a} + t\vec{b} \quad s, t \in \mathbb{R}$

Linear Independence of Two or Three Vectors

geometric condition	$\vec{a} \nparallel \vec{b}$	$\vec{a}, \vec{b}, \vec{c}$ are not coplanar
algebraic conditions	1. If $m\vec{a} + k\vec{b} = \vec{0}$ then $m = k = 0$, or 2. no s exists such that $\vec{b} = s\vec{a}, \quad s \in \mathbb{R}$	If $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$ then $m = k = p = 0$

Basis Vectors

In $\mathbb{V}_2$ (the set of 2-space vectors), if $\vec{a}$ and $\vec{b}$ are linearly independent, then any other vector in $\mathbb{V}_2$ can be expressed as a *linear combination* of $\vec{a}$ and $\vec{b}$.

The vectors $\vec{a}$ and $\vec{b}$ form a *basis for* $\mathbb{V}_2$. In particular, the unit vectors $\vec{i} = \overrightarrow{(1,0)}$ and $\vec{j} = \overrightarrow{(0,1)}$ form a basis in $\mathbb{V}_2$.

In $\mathbb{V}_3$ (the set of 3-space vectors), if $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly independent, then any other vector in $\mathbb{V}_3$ can be expressed as a *linear combination* of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ form a *basis for* $\mathbb{V}_3$.In particular, the unit vectors $\vec{i} = \overrightarrow{(1,0,0)}$, $\vec{j} = \overrightarrow{(0,1,0)}$, and $\vec{k} = \overrightarrow{(0,0,1)}$ form a basis in $\mathbb{V}_3$.

Collinear Points

1. If any two of the vectors, $\overrightarrow{PD}$, $\overrightarrow{PR}$, and $\overrightarrow{DR}$ are linearly dependent, then points P, D, and R are collinear.
2. If $\overrightarrow{OD} = s\overrightarrow{OP} + t\overrightarrow{OR}$, where $s + t = 1$, then P, D, and R are collinear points.
3. If point D divides the line segment PR in the ratio $m:k$, and O is any fixed point, then
$$\overrightarrow{OD} = \frac{k}{m+k}\overrightarrow{OP} + \frac{m}{m+k}\overrightarrow{OR}$$

For internal division m and k are positive. For external division the smaller of m and k is negative.

Coplanar Points

For the four points P, Q, R, S to be coplanar, three vectors (chosen with a common origin) must be linearly dependent.

Using Vectors in Euclidean Geometry

A method of solving problems involving parallel or collinear segments where the ratio between the lengths of some segments is required can be found on page 103.

Inventory

Answer the following by filling in the blanks.

1. $\vec{a}$ and $\vec{b}$ are linearly dependent. Thus, geometrically, $\vec{a}$ and $\vec{b}$ are ______, and a scalar k exists such that $\vec{b} =$ ______, and scalars s and t exist such that $s\vec{a} +$ ______ $= \vec{0}$.

2. $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent. Thus, geometrically, $\vec{a}$, $\vec{b}$, and $\vec{c}$ are ______, and at least one of $\vec{a}$, $\vec{b}$, and $\vec{c}$ can be written as a linear ______ of the other two; for example, $\vec{c} =$ ______. Also, scalars m, k, and p exist, not all ______, such that ______ $= \vec{0}$.

3. If $\vec{a}$ and $\vec{b}$ are not collinear then $\vec{a}$ and $\vec{b}$ are linearly ______.

4. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are not coplanar, then $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly ______.

5. a) If $\vec{a}$ and $\vec{b}$ are linearly independent and $m\vec{a} + k\vec{b} = \vec{0}$, then $k =$ ______ and $m =$ ______.
   b) If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly independent and $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$, then $k =$ ______, $m =$ ______, and $p =$ ______.

6. a) Vectors $\overrightarrow{(3,5)}$ and ______ are linearly dependent.
   b) Vectors $\overrightarrow{(3,5)}$ and ______ are linearly independent.
   c) Vectors $\overrightarrow{(2,3,1)}$ and ______ are linearly dependent.
   d) Vectors $\overrightarrow{(2,3,1)}$ and ______ are linearly independent.

7. If $m\overrightarrow{(1,2)} + k\overrightarrow{(3,4)} = \vec{0}$, then $m + 3k =$ ______, and ______ $=$ ______.

8. If $\vec{a} = k\vec{b}$, then $\vec{a}$ and $\vec{b}$ are ______.

9. If scalars a, b, and c exist, not all 0, such that $a\vec{x} + b\vec{y} + c\vec{z} = \vec{0}$, then $\vec{x}$, $\vec{y}$, and $\vec{z}$ are ______.

10. a) Point Q divides segment PR internally in the ratio $5:7$. Thus, $\overrightarrow{OQ} =$ ______ $\overrightarrow{OP} +$ ______ $\overrightarrow{OR}$.
    b) Point Q divides segment PR externally in the ratio $5:7$. Thus, $\overrightarrow{OQ} =$ ______ $\overrightarrow{OP} +$ ______ $\overrightarrow{OR}$.

11. a) $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly dependent. Thus, A, B, and C are ______.
    b) $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ are linearly dependent. Thus, P, Q, R, and S are ______.

12. In 2-space every vector is a linear combination of each pair of ______ vectors.

13. To form a basis in $\mathbb{V}_2$ you need ______ vectors that are linearly ______.

14. To form a basis in $\mathbb{V}_3$ you need ______ vectors that are linearly ______.

Review Exercises

1. $\vec{a}$ and $\vec{b}$ are linearly dependent vectors.
   a) What is the geometric relationship between $\vec{a}$ and $\vec{b}$?
   b) State two algebraic equations that are true relating $\vec{a}$ and $\vec{b}$.
   c) What conditions, if any, are imposed on the scalars in the equations in b)?

2. Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent.
   a) What is the geometric relationship among $\vec{a}$, $\vec{b}$, and $\vec{c}$?
   b) State two algebraic equations that are true relating $\vec{a}$, $\vec{b}$, and $\vec{c}$.
   c) What conditions, if any, are imposed on the scalars in the equations in b)?

3. $\vec{k}$ and $\vec{t}$ are linearly independent vectors and $a\vec{k} + b\vec{t} = \vec{0}$. What conclusion can you draw about the scalars a and b?

4. Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are not coplanar, and $s\vec{a} + t\vec{b} + r\vec{c} = \vec{0}$. What, if anything, is true about the scalars s, t, and r?

5. Given that $\vec{a} \parallel \vec{b}$, and $\vec{b} \nparallel \vec{c}$, and $\vec{d} = 4\vec{a}$, which of the following vectors are linearly dependent with $\vec{a}$?
   a) $2\vec{a}$ b) $-8\vec{b}$ c) $2\vec{c}$ d) $-7\vec{d}$

6. Vectors $\vec{p}$ and $\vec{q}$ are linearly independent.
   $\vec{x} = 3\vec{p}$, $\vec{y} = -2\vec{q}$, $\vec{z} = \frac{1}{2}\vec{q}$, $\vec{w} = 3\vec{y}$
   a) Which of $\vec{x}$, $\vec{y}$, $\vec{z}$, and $\vec{w}$ are not parallel?
   b) Which among the vectors $\vec{x}$, $\vec{p}$, $\vec{q}$, and $\vec{z}$ are linearly dependent with $\vec{w}$?

7. Vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$ are linearly independent. Which of $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ are linearly dependent with $\vec{x}$ and $\vec{y}$?
   $\vec{a} = 4\vec{x} + 5\vec{y}$, $\vec{b} = 4\vec{a} + 3\vec{x}$
   $\vec{c} = 3\vec{x} + 7\vec{z}$, $\vec{d} = 7\vec{a} + 2\vec{b}$

8. Vectors $\vec{a}$ and $\vec{b}$ are not collinear; $\vec{c}$ is not coplanar with $\vec{a}$ and $\vec{b}$. Which of the following vectors are coplanar with $\vec{a}$ and $\vec{b}$?
   $\vec{z} = 5\vec{a} + 3\vec{b}$, $\vec{y} = 2\vec{a} + 4\vec{c}$,
   $\vec{h} = \vec{a} + 2\vec{b}$, $\vec{w} = 3\vec{z} + 4\vec{h}$

9. Given $\vec{a} = \overrightarrow{(3,5)}$.
   a) Write three vectors linearly dependent with $\vec{a}$.
   b) Write one vector linearly independent with $\vec{a}$.

10. Given $\vec{b} = \overrightarrow{(7,1,-2)}$.
    a) Write three vectors collinear with $\vec{b}$.
    b) Write two vectors linearly independent with $\vec{b}$.

11. Vectors $\vec{a}$ and $\vec{b}$ are linearly independent. Vectors $\vec{c}$ and $\vec{d}$ are such that $\vec{c} = 2\vec{a} + 5\vec{b}$ and $\vec{d} = 3\vec{a} - 2\vec{c}$.
    a) Use a geometric argument to show that $\vec{c}$ and $\vec{d}$ lie in the plane of $\vec{a}$ and $\vec{b}$.
    b) Use an algebraic argument to show that $\vec{c}$ and $\vec{d}$ lie in the plane of $\vec{a}$ and $\vec{b}$.

12. Which of the following pairs of vectors are linearly dependent? Justify your answer.
    a) $\overrightarrow{(2,5)}$, $\overrightarrow{(5,2)}$
    b) $\overrightarrow{(-3,2)}$, $\overrightarrow{(1.5,-1)}$
    c) $\overrightarrow{(3,1,2)}$, $\overrightarrow{(9,3,6)}$
    d) $\overrightarrow{(4,2,1)}$, $\overrightarrow{(2,1,1)}$

13. Express the vector $\vec{v} = \overrightarrow{(11,-2)}$ as a linear combination of $\vec{a} = \overrightarrow{(1,-2)}$ and $\vec{b} = \overrightarrow{(2,1)}$.

14. Express the vector $\vec{v} = \overrightarrow{(-5,16,5)}$ as a linear combination of $\vec{a} = \overrightarrow{(1,2,3)}$, $\vec{b} = \overrightarrow{(4,0,1)}$, and $\vec{c} = \overrightarrow{(-1,4,0)}$.

15. Establish whether or not the vectors $\overrightarrow{(2,1,0)}$, $\overrightarrow{(3,1,1,)}$, and $\overrightarrow{(1,0,2)}$ are coplanar.

16. Scalars m and k exist, neither equal to 0, such that $m\vec{a} + k\vec{b} = \vec{0}$. Are $\vec{a}$ and $\vec{b}$ necessarily linearly dependent?

17. a) Given three points A, B, C such that $\overrightarrow{AB} = 6\overrightarrow{BC}$. Explain why you can say that points A, B, and C lie along the same line.
b) Given four points A, B, C, and D such that $\overrightarrow{AB} = 2\overrightarrow{AC} + 4\overrightarrow{BD}$. Explain why you can say the four points are coplanar.

18. a) Vectors $\vec{a}$ and $\vec{b}$ are basis vectors for $\mathbb{V}_2$. Explain what this means.
b) Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are basis vectors for $\mathbb{V}_3$. Explain what this means.

19. Determine whether or not the three vectors in each of the following are linearly dependent. In each case state the geometric significance of the result.
a) $\overrightarrow{(6,0,2)}$, $\overrightarrow{(-3,1,1)}$, and $\overrightarrow{(-1,1,2)}$
b) $\overrightarrow{(1,2,3)}$, $\overrightarrow{(6,11,4)}$, and $\overrightarrow{(0,1,14)}$
c) $\overrightarrow{(4,1,9)}$, $\overrightarrow{(-3,1,1)}$, and $\overrightarrow{(6,3,29)}$

20. Given the vectors $\vec{a} = \overrightarrow{(-2,3)}$ and $\vec{b} = \overrightarrow{(3,5)}$.
a) Prove the two vectors form a basis for $\mathbb{V}_2$.
b) Express the vector $\overrightarrow{(-2,22)}$ as a linear combination of $\vec{a}$ and $\vec{b}$.

21. Given the vectors $\vec{a} = \overrightarrow{(0,1,5)}$, $\vec{b} = \overrightarrow{(2,1,-4)}$ and $\vec{c} = \overrightarrow{(6,4,0)}$
a) Prove the three vectors form a basis for $\mathbb{V}_3$.
b) Express the vector $\overrightarrow{(11,9,-1)}$ as a linear combination of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

22. The vectors $\vec{a} = \overrightarrow{(-4,-1,0)}$, $\vec{b} = \overrightarrow{(-1,5,1)}$, and $\vec{c} = \overrightarrow{(5,17,k)}$ are linearly dependent. Find the value of k.

23. In each of the following, find the vector $\overrightarrow{OQ}$ such that point Q divides the line segment PR internally in the indicated ratio, where O is any fixed point.
a) 11:5
b) 5:7
c) 7:1
d) 6:11

24. In each of the following, find the vector $\overrightarrow{OT}$ such that point T divides the line segment AB externally in the indicated ratio, where O is any fixed point.
a) 1:5 b) 7:3 c) 4:9

25. The point A divides the segment PQ externally in the ratio 4:5. The point B divides AQ internally in the ratio 3:2. If O is any fixed point, then express $\overrightarrow{OB}$ in terms of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.

26. Find the coordinates of the points that divide the line segment joining points $P(2,5,8)$ and $R(-4,1,5)$ in the indicated ratios.
a) 3:1, internally
b) 4:7, externally

27. The point Q divides the line segment PR externally in the ratio 1:2. The point A divides the segment PQ internally in the ratio 4:3. The point T divides PA externally in the ratio 5:6. If O is any point, then express $\overrightarrow{OT}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OR}$.

28. a) State a vector condition for three points P, Q, and R to be collinear.
b) State a vector condition for four points A, B, C, and D to be coplanar.

29. In each of the following decide whether or not the three points are collinear.
a) $A(0,3,2)$, $B(1,5,4)$, $C(3,9,8)$
b) $P(4,1,6)$, $Q(-2,1,-5)$, $R(0,1,2)$

30. In each of the following decide whether or not the four points are coplanar.
a) $A(1,4,-5)$, $B(2,12,-8)$, $C(4,6,-4)$, $D(5,3,-2)$
b) $P(3,2,1)$, $Q(0,2,-1)$, $R(1,0,4)$, $S(0,-2,1)$

31. If O is any point, then explain why the points P, Q, and R are collinear where
$$\overrightarrow{OQ} = \frac{5}{7}\overrightarrow{OP} + \frac{2}{7}\overrightarrow{OR}.$$
State the ratio into which point Q divides segment PR.

32. Points P, Q, and R are collinear and O is any point such that $\overrightarrow{OQ} = 2m\overrightarrow{OP} + k\overrightarrow{OR}$, and $4m + 3k = 5$. Find the values of k and m.

33. $\vec{a}$ and $\vec{b}$ are linearly independent. O, A, B, and C are points such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, and $\overrightarrow{OC} = 6\vec{a} - 5\vec{b}$. Prove that points A, B, and C are collinear.

34. O, A, B, C, and Z are five points in 3-space such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$, and $\overrightarrow{OZ} = 4\vec{a} + 4\vec{b} - 7\vec{c}$.
    a) Express $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AZ}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$.
    b) Prove that points A, B, C, and Z are coplanar.

Use vector methods to solve problems 35–39.

35. $ABCD$ is a parallelogram. E is the point that divides side AD in the ratio $4:7$. Segments AC and BE intersect at point F. Find the ratio into which AC divides BE.

36. In a parallelogram $ABCD$, H is the midpoint of AD, and E divides BC in the ratio $3:2$. If BH and AE intersect at M, find the ratio $AM:ME$.

37. In $\triangle ABC$, E is the point that divides side AB into the ratio $3:2$. Point F is on side AC such that segment EF is parallel to side BC. Into what ratio does F divide side AC?

38. In $\triangle ABC$, D divides AB in the ratio $3:2$ and E divides AC in the ratio $5:4$. BE and CD intersect at point F. Find the ratios into which F divides each of BE and CD.

39. In parallelogram $PQRS$, A divides PQ in the ratio $2:1$, and B divides SR in the ratio $3:4$. Segments PR and AB intersect at C. Find the ratio into which C divides segment PR.

40. Vector $\vec{c}$ is a linear combination of the vectors $\vec{b}$ and $\vec{d}$, and $\vec{c} \neq \vec{0}$. Vectors $\vec{a}$, $\vec{b}$, and $\vec{d}$ are linearly independent. Prove that $\vec{a}$ and $\vec{d}$ cannot be linearly dependent.

41. Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly independent. O, A, B, C, and Z are points such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$, and $\overrightarrow{OZ} = -4\vec{a} + 2\vec{b} + 3\vec{c}$.
    a) Express $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AZ}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$.
    b) Prove that $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AZ}$ are linearly dependent.
    c) Draw conclusions about the geometric relationship among the points A, B, C, and Z.

42. The position vectors of A and B are $\vec{i} - 2\vec{j} + \vec{k}$ and $5\vec{i} + 4\vec{j} - 7\vec{k}$ respectively. The point P lies between A and B and is such that $\overrightarrow{AP} = 2\overrightarrow{PB}$. Find the position vector of P.
(83 H)

43. The position vectors of A, B and C are $2\vec{i} - \vec{j} + 3\vec{k}$, $12\vec{i} + 4\vec{j} - 7\vec{k}$ and $6\vec{i} + \vec{j} - \vec{k}$ respectively. Given that $\overrightarrow{AC} = \lambda\overrightarrow{AB}$, find the value of λ.
(87 SMS)

44. R is a point on the line PQ where P has coordinates $(2,7)$ and Q has coordinates $(-2,3)$. If R divides PQ in the ratio $3:-2$ then the coordinates of R are
A. $(-10,-5)$ **B.** $(10,-5)$ **C.** $(-2,-1)$
D. $(2,1)$ **E.** $(10,16)$
(79 S)

45. With respect to the standard basis of $\mathbb{R}^3$ the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are defined by $\mathbf{a} = \overline{(1,2,3)}$, $\mathbf{b} = \overline{(0,1,3)}$, $\mathbf{c} = \overline{(\lambda,0,1)}$ where $\lambda \in \mathbb{R}$. Which of the following statements is true?
A. $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent.
B. $\mathbf{a}$ and $\mathbf{c}$ are linearly dependent.
C. $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent for all values of λ.
D. $\mathbf{a}$ and $\mathbf{b}$ form a basis of $\mathbb{R}^3$.
E. $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ form a basis of $\mathbb{R}^3$ if $\lambda = 1$.
(81 H)

VECTORS, MATRICES
and
COMPLEX NUMBERS

with
International Baccalaureate
questions

Jean-Paul GINESTIER
and
John EGSGARD

CHAPTER THREE

THE MULTIPLICATION OF VECTORS

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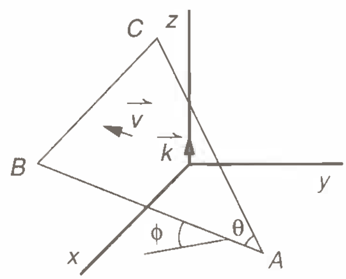
CHAPTER THREE

The Multiplication of Vectors

Problem

a) Given the points $A(1,3,-1)$, $B(2,-5,3)$, and $C(4,-1,6)$, you are asked to find the angle BAC or θ. Note that, although the diagram may help, it does not lend itself to discovering a simple solution through elementary trigonometry.



There is a way to proceed, as follows.

1. Calculate the lengths of AB, AC, and BC, that is, the lengths of the sides of the triangles ABC.
2. Use the cosine law (see page 542) in the triangle ABC to find the angle BAC.

However, this involves a lot of arithmetic. You shall see that by defining an operation called the **dot product** of vectors, a remarkably short and elegant method of calculating this angle can be devised.

b) Now consider the plane defined by the points A, B and C. You are asked to determine a direction, or a vector $\vec{v}$, that is perpendicular to the plane ABC. This means finding a vector that is perpendicular to every line in the plane ABC. It can be shown that it is sufficient to find a vector perpendicular to two linearly independent vectors coplanar with ABC. However, the task is still not simple. You will see that the **cross product** of vectors provides a simple way of determining such a vector $\vec{v}$.

c) You are now asked to calculate the angle between AB and the xy-plane. Recall from chapter 1, the *In Search of* on page 17, that the angle between a line and a plane is the smallest angle that can be defined between the line and a line in the plane. (It turns out to be the angle between the line and its 'perpendicular projection' in the plane.) Let this angle be ϕ. Once again, note that finding ϕ is not obvious, even with a diagram. You will see that the simplest way to determine this angle is by using the dot product between the vector $\overrightarrow{AB}$ and a vector that is perpendicular to the xy-plane (called a **normal** vector to the plane). In the diagram $\vec{k}$ is such a vector.

d) A more challenging task to undertake with ordinary trigonometry would be to calculate the angle between two planes, such as ABC and the xy-plane. This can be accomplished by finding a normal to each plane with the cross product, then calculating the angle α between the normals.

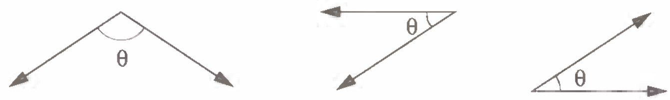
The types of problem described in c) and d) will be investigated further in chapter 6. Once you have learned to multiply vectors, you will appreciate that vector analysis is a very powerful tool that brings 3-space geometric problems to a level hardly more difficult than problems in 2-space. You will be using products of vectors extensively in the rest of your work on vectors in this book.

3.1 Projections and Components

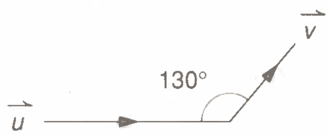
Before investigating the products of vectors, you need to know how a vector can be projected onto another. This projection will depend on the angle between the vectors.

The Angle between Two Vectors

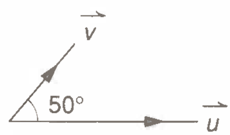
is defined as the angle θ between the vectors when they are drawn with a common tail. Note that $0° \leqslant \theta \leqslant 180°$.



Example 1 What is the angle between the vectors $\vec{u}$ and $\vec{v}$ shown?



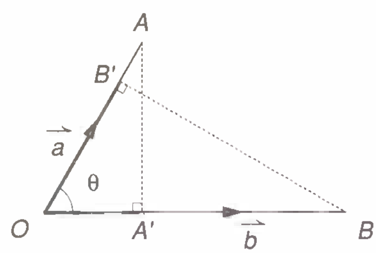
Solution Draw the vectors again so that they have a common tail.



The angle between $\vec{u}$ and $\vec{v}$ is $180° - 130° = 50°$. ■

Projections

Given the vectors $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$ such that θ is the angle between $\vec{a}$ and $\vec{b}$.

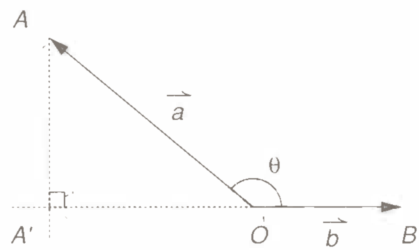


Let A' be the foot of the perpendicular from A to OB,
and B' be the foot of the perpendicular from B to OA.
Then the vector $\overrightarrow{OA'}$ is known as the **orthogonal projection**,
or **projection, of $\vec{a}$ on $\vec{b}$**, and
the vector $\overrightarrow{OB'}$ is known as the orthogonal projection,
or projection, of $\vec{b}$ on $\vec{a}$.

Note: The projection of $\vec{a}$ on $\vec{b}$ is parallel to $\vec{b}$, and the projection of $\vec{b}$ on $\vec{a}$ is parallel to $\vec{a}$.

Now $|\overrightarrow{OA'}| = ||\overrightarrow{OA}| \cos \theta| = ||\vec{a}| \cos \theta|$.

If $\theta < 90°$ it should be clear that the direction of vector $\overrightarrow{OA'}$ is the same as the direction of $\vec{b}$. In the case when $\theta > 90°$, the diagram is as shown.



If you now draw a perpendicular to OB from A, you find that OB needs to be extended beyond O. (This extension is described by saying that "BO is produced".)

Now $\overrightarrow{OA'}$ is still parallel to $\vec{b}$, but is in *the opposite direction*.

The fact that $\cos \theta$ is negative in this case is critical in the definition that follows.

To ensure that the projection of $\vec{a}$ on $\vec{b}$ is a vector, you will need a unit vector in the direction of $\vec{b}$, namely $\vec{e}_b = \frac{1}{|\vec{b}|} \vec{b}$.

DEFINITION

The projection of $\vec{a}$ on $\vec{b}$ is the vector

$$\vec{p} = |\vec{a}| \cos \theta \, \vec{e}_b = \frac{|\vec{a}| \cos \theta \, \vec{b}}{|\vec{b}|}$$

Observe that this definition does give the correct direction for $\overrightarrow{OA'}$.

If $\theta < 90°$, then $\cos \theta > 0$, and $\overrightarrow{OA'}$ has the direction of $\vec{b}$, but if $\theta > 90°$, then $\cos \theta < 0$, and $\overrightarrow{OA'}$ has the direction opposite to $\vec{b}$.

Note: The projection of $\vec{a}$ on $\vec{b}$ does not depend on the length of $\vec{b}$. For example, the projection of $\vec{a}$ on $3\vec{b}$ (which has the same direction as $\vec{b}$), is

$$\frac{|\vec{a}| \cos \theta}{|3\vec{b}|} 3\vec{b} = \frac{|\vec{a}| \cos \theta}{|\vec{b}|} \vec{b} = \overrightarrow{OA'}.$$

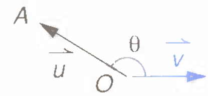
Alternatively, recall from section 1.8 that $\vec{e}_b = \vec{e}_{3b}$.

Example 2 The angle between vectors $\vec{u}$ and $\vec{v}$ is θ, where $\theta > 90°$. Find an expression for

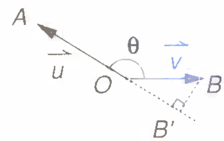
a) the projection of $\vec{v}$ on $\vec{u}$,

b) the projection of $3\vec{v}$ on $\vec{u}$.

Draw each projection.



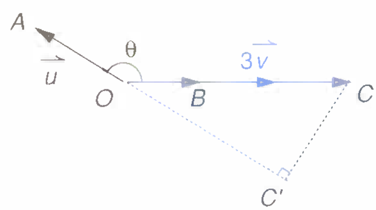
Solution a) Draw the vectors with a common tail so that $\overrightarrow{OA} = \vec{u}$, $\overrightarrow{OB} = \vec{v}$.
Let the projection of $\vec{v}$ on $\vec{u}$ be $\overrightarrow{OB'}$.
B' is the intersection of AO produced and the perpendicular to $\vec{u}$ from B.



The required projection is $\overrightarrow{OB'} = |\vec{v}| \cos\theta\, \vec{e}_u$.

(Notice that since $\theta > 90°$, $\overrightarrow{OB'}$ will have a direction opposite that of $\overrightarrow{OA}$. This is confirmed by the diagram.)

b) Let the projection of $3\vec{v}$ on $\vec{u}$ be $\overrightarrow{OC'}$.



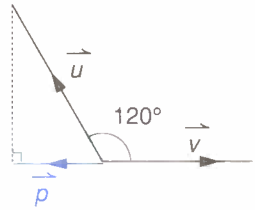
Thus, $\overrightarrow{OC'} = |3\vec{v}| \cos\theta\, \vec{e}_u = 3(|\vec{v}| \cos\theta\, \vec{e}_u) = 3\overrightarrow{OB'}$ ■

Components

In the definition of the orthogonal projection of $\vec{a}$ on $\vec{b}$, the scalar $c = |\vec{a}| \cos\theta$ which multiplies the unit vector $\vec{e}_b$ is called the **component of $\vec{a}$ on $\vec{b}$**, or **component of $\vec{a}$ in the direction of $\vec{b}$.**

Example 3 Vectors $\vec{u}$ and $\vec{v}$ make an angle of 120° with each other. If $|\vec{u}| = 6$ and $|\vec{v}| = 5$, calculate the projection of $\vec{u}$ on $\vec{v}$, and the component of $\vec{u}$ on $\vec{v}$.

Solution



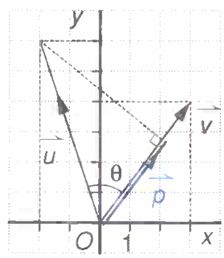
If the unit vector in the direction of $\vec{v}$ is $\vec{e}_v$,
then the projection of $\vec{u}$ on $\vec{v}$ is
$\vec{p} = |\vec{u}| \cos 120° \, \vec{e}_v = 6(-0.5)\, \vec{e}_v = -3\vec{e}_v$.
Thus the component of $\vec{u}$ in the direction of $\vec{v}$ is -3. ■

Note: The length of $\vec{v}$ is *irrelevant*, since there is only one unit vector in the direction of $\vec{v}$.

Example 4

The angle θ between vectors $\vec{u} = \overrightarrow{(-2,6)}$ and $\vec{v} = \overrightarrow{(3,4)}$ is such that $\cos\theta = 0.5692$. Find each of the following, correct to 3 significant digits.

a) the component of $\vec{u}$ on $\vec{v}$
b) the projection of $\vec{u}$ on $\vec{v}$



Solution

a) Let the component of $\vec{u}$ on $\vec{v}$ be c.
Then $c = |\vec{u}| \cos\theta = \sqrt{(-2)^2 + 6^2}(0.5692)$
$= \sqrt{40}(0.5692) = 3.599\ldots$ or 3.60,
correct to 3 significant digits.

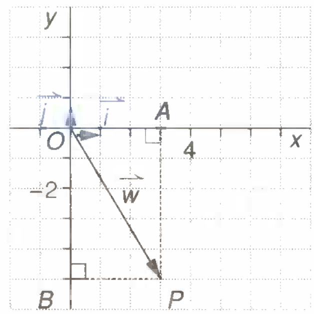
b) Let the projection of $\vec{u}$ on $\vec{v}$ be $\vec{p}$.
Then $\vec{p} = c\vec{e}_v$
where the unit vector in the direction of $\vec{v}$,
$$\vec{e}_v = \frac{1}{|\vec{v}|}\vec{v} = \frac{1}{\sqrt{3^2 + 4^2}}\overrightarrow{(3,4)} = \frac{1}{5}\overrightarrow{(3,4)} = \overrightarrow{(0.6,0.8)}$$
so $\vec{p} = (3.60)\overrightarrow{(0.6,0.8)} = \overrightarrow{(2.16,2.88)}$. ■

Note: $\vec{p}$ is parallel to $\vec{v}$.

You will see from the following example that the definition of *component* in this section agrees with the previous meaning given to the term.

Example 5 Find the projections and components of the vector $\vec{w} = \overrightarrow{(3,-5)}$ in the direction of $\vec{i}$, and in the direction of $\vec{j}$.

Solution



$\vec{w} = \overrightarrow{OP} = \overrightarrow{(3,-5)}$ is the position vector of the point (3,–5).
Draw a perpendicular from P to A on the x-axis, and to B on the y-axis.

Now the length of OA is 3, and since $\overrightarrow{OA}$ is in the same direction as $\vec{i}$, the projection of $\overrightarrow{(3,-5)}$ on $\vec{i}$ is $3\vec{i}$, and the component of $\overrightarrow{(3,-5)}$ on $\vec{i}$ is 3.

The length of OB is 5, but since $\overrightarrow{OB}$ and $\vec{j}$ are in *opposite* directions, the projection of $\overrightarrow{(3,-5)}$ on $\vec{j}$ is $-5\vec{j}$, and the component of $\overrightarrow{(3,-5)}$ on $\vec{j}$ is -5. ■

Alternative Solution

Recall that in section 1.9, you learned that $\overrightarrow{(3,-5)} = 3\vec{i} - 5\vec{j}$.
Thus,
$3\vec{i}$ and $-5\vec{j}$ are the projections of $\overrightarrow{(3,-5)}$ onto $\vec{i}$ and $\vec{j}$ respectively;
3 and -5 are the components of $\overrightarrow{(3,-5)}$ on $\vec{i}$ and $\vec{j}$ respectively.

Resolution of a Vector

The components in Example 5 are sometimes called **rectangular components**, because they refer to mutually orthogonal directions.

Writing a vector in terms of its projections on mutually orthogonal directions is called *resolving* the vector in those directions.

Thus, in Example 5, by writing $\vec{w} = 3\vec{i} - 5\vec{j}$, you are resolving $\vec{w}$ in the directions of $\vec{i}$ and $\vec{j}$.

You can also say that you are *resolving* a vector when you express that vector as a linear combination of an orthonormal basis.

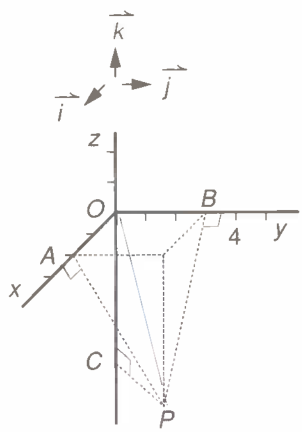
The word 'resolution' is not necessarily confined to cases of mutually orthogonal directions. In this book, however, it shall always refer to mutually orthogonal directions in order to avoid confusion. It is unfortunate that the vocabulary pertaining to the ideas of this section is not standardized in all books on vectors. Some books refer to our projections as 'components', and consider 'projections' to be lengths, that is, positive scalars.
You must beware of this.

Example 6 Resolve the vector $\vec{v} = \overrightarrow{(2,3,-5)}$ in the directions of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

Solution From section 1.9, $\overrightarrow{(2,3,-5)} = 2\vec{i} + 3\vec{j} - 5\vec{k}$. ■

Note: $2\vec{i}$, $3\vec{j}$, $-5\vec{k}$ are the projections of $\vec{v}$, and 2, 3, −5 are the components of $\vec{v}$ in the required directions.

Notice also that, if $\vec{v} = \overrightarrow{OP}$, the projection of $\vec{v}$ in the direction of $\vec{i}$ is $\overrightarrow{OA}$ where PA is perpendicular to the x-axis, as shown in the diagram. Similarly, the projections of $\vec{v}$ in the directions of $\vec{j}$ and $\vec{k}$ are $\overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively.



SUMMARY

The projection of $\vec{a}$ on $\vec{b}$ is the vector

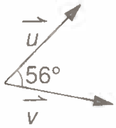
$$\vec{p} = |\vec{a}| \cos\theta \, \vec{e}_b = \frac{|\vec{a}| \cos\theta \, \vec{b}}{|\vec{b}|}$$

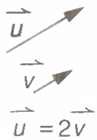
The scalar $c = |\vec{a}| \cos\theta$ that multiplies the unit vector $\vec{e}_b$ is called the component of $\vec{a}$ on $\vec{b}$, or the component of $\vec{a}$ in the direction of $\vec{b}$.

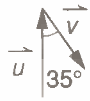
Writing $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$ is called *resolving* $\vec{v}$ in the directions of $\vec{i}$, $\vec{j}$, and $\vec{k}$. (x, y, and z are the components of $\vec{v}$, and $x\vec{i}$, $y\vec{j}$, and $z\vec{k}$ are the projections of $\vec{v}$ in the directions of $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively.)

3.1 Exercises

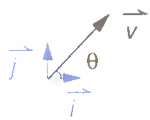
1. State the angle between the following pairs of vectors.

a) 

b) 

c) 

2. Given that the following vectors are in $\mathbb{V}_2$, and that θ is the angle between each vector and $\vec{i}$, find the component of each vector on $\vec{i}$ and $\vec{j}$ (correct to 3 decimal places).
   a) $|\vec{a}| = 5, \theta = 40°$
   b) $|\vec{b}| = 7, \theta = 110°$
   c) $|\vec{c}| = 13, \theta = 90°$.



3. Vectors $\vec{u}$ and $\vec{v}$ make an angle of 68° with each other. If $|\vec{u}| = 4$ and $|\vec{v}| = 3$, find the following.
   a) the component of $\vec{u}$ on $\vec{v}$ and the projection of $\vec{u}$ on $\vec{v}$
   b) the component of $\vec{v}$ on $\vec{u}$ and the projection of $\vec{v}$ on $\vec{u}$

4. Calculate the component of $\vec{i}$ on $\vec{v}$, given that the angle between $\vec{i}$ and $\vec{v}$ is 110°.

5. State the projections and the components of the following in the directions of $\vec{i}$ and $\vec{j}$.
   a) $\overrightarrow{(2,-3)}$ b) $\overrightarrow{(1,0)}$ c) $3\overrightarrow{(-5,1)}$

6. State the projections and the components of the following vectors in the directions of $\vec{i}$, $\vec{j}$, and $\vec{k}$.
   a) $\overrightarrow{(1,-4,1)}$ b) $\overrightarrow{(2,0,3)}$ c) $-2\overrightarrow{(1,1,0)}$

7. Resolve the following vectors on $\vec{i}, \vec{j}, \vec{k}$.
   $\vec{u} = \overrightarrow{(4,5,0)}$ $\vec{v} = \overrightarrow{(-2,-3,1)}$

8. Given $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$, state the following.
   a) the component of $\vec{v}$ on $\vec{i}$
   b) the projection of $\vec{v}$ on $\vec{j}$
   c) the projection of $\vec{v}$ on $-\vec{k}$

9. Calculate the component of the following on $\vec{v} = \overrightarrow{(1,1)}$.
   a) $\vec{i}$ b) $\vec{j}$

10. If the angle between $\vec{u}$ and $\vec{v}$ is θ, show that $|\vec{u}|$ (the component of $\vec{v}$ on $\vec{u}$) = ____ $|\vec{v}|$ (the component of $\vec{u}$ on $\vec{v}$).

11. Find the following.
    a) the component of $\overrightarrow{(2,-5)}$ on $\vec{i}$
    b) the component of $\vec{i}$ on $\overrightarrow{(2,-5)}$
    c) the projection of $\overrightarrow{(-3,4)}$ on $\vec{j}$
    d) the projection of $\vec{j}$ on $\overrightarrow{(-3,4)}$

12. The component of $\vec{u}$ in the direction of $\vec{v}$ is zero, where $\vec{v} \neq 0$. What can you deduce about $\vec{u}$ and/or $\vec{v}$?

13. Given two non-zero vectors $\vec{u}$ and $\vec{v}$, what can you deduce about $\vec{u}$ and $\vec{v}$ where
    a) the component of $\vec{u}$ on $\vec{v}$ is equal to the component of $\vec{v}$ on $\vec{u}$?
    b) the projection of $\vec{u}$ on $\vec{v}$ is equal to the projection of $\vec{v}$ on $\vec{u}$?

14. The vector $\sqrt{2}\,\vec{i}$ is resolved into two equal rectangular components. What are they?

15. The vector $\overrightarrow{(4,5)}$ has components a and $2a$ when resolved along two perpendicular lines. Calculate the value of a.

16. The vectors $\vec{u} = \overrightarrow{(1,\sqrt{3})}$ and $\vec{v} = \overrightarrow{(-2\sqrt{3},6)}$ make an angle of 60° with each other. Find each of the following.
    a) the component of $\vec{u}$ on $\vec{v}$
    b) the projection of $\vec{u}$ on $\vec{v}$

17. The projection of $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ on a plane Π is defined to be the vector $\vec{p}$, where $\vec{p}$ is parallel to Π and $\vec{v} - \vec{p}$ is perpendicular to Π.
    If $\vec{v} = 5\vec{i} - 12\vec{j} + 2\vec{k}$, calculate
    a) the component of $\vec{v}$ on the xy-plane
    b) the projection of $\vec{v}$ on the yz-plane.

3.2 The Dot Product

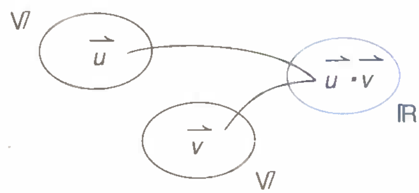
So far you have learned to add and subtract vectors. You have seen that it is possible to multiply a vector $\vec{u}$ by a real number (scalar) to obtain a vector parallel to vector $\vec{u}$. All of these operations produce another vector.

You are now ready to find out how to multiply vectors. There are actually two kinds of vector products, symbolized by $\vec{u} \cdot \vec{v}$ and $\vec{u} \times \vec{v}$. The first product, $\vec{u} \cdot \vec{v}$, is a scalar, while the second, $\vec{u} \times \vec{v}$, is a vector.

The product $\vec{u} \cdot \vec{v}$ is called the 'dot' product, or 'scalar' product of two vectors. The **dot product** takes two *vectors*, $\vec{u}$ and $\vec{v} \in \mathbb{V}$, and returns a *real number*, as follows, where θ is the angle between $\vec{u}$ and $\vec{v}$.

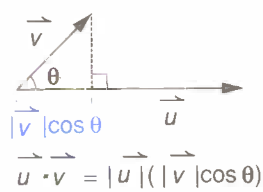
DEFINITION

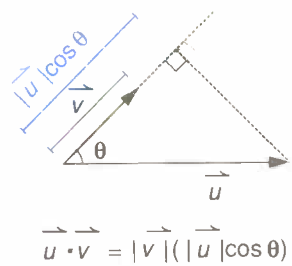
$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta,\ 0^\circ \leqslant \theta \leqslant 180^\circ$$



Note: You *must* use a dot (·) when writing out this product.

Geometrically, you can see that the dot product is the product of the *length of one vector* with the *component of the other vector* along the first. The result $\vec{u} \cdot \vec{v}$ is indeed a scalar. (Be careful about this!)



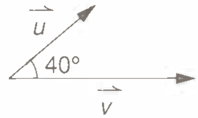


If the component of one of the vectors is directed *opposite* the other vector, the dot product will be negative. If you look back at the definition, you will note that this occurs when the angle θ is greater than 90°; indeed, in that case, the cosine of θ is negative.

The dot product has some very interesting and powerful applications, particularly in trigonometry, which you will be discovering in the next sections.

Example 1 Find $\vec{u}\cdot\vec{v}$ if $|\vec{u}| = 3$, $|\vec{v}| = 5$, $\theta = 40°$, and draw a diagram showing $\vec{u}$ and $\vec{v}$.

Solution
$$\begin{aligned}\vec{u}\cdot\vec{v} &= |\vec{u}||\vec{v}|\cos\theta\\ &= (3)(5)\cos 40°\\ &= (15)(0.7660\ldots)\\ &\doteq 11.5\end{aligned}$$

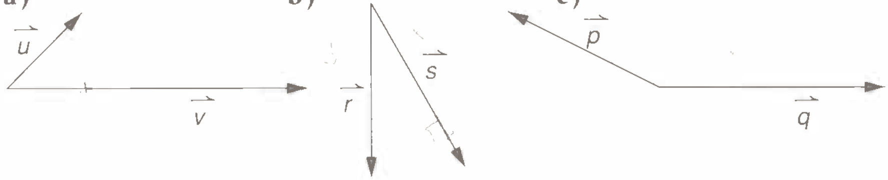


■

(Notice that you do *not* know the precise direction of the vectors here; you merely know that they are at 40° to each other. However, you still have enough information to find the dot product.)

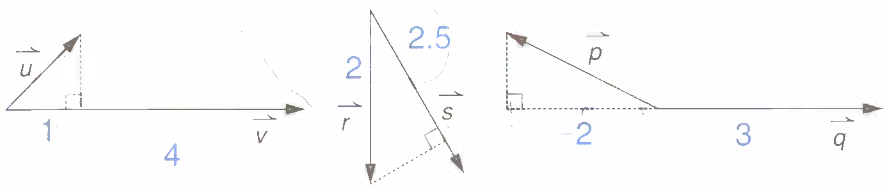
Example 2 Find the dot product of the following vectors *by measuring* the component of the first vector along the second vector, and *by measuring* the second vector. Use the centimetre as unit.

a) b) c)



$\vec{u}\cdot\vec{v}$ $\vec{r}\cdot\vec{s}$ $\vec{p}\cdot\vec{q}$

Solution a) b) c)



$\vec{u}\cdot\vec{v} = 1 \times 4 = 4$ $\vec{r}\cdot\vec{s} = 2 \times 2.5 = 5$ $\vec{p}\cdot\vec{q} = -2 \times 3 = -6$ ■

Note: You are measuring $(|\vec{u}|\cos\theta)$ and $|\vec{v}|$, etc.
But $|\vec{u}|\cos\theta|\vec{v}| = |\vec{u}||\vec{v}|\cos\theta$.

The following example should assist you in discovering some useful properties of the dot product.

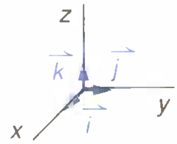
Example 3 Calculate the following dot products

a) $\vec{w}\cdot\vec{w}$, where $|\vec{w}| = 12$.
b) $\vec{i}\cdot\vec{k}$

Solution a) $\vec{w}\cdot\vec{w} = 12 \times 12 \times \cos 0° = 12^2 = 144.$

PROPERTY The dot product of a vector with itself is the square of its length.

b) $\vec{i}\cdot\vec{k} = 1 \times 1 \times \cos 90° = 0.$



PROPERTY The dot product of two perpendicular vectors is zero. ■

In the exercises, you will be proving these and other properties.
The next example should also lead you to the discovery of a property.

Example 4 Given $\vec{u} = \overrightarrow{(1,0)}$, $\vec{v} = \overrightarrow{(-1,\sqrt{3})}$, calculate the following dot products.

a) $\vec{u}\cdot\vec{v}$ b) $(6\vec{u})\cdot(2\vec{v})$

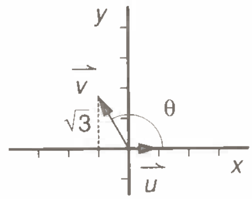
Solution a) $|\vec{u}| = \sqrt{1^2 + 0^2} = 1$

$|\vec{v}| = \sqrt{(-1)^2 + \sqrt{3}^2} = 2.$

If θ is the angle between $\vec{u}$ and $\vec{v}$,

then $\tan\theta = \dfrac{\sqrt{3}}{-1} = -\sqrt{3}$, giving $\theta = 120°$.

Thus $\vec{u}\cdot\vec{v} = (1)(2)\cos 120° = (2)(-0.5) = -1.$

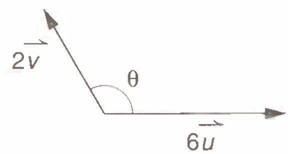


b) $|6\vec{u}| = |6||\vec{u}| = (6)(1) = 6$

$|2\vec{v}| = |2||\vec{v}| = (2)(2) = 4$

The angle between $\vec{u}$ and $\vec{v}$
is the same as the angle between
$6\vec{u}$ and $2\vec{v}$, namely $120°$.

Thus $(\overrightarrow{6u})\cdot(\overrightarrow{2v}) = (6)(4)\cos 120° = -12.$ ■



Notice that the results obtained in this example indicate that
$(6\vec{u})\cdot(2\vec{v}) = (6 \times 2)(\vec{u}\cdot\vec{v})$, which illustrates the following property.

PROPERTY For any vectors $\vec{u}$, $\vec{v}$, and scalars m, n

$(m\,\vec{u})\cdot(n\,\vec{v}) = (mn)(\vec{u}\cdot\vec{v})$

In the next section, you will also discover an alternative method of finding the dot product of two vectors expressed in component form; it will allow you to do a question like Example 4 above more quickly.

SUMMARY The dot product of vectors $\vec{u}$ and $\vec{v}$, having an angle θ between them when drawn with a common tail, is the scalar

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta \quad (0° \leqslant \theta \leqslant 180°)$

The dot product equals:
(length of one vector) (component of the other vector along the first)

3.2 Exercises

1. Calculate the dot product of the following pairs of vectors correct to 3 significant digits, given that θ is the angle between them.
   a) $|\vec{u}| = 4, |\vec{v}| = 6, \theta = 60°$
   b) $|\vec{w}| = 7, |\vec{t}| = 3, \theta = 27°$
   c) $|\vec{a}| = 6, |\vec{b}| = 7.5, \theta = 90°$
   d) $|\vec{c}| = 8, |\vec{d}| = 4, \theta = 0°$
   e) $|\vec{e}| = 12, |\vec{f}| = 15, \theta = 91°$

2. Calculate (approximately) the dot products of the following vectors by measurement of lengths and projections, using the centimetre as a unit.
   a)
   
   b)
   
   c)
   

3. Calculate the following dot products.
   a) $\vec{j} \cdot \vec{i}$ b) $\vec{i} \cdot \vec{i}$ c) $\vec{k} \cdot (-\vec{k})$

4. Given that $\vec{u}, \vec{v}, \vec{w}$ are vectors, and $m, n \in \mathbb{R}$, state which of the following are vectors, and which are scalars.
   a) $2\vec{v}$
   b) $|2\vec{v}|$
   c) $\vec{v}(m - n)$
   d) $m(\vec{u} + \vec{v})$
   e) $\vec{u} \cdot \vec{v}$
   f) $\vec{u} \cdot (\vec{v} + \vec{w})$
   g) $\vec{v} + \vec{u}$
   h) $\vec{v} - \vec{u}$
   i) $-m\,\vec{u}$

5. Prove that the dot product of two unit vectors is equal to the cosine of the angle between them.

6. Prove that the dot product of a vector with itself is equal to the square of its length.

7. Prove that the dot product of two perpendicular vectors is zero.

8. Given vectors $\vec{u}, \vec{v}$ at an angle θ to each other, and scalar m, prove that $(m\,\vec{u}) \cdot \vec{v} = m(\vec{u} \cdot \vec{v})$ in the following cases.
   a) m is positive and θ is acute
   b) m is negative and θ is acute
   c) m is positive and θ is obtuse
   d) m is negative and θ is obtuse

9. Using the results of question 8, prove that, if n is another scalar,
   $(m\,\vec{u}) \cdot (n\,\vec{v}) = (mn)\,\vec{u} \cdot \vec{v}$.

10. If $\vec{u} \cdot \vec{v} = 0$, is the angle between $\vec{u}$ and $\vec{v}$ necessarily 90°?

11. ABC is an equilateral triangle whose sides have length 10 units. Calculate the following.
    a) $\overrightarrow{AB} \cdot \overrightarrow{AC}$ b) $\overrightarrow{AB} \cdot \overrightarrow{BC}$

12. Given that $|\vec{p}| = 10$ and $|\vec{q}| = 3$, calculate the angle θ between the vectors $\vec{p}$ and $\vec{q}$ in the following cases. Give your answers to the nearest degree.
    a) $\vec{p} \cdot \vec{q} = 30$ b) $\vec{p} \cdot \vec{q} = -5$ c) $\vec{p} \cdot \vec{q} = 0$

13. Given any three vectors $\vec{u}, \vec{v}, \vec{w}$, which of the following expressions are meaningful? Justify your answers.
    a) $\vec{u} + (\vec{v} \cdot \vec{w})$
    b) $(\vec{u} + \vec{v}) \cdot \vec{w}$
    c) $\vec{u} \cdot \vec{v} \cdot \vec{w}$
    d) $\vec{u} \cdot (\vec{v} \cdot \vec{w})$
    e) $(\vec{u} \cdot \vec{v})\vec{w}$
    f) $\vec{u}(\vec{v} \cdot \vec{w})$

14. a) Given that vectors $\vec{a}$ and $\vec{b}$ of $\mathbb{V}_2$ make angles of 45° and 60° respectively with $\vec{i}$, where $|\vec{a}| = 4\sqrt{2}$, and $|\vec{b}| = 8$, find the exact value of $\vec{a} \cdot \vec{i}$ and $\vec{b} \cdot \vec{i}$.
    b) Use your result to part a) to comment on the following.
    If $\vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w}$, is it necessarily true that $\vec{u} = \vec{v}$?

15. Prove that for any vectors $\vec{u}$ and $\vec{v}$,
    $|\vec{u} \cdot \vec{v}| \leqslant |\vec{u}|\,|\vec{v}|$
    When does the equality hold?

3.3 Properties of the Dot Product

Commutativity

By definition, $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}| \cos\theta$.

Thus, $\vec{v}\cdot\vec{u} = |\vec{v}||\vec{u}| \cos\theta$, which is the same real number as $\vec{u}\cdot\vec{v}$.

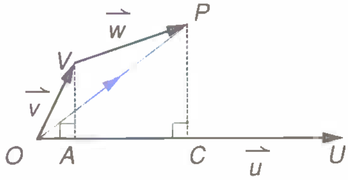
Hence, for any vectors $\vec{u}$ and $\vec{v}$,

$\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$, *the dot product is commutative.*

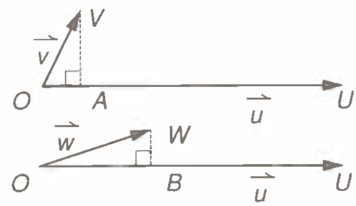
Distributivity over Addition

Look at the diagrams to help you see the results of the following expressions. The vectors are drawn so that $\vec{u} = \overrightarrow{OU}$, $\vec{v} = \overrightarrow{OV}$, $\vec{w} = \overrightarrow{OW}$, and $\vec{v} + \vec{w} = \overrightarrow{OP}$.

$x = \vec{u}\cdot(\vec{v} + \vec{w})$



$y = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$



$$x = \overrightarrow{OU}\cdot\overrightarrow{OP}$$
$$= |\overrightarrow{OU}| \text{ (component of } \overrightarrow{OP} \text{ on } \overrightarrow{OU})$$
$$= (OU)(OC)$$

$$y = \overrightarrow{OU}\cdot\overrightarrow{OV} + \overrightarrow{OU}\cdot\overrightarrow{OW}$$
$$= |\overrightarrow{OU}|\,(OA) + |\overrightarrow{OU}|\,(OB)$$
$$= (OU)(OA + OB)$$
$$= (OU)(OA + AC) = (OU)(OC)$$

This indicates the following property.

For any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, $\vec{u}\cdot(\vec{v} + \vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$

That is, *the dot product is distributive over vector addition.*

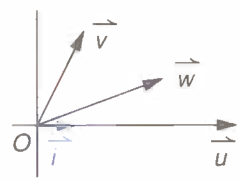
Algebraic Proof of the Distributivity Property

Draw $\vec{u}$ along the positive x-axis, that is, in the direction of $\vec{i}$, with its tail at (0,0).

Let $\vec{v} = \overrightarrow{(v_1,v_2)}$ and $\vec{w} = \overrightarrow{(w_1,w_2)}$, thus $\vec{v} + \vec{w} = \overrightarrow{(v_1 + w_1, v_2 + w_2)}$.

Now $\vec{u}\cdot\vec{v} = |\vec{u}|$ (the component of $\vec{v}$ on $\vec{u}$)
$= |\vec{u}|$ (the component of $\vec{v}$ on $\vec{i}$) $= |\vec{u}|\,(v_1)$

and $\vec{u}\cdot\vec{w} = |\vec{u}|$ (the component of $\vec{w}$ on $\vec{u}$)
$= |\vec{u}|$ (the component of $\vec{w}$ on $\vec{i}$) $= |\vec{u}|\,(w_1)$



therefore $\vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w} = |\vec{u}|(v_1) + |\vec{u}|(w_1) = |\vec{u}|(v_1 + w_1)$

but $\vec{u}\cdot(\vec{v} + \vec{w}) = |\vec{u}|$ (the component of $[\vec{v} + \vec{w}]$ on $\vec{u}$)
$= |\vec{u}|$ (the component of $[\vec{v} + \vec{w}]$ on $\vec{i}$) $= |\vec{u}|\,(v_1 + w_1)$

Therefore, $\vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w} = \vec{u}\cdot(\vec{v} + \vec{w})$ as required.

There follows a summary of the major properties of the dot product. The first two properties have been proved above. You have demonstrated properties 3, 4 and 5 in the examples and problems of the last section.

PROPERTIES

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ — commutative
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ — distributive over +
3. $(k\vec{u}) \cdot (m\vec{v}) = (km)(\vec{u} \cdot \vec{v})$
4. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$
5. If $\vec{u}$ is perpendicular to $\vec{v}$, then $\vec{u} \cdot \vec{v} = 0$.
6. $\vec{i} \cdot \vec{i} = 1$, $\vec{j} \cdot \vec{j} = 1$, $\vec{k} \cdot \vec{k} = 1$
7. $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{i} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{j} = \vec{k} \cdot \vec{i} = \vec{i} \cdot \vec{k} = 0$

Property 6 follows from property 4 (recall that $\vec{i}$, $\vec{j}$, and $\vec{k}$ are unit vectors), and property 7 follows from property 5.

NOTATION

It is also possible to write the dot product $\vec{u} \cdot \vec{u}$ as $\vec{u}^2$.

Now you are ready to use these properties to find a formula for calculating the dot product of vectors in component form. Indeed, if $\vec{u} = u_1\vec{i} + u_2\vec{j}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j}$,

$$\begin{aligned}
\vec{u} \cdot \vec{v} &= (u_1\vec{i} + u_2\vec{j}) \cdot (v_1\vec{i} + v_2\vec{j}) \\
&= (u_1\vec{i}) \cdot (v_1\vec{i}) + (u_1\vec{i}) \cdot (v_2\vec{j}) + (u_2\vec{j}) \cdot (v_1\vec{i}) + (u_2\vec{j}) \cdot (v_2\vec{j}) && \text{property 2} \\
&= (u_1v_1)(\vec{i} \cdot \vec{i}) + (u_1v_2)(\vec{i} \cdot \vec{j}) + (u_2v_1)(\vec{j} \cdot \vec{i}) + (u_2v_2)(\vec{j} \cdot \vec{j}) && \text{property 3} \\
&= (u_1v_1)(1) + (u_1v_2)(0) + (u_2v_1)(0) + (u_2v_2)(1) && \text{properties 6 and 7} \\
&= u_1v_1 + u_2v_2
\end{aligned}$$

FORMULA

So $\vec{u} = \overrightarrow{(u_1,u_2)}$ and $\vec{v} = \overrightarrow{(v_1,v_2)} \Rightarrow \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2$

In the exercises, you will be demonstrating in a similar manner that in $\mathbb{V}_3$

FORMULA

$\vec{u} = \overrightarrow{(u_1,u_2,u_3)}$ and $\vec{v} = \overrightarrow{(v_1,v_2,v_3)} \Rightarrow \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$

Note: The result in each case is indeed a real number (or scalar) as expected.

Now you can use these new formulas to redo Example 4 of section 3.2.

Example 1

Find the dot product in the following cases.

a) $\vec{u} = \overrightarrow{(1,0)}$, $\vec{v} = \overrightarrow{(-1,\sqrt{3})}$ b) $6\vec{u} = \overrightarrow{(6,0)}$, $2\vec{v} = \overrightarrow{(-2,2\sqrt{3})}$

Solution

a) $\vec{u} \cdot \vec{v} = (1)(-1) + (0)(\sqrt{3}) = -1$ b) $\vec{u} \cdot \vec{v} = (6)(-2) + (0)(2\sqrt{3}) = -12$ ■

Notice that the method is a lot quicker, if the vectors' components are known. The fact that you now have two methods of calculating dot products will help you to make more discoveries.

Example 2 Find the angle θ between $\vec{u} = \overrightarrow{(1,2,5)}$ and $\vec{v} = \overrightarrow{(-1,-3,4)}$.

Solution The definition of the dot product states $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta$.
This is an opportunity to use both methods of calculating the dot product.

$$\overrightarrow{(1,2,5)} \cdot \overrightarrow{(-1,-3,4)} = |\overrightarrow{(1,2,5)}||\overrightarrow{(-1,-3,4)}| \cos \theta$$
$$(1)(-1) + (2)(-3) + (5)(4) = \sqrt{1^2 + 2^2 + 5^2}\sqrt{(-1)^2 + (-3)^2 + 4^2} \cos \theta$$
$$-1 - 6 + 20 = \sqrt{30}\sqrt{26} \cos \theta$$

Thus $\cos \theta = \dfrac{13}{\sqrt{(30 \times 26)}} = 0.46547\ldots$ so that $\theta \doteq 62°$. ■

(Recall that this is the angle between the vectors *when they are drawn with a common tail*.)

Example 3 Find the value of x if $\overrightarrow{(-2,0,-6)} \cdot \overrightarrow{(1,3,x)} = 10$.

Solution The dot product is $(-2)(1) + (0)(3) + (-6)(x) = 10$
hence $-2 - 6x = 10$ or $x = -2$. ■

Example 4 If $\vec{u} = \overrightarrow{(3,-1)}$, find a two-dimensional vector perpendicular to $\vec{u}$.

Solution Let $\vec{v} = \overrightarrow{(x,y)}$ be perpendicular to $\vec{u}$.

Then $\overrightarrow{(3,-1)} \cdot \overrightarrow{(x,y)} = 0$
$3x + (-y) = 0$ or $y = 3x$.

If x is any real number, say $x = k$, then $y = 3k$.
So $\vec{v} = \overrightarrow{(k,3k)}$ is perpendicular to $\vec{u}$, no matter what the value of k. For instance, if $k = 2$, then $\vec{v} = \overrightarrow{(2,6)}$, which is perpendicular to $\vec{u}$. ■

The formulas on page 126 calculate the dot product of vectors expressed in components with the basis vectors $\vec{i},\vec{j}$ or $\vec{i},\vec{j},\vec{k}$. Would similar results hold for *any basis*? In the demonstration on page 126, you used facts such as $\vec{i} \cdot \vec{i} = 1$ and $\vec{i} \cdot \vec{j} = 0$ (properties 6 and 7). If you were using *any* basis, this would not necessarily be true, so the answer is NO.

The formulas stated are true because:

1. $\vec{i}, \vec{j}, \vec{k}$ are *unit vectors*, and
2. $\vec{i}, \vec{j}, \vec{k}$ are *mutually perpendicular* or orthogonal.

A basis which has these two qualities is called an **orthonormal basis.** Similar formulas for the calculation of the dot product would hold true only in an orthonormal basis.

SUMMARY

A basis of a vector space is orthonormal if

1. the basis vectors are all unit vectors
2. the basis vectors are all mutually perpendicular

In an orthonormal basis of $\mathbb{V}_2$,
$\vec{u} = \overrightarrow{(u_1,u_2)}$ and $\vec{v} = \overrightarrow{(v_1,v_2)}$
$\Rightarrow \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2$

In an orthonormal basis of $\mathbb{V}_3$,
$\vec{u} = \overrightarrow{(u_1,u_2u_3)}$ and $\vec{v} = \overrightarrow{(v_1,v_2v_3)}$
$\Rightarrow \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$

3.3 Exercises

1. Find the dot product $\vec{u} \cdot \vec{v}$ if
   a) $\vec{u} = \overrightarrow{(1,2,4)}$ $\quad \vec{v} = \overrightarrow{(2,0,3)}$
   b) $\vec{u} = \overrightarrow{(4,5)}$ $\quad \vec{v} = \overrightarrow{(2,-4)}$
   c) $\vec{u} = \overrightarrow{(4,5)}$ $\quad \vec{v} = \overrightarrow{(-5,4)}$
   d) $\vec{u} = \overrightarrow{(1,0,0)}$ $\quad \vec{v} = \overrightarrow{(6,17,-32)}$
   e) $\vec{u} = \overrightarrow{(-1,3,4)}$ $\quad \vec{v} = \overrightarrow{(-2,2,-2)}$

   What can you say about the vectors in parts c) and e)?

2. If $\vec{e}$ is a unit vector, what is $\vec{e} \cdot \vec{e}$?

3. Use the component formula to calculate the following in $\mathbb{V}_3$.
   a) $\vec{i} \cdot \vec{k}$ $\quad$ b) $\vec{j} \cdot \vec{j}$

4. Find the value of t if the vectors $\vec{u} = \overrightarrow{(3,t,-2)}$ and $\vec{v} = \overrightarrow{(4,-1,5)}$ are perpendicular.

5. Find the angle between the following pairs of vectors. Give your answers to the nearest degree.
   a) $\vec{a} = \overrightarrow{(3,7)}$ $\quad \vec{b} = \overrightarrow{(2,1)}$
   b) $\vec{a} = \overrightarrow{(-5,0)}$ $\quad \vec{b} = \overrightarrow{(0,1)}$
   c) $\vec{a} = \overrightarrow{(-5,1)}$ $\quad \vec{b} = \overrightarrow{(1,2)}$
   d) $\vec{a} = \overrightarrow{(2,-3,1)}$ $\quad \vec{b} = \overrightarrow{(5,0,-6)}$

6. Calculate k given that
   a) $\overrightarrow{(3,9)} \cdot \overrightarrow{(k,-1)} = 0$
   b) $\overrightarrow{(3,k,2)} \cdot \overrightarrow{(0,5,4,)} = 6$

7. a) Calculate the two values of k if $\vec{u} \cdot \vec{v} = 3$ where $\vec{u} = \overrightarrow{(k,1,4)}$ and $\vec{v} = \overrightarrow{(k,2k,-3)}$.
   b) For each of the values of k found in a), calculate the angle between $\vec{u}$ and $\vec{v}$.

8. Calculate $\vec{v} \cdot \vec{v}$ in the following cases.
   a) $\vec{v} = \overrightarrow{(3,-2)}$ $\quad$ c) $\vec{v} = \overrightarrow{(x,y,z)}$
   b) $\vec{v} = \overrightarrow{(-1,4,3)}$

9. a) Find a vector $\vec{p}$ perpendicular to the vector $\vec{q} = \overrightarrow{(4,-5)}$.
   b) Normalize $\vec{p}$.

10. Given the vectors $\vec{u} = \overrightarrow{(1,-3,2)}$, $\vec{v} = \overrightarrow{(-4,1,1)}$, and $\vec{w} = \overrightarrow{(2,0,5)}$, calculate the following.
    a) $2\vec{u} \cdot \vec{v}$
    b) $(\vec{u} + \vec{v}) \cdot \vec{w}$
    c) $-4(\vec{v} \cdot \vec{w})$
    d) $\vec{u} \cdot \vec{v} - \vec{w} \cdot \vec{v}$
    e) $(2\vec{u} - \vec{v}) \cdot (2\vec{u} + \vec{v})$

11. If $\vec{v} \neq \vec{0}$, prove that the angle between $\vec{v}$ and $-\vec{v}$ is 180°.

12. For any vector $\vec{v}$ of $\mathbb{V}_3$, prove that $(\vec{v} \cdot \vec{i})\vec{i} + (\vec{v} \cdot \vec{j})\vec{j} + (\vec{v} \cdot \vec{k})\vec{k} = \vec{v}$

13. $\vec{p}$ and $\vec{q}$ are unit vectors at an angle of 60° with each other.
    a) Calculate $(\vec{p} - 3\vec{q}) \cdot (\vec{p} - 3\vec{q})$.
    b) Hence find the unit vector in the direction of $\vec{p} - 3\vec{q}$.

14. $\vec{u}$, $\vec{v}$, and $\vec{w}$ are three distinct non-zero vectors. $\vec{v} \perp$ to both $\vec{u}$ and $\vec{w}$.
    a) If $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{v} \cdot (\vec{u} - \vec{w})$, prove that $\vec{w}$ is perpendicular to $(\vec{u} + \vec{v})$.
    b) If $(\vec{u} \cdot \vec{v})\vec{w} = (\vec{v} \cdot \vec{w})\vec{u}$, prove that $\vec{u} \| \vec{w}$.

15. The angle between the vectors $\vec{a}$ and $\vec{b}$ is θ where $\cos\theta = \frac{3}{7}$. If $\vec{a} = \overrightarrow{(2,3,-1)}$ and $\vec{b} = \overrightarrow{(-1,k,1)}$, find the possible values of k, correct to 3 decimal places.

16. A triangle is such that its three sides represent the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$. By expressing $\vec{c}$ in terms of $\vec{a}$ and $\vec{b}$, prove the cosine law. That is, prove that $|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}| \cos C$, where C is the angle between $\vec{a}$ and $\vec{b}$.

17. The position vectors of points A, B, C relative to the origin O are $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively. AB is perpendicular to OC and BC is perpendicular to OA. Prove that OB and AC are perpendicular.

3.4 Applications: The Dot Product and Trigonometry

In this section you will use the dot product to prove some trigonometric results. These proofs are very much neater than the proofs you have seen before, due to the power of vector algebra.

Finding Components

You can use the dot product to create another formula to obtain the component of a vector in the direction of another vector. Indeed, if the angle between $\vec{u}$ and $\vec{v}$ is θ, then the component of $\vec{u}$ on $\vec{v}$ is $|\vec{u}| \cos \theta$.

But $\quad |\vec{u}||\vec{v}| \cos \theta = \vec{u} \cdot \vec{v}$

Dividing each side by the scalar $|\vec{v}|$ gives

$$|\vec{u}| \cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$$

Thus the **component of $\vec{u}$ on $\vec{v}$**, $c = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \vec{u} \cdot \vec{e}_v$,

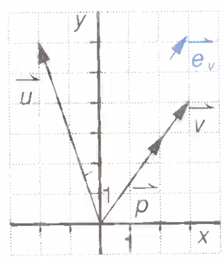
where $\vec{e}_v$ is the unit vector in the direction of $\vec{v}$.

Thus, the **projection of $\vec{u}$ on $\vec{v}$**, $\vec{p} = (\vec{u} \cdot \vec{e}_v)\, \vec{e}_v$

The following example matches Example 4 of section 3.1.

Example Given $\vec{u} = \overrightarrow{(-2,6)}$ and $\vec{v} = \overrightarrow{(3,4)}$, find the following.

a) the component of $\vec{u}$ on $\vec{v}$ b) the projection of $\vec{u}$ on $\vec{v}$



Solution a) The unit vector $\vec{e}_v = \dfrac{1}{|\vec{v}|}\vec{v} = \dfrac{1}{\sqrt{3^2 + 4^2}}\overrightarrow{(3,4)} = \dfrac{1}{5}\overrightarrow{(3,4)} = \overrightarrow{(0.6,0.8)}$

Thus the component of $\vec{u}$ on $\vec{v}$ is $c = \overrightarrow{(-2,6)} \cdot \overrightarrow{(0.6,0.8)} = 3.6$

b) The projection of $\vec{u}$ on $\vec{v}$ is thus

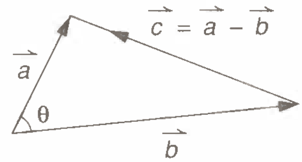
$\vec{p} = (3.6)\vec{e}_v = 3.6\overrightarrow{(0.6,0.8)} = \overrightarrow{(2.16,2.88)}$ ■

Note: It is not necessary to know the angle between $\vec{u}$ and $\vec{v}$, as in 3.1.

You will find other applications of the dot product in chapter 4, but some of these are interesting and spectacular enough to be introduced now.

A Proof of the Cosine Law using the Dot Product

In the figure, any vectors $\vec{a}$ and $\vec{b}$ are sketched with their tails in common.



The angle between $\vec{a}$ and $\vec{b}$ is θ.

The vector $\vec{c}$ which 'closes the triangle' as shown is equal to $\vec{a} - \vec{b}$. Now

$\vec{c}\cdot\vec{c} = (\vec{a} - \vec{b})\cdot(\vec{a} - \vec{b})$

$= \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$ using property 2

$|\vec{c}|^2 = |\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{b}|^2$ using properties 1 and 4

$|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$ using the definition

which is already the cosine law!

From this fast proof you might appreciate the power of the dot product, and the vastness of the possible applications of vectors to mathematics. If you look back at a traditional proof of the cosine law, you will see how much more concise this one is.

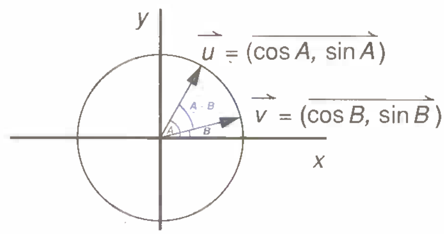
Finding Compound Angle Identities

Another fundamental result in trigonometry is the set of compound angle identities, that is, the formulas for the sine, cosine, and tangent of the sum and difference of two angles (see page 542).

Finding an expression for, say, $\cos(A - B)$ in terms of trigonometrical ratios of A or B only, by applying basic trigonometry, is a much longer process.

Once again, the dot product of vectors will allow you to arrive at a result remarkably quickly.

Consider two unit vectors $\vec{u}$ and $\vec{v}$, making angles A and B with the horizontal respectively, as shown.



In component form, the vectors are $\vec{u} = \overrightarrow{(\cos A, \sin A)}$ and $\vec{v} = \overrightarrow{(\cos B, \sin B)}$.

$|\vec{u}| = \sqrt{\cos^2 A + \sin^2 A} = 1$, as expected, and $|\vec{v}| = 1$. Now

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(A - B) = (1)(1)\cos(A - B) = \cos(A - B)$ ①

but by using components

$\vec{u}\cdot\vec{v} = \overrightarrow{(\cos A, \sin A)}\cdot\overrightarrow{(\cos B, \sin B)} = \cos A\cos B + \sin A\sin B.$ ②

From ① and ②, $\cos(A - B) = \cos A\cos B + \sin A\sin B$! !

The speed and conciseness of this proof, compared to traditional proofs, is even more striking than the comparison in the case of the cosine law.

3.4 Exercises

1. a) Calculate, to the nearest degree, the angle between the vector $\vec{v} = \overrightarrow{(2,3,-5)}$ and the three coordinate axes.
   b) What is the component of $\vec{v}$ on each of $\vec{i}$, $\vec{j}$, and $\vec{k}$?

2. Calculate, to the nearest degree, the angle between $\overrightarrow{PQ}$ and $\vec{i}$, if P and Q are the following points.
   a) $P(2,1)$, $Q(3,7)$
   b) $P(-2,0)$, $Q(4,5)$
   c) $P(-2,-2,1)$, $Q(4,3,2)$
   d) $P(5,3,1)$, $Q(1,-1,1)$

3. Using the points of question 2, state the component of $\overrightarrow{PQ}$ on $\vec{i}$ in each case.

4. Calculate the component of $\vec{u}$ on $\vec{v}$, and the component of $\vec{v}$ on $\vec{u}$, in the following cases.
   a) $\vec{u} = \overrightarrow{(-3,5)}$, $\vec{v} = \overrightarrow{(1,2)}$
   b) $\vec{u} = \overrightarrow{(-1,1,1)}$, $\vec{v} = \overrightarrow{(2,4,-5)}$

5. Show that the triangle ABC with vertices $A(2,2,2)$, $B(-2,-4,0)$, $C(0,-12,2)$ has an obtuse angle at B.

6. Determine the angles of the triangle PQR in the following cases, to the nearest degree.
   a) $P(3,-1)$, $Q(4,4)$, $R(-2,-3)$
   b) $P(2,0,1)$, $Q(5,1,-3)$, $R(-4,2,7)$

The next three questions match questions 7, 9, and 13 of 3.1 Exercises. Use the dot product to find the solutions here.

7. Resolve the following vectors on $\vec{i}$, $\vec{j}$, and $\vec{k}$.
   $\vec{u} = \overrightarrow{(4,5,0)}$ $\qquad$ $\vec{v} = \overrightarrow{(-2,-3,1)}$

8. Calculate the projection of the following on $\vec{v} = \overrightarrow{(1,1)}$.
   a) $\vec{i}$ $\qquad$ b) $\vec{j}$

9. Given two non-zero vectors $\vec{u}$ and $\vec{v}$, what can you deduce about $\vec{u}$ and $\vec{v}$ where
   a) the component of $\vec{u}$ on $\vec{v}$ is equal to the component of $\vec{v}$ on $\vec{u}$?
   b) the projection of $\vec{u}$ on $\vec{v}$ is equal to the projection of $\vec{v}$ on $\vec{u}$?

10. Given the vectors $\vec{u} = \overrightarrow{(2,10)}$ and $\vec{v} = \overrightarrow{(-3,-2)}$, find each of the following, correct to 3 significant digits.
    a) the component of $\vec{u}$ on $\vec{v}$
    b) the projection of $\vec{u}$ on $\vec{v}$

11. Resolve the vector $\vec{v} = \overrightarrow{(-2,3)}$ onto the vectors $\vec{a} = \overrightarrow{(1,1)}$ and $\vec{b} = \overrightarrow{(-1,1)}$.

12. Use the dot product to determine whether or not the following points determine a right-angled triangle.
    a) $A(2,1)$, $B(6,5)$, $C(3,0)$
    b) $A(2,1)$, $B(3,-1)$, $C(6,5)$
    c) $A(1,-1,5)$, $B(2,3,-4)$, $C(3,5,-3)$

13. A circle of centre O has a diameter PR. C is any other point on the circle.
    a) If $\overrightarrow{OR} = \vec{r}$, state the vector $\overrightarrow{OP}$ in terms of $\vec{r}$.
    b) If $\overrightarrow{OC} = \vec{c}$, express the vectors $\overrightarrow{RC}$ and $\overrightarrow{PC}$ in terms of $\vec{r}$ and $\vec{c}$.
    c) Calculate the dot product $\overrightarrow{RC} \cdot \overrightarrow{PC}$.
    d) Hence deduce the value of an angle inscribed in a semi-circle.

14. A fact about a circle is that any angle inscribed in a given segment of a circle is constant. A converse of this theorem can be described as follows. If A, B, C, and D are four points such that $\measuredangle ABD = \measuredangle ACD$, then A, B, C, and D lie on a circle. Use this fact to prove that the following four points lie on a circle. (Such points are called cyclic or concyclic).

    $A(-2,-2)$, $B(-1,5)$, $C(6,4)$, $D(7,1)$.

3.5 The Cross Product

The second product of two vectors $\vec{u}$ and $\vec{v}$ is written $\vec{u} \times \vec{v}$, and is called the **cross product** or **vector product** of $\vec{u}$ and $\vec{v}$.

The cross product is a regular binary operation, unlike the dot product which gives you a scalar result. In other words, the result of the cross product of two vectors is also a vector.

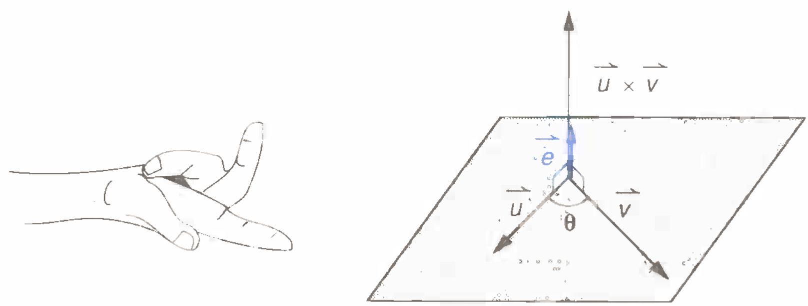
When writing this product, you *must* use a 'cross' ($\times$) as shown:

DEFINITION

$\vec{u} \times \vec{v} = (|\vec{u}||\vec{v}| \sin \theta)\vec{e}$

where θ is the angle between the two vectors when they are drawn with a common tail, and $\vec{e}$ is a unit vector *perpendicular* to both $\vec{u}$ and $\vec{v}$ chosen in such a way that the triple $[\vec{u}, \vec{v}, \vec{e}]$ forms a right-handed system.

(See chapter 1, section 2.
Using your right hand, the directions of $\vec{u}$, $\vec{v}$, and $\vec{e}$ are represented by your thumb, your first finger and your second finger respectively.)



The diagram shows the direction of $\vec{e}$ and hence of $\vec{u} \times \vec{v}$.

PROPERTY

If $\vec{u}$ and $\vec{v}$ were interchanged, then $\vec{e}$ would be pointing in precisely the opposite direction. (Again, check this with your right hand.) This indicates that $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$. This will be confirmed in the following example.

Note: Since a three-dimensional system is required for its definition, the cross product *is meaningless in* $\mathbb{V}_2$.

Example 1 In each of the following, find the cross products, if possible.

a) If $|\vec{u}| = 4$, $|\vec{v}| = 5$, $\theta = 30°$, find $\vec{u} \times \vec{v}$.

b) If $\vec{u} = \overrightarrow{(2,5)}$, $\vec{v} = \overrightarrow{(-1,1)}$, find $\vec{u} \times \vec{v}$.

c) If $\vec{u} = \overrightarrow{(2,5,0)}$, $\vec{v} = \overrightarrow{(-1,1,0)}$, find both $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$.

Solution a) $\vec{u}\times\vec{v} = (|\vec{u}||\vec{v}|\sin\theta)\vec{e} = (4\times5\times\sin 30°)\vec{e} = (20)(0.5)\vec{e} = 10\vec{e}$

Note: *The direction of $\vec{e}$, and hence of $\vec{u}\times\vec{v}$, is not known*, unless the exact directions of $\vec{u}$ and of $\vec{v}$ are known. In this example, you do *not* have that information.

b) $\vec{u}\times\vec{v}$ is not defined in $\mathbb{V}_2$, so this vector product is *impossible*.

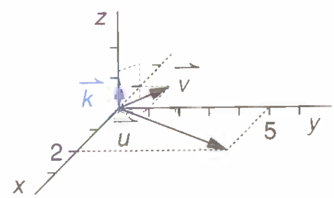
c) $|\vec{u}| = \sqrt{2^2+5^2+0^2} = \sqrt{29}$, and $|\vec{v}| = \sqrt{(-1)^2+1^2+0^2} = \sqrt{2}$.

You can obtain the angle θ between $\vec{u}$ and $\vec{v}$ by using the dot product.

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$, so $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$

$$\cos\theta = \frac{\overrightarrow{(2,5,0)}\cdot\overrightarrow{(-1,1,0)}}{\sqrt{29}\sqrt{2}}$$

$$= \frac{(2)(-1)+(5)(1)+(0)(0)}{\sqrt{58}} = \frac{3}{\sqrt{58}} = 0.3939\ldots \Rightarrow \theta \doteq 67°.$$



Thus $\vec{u}\times\vec{v} = |\vec{u}||\vec{v}|\sin\theta\vec{e}$

$= \sqrt{29}\sqrt{2}\sin 67°\ \vec{e}$

$= (\sqrt{58})(0.9191\ldots) = 7\vec{e}$.

But vectors $\vec{u}$ and $\vec{v}$ are in the *xy*-plane, as the diagram shows.
The triple $[\vec{u}, \vec{v}, \vec{k}]$ forms a right-handed system.
Thus, $\vec{e} = \vec{k}$, therefore $\vec{u}\times\vec{v} = 7\vec{k}$, or $\overrightarrow{(0,0,7)}$.

If $\vec{u}$ and $\vec{v}$ are interchanged, the second finger of your right hand points in the *opposite direction* to the direction of $\vec{k}$, namely in the direction of $-\vec{k}$.

Thus $\vec{v}\times\vec{u} = |\vec{v}||\vec{u}|\sin\theta\ (-\vec{k})$

$= \sqrt{2}\sqrt{29}\sin 113°\ (-\vec{k})$

$= -7\vec{k}$. ■

Geometrical interpretation of the Cross Product

Consider a parallelogram *ABCD* where $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{AD} = \vec{v}$. Let *H* be the foot of the perpendicular from *D* to *AB*.

If θ is the angle between $\vec{u}$ and $\vec{v}$, then $DH = |\vec{v}|\sin\theta$.

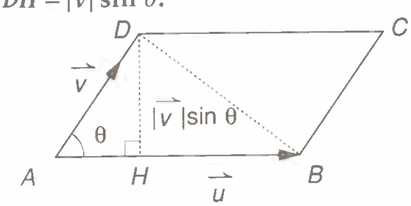
The area of this parallelogram is

(base)(height)

$= (AB)(DH)$

$= |\vec{u}|(|\vec{v}|\sin\theta)$

$= |\vec{u}\times\vec{v}|$.



In other words, the magnitude of $\vec{u}\times\vec{v}$ is the area of the parallelogram whose sides are represented by the vectors $\vec{u}$ and $\vec{v}$.

Example 2 A parallelogram $ABCD$ is such that $|\vec{AB}| = 6$, $|\vec{AD}| = 5$, and $\measuredangle BAD$ is 28°. Calculate the area of the parallelogram, correct to 1 decimal place.

Solution The area of $ABCD$ is
$|\vec{AB} \times \vec{AD}| = |\vec{AB}||\vec{AD}| \sin 28° = (6)(5)(0.4694\ldots) \doteq 14.1.$ ■

The following examples should help you to discover further properties of the cross product.

Example 3 Calculate a) $\vec{i} \times \vec{i}$, b) $\vec{i} \times \vec{j}$ c) $\vec{k} \times \vec{j}$

Solution a) $\vec{i} \times \vec{i} = (1)(1)(\sin 0°)\vec{e} = 0\vec{e} = \vec{0}$

Note: You cannot find a *unique* direction for $\vec{e}$, because there are many vectors perpendicular to $\vec{i}$ in $\mathbb{V}_3$. However, this is not important, since the coefficient of this vector is 0. Thus the result is the zero vector, $\vec{0}$, which is considered to have *any* direction.

PROPERTY

The cross product of a vector with itself is the zero vector.

b) $\vec{i} \times \vec{j} = (1)(1)(\sin 90°)\vec{k} = \vec{k}$
c) $\vec{k} \times \vec{j} = (1)(1)(\sin 90°)(-\vec{i}) = -\vec{i}$ ■ right-handed system

PROPERTY

The cross product of any two distinct standard basis vectors of $\mathbb{V}_3$ gives ±(the third standard basis vector).

Example 4 Calculate $(3\vec{i}) \times (4\vec{j})$.

Solution $(3\vec{i}) \times (4\vec{j}) = (3)(4)(\sin 90°)\vec{k} = 12\vec{k}.$ ■

Compare the result of this example to that of Example 3b).

PROPERTY

For any vectors $\vec{u}$, $\vec{v}$, and scalars m, n, $(m\vec{u}) \times (n\vec{v}) = (mn)\vec{u} \times \vec{v}$.

You might note that the vectors in the examples of this section were in one of the major planes of $\mathbb{V}_3$. If they had not been, you would have had difficulty in determining the direction of $\vec{e}$. The method of the next section will allow you to find the cross product of *any* two vectors of $\mathbb{V}_3$.

SUMMARY

The cross product of vectors $\vec{u}$ and $\vec{v}$ is the vector $\vec{u} \times \vec{v} = |\vec{u}||\vec{v}| \sin \theta\, \vec{e}$, where $\vec{e}$ is a unit vector perpendicular to $\vec{u}$ and $\vec{v}$ such that the triple $[\vec{u}, \vec{v}, \vec{e}]$ forms a right-handed system.

The magnitude of $\vec{u} \times \vec{v}$ is the area of the parallelogram whose sides are represented by the vectors $\vec{u}$ and $\vec{v}$.

$\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$.

3.5 Exercises

1. Calculate $\vec{u} \times \vec{v}$ for each of the following, where $\vec{u}$ and $\vec{v}$ are vectors of $\mathbb{V}_3$. Specify the direction as precisely as you can in each case.
   a) $|\vec{u}| = 5, |\vec{v}| = 2$, and the angle between $\vec{u}$ and $\vec{v}$ is 30°
   b) $|\vec{u}| = 12, |\vec{v}| = 7$, and the angle between $\vec{u}$ and $\vec{v}$ is 150°
   c) $|\vec{u}| = 3, |\vec{v}| = 6$, and $\vec{u} \cdot \vec{v} = 8$
   d) $|\vec{u}| = 4, |\vec{v}| = 5$, and $\vec{u} \cdot \vec{v} = -10$
   e) $\vec{u} = \overrightarrow{(0,6,3)}$ and $\vec{v} = \overrightarrow{(0,-1,5)}$
   f) $\vec{u} = \overrightarrow{(0,-1,5)}$ and $\vec{v} = \overrightarrow{(0,6,3)}$

2. Calculate
   a) $\vec{j} \times \vec{j}$
   b) $\vec{j} \times \vec{k}$
   c) $\vec{j} \times \vec{i}$.

3. A parallelogram $ABCD$ is such that $|\overrightarrow{AB}| = 10, |\overrightarrow{AD}| = 4$, and the angle BAD is 42°. Calculate the area of the parallelogram, correct to 1 decimal place.

4. A triangle ABC is such that $|\overrightarrow{AB}| = 15$, $|\overrightarrow{AC}| = 12$, and $\overrightarrow{AB} \cdot \overrightarrow{AC} = -100$. Calculate the area of the triangle, correct to 1 decimal place.

5. Given any three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ of $\mathbb{V}_3$, determine which of the following expressions are meaningful.
   a) $\vec{u} \times (\vec{v} \cdot \vec{w})$
   b) $\vec{u} \times (\vec{v} + \vec{w})$
   c) $\vec{u} \times (\vec{v} - \vec{w})$
   d) $(\vec{u} \times \vec{v})\vec{w}$
   e) $(\vec{u} \times \vec{v}) \times \vec{w}$
   f) $(\vec{u} \cdot \vec{v}) + (\vec{u} \times \vec{v})$

6. Prove that the cross product of a vector with itself is the zero vector.

7. Prove that, for any vectors $\vec{u}$ and $\vec{v}$ of $\mathbb{V}_3$, $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.

8. Suppose m is any scalar, and $[\vec{u}, \vec{v}, \vec{e}]$ is any right-handed system of vectors, where $\vec{e}$ is a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$. Prove that $(m\,\vec{u}) \times \vec{v} = m(\vec{u} \times \vec{v})$ in the following cases.
   a) m is positive
   b) m is negative

9. Using the results of question 8, prove that, if n is another scalar, $(m\,\vec{u}) \times (n\,\vec{v}) = (mn)(\vec{u} \times \vec{v})$

10. Prove that $\vec{u} \times \vec{v} = \vec{0}$ if and only if $\vec{u}$ and $\vec{v}$ are linearly dependent.

11. Consider the standard basis vectors $\vec{i}$, $\vec{k}$, and the vector $\vec{u} = \overrightarrow{(1,1,0)}$ of $\mathbb{V}_3$.
   a) Explain why $(\vec{i} \times \vec{u}) \times \vec{k} = \vec{0}$.
   b) Does $\vec{i} \times (\vec{u} \times \vec{k}) = \vec{0}$?
   c) State whether or not the cross product is associative.

12. Prove that, for any vectors $\vec{u}$ and $\vec{v}$ of $\mathbb{V}_3$, $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$.

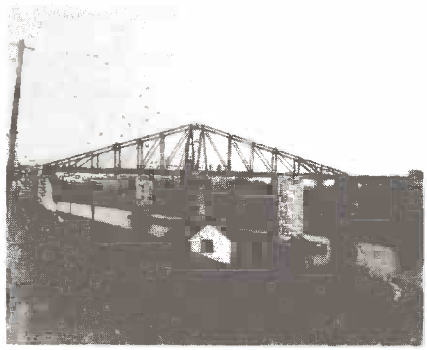
13. The standard basis vectors of $\mathbb{V}_3$ are $\vec{i}$, $\vec{j}$, and $\vec{k}$. Prove that the cross product of any two distinct standard basis vectors in $\mathbb{V}_3$ gives $\pm$(the third standard basis vector). (You may use a general argument, or treat this case by case.)

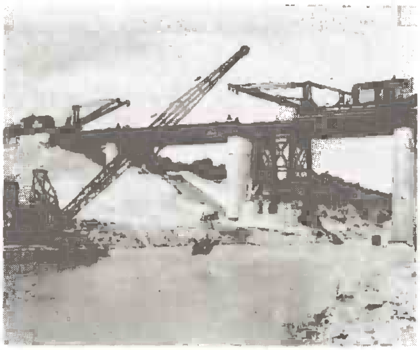
14. Given that the vectors $\vec{u}$ and $\vec{v}$ of $\mathbb{V}_3$ make an angle θ with each other, prove the following.
   a) $(\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v}) - (\vec{u} \cdot \vec{v})^2 = |\vec{u}|^2 |\vec{v}|^2 \sin^2\theta$
   b) $|\vec{u} \times \vec{v}| = \sqrt{(\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v}) - (\vec{u} \cdot \vec{v})^2}$

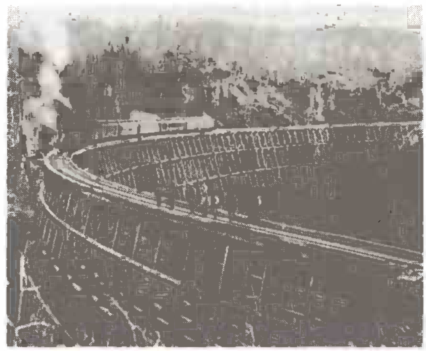
15. a) If $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, does it necessarily follow that $\vec{b} = \vec{c}$?
   b) If $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ *and* $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$, does it necessarily follow that $\vec{b} = \vec{c}$?

16. Verify that $(\vec{i} + \vec{j}) \times \vec{k} = \vec{i} \times \vec{k} + \vec{j} \times \vec{k}$. (This illustrates that the cross product is distributive over vector addition.)

Bridges







Bridges have been in existence for a long time. The first people on earth probably used a fallen tree to go from one side of a stream to the other.

In the 18th, 19th and the early 20th centuries much of the work done in designing bridges was based on the success of previously constructed bridges. If an earlier construction had no problems, bigger, larger, and more elaborate versions of the old structures were erected. When a bridge did fail, then engineers would locate the source of the problem and add more safety features in future structures.

With the advent of the railroad in Canada and the United States in the 19th century much work was done developing strong truss structures that would enable the construction of longer bridges. One problem that had to be overcome was caused by the vibrations set off in a bridge by a train passing over the bridge.

Designers were not always successful. In 1877 a bridge in Ashtabula, Ohio, failed as a train passed over it. 90 people were killed. Two years later, a train consisting of a locomotive and six passenger coaches fell from a collapsing bridge into the Firth of Tay in Scotland, killing over 100 passengers. During the period from 1870 to 1890, truss bridges in the United States failed at the rate of 25 per year. Something needed to be done.

In 1934, C.E. Inglis completed a long study called *A Mathematical Treatise on Vibrations in Railway Bridges*. In this treatise he wrote, "Mathematical analysis is required to indicate the lines along which experiments should proceed, and experiment, in its turn, is necessary to check the validity of theoretical predictions and to prevent mathematics running off the scent and barking, so to speak, up the wrong tree."

Nowadays one expects engineers to make use of mathematics in their designs of bridges. Indeed, calculations and simulations based on appropriate mathematical models form the basis of structural engineering. The use of theory, rather than experience, to design bridges, has been in part motivated by the desire for safety, and to use new materials that will cut costs and speed construction.

Theory has allowed designers of bridges to come closer to the limits of safety while still maintaining a large margin of safety. But nevertheless, designers still run the risk of structural failure, especially if they do not take into account all physical situations.

One notable failure was 'Galloping Gertie'. Galloping Gertie was the nickname given to a bridge in the United States built to cross the Puget Sound in Tacoma, Washington.

From the time it opened on July 1, 1940, the bridge achieved a certain popularity and notoriety due to its tendency to sway in the wind. People would drive over the bridge just to get a roller coaster feeling. But all was not well. Just a few months later, on November 7, 1940, the bridge collapsed into the Puget Sound.

After the collapse of this bridge, new studies were made to try to prevent similar disasters. In the case of Gertie, the static analysis of the bridge had been done correctly but proper attention had not been paid to aerodynamical considerations.

Now mathematicians, computer scientists, and engineers act together with architects to design bridges. Computer simulations of the design features and workings of a bridge can be produced graphically on a computer screen. These new bridges should be beautiful, functional bridges that do not collapse.

3.6 Properties of the Cross Product

In the examples and exercises of the last section, you proved the first four of the following properties of the cross product.

For any vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ and any scalar m,

PROPERTIES

1. $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$ [not commutative]
2. $\vec{u}\times\vec{u} = \vec{0}$
3. $\vec{i}\times\vec{j} = \vec{k}$, $\vec{j}\times\vec{k} = \vec{i}$, $\vec{k}\times\vec{i} = \vec{j}$
4. $(m\,\vec{u})\times\vec{v} = m(\vec{u}\times\vec{v})$
5. $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$ [distributive over +]

This last property will be proved later in this section. Before this can be done, you need to investigate the following product.

Triple Scalar Product

The expression $(\vec{a}\times\vec{b})\cdot\vec{c}$ is known as a **triple scalar product.**

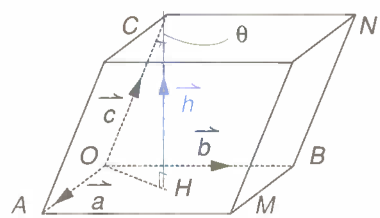
Note 1 $(\vec{a}\times\vec{b})\cdot\vec{c}$ is meaningful, since $\vec{a}\times\vec{b}$ and $\vec{c}$ are both vectors. Thus the dot product *can* be obtained, giving a *scalar* as a final result.

2 "$\vec{a}\times(\vec{b}\cdot\vec{c})$" is meaningless. You cannot perform a cross product with a vector and a scalar.

Consider a parallelepiped whose sides emanating from O are represented by vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, where $\vec{a}$, $\vec{b}$, $\vec{c}$ form a right-handed system. Let the height of the parallelepiped be represented by the vector $\overrightarrow{HC} = \vec{h}$.

The volume V of the parallelepiped can be calculated as follows.

$$
\begin{aligned}
V &= (\text{base area})\,(\text{height})\\
&= (OAMB)(\overrightarrow{HC})\\
&= |\vec{a}\times\vec{b}||\vec{h}|\\
&= |\vec{a}\times\vec{b}||\vec{c}|\cos\theta|\\
&= |(\vec{a}\times\vec{b})\cdot\vec{c}| \qquad ①
\end{aligned}
$$



Since the vector $\vec{a}\times\vec{b}$ has the same direction as $\vec{h}$ (which is perpendicular to the base), the angle between $\vec{a}\times\vec{b}$ and $\vec{c}$ is the same as the angle between $\vec{h}$ and $\vec{c}$, namely θ. Thus the dot product $(\vec{a}\times\vec{b})\cdot\vec{c}$ is equal, by definition, to $|\vec{a}\times\vec{b}||\vec{c}|\cos\theta|$. Furthermore, since $[\vec{a},\vec{b},\vec{c}]$ is a right-handed system, the angle θ is acute ($0^\circ < \theta \leqslant 90^\circ$), so $\cos\theta$ is positive. Thus, the dot product is positive, and the absolute value signs of line ① are not required.

Therefore $V = (\vec{a}\times\vec{b})\cdot\vec{c}$.

Similarly, by using parallelogram *OBNC* instead of parallelogram *OAMB* as a base, you can calculate the volume of the parallelepiped as

$V = (\vec{b} \times \vec{c}) \cdot \vec{a}$,

and since the dot product is commutative,

$V = \vec{a} \cdot (\vec{b} \times \vec{c})$.

(Notice that $[\vec{b}, \vec{c}, \vec{a}]$ or $[\vec{a}, \vec{b}, \vec{c}]$ are both right-handed systems.)

Thus, $V = (\vec{a} \times \vec{b}) \cdot \vec{c}$ and $V = \vec{a} \cdot (\vec{b} \times \vec{c})$.

From this argument, the following property of triple scalar products can be deduced.

PROPERTY

$$\vec{a} \times \vec{b} \cdot \vec{c} = \vec{a} \cdot \vec{b} \times \vec{c}$$

In other words, *as long as the order of the vectors remains the same, the dot and the cross of a triple scalar product can be interchanged*!

Notice that brackets are not essential, since either expression is meaningful only if the cross product is performed *first*.

Proof of the Distributivity of × over + (property 5)

Consider $\vec{r} = \vec{u} \times (\vec{v} + \vec{w}) - \vec{u} \times \vec{v} - \vec{u} \times \vec{w}$, where $\vec{u}$, $\vec{v}$, and $\vec{w}$ are any vectors. Then

$$\begin{aligned}
\vec{r} \cdot \vec{r} &= \vec{r} \cdot [\vec{u} \times (\vec{v} + \vec{w}) - \vec{u} \times \vec{v} - \vec{u} \times \vec{w}] \\
&= \underline{\vec{r} \cdot \vec{u}} \times \underline{(\vec{v} + \vec{w})} - \underline{\vec{r} \cdot \vec{u}} \times \underline{\vec{v}} - \underline{\vec{r} \cdot \vec{u}} \times \vec{w} && \text{dot product is distributive} \\
&= \vec{r} \times \vec{u} \cdot (\vec{v} + \vec{w}) - \vec{r} \times \vec{u} \cdot \vec{v} - \vec{r} \times \vec{u} \cdot \vec{w} && \text{triple scalar product property} \\
&= \vec{r} \times \vec{u} \cdot \vec{v} + \vec{r} \times \vec{u} \cdot \vec{w} - \vec{r} \times \vec{u} \cdot \vec{v} - \vec{r} \times \vec{u} \cdot \vec{w} && \text{dot product is distributive} \\
&= 0.
\end{aligned}$$

Thus $\vec{r} = \vec{0}$, that is,

$$\begin{aligned}
\vec{u} \times (\vec{v} + \vec{w}) - \vec{u} \times \vec{v} - \vec{u} \times \vec{w} &= \vec{0}, \text{ thus} \\
\vec{u} \times (\vec{v} + \vec{w}) &= \vec{u} \times \vec{v} + \vec{u} \times \vec{w}
\end{aligned}$$

which completes the proof.

Now by using the properties of the cross product listed at the beginning of this section, you can find a formula for calculating the cross product of vectors expressed in component form in the orthonormal basis $\vec{i}$, $\vec{j}$, $\vec{k}$.

If $\vec{u} = \overrightarrow{(u_1, u_2, u_3)}$ and $\vec{v} = \overrightarrow{(v_1, v_2, v_3)}$,

$$\begin{aligned}
\vec{u} \times \vec{v} &= (u_1\vec{i} + u_2\vec{j} + u_3\vec{k}) \times (v_1\vec{i} + v_2\vec{j} + v_3\vec{k}) \\
&= u_1v_1(\vec{i} \times \vec{i}) + u_1v_2(\vec{i} \times \vec{j}) + u_1v_3(\vec{i} \times \vec{k}) \\
&\quad + u_2v_1(\vec{j} \times \vec{i}) + u_2v_2(\vec{j} \times \vec{j}) + u_2v_3(\vec{j} \times \vec{k}) && \text{properties} \\
&\quad + u_3v_1(\vec{k} \times \vec{i}) + u_3v_2(\vec{k} \times \vec{j}) + u_3v_3(\vec{k} \times \vec{k}) && \text{4 and 5} \\
&= u_1v_2\vec{k} - u_1v_3\vec{j} - u_2v_1\vec{k} + u_2v_3\vec{i} + u_3v_1\vec{j} - u_3v_2\vec{i} && \text{properties 2 and 3}
\end{aligned}$$

FORMULA

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}$$

This is a particularly difficult formula to remember. It is easier to recall it in the form of a 3×3 determinant.

The 3×3 determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

where a 2×2 determinant, such as $\begin{vmatrix} e & f \\ g & h \end{vmatrix} = eh - fg$.

You will be learning more about determinants in chapter 7.

If $\vec{u} = \overrightarrow{(u_1,u_2,u_3)}$ and $\vec{v} = \overrightarrow{(v_1,v_2,v_3)}$

FORMULA

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Note: The result is indeed a vector.

Because $\vec{u}\times\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$, the cross product can be used to find a vector perpendicular to two given vectors.

Example 1 In each of the following cases, use the cross product to find a vector perpendicular to both $\vec{u}$ and $\vec{v}$.

a) $\vec{u} = \overrightarrow{(2,5,0)}$ and $\vec{v} = \overrightarrow{(-1,1,0)}$

b) $\vec{u} = \overrightarrow{(1,2,3)}$ and $\vec{v} = \overrightarrow{(-2,5,6)}$

Solution

a) $$\vec{u}\times\vec{v} = \overrightarrow{(2,5,0)}\times\overrightarrow{(-1,1,0)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 5 & 0 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= \vec{i}[(5)(0) - (0)(1)] - \vec{j}[(2)(0) - (0)(-1)] + \vec{k}[(2)(1) - (5)(-1)]$$

$$= \vec{i}[0 - 0] - \vec{j}[0 - 0] + \vec{k}[2 - (-5)]$$

$$= 7\vec{k}$$

Note: The result is indeed perpendicular to both $\vec{u}$ and $\vec{v}$, which are in the xy-plane.

(Compare this to the solution of Example 1 c) in the previous section.)

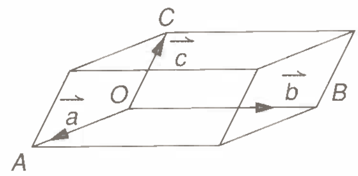
b) $$\vec{u}\times\vec{v} = \overrightarrow{(1,2,3)}\times\overrightarrow{(-2,5,6)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ -2 & 5 & 6 \end{vmatrix}$$

$$= \vec{i}[(2)(6) - (3)(5)] - \vec{j}[(1)(6) - (3)(-2)] + \vec{k}[(1)(5) - (2)(-2)]$$

$$= \vec{i}[12 - 15] - \vec{j}[6 - (-6)] + \vec{k}[5 - (-4)]$$

$$= \overrightarrow{(-3,-12,9)}$$ ■

Example 2 Given $\vec{a} = \overrightarrow{(2,1,1)}$, $\vec{b} = \overrightarrow{(0,-1,1)}$, and $\vec{c} = \overrightarrow{(-1,3,0)}$, use both versions of the triple scalar product to find the volume of the parallelepiped shown.



Solution A volume must be positive. To avoid checking whether or not $[\vec{a},\vec{b},\vec{c}]$ forms a right-handed system, use the absolute value of the triple scalar product.

The volume is either $|(\vec{a}\times\vec{b})\cdot\vec{c}|$ or $|\vec{a}\cdot(\vec{b}\times\vec{c})|$.

$$\text{Now } \vec{a}\times\vec{b} = \overrightarrow{(2,1,1)}\times\overrightarrow{(0,-1,1)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix}$$
$$= \vec{i}(1+1) - \vec{j}(2-0) + \vec{k}(-2-0)$$
$$= 2\vec{i} - 2\vec{j} - 2\vec{k}$$

and so $|(\vec{a}\times\vec{b})\cdot\vec{c}| = |\overrightarrow{(2,-2,-2)}\cdot\overrightarrow{(-1,3,0)}| = |-2 + (-6) + 0| = 8$.

Alternatively,

$$\vec{b}\times\vec{c} = \overrightarrow{(0,-1,1)}\times\overrightarrow{(-1,3,0)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 1 \\ -1 & 3 & 0 \end{vmatrix}$$
$$= \vec{i}(0-3) - \vec{j}(0+1) + \vec{k}(0-1)$$
$$= -3\vec{i} - \vec{j} - \vec{k}$$

and so $|\vec{a}\cdot(\vec{b}\times\vec{c})| = |\overrightarrow{(2,1,1)}\cdot\overrightarrow{(-3,-1,-1)}| = |-6-1-1| = 8$.

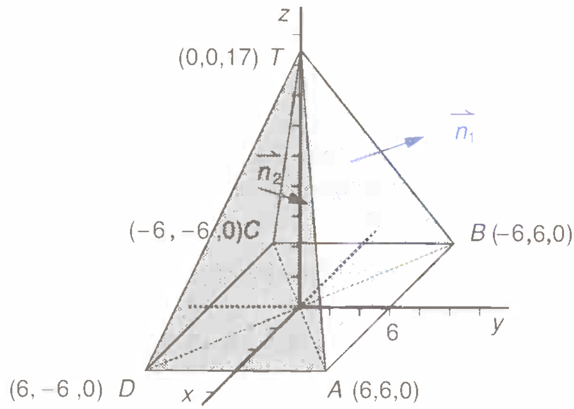
The same volume is obtained in both cases. ■

Example 3 Prove that $\vec{a}\times\vec{b}\cdot\vec{c} = 0$ if and only if $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent.

Solution

1. Given $\vec{a}\times\vec{b}\cdot\vec{c} = 0$, prove that $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly dependent.
   If $\vec{a}\times\vec{b}\cdot\vec{c} = 0$ then $\vec{a}\times\vec{b}$ is perpendicular to $\vec{c}$.
   But $\vec{a}\times\vec{b}$ is also perpendicular to both $\vec{a}$ and $\vec{b}$,
   so $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar.
2. Given $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly dependent, prove that $\vec{a}\times\vec{b}\cdot\vec{c} = 0$.
   If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar,
   $\vec{a}\times\vec{b}$ is perpendicular to the plane of $\vec{a}$ and $\vec{b}$,
   and since $\vec{c}$ is in the plane of $\vec{a}$ and $\vec{b}$,
   $\vec{c}$ is also perpendicular to $\vec{a}\times\vec{b}$.
   Hence $\vec{a}\times\vec{b}\cdot\vec{c} = 0$. ■

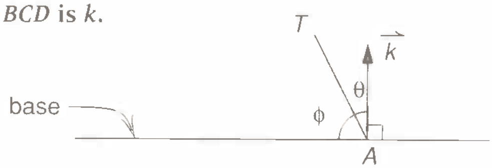
Example 4 A right square pyramid $ABCDT$ whose base has side 12 units and whose height is 17 units is positioned in 3-space with coordinates $A(6,6,0)$, $B(-6,6,0)$, $C(-6,-6,0)$, $D(6,-6,0)$, and $T(0,0,17)$. Calculate the following, giving your answers to the nearest degree.



a) the angle between the edge AT and the base $ABCD$

b) the angle between the planes ABT and ADT

Solution a) A normal vector to the base $ABCD$ is $\vec{k}$.



The diagram shows that you must find ϕ, the complement of the angle θ between $\overrightarrow{AT}$ and $\vec{k}$.

First calculate the angle θ.

$\overrightarrow{AT} = \overrightarrow{OT} - \overrightarrow{OA} = \overrightarrow{(0,0,17)} - \overrightarrow{(6,6,0)} = \overrightarrow{(-6,-6,17)}$

By the dot product

$$\overrightarrow{AT} \cdot \vec{k} = |\overrightarrow{AT}||\vec{k}| \cos\theta$$

$$\overrightarrow{(-6,-6,17)} \cdot \overrightarrow{(0,0,1)} = \sqrt{(-6)^2 + (-6)^2 + 17^2}\,(1)(\cos\theta)$$

$$17 = \sqrt{361}\cos\theta, \text{ or}$$

$$\cos\theta = \frac{17}{19} = 0.8947\ldots \Rightarrow \theta \doteq 27^\circ$$

Thus the required angle is $\phi = 90^\circ - \theta \doteq 63^\circ$

b) In order to find the angle between the planes, first find the angle between normal vectors to each plane.

A normal vector for plane ABT is $\vec{n}_1 = \overrightarrow{TA} \times \overrightarrow{TB}$

A normal vector for plane ADT is $\vec{n}_2 = \overrightarrow{TA} \times \overrightarrow{TD}$

(Note that, by the right-handed rule of the cross product, $\vec{n}_1$ points outside the pyramid, but $\vec{n}_2$ points inside.)

Now from a), $\overrightarrow{TA} = -\overrightarrow{AT} = \overrightarrow{(6,6,-17)}$.
Also, $\overrightarrow{TB} = \overrightarrow{OB} - \overrightarrow{OT} = \overrightarrow{(-6,6,-17)}$ and $\overrightarrow{TD} = \overrightarrow{OD} - \overrightarrow{OT} = \overrightarrow{(6,-6,-17)}$

$$\text{Thus, } \vec{n}_1 = \overrightarrow{(6,6,-17)} \times \overrightarrow{(-6,6,-17)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 6 & -17 \\ -6 & 6 & -17 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(-204) + \vec{k}(72)$$
$$= \overrightarrow{(0,204,72)} = 12\overrightarrow{(0,17,6)}$$

$$\text{Also } \vec{n}_2 = \overrightarrow{(6,6,-17)} \times \overrightarrow{(6,-6,-17)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 6 & -17 \\ 6 & -6 & -17 \end{vmatrix}$$

$$= \vec{i}(-204) - \vec{j}(0) + \vec{k}(-72)$$
$$= \overrightarrow{(-204,0,-72)} = -12\overrightarrow{(17,0,6)}$$

Let the angle between $\vec{n}_1$ and $\vec{n}_2$ be α.

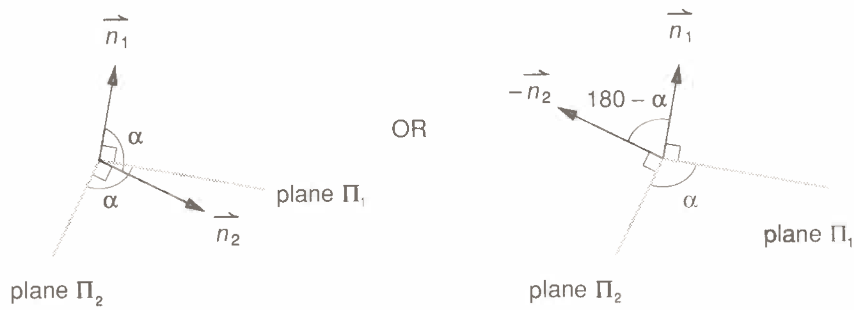
$$\text{Then } \vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2| \cos\alpha$$

$$12\overrightarrow{(0,17,6)} \cdot [-12\overrightarrow{(17,0,6)}] = 12\sqrt{17^2 + 6^2}\, 12\sqrt{17^2 + 6^2} \cos\alpha$$

$$-144(36) = 144(\sqrt{325})^2 \cos\alpha$$

$$\cos\alpha = -\frac{36}{325} = -0.1107\ldots \Rightarrow \alpha \doteq 96°.$$

This is the required angle between the planes. ■



Observe from the diagrams that the angle α between two planes could be either *equal* to the angle between two normals, or equal to the *supplement* of this angle.

If the planes Π_1 and Π_2 are infinite mathematical planes, there are actually two possible angles between them, α, and $180° - \alpha$.

However, if the two planes refer to real physical objects such as the pyramid of Example 4, you must decide which of the two angles is appropriate to describe the physical situation.

The Angle between Two Vectors

If you are given two non-zero vectors $\vec{u}$ and $\vec{v}$ in component form, recall (Example 2, section 3.3) that you can calculate the angle between them by using the dot product.

The cross product, $\vec{u}\times\vec{v} = |\vec{u}||\vec{v}|\sin\theta\,\vec{e}$, where θ is the required angle and $\vec{e}$ is a unit vector so that $\vec{u}$, $\vec{v}$, $\vec{e}$ form a right-handed system, can also be used to calculate θ.

To find $\sin\theta$, you can equate the *lengths* of the above vectors. Thus, $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$, since $\sin\theta$ is never negative (because $0^\circ \leqslant \theta \leqslant 180^\circ$).

Hence, $\sin\theta = \dfrac{|\vec{u}\times\vec{v}|}{|\vec{u}||\vec{v}|}$

However, this will produce two solutions for θ, in the range $0^\circ \leqslant \theta \leqslant 180^\circ$. One solution is the angle between the vectors, and the other is the supplement of this angle. You must select the correct angle.

The dot product is more useful than the cross product to determine the angle between two vectors, because it produces only one solution (the correct solution) in the range $0^\circ \leqslant \theta \leqslant 180^\circ$.

SUMMARY

The triple scalar product property

$\vec{a}\times\vec{b}\cdot\vec{c} = \vec{a}\cdot\vec{b}\times\vec{c}$

If $\vec{u} = \overrightarrow{(u_1,u_2,u_3)}$ and $\vec{v} = \overrightarrow{(v_1,v_2,v_3)}$, then

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}$$

3.6 Exercises

1. Calculate $\vec{u} \times \vec{v}$ in the following cases.
   a) $\vec{u} = \vec{i} = (1,0,0)$ and $\vec{v} = \vec{j} = (0,1,0)$
   b) $\vec{u} = (2,3,5)$ and $\vec{v} = (1,0,4)$
   c) $\vec{u} = (-1,8,-3)$ and $\vec{v} = (2,-2,-5)$
   d) $\vec{u} = (1,-3,2)$ and $\vec{v} = (2,-6,4)$

2. Find a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$ for the vectors given in question 1.

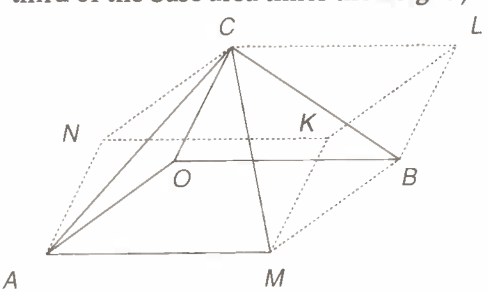
3. Simplify the following.
   a) $\vec{p} \times (\vec{p} + \vec{q})$
   b) $(\vec{p} + \vec{q}) \times (\vec{p} + \vec{q})$
   c) $\vec{p} \cdot \vec{q} \times \vec{p}$
   d) $\vec{p} \times (\vec{q} + \vec{r}) \cdot \vec{q}$

4. Find the area of the parallelogram *ABCD* if $\overrightarrow{AB} = (1,9,2)$, and $\overrightarrow{AD} = (-2,1,7)$.

5. Find the area of the triangle *ABC* in the following cases.
   a) $\overrightarrow{AB} = (1,2,-3)$, $\overrightarrow{AC} = (4,4,1)$
   b) the vertices are $A(12,5,7)$, $B(4,10,13)$, and $C(8,-3,-1)$.

6. Three sides of a parallelepiped represent the vectors $\vec{u} = (3,2,1,)$, $\vec{v} = (-4,0,2)$, and $\vec{w} = (5,1,-1)$. Calculate the volume of parallelepiped.

7. The parallelepiped *OAMBLCNK* shown is such that $\overrightarrow{OA} = (1,4,4)$, $\overrightarrow{OB} = (-1,2,1)$, and $\overrightarrow{OC} = (6,-2,-3)$. Calculate the volume of the pyramid *OAMBC*.
   (Formula: the volume of a pyramid is one third of the base area times the height.)



8. In the pyramid *OAMBC* of question 7, show that $\overrightarrow{OM} = (0,6,5)$. Then calculate the following, giving your answers correct to the nearest degree.
   a) the angle between *BC* and the base *OAMB*
   b) the angle between *MC* and the base *OAMB*
   c) the angle between the planes *MBC* and *OAMB*
   d) the angle between the planes *MBC* and *MAC*

9. $|\vec{u}| = 5, |\vec{v}| = 6, |\vec{u} \times \vec{v}| = 21$. Determine whether or not this information is sufficient to find the angle θ between $\vec{u}$ and $\vec{v}$.

10. Given $\vec{u} = (4,-6,0)$, $\vec{v} = (2,1,1)$, and $\vec{w} = (1,3,-5)$.
    a) Calculate $(\vec{u} \times \vec{v}) \times \vec{w}$ and $\vec{u} \times (\vec{v} \times \vec{w})$. (These are known as **triple vector products**).
    b) Draw a conclusion about the associativity of the cross product.

11. Confirm the result of 3.5 Exercises, question 11, as follows. Given $\vec{i} = (1,0,0)$, $\vec{u} = (1,1,0)$, and $\vec{k} = (0,0,1)$, show that $(\vec{i} \times \vec{u}) \times \vec{k} = \vec{0}$, but $\vec{i} \times (\vec{u} \times \vec{k}) \neq \vec{0}$.

12. Prove that the three $\mathbb{V}_3$ vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are non-coplanar if and only if $\vec{u} \cdot \vec{v} \times \vec{w} \neq 0$.

13. Use the cross product to show that the vectors $\vec{a} = (2,-1,3)$, $\vec{b} = (1,2,0)$, and $\vec{c} = (1,-13,9)$ are coplanar (hence linearly dependent).

14. Show that $\frac{1}{2}(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \vec{b} \times \vec{a}$.

15. $A(2,-1,4)$, $B(3,1,-5)$, and $C(1,1,1)$ are three points in a plane Π. Use the cross product to find a vector perpendicular to the plane Π.

Summary

- The angle θ between two vectors $\vec{a}$ and $\vec{b}$ is the angle formed when the vectors are drawn with a common tail.
- The dot product of $\vec{a}$ and $\vec{b}$ is the scalar
  $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$
- The orthogonal projection of $\vec{a}$ on $\vec{b}$ is the vector
  $$\vec{p} = |\vec{a}|\cos\theta\frac{\vec{b}}{|\vec{b}|} = |\vec{a}|\cos\theta\,\vec{e}_b = (\vec{a}\cdot\vec{e}_b)\,\vec{e}_b$$

where $\vec{e}_b$ is the unit vector in the direction of $\vec{b}$.

- The component of $\vec{a}$ on $\vec{b}$ is the scalar coefficient of the unit vector in the projection,
  $$c = |\vec{a}|\cos\theta = \vec{a}\cdot\vec{e}_b = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$
- *Properties of the dot product*
  1. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$
  2. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$
  3. $(k\vec{a})\cdot\vec{b} = k(\vec{a}\cdot\vec{b})$
  4. $\vec{a}\cdot\vec{a} = |\vec{a}|^2$
  5. If $\vec{a}$ is perpendicular to $\vec{b}$, then $\vec{a}\cdot\vec{b} = 0$.
  6. $\vec{i}\cdot\vec{i} = 1, \vec{j}\cdot\vec{j} = 1, \vec{k}\cdot\vec{k} = 1$
  7. $\vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{i} = \vec{j}\cdot\vec{k} = \vec{k}\cdot\vec{j} = \vec{k}\cdot\vec{i} = \vec{i}\cdot\vec{k} = 0$
- In $\mathbb{V}_2$, $\vec{a} = \overrightarrow{(a_1,a_2)}$ and
  $\vec{b} = \overrightarrow{(b_1,b_2)}$
  $\Rightarrow \vec{a}\cdot\vec{b} = a_1b_2 + a_2b_2$
- In $\mathbb{V}_3$, $\vec{a} = \overrightarrow{(a_1,a_2,a_3)}$ and
  $\vec{b} = \overrightarrow{(b_1,b_2,b_3)}$
  $\Rightarrow \vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
- Vectors that are all unit, and mutually perpendicular, are said to form an orthonormal basis of the vector space.
- The cross product of vectors $\vec{a}$ and $\vec{b}$ is the vector
  $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\vec{e},$
  where $\vec{e}$ is a unit vector perpendicular to $\vec{a}$ and $\vec{b}$, such that the triple $[\vec{a},\vec{b},\vec{e}]$ forms a right-handed system.
- *Properties of the cross product*
  1. $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$
  2. $\vec{a}\times\vec{a} = \vec{0}$
  3. $\vec{i}\times\vec{j} = \vec{k}, \vec{j}\times\vec{k} = \vec{i}, \vec{k}\times\vec{i} = \vec{j}$
  4. $(k\vec{a})\times\vec{b} = k(\vec{a}\times\vec{b})$
  5. $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$

- The triple scalar product property
  $\vec{a} \times \vec{b} \cdot \vec{c} = \vec{a} \cdot \vec{b} \times \vec{c}$
- If $\vec{a} = \overrightarrow{(a_1,a_2,a_3)}$ and $\vec{b} = \overrightarrow{(b_1,b_2,b_3)}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \text{ where } \begin{vmatrix} c & f \\ e & d \end{vmatrix} = cd - fe.$$

Inventory

1. The angle between two vectors is defined as the angle between them when they are drawn with a common ______.
2. The projection of the vector $\overrightarrow{(2,-3)}$ on the x-axis is ______.
3. The component of the vector $\overrightarrow{(2,-3)}$ on the y-axis is ______.
4. Writing the vectors $\overrightarrow{(2,-3)}$ as $2\vec{i} - 3\vec{j}$ is called ______ the vector in the directions of $\vec{i}$ and $\vec{j}$.
5. The dot product combines two vectors to produce a ______.
6. The cross product combines two vectors to produce a ______.
7. The ______ product is not defined in $\mathbb{V}_2$.
8. The dot product of $\vec{u}$ and $\vec{v}$ equals
   (length of $\vec{u}$) (component of ______ on ______).
9. The magnitude of the cross product of $\vec{u}$ and $\vec{v}$ equals the area of the ______ whose adjacent sides represent the vectors ______ and ______.
10. The ______ product is not commutative.
11. The dot product of a unit vector with itself equals ______.
12. The cross product of a unit vector with itself equals ______.
13. The dot product of two perpendicular vector equals ______.
14. If the angle between two vectors is obtuse, then the dot product of those two vectors is ______, and vice-versa.
15. The expression $\vec{a} \times \vec{b} \cdot \vec{c}$ is known as a ______ ______ ______.
16. The expression $\vec{a} \times \vec{b} \cdot \vec{c}$ is equal to ______.
17. The product $\vec{u} \times \vec{v}$ yields a vector that is ______ to both $\vec{u}$ and $\vec{v}$.

Review Exercises

1. State the projections and the components of the following vectors in the directions of $\vec{i}$, $\vec{j}$ and $\vec{k}$.
   a) $\overrightarrow{(6,-5,-3)}$
   b) $\overrightarrow{(-\sqrt{2},1,\sqrt{3})}$
   c) $-3\overrightarrow{(1,-1,2)}$

2. Given that the following vectors are in $\mathbb{V}_2$, and that θ is the angle between each vector and $\vec{i}$, find the component of each vector on $\vec{i}$ and $\vec{j}$. Express your answers correct to 3 decimal places.
   a) $|\vec{a}| = 2$, $\theta = 55°$
   b) $|\vec{b}| = 4$, $\theta = 161°$
   c) $|\vec{c}| = 5$, $\theta = 180°$.

3. What can you deduce about $\vec{u}$ and/or $\vec{v}$ in the following cases?
   a) The component of $\vec{u}$ in the direction of $\vec{v}$ is equal to $|\vec{u}|$.
   b) The component of $\vec{u}$ in the direction of $\vec{v}$ is equal to $-|\vec{u}|$.

4. Calculate the dot product of the following pairs of vectors, given that θ is the angle between them.
   a) $|\vec{u}| = 7, |\vec{v}| = 1, \theta = 35°$
   b) $|\vec{w}| = 2, |\vec{t}| = 5, \theta = 120°$
   c) $|\vec{a}| = 10, |\vec{b}| = 10, \theta = 157°$

5. In question 4, calculate the component of the first vector on the second in each case.

6. *ABCDEF* is a regular hexagon whose sides have length 2 units. Calculate the following.
   a) $\overrightarrow{AB}\cdot\overrightarrow{BC}$
   b) $\overrightarrow{AF}\cdot\overrightarrow{FE}$
   c) $\overrightarrow{AF}\cdot\overrightarrow{BC}$
   d) $\overrightarrow{AB}\cdot\overrightarrow{ED}$
   e) $\overrightarrow{AB}\cdot\overrightarrow{BD}$
   f) $\overrightarrow{AF}\cdot\overrightarrow{BE}$

7. Find the dot product $\vec{u}\cdot\vec{v}$ if
   a) $\vec{u} = \overrightarrow{(2,4)}$ $\quad \vec{v} = \overrightarrow{(0,3)}$
   b) $\vec{u} = \overrightarrow{(-5,0,12)}$ $\quad \vec{v} = \overrightarrow{(3,-4,-2)}$
   c) $\vec{u} = \overrightarrow{(2,7)}$ $\quad \vec{v} = \overrightarrow{(21,-6)}$
   d) $\vec{u} = \overrightarrow{(1,2,5)}$ $\quad \vec{v} = \overrightarrow{(3,1,-1)}$

8. Find the value of k if the vectors $\vec{u} = \overrightarrow{(-8,6,7)}$ and $\vec{v} = \overrightarrow{(k,-1,2)}$ are perpendicular.

9. Prove that, for any vectors $\vec{u}$ and $\vec{v}$,
   a) $(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$
   $= |\vec{u}|^2 + 2\vec{u}\cdot\vec{v} + |\vec{v}|^2$
   b) $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = |\vec{u}|^2 - |\vec{v}|^2$

10. Apply the result of question 9 a) to two perpendicular vectors $\vec{u}$ and $\vec{v}$ to prove the theorem of Pythagoras.

11. Two adjacent sides of a rhombus represent the vectors $\vec{u}$ and $\vec{v}$. Using the result of question 9 b), prove that the diagonals of a rhombus are perpendicular.

12. $\vec{a}$ and $\vec{b}$ are unit vectors at an angle of 135° to each other. Use the dot product to find an exact expression for the unit vector in the direction of $2\vec{a} - \vec{b}$. (See the table of trigonometric ratios on page 543.)

13. Given the vectors $\vec{u} = \overrightarrow{(2,-1)}$ and $\vec{v} = \overrightarrow{(-2,-3)}$, find each of the following.
    a) the angle between $\vec{u}$ and $\vec{v}$
    b) the component of $\vec{u}$ on $\vec{v}$
    c) the projection of $\vec{u}$ on $\vec{v}$
    d) the component of $\vec{v}$ on $\vec{u}$
    e) the projection of $\vec{v}$ on $\vec{u}$

14. The angle between the vectors $\vec{u}$ and $\vec{v}$ is θ where $\cos\theta = \frac{1}{\sqrt{6}}$. If $\vec{u} = \overrightarrow{(2,-1,1)}$ and $\vec{v} = \overrightarrow{(a,3,4)}$, find the possible values of a.

15. a) Prove that $\vec{u} = \vec{v}$ if and only if $\vec{u} \cdot \vec{p} = \vec{v} \cdot \vec{p}$ for *every* vector $\vec{p}$.
b) Hence prove that in $\mathbb{V}_2$ it is sufficient to verify this relationship for *two* linearly independent vectors $\vec{p}_1$ and $\vec{p}_2$.
c) How many vectors would be needed to verify the relationship in $\mathbb{V}_3$?

16. Determine the angles of the triangle *PQR* in the following cases.
a) $P(-2,4)$, $Q(7,-9)$, $R(0,3)$
b) $P(5,4,1)$, $Q(8,-1,-3)$, $R(9,4,4)$

17. *OAB* is a triangle with $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$. *M* is the midpoint of *OA*, and *N* is the midpoint of *OB*.
a) Express $\overrightarrow{AN}$ and $\overrightarrow{BM}$ in terms of $\vec{a}$ and $\vec{b}$.
b) If $|\overrightarrow{AN}| = |\overrightarrow{BM}|$, prove that $|\vec{a}| = |\vec{b}|$.

(This proves that if two medians of a triangle are equal in length, then the triangle is isosceles.)

18. *OABC* is a parallelogram with $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OC} = \vec{c}$. Evaluate $\overrightarrow{AC} \cdot \overrightarrow{AC} + \overrightarrow{OB} \cdot \overrightarrow{OB}$ in terms of $\vec{a}$ and $\vec{c}$ to prove the following theorem.
The sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the sides.

19. Calculate $\vec{u} \times \vec{v}$ for each of the following, where $\vec{u}$ and $\vec{v}$ are vectors of $\mathbb{V}_3$.
a) $|\vec{u}| = 3$, $|\vec{v}| = 2$, and the angle between $\vec{u}$ and $\vec{v}$ is 25°.
b) $|\vec{u}| = 4$, $|\vec{v}| = 1$, and the angle between $\vec{u}$ and $\vec{v}$ is 110°.
c) $|\vec{u}| = 5$, $|\vec{v}| = 6$, and $\vec{u} \cdot \vec{v} = -5$.
d) $\vec{u} = \overrightarrow{(9,-2,4)}$ and $\vec{v} = \overrightarrow{(3,-1,0)}$.

20. Find the two unit vectors that are perpendicular to $\vec{a} = \overrightarrow{(1,6,8)}$ and $\vec{b} = \overrightarrow{(4,-2,-5)}$.

21. Calculate the area of the triangle whose vertices are $P(10,-3,9)$, $Q(-1,4,2)$, and $R(0,5,-6)$.

22. If $\vec{a} = \overrightarrow{(-1,3,6)}$ and $\vec{b} = \overrightarrow{(-2,-2,5)}$, determine whether or not it is possible to find the angle θ between $\vec{a}$ and $\vec{b}$ by using the cross product exclusively.

23. In each of the following, use the triple scalar product $(\vec{a} \times \vec{b}) \cdot \vec{c}$ to determine whether or not the three vectors are linearly dependent.
a) $\overrightarrow{(0,1,3)}$, $\overrightarrow{(-3,5,2)}$, and $\overrightarrow{(-6,11,7)}$
b) $\overrightarrow{(1,2,3)}$, $\overrightarrow{(-3,0,4)}$, and $\overrightarrow{(-1,4,6)}$
c) $\overrightarrow{(4,1,8)}$, $\overrightarrow{(-2,1,0)}$, and $\overrightarrow{(0,3,16)}$
d) $\overrightarrow{(1,2,4)}$, $\overrightarrow{(2,-3,-1)}$, and $\overrightarrow{(-1,-9,-13)}$
e) $\overrightarrow{(3,5,1)}$, $\overrightarrow{(2,-2,-1)}$, and $\overrightarrow{(-4,-4,0)}$

24. Choose specific vectors in $\mathbb{V}_3$ to show that the cross product is not associative. That is, show by counterexample that $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$.

25. a) Find a vector $\vec{p}$ perpendicular to $\vec{q} = \overrightarrow{(4,-1,-2)}$ and perpendicular to $\vec{r} = \overrightarrow{(0,3,1)}$.
b) Normalize $\vec{p}$.

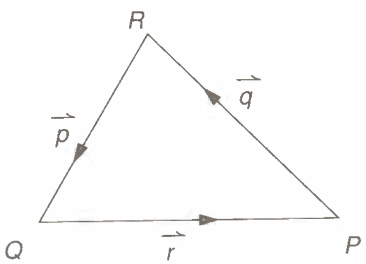
26. Given a vector $\vec{v} \neq \vec{0}$ and a scalar k, use the dot product to calculate the angle between $\vec{v}$ and $k\vec{v}$ in the following cases.
a) $k > 0$ b) $k < 0$

27. In question 26, discuss the case where $k = 0$.

28. Given any three vectors $\vec{p}$, $\vec{q}$, $\vec{r}$ of $\mathbb{V}_3$, prove the following.
a) $\vec{p} \cdot \vec{q} \times \vec{r} = \vec{q} \cdot \vec{r} \times \vec{p} = \vec{r} \cdot \vec{p} \times \vec{q}$ (This is known as the **cyclic property** of triple scalar products.)
b) $\vec{p} \cdot \vec{q} \times \vec{r} = -\vec{q} \cdot \vec{p} \times \vec{r}$.

29. a) Show that $(\vec{u}\times\vec{v})\times\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$. That is, show that $(\vec{u}\times\vec{v})\times\vec{w}=k\vec{u}+s\vec{v}$, where k and s are scalars.
   b) Show similarly that $\vec{u}\times(\vec{v}\times\vec{w})=m\vec{v}+n\vec{w}$, where m and n are scalars.
   c) Using the results of parts a) and b), describe the triples $\vec{u}, \vec{v}, \vec{w}$ for which the cross product is associative.

30. Given the triangle PQR, whose sides represent the vectors $\vec{p}$, $\vec{q}$, and $\vec{r}$ as shown on the diagram.



   a) Prove that $\vec{p}+\vec{q}+\vec{r}=\vec{0}$.
   b) Write and then simplify the relation obtained by carrying out the cross product of $\vec{p}$ with each side of the relation in a).
   c) Repeat step b) by using $\vec{q}$, then $\vec{r}$.
   d) From your results, prove the sine law in triangle PQR (see page 542).

31. The vertices of the triangle ABC have position vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ respectively from an origin O. Prove that the area of ABC is $\frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$.

32. A plane contains the points A, B, and C whose position vectors are $\vec{a}$, $\vec{b}$, and $\vec{c}$ respectively. Prove that the vector $\vec{n}=\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$ is normal to the plane.

33. Let $\vec{i}, \vec{j}, \vec{k}$ be unit vectors along the coordinate axes of a three-dimensional rectangular Cartesian coordinate system, and let $\vec{a}, \vec{b}, \vec{c}$ be defined by the equations
$\vec{a}=-\vec{i}+\vec{j}+\vec{k}, \quad \vec{b}=\vec{i}-\vec{j}+\vec{k},$
$\vec{c}=\vec{i}+\vec{j}-\vec{k}.$
   a) Find the angle between the vectors $\vec{a}$ and $\vec{b}$, giving your answer in degrees, correct to 1 decimal place.
   b) Given that O is the origin and $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the points A, B, C respectively, find
      i) the lengths of the sides of the triangle OAB, leaving your answers in surd form, and
      ii) the magnitudes of its angles.
   c) What are the lengths of the sides of the triangle ABC and the magnitudes of its angles?
   d) Find the position vector of the mid-point D of $[BC]$, and write down the position vectors of the mid-points E, F of $[CA]$ and $[AB]$ respectively. Hence find the volume of the tetrahedron $ODEF$.
   e) By considering the ratio of the areas of the triangles ABC and DEF, or otherwise, prove that the volume of the tetrahedron $OABC$ is $\frac{2}{3}$ units³.

(84 S)

34. In a rectangular Cartesian coordinate system the points O, A, B and C have coordinates (0,0), (4,8), (4,–2) and (–8,–6) respectively.
   a) Show the points O, A, B and C in a sketch, taking $\frac{1}{2}$ cm as a unit.
   b) Prove that $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are perpendicular.
   c) Given $\overrightarrow{OD}=\overrightarrow{OB}+\overrightarrow{OC}$, find the coordinates of the point D.

d) The point C' lies on the line (AB) so that $\overrightarrow{OC'} = \alpha\overrightarrow{OC}$. Calculate the value of α.

e) i) Given that the point E lies on the y-axis and $\overrightarrow{OA} \cdot \overrightarrow{AE} = 0$, calculate the coordinates of the point E.

ii) It is further given that $\overrightarrow{OF} = \overrightarrow{OE} - \overrightarrow{OA}$. Calculate the coordinates of the point F.

f) Calculate

i) the area of the rectangle $OAEF$, and

ii) the area of the parallelogram $OBDC$.

(85 SMS)

35. The point O is the centre of the circle drawn through the vertices of the triangle ABC. With respect to the point O as origin the position vectors of the points A, B and C are $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively, so that $|\vec{a}| = |\vec{b}| = |\vec{c}|$. The points G and H are such that $\overrightarrow{OG} = \frac{1}{3}(\vec{a} + \vec{b} + \vec{c})$ and $\overrightarrow{OH} = \vec{a} + \vec{b} + \vec{c}$

i) Prove that $\overrightarrow{AG} = \frac{1}{3}(\vec{b} + \vec{c} - 2\vec{a})$

ii) Prove that (AG) passes through the midpoint L of $[BC]$ and find the ratio $AG : GL$.

iii) Deduce, from the result obtained in part (ii), that the three medians of triangle ABC, i.e. the lines joining a vertex to the midpoint of the opposite side, pass through G.

iv) Prove that (AH) is perpendicular to (BC).

v) Deduce, from the result obtained in part (iv), that the three perpendiculars, from the vertices of triangle ABC to the opposite sides, pass through H.

(85 H)

36. In a rectangular Cartesian coordinate system the points A, B and C have coordinates $(-6,1)$, $(-2,4)$ and $(1,0)$ respectively.

i) a) Given the column vector $\overrightarrow{AB} = \begin{pmatrix} p \\ q \end{pmatrix}$, find the values of p and q.

b) Given $\overrightarrow{AB} = \overrightarrow{DC}$, find the coordinates of the point D.

c) Show that I, the midpoint of $[AC]$, is also the midpoint of $[BD]$.

d) Show that $ABCD$ is a square.

ii) Calculate the coordinates of the points E and F such that $\overrightarrow{BE} = \frac{1}{4}\overrightarrow{BI}$ and $\overrightarrow{CF} = 2\overrightarrow{CE}$.

iii) a) By finding the column vectors for each of $\overrightarrow{AF}$ and $\overrightarrow{DB}$ show that $\overrightarrow{AF} = \frac{3}{4}\overrightarrow{DB}$.

b) Continue the argument $\overrightarrow{AF} = \overrightarrow{AC} + \overrightarrow{CF} = 2(\overrightarrow{IC} + \overrightarrow{CE})$ to confirm that $\overrightarrow{AF} = \frac{3}{4}\overrightarrow{DB}$.

(84 SMS)

37. Given that

$\vec{p} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, $\vec{q} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

and that

$\vec{s} = k\vec{p} - (\vec{p} \cdot \vec{q})\vec{q}$, where $k \in \mathbb{R}$,

find

a) the value of $\vec{p} \cdot \vec{q}$, and

b) the value of the constant k such that the directions of $\vec{s}$ and $\vec{q}$ are at right angles.

(87 S)

VECTORS, MATRICES
and
COMPLEX NUMBERS

with
International Baccalaureate
questions

Jean-Paul GINESTIER
and
John EGSGARD

CHAPTER FOUR

APPLICATIONS OF VECTORS

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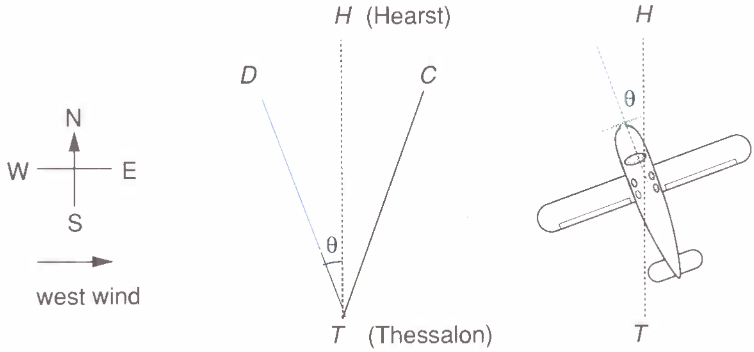
CHAPTER FOUR

Applications of Vectors

You are in a small airplane in the town of Thessalon, Ontario, ready to fly to Hearst, a distance of 400 km due north. The plane can fly at 200 km/h, and so you would expect the trip to take two hours. However, the 200 km/h is the plane's *airspeed*, in other words, its speed relative to the air around it. This is not necessarily the same as the plane's *groundspeed*.

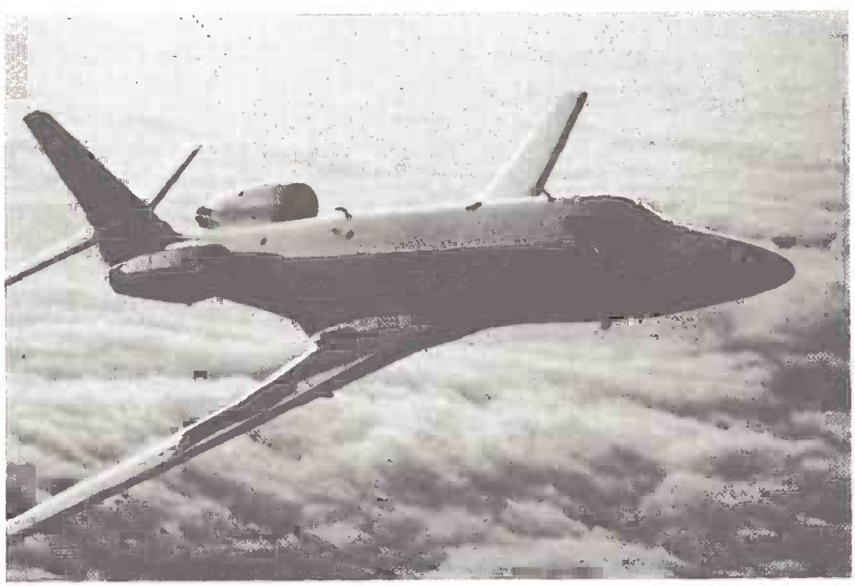
Naturally, if there is a strong wind against you (known as a headwind), you would expect the trip to take longer. Similarly, if there is a strong wind behind you (known as a tailwind), your trip would be shorter.

On this day, a wind is blowing at 50 km/h from the west, and your pilot cannot simply head north. Indeed, if she points the plane due north, along the line *TH*, the wind will blow the airplane along a line such as *TC*. Clearly, in order to reach Hearst, she will need to point the airplane in a direction to the west of line *TH*, parallel to a line such as *TD*. The plane should then continue to fly partly 'sideways', as shown in the diagram, and the wind will keep pushing it so that it remains over the line *TH*.



Her problem will be to determine the angle θ between *TH* and *TD*. The stronger the wind from the west, the greater angle θ must be; the greater the angle θ, the more her groundspeed will be reduced. She will also need to determine this groundspeed, v, to estimate the time of arrival in Hearst.

To find these two numbers, θ and v, she will use a mathematical model that will be able to represent both the speed and the direction of the airplane. Vectors provide an excellent model for this type of situation, and for other problems in physics, as you shall see in this chapter.



4.1 Forces as Vectors

Since a vector has magnitude and direction, and a force also has magnitude and direction, the theory of vectors can be applied to forces.

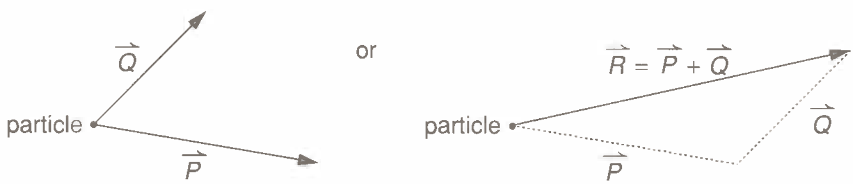
In fact, the theory of vectors grew out of investigations on forces. (See the introduction to chapter 1.)

The branch of physics called mechanics is divided into two main sections, called **statics** and **dynamics**. The latter is a study of how objects change their motion under the action of forces, while statics examines the relationship of forces acting upon stationary (or other non-accelerating) objects.

You will be taking an elementary look at **statics of a particle** in this section. (A particle is the name used for an object small enough to be considered as a point.)

Forces are measured in **newtons** (symbol N). In general, the **gravitational force on a mass of** m **kg is** mg **N**, where g is the acceleration due to gravity. On earth, $g \doteq 9.8$ m/s^2. Thus, the force due to gravity on a mass of 1 kg is about 9.8 N. Alternatively, a mass of 1 kg is said to **weigh** 9.8 N.

The fundamental reason which allows you to apply vector theory to forces is that two forces can be combined in the same way that vectors are *added*. In other words, if a particle is being acted upon by two forces $\vec{P}$ and $\vec{Q}$ as shown, these will have the *same effect* on the particle as the force $\vec{R} = \vec{P} + \vec{Q}$.



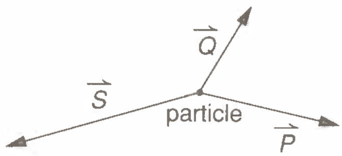
$\vec{R}$ is called the **resultant** of $\vec{P}$ and $\vec{Q}$.

This is also true for any number of forces. That is, the *resultant* of a number of forces is the *vector sum* of those forces.

The effect will be to pull the particle in the direction of $\vec{R}$, and thus accelerate it, or increase its speed.

If the particle does not change its motion, or its state of rest, then the particle remains in a **state of equilibrium**. The particle must then be acted upon simultaneously by a force which counteracts or cancels the effect of $\vec{R}$. Consider the force $\vec{S}$, acting in the opposite direction to that of $\vec{R}$, but with the same magnitude.

You would then have the following situation:



or

The particle is *in equilibrium* under the effect of $\vec{P}$, $\vec{Q}$, and $\vec{S}$, or under the effect of $\vec{R}$ and $\vec{S}$.

$\vec{S}$ is called the **equilibrant** of $\vec{R}$.
Conversely, $\vec{R}$ is called the equilibrant of $\vec{S}$.

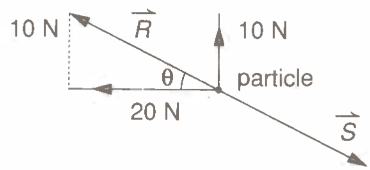
Note: $\vec{S} + \vec{R} = \vec{0}$ or $\vec{S} = -\vec{R}$
$\vec{P} + \vec{Q} + \vec{S} = \vec{0}$

DEFINITION Whenever a particle is in equilibrium, the vector sum of all forces acting upon it is $\vec{0}$.

In the examples of this chapter, unless it is specified otherwise, all magnitudes of forces will be calculated to 3 significant digits, and all angles will be calculated to the nearest degree.

Example 1 A force of 20 N acts west, and a force of 10 N acts north on a particle. Find the direction and magnitude of the resultant and the equilibrant of these two forces.

Solution



Let the resultant be $\vec{R}$. Then by the theorem of Pythagoras,
$|\vec{R}|^2 = 20^2 + 10^2 \Rightarrow |\vec{R}| = \sqrt{500}$ or $|\vec{R}| \doteq 22.4$

The angle θ shown is such that

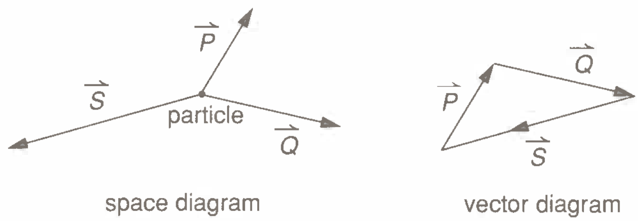
$$\tan\theta = \frac{10}{20} = \frac{1}{2} \Rightarrow \theta \doteq 27°$$

Therefore, the resultant is the force $\vec{R}$ of 22.4 N, acting at bearing $270° + 27° = 297°$ on the particle.

From the diagram, you can see that the equilibrant is thus $\vec{S}$, a force of 22.4 N acting at bearing $90° + 27° = 117°$ on the particle. ■

Vector Diagrams

The diagrams you have seen so far in this section are called **space diagrams** or **position diagrams**, because they try to portray the reality of the particle being tugged at or pushed from different directions. Since vectors can be drawn anywhere, however, you can also represent the forces $\vec{P}$, $\vec{Q}$, $\vec{S}$ by joining the tip of one vector to the tail of another.



space diagram vector diagram

This often clarifies relationships between vectors, and facilitates calculations, as you will see in the forthcoming examples. The diagram on the right is known as a **vector diagram**.

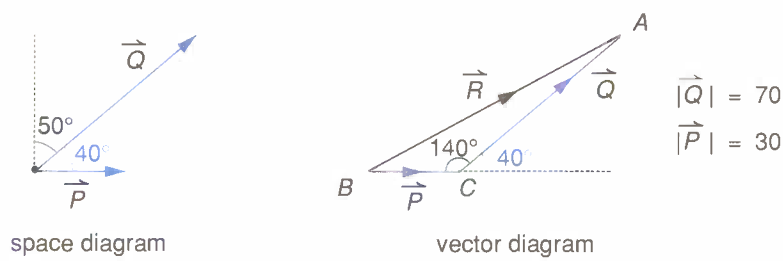
The vector diagram of forces in equilibrium will always be a closed polygon, since the sum of all the vectors is $\vec{0}$.

The particle itself is not represented in the vector diagram.

Force vectors are sometimes described as *fixed* vectors (as opposed to *free* vectors), because they must act on a particular point.

Example 2 Two forces $\vec{P}$ and $\vec{Q}$ act on a particle. $\vec{P}$ points due east and has a magnitude of 30 N. $\vec{Q}$ points on bearing 050° and has a magnitude of 70 N. Find the resultant of $\vec{P}$ and $\vec{Q}$.

Solution



space diagram vector diagram

The space diagram indicates how $\vec{P}$ and $\vec{Q}$ act on the particle.
The vector diagram, denoted triangle *ABC*, shows the resultant $\vec{R}$ and the relationship $\vec{P} + \vec{Q} = \vec{R}$.

Notice that the angle between the forces on the space diagram is $90° - 50° = 40°$.

Notice further that the angle *ACB* in the vector diagram is $180° - 40° = 140°$.

By the cosine law in the vector diagram

$$|\overrightarrow{AB}|^2 = |\overrightarrow{BC}|^2 + |\overrightarrow{AC}|^2 - 2|\overrightarrow{BC}||\overrightarrow{AC}| \cos (\measuredangle ACB)$$

$$\text{thus } |\vec{R}|^2 = |\vec{P}|^2 + |\vec{Q}|^2 - 2|\vec{P}||\vec{Q}| \cos 140^\circ$$
$$= 30^2 + 70^2 - (2)(30)(70)(-0.766\ldots)$$
$$= 900 + 4900 + 3217.3\ldots$$
$$= 9017.3\ldots$$

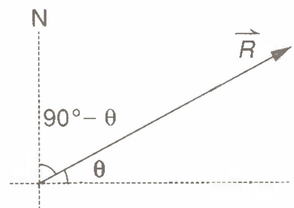
$$\text{thus} \quad |\vec{R}| = \sqrt{9017.3\ldots} = 94.959\ldots \doteq 95.0$$

The direction of $\vec{R}$ can be obtained by using angle B.
By the sine law in the vector diagram

$$\frac{|\overrightarrow{BC}|}{\sin A} = \frac{|\overrightarrow{CA}|}{\sin B} = \frac{|\overrightarrow{AB}|}{\sin C}$$

You need only use the second equation. Let angle $B = \theta°$.

$$\text{Thus } \frac{|\vec{Q}|}{\sin \theta} = \frac{|\vec{R}|}{\sin 140^\circ}$$



$$\sin \theta = \frac{|\vec{Q}| \sin 140^\circ}{|\vec{R}|}$$
$$= \frac{(70)(0.6427\ldots)}{94.959}$$
$$= 0.4738\ldots$$

Thus $\theta \doteq 28°$.

The bearing of $\vec{R}$ is $90° - 28° = 62°$.

The resultant of $\vec{P}$ and $\vec{Q}$ is a force $\vec{R}$ of magnitude 95.0 N, acting on bearing 062°. ■

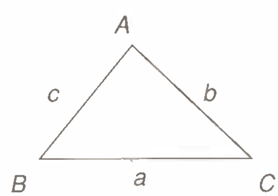
In this chapter, you will be making extensive use of the theorem of Pythagoras, the sine law and the cosine law, as in the above examples. To avoid cluttering the diagrams with unnecessary letters, the formulas for these laws will not be fully quoted at each instance of their use. You can refer to the formulas for these laws on page 542.

Solution of Triangles

The cosine law
$a^2 = b^2 + c^2 - 2bc \cos A$

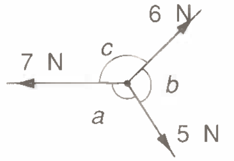
The sine law

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

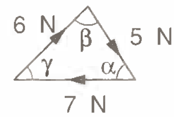


Example 3 Three forces of 5 N, 6 N, and 7 N act simultaneously on a particle, which remains in a state of equilibrium. What are the angles between the forces?

Solution



space diagram



vector diagram

Since the forces are in equilibrium, the vector diagram is a triangle. You can use this triangle to perform the calculations. The angles α, β, and γ which you will find are the *supplements* of the required angles a, b, and c respectively, between the forces.

Using the cosine law to find α:

$$6^2 = 5^2 + 7^2 - (2)(5)(7)\cos\alpha$$

$$\cos\alpha = \frac{25 + 49 - 36}{70} = \frac{38}{70} = 0.5428\ldots \Rightarrow \alpha \doteq 57.12^\circ$$

Using the cosine law again for β:

$$7^2 = 5^2 + 6^2 - (2)(5)(6)\cos\beta$$

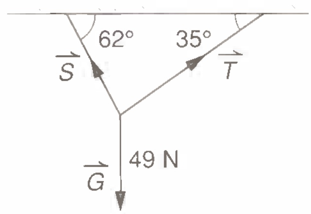
$$\cos\beta = \frac{25 + 36 - 49}{60} = \frac{12}{60} = 0.2 \Rightarrow \beta \doteq 78.46^\circ$$

Thus $\gamma \doteq 180 - (57.12 + 78.46) = 44.42^\circ$

Thus the angles α, β, and γ, to the nearest degree, are respectively 57°, 78°, and 44°, and hence the required angles are:
between the 5 N and 7 N forces, $a = 123^\circ$,
between the 5 N and 6 N forces, $b = 102^\circ$,
between the 6 N and 7 N forces, $c = 136^\circ$. ■

(Observe that, by approximating each angle to nearest degree, in this case $a + b + c \neq 360^\circ$.)

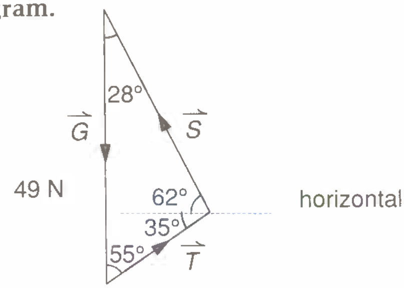
Example 4 A particle of mass 5 kg is suspended from a horizontal ceiling by two strings making angles of 35° and 62° with the ceiling. Calculate the tensions in these strings.



space diagram

Solution Note: The force of gravity on a mass of 5 kg is (5)(9.8) = 49 newtons, acting vertically downward. Call this force $\vec{G}$. Call the tensions in the strings $\vec{S}$ and $\vec{T}$, as shown in the diagram.

Now the particle is in equilibrium, so $\vec{S} + \vec{T} + \vec{G} = \vec{0}$, and this gives the following vector diagram.



By the sine law, $\dfrac{|\vec{T}|}{\sin 28^\circ} = \dfrac{|\vec{S}|}{\sin 55^\circ} = \dfrac{|\vec{G}|}{\sin 97^\circ}$

Also, $|\vec{G}| = 49$,

thus $|\vec{T}| = \dfrac{(49)(\sin 28^\circ)}{\sin 97^\circ} = \dfrac{23.00\ldots}{0.9925\ldots} \doteq 23.2$

and $|\vec{S}| = \dfrac{(49)(\sin 55^\circ)}{\sin 97^\circ} = \dfrac{4.13\ldots}{0.9925} \doteq 40.4$

The tensions in the strings are thus 23.2 N and 40.4 N. ■

Note: An accurately drawn vector diagram would allow you to find the solution by *drawing to scale* instead of carrying out calculations. Indeed, if the length of $\vec{G}$ is drawn to scale as 49 mm, and the angles are correctly drawn, then the lengths of $\vec{S}$ and $\vec{T}$, in mm, will provide the required tensions!

Also the vector diagram seems to indicate that $|\vec{S}| > |\vec{T}|$, whereas the opposite appears to be the case in the space diagram. This is because the space diagram is showing *lengths of strings, not vectors*. The vector diagram is the correct representation of the forces involved, and indeed, the greater tension will occur in the shorter string.

SUMMARY

The *resultant*, $\vec{R}$, of a number of forces is the *vector sum* of those forces. The *equilibrant* of those forces is $-\vec{R}$.

A particle is in *equilibrium* when the vector sum of all forces acting upon it is $\vec{0}$.

The gravitational force on a particle of mass m kg is mg N, where g is the acceleration due to gravity.
(On earth, $g \doteq 9.8$ m/s^2.)

4.1 Exercises

Unless directed otherwise, give all magnitudes in newtons, correct to 3 significant digits, and all angles correct to the nearest degree.
Use $g \doteq 9.8$ m/s^2.

1. Two forces $\vec{P}$ and $\vec{Q}$ act on a particle. Find the resultant and the equilibrant of $\vec{P}$ and $\vec{Q}$ in the following cases.
   a) $|\vec{P}| = 12$ and $\vec{P}$ acts due east
   $|\vec{Q}| = 5$ and $\vec{Q}$ acts due north
   b) $|\vec{P}| = 8$ and $\vec{P}$ acts on bearing 045°
   $|\vec{Q}| = 15$ and $\vec{Q}$ acts on bearing 135°
   c) $|\vec{P}| = 10$ and $\vec{P}$ acts due west
   $|\vec{Q}| = 13$ and $\vec{Q}$ acts on bearing 350°
   d) $|\vec{P}| = 20$ and $\vec{P}$ acts on bearing 058°
   $|\vec{Q}| = 35$ and $\vec{Q}$ acts on bearing 328°
   e) $|\vec{P}| = 510$ and $\vec{P}$ acts on bearing 232°
   $|\vec{Q}| = 425$ and $\vec{Q}$ acts on bearing 105°

2. Two forces $\vec{P}$ and $\vec{Q}$ of magnitudes 30 N and 60 N respectively act on a particle. The angle between $\vec{P}$ and $\vec{Q}$ is 75°. Calculate the magnitude and the direction of the resultant of $\vec{P}$ and $\vec{Q}$.

3. A particle is being pulled by two forces $\vec{F}_1$ and $\vec{F}_2$. $\vec{F}_1$ acts eastward and its magnitude is 250 N. $\vec{F}_2$ acts at a bearing of 10°, and its magnitude is 120 N. Find the magnitude and the direction of the resultant force.

4. Three forces of 10 N, 24 N, and 26 N act simultaneously on a particle, which remains in a state of equilibrium. Calculate the angles between the forces.

5. Repeat question 4 if the forces have magnitudes 15 N, 11 N, and 23 N.

6. The resultant of $\vec{P}$ and $\vec{Q}$ is the force $\vec{F}$, of magnitude 80 N, acting on bearing 070°. $\vec{P}$ has magnitude 25 N and acts due east. Find the magnitude and direction of $\vec{Q}$.

7. Two perpendicular forces of equal magnitude act on a particle. If the resultant has magnitude 100 N, calculate the magnitude of the perpendicular forces.

8. Two forces of equal magnitude acting at an angle of 130° to each other have a resultant of magnitude 42 N. Calculate the magnitude of the two forces.

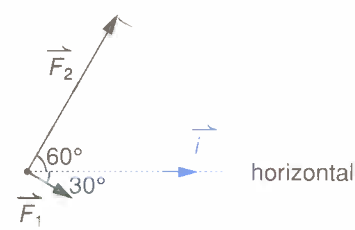
9. A particle of mass 10 kg is suspended from a horizontal ceiling by two strings making angles of 40° and 50° with the ceiling. Calculate the tensions in these strings.

10. Repeat question 9 if the strings make angles of 29° and 41° with the ceiling.

11. A particle of mass 5 kg is suspended by cords from two points A and B on a horizontal ceiling such that $AB = 100$ cm. The lengths of the cords are 80 cm and 70 cm. Calculate the tension in each cord.

12. A force $\vec{P}$ of magnitude 10 N acts along the bearing 060°. Calculate the smallest possible magnitude of a force $\vec{Q}$ such that the resultant of $\vec{P}$ and $\vec{Q}$ acts along the bearing 090°.

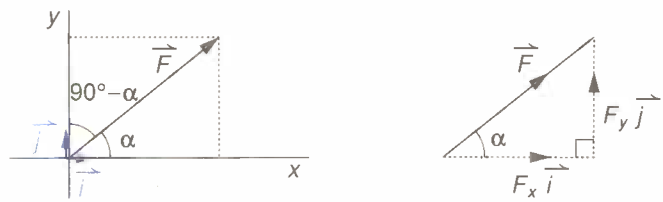
13. Two forces $\vec{F}_1$ and $\vec{F}_2$, of magnitudes 7 N and 24 N, act on a particle. $\vec{F}_1$ acts at 30° to the vector $\vec{i}$, and $\vec{F}_2$ acts at 60° to the vector $\vec{i}$, as shown in the figure. Another force $\vec{F}_3$ of magnitude 30 N acts simultaneously on the particle. Calculate the direction of $\vec{F}_3$ so that the resultant of the three forces is in the direction of $\vec{i}$. (Note: The three forces would then cause the particle to move in the direction of $\vec{i}$.)



4.2 Resolution of Forces

In the last section you learned how to use triangles to find the resultant of two forces. In this section you will learn a second method to solve such problems. This method is the **resolution of forces**. One of its advantages is that it allows you to find the resultant of more than two forces without extra difficulty.

Recall from chapter 3 that a vector $\vec{F}$, making an angle α with $\vec{i}$, can be resolved into two vectors $F_x\vec{i}$ and $F_y\vec{j}$ acting parallel to the x-axis and parallel to the y-axis respectively.



$$\vec{F} = F_x\vec{i} + F_y\vec{j}$$

$F_x\vec{i}$ and $F_y\vec{j}$ are called the *projections* of $\vec{F}$ onto $\vec{i}$ and $\vec{j}$ respectively.

Since $\{\vec{i},\vec{j}\}$ is an orthonormal basis, the scalar multiples F_x and F_y are called the *components* of $\vec{F}$ onto $\vec{i}$ and $\vec{j}$. Recall further that components can be obtained by using the dot product.

$F_x = \vec{F}\cdot\vec{i} = |\vec{F}||\vec{i}|\cos\alpha = |\vec{F}|\cos\alpha$, since $|\vec{i}| = 1$.

$F_y = \vec{F}\cdot\vec{j} = |\vec{F}||\vec{j}|\cos(90^\circ - \alpha) = |\vec{F}|\sin\alpha$, since $|\vec{j}| = 1$ and $\cos(90^\circ - \alpha) = \sin\alpha$.

Alternatively, you can see from the right triangle that

$$\cos\alpha = \frac{F_x}{|\vec{F}|} \text{ and } \sin\alpha = \frac{F_y}{|\vec{F}|},$$

giving the same results for F_x and F_y.

FORMULA

A vector $\vec{F}$ making an angle of α with $\vec{i}$ is resolved on $\vec{i}$ and $\vec{j}$ as follows.

$$\vec{F} = |\vec{F}|\cos\alpha\ \vec{i} + |\vec{F}|\sin\alpha\ \vec{j}$$

The following similar result is true in 3-space, and you will have an opportunity to prove it in the exercises.

FORMULA

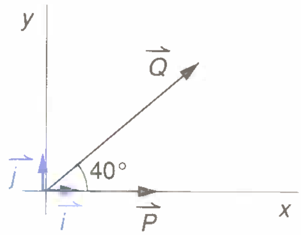
A vector $\vec{F}$ making angles of α, β and γ with $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively is resolved on $\vec{i}$, $\vec{j}$, and $\vec{k}$ as follows.

$$\vec{F} = |\vec{F}|\cos\alpha\ \vec{i} + |\vec{F}|\cos\beta\ \vec{j} + |\vec{F}|\cos\gamma\ \vec{k}$$

The following example matches Example 2 of the previous section, although it is worded differently. Thus you will be able to compare methods.

Example 1 Two forces $\vec{P}$ and $\vec{Q}$ act on a particle and make an angle of 40° with each other. If $|\vec{P}| = 30$ N and $|\vec{Q}| = 70$ N, find the resultant force $\vec{R}$.

Solution Vectors may be drawn anywhere. Draw $\vec{P}$ with its tail at the origin of a two-dimensional coordinate system so that $\vec{P}$ points in the direction of the positive x-axis (that is, in the direction of $\vec{i}$). This is shown in the diagram.



To find the resultant $\vec{R} = \vec{P} + \vec{Q}$, first resolve $\vec{P}$ and $\vec{Q}$ on $\vec{i}$ and $\vec{j}$.

$\vec{P}$: Since $|\vec{P}| = 30$, $\vec{P} = 30\vec{i}$

$\vec{Q}$:

$$\begin{aligned}\vec{Q} &= |\vec{Q}| \cos 40° \ \vec{i} + |\vec{Q}| \sin 40° \vec{j} \\ &= (70)(0.7660\ldots)\vec{i} + (70)(0.6427\ldots)\ \vec{j} \\ &\doteq 53.623\ \vec{i} + 44.995\ \vec{j}\end{aligned}$$

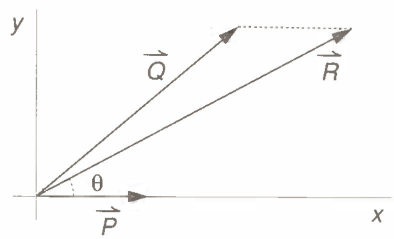
Now

$$\begin{aligned}\vec{R} &= \vec{P} + \vec{Q} \\ &= (30\vec{i}) + (53.623\vec{i} + 44.995\vec{j}) \\ &= 83.623\vec{i} + 44.995\vec{j}\end{aligned}$$

Magnitude of $\vec{R}$

$$|\vec{R}| = \sqrt{83.623^2 + 44.995^2} = \sqrt{9017.3\ldots} \doteq 95.0$$

Direction of $\vec{R}$



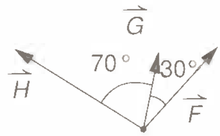
Let θ be the angle between $\vec{R}$ and $\vec{i}$.

Then $\tan \theta = \dfrac{44.995}{83.623} = 0.5380\ldots$, therefore $\theta \doteq 28°$

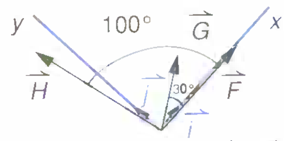
Thus the resultant of $\vec{P}$ and $\vec{Q}$ has a magnitude of 95.0 N and makes an angle of 28° with the force $\vec{P}$
(or an angle of 40° −28° = 22° with force $\vec{Q}$).

The same results are obtained as for Example 2 of the previous section. ■

Example 2 Three coplanar forces $\vec{F}$, $\vec{G}$, and $\vec{H}$ of magnitudes 15 N, 10 N, and 20 N respectively, act on a particle as shown in the diagram. Find the resultant of the three forces.



Solution Vectors can be drawn anywhere. Draw $\vec{F}$ with its tail at the origin of a two-dimensional coordinate system so that $\vec{F}$ points in the direction of $\vec{i}$. This is shown in the diagram. Note that the angle between $\vec{G}$ and $\vec{i}$ is 30°, and the angle between $\vec{H}$ and $\vec{i}$ is 100°.



Then the resultant of the three forces is $\vec{R} = \vec{F} + \vec{G} + \vec{H}$.
Now resolve $\vec{F}$, $\vec{G}$, and $\vec{H}$ on $\vec{i}$ and $\vec{j}$.

$\vec{F}$: Since $|\vec{F}| = 15, \vec{F} = 15\vec{i}$

$\vec{G}$:
$$\begin{aligned}\vec{G} &= |\vec{G}| \cos 30° \ \vec{i} + |\vec{G}| \sin 30° \ \vec{j} \\ &= (10)(0.8660\ldots)\ \vec{i} + (10)(0.5)\ \vec{j} \\ &\doteq 8.660\vec{i} + 5\vec{j}\end{aligned}$$

$\vec{H}$:
$$\begin{aligned}\vec{H} &= |\vec{H}| \cos 100° \ \vec{i} + |\vec{H}| \sin 100° \ \vec{j} \\ &= (20)(-0.1736\ldots)\ \vec{i} + (20)(0.9848\ldots)\ \vec{j} \\ &\doteq -3.472\vec{i} + 19.696\vec{j}\end{aligned}$$

So
$$\begin{aligned}\vec{R} = \vec{F} + \vec{G} + \vec{H} &= (15\vec{i}) + (8.66\vec{i} + 5\vec{j}) + (-3.472\vec{i} + 19.696\vec{j}) \\ &= 20.188\vec{i} + 24.696\vec{j}\end{aligned}$$

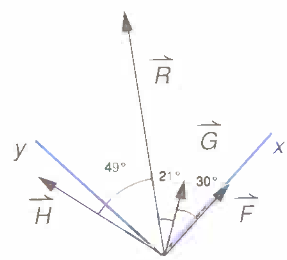
Magnitude of $\vec{R}$

$|\vec{R}| = \sqrt{20.188^2 + 24.696^2} = \sqrt{1017.44\ldots} \doteq 31.9$

Direction of $\vec{R}$

Let θ be the angle between $\vec{R}$ and $\vec{i}$.

Then $\tan \theta = \dfrac{24.696}{20.188} = 1.2233\ldots$, thus $\theta \doteq 51°$

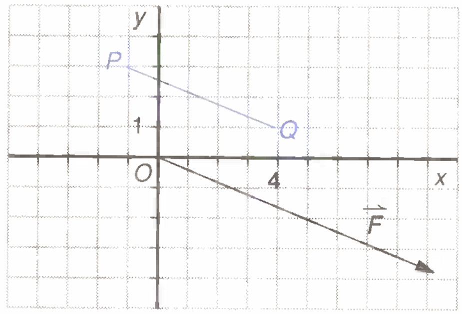


Therefore, the resultant of forces $\vec{F}$, $\vec{G}$ and $\vec{H}$ has magnitude 31.9 N and makes an angle of 51° with the force $\vec{F}$ (or an angle of 51° − 30° = 21° with the force $\vec{G}$, or an angle of 100° − 51° = 49° with the force $\vec{H}$). ■

Some vector problems on forces are formulated in component form, and it is convenient to treat them by using components, as in the next example.

Example 3 Given points $P(-1,3)$ and $Q(4,1)$, a force $\vec{F}$ of 10 N acts in the direction of $\overrightarrow{PQ}$. Resolve $\vec{F}$ parallel to the coordinate axes.

Solution



The force $\vec{F}$ points in the same direction as

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \overrightarrow{(4,1)} - \overrightarrow{(-1,3)} = \overrightarrow{(5,-2)}$,

so $\vec{F} = k\overrightarrow{(5,-2)}$ where k is a positive number.

Now $|\vec{F}| = |k\overrightarrow{(5,-2)}| = k|\overrightarrow{(5,-2)}| = k\sqrt{5^2 + 2^2} = k\sqrt{29}$

but you know that $|\vec{F}| = 10$,

so $$k\sqrt{29} = 10 \text{ or } k = \frac{10}{\sqrt{29}}$$

Thus the force $$\vec{F} = \frac{10}{\sqrt{29}}\overrightarrow{(5,-2)}$$

in other words, the force can be resolved as

$$\vec{F} = \frac{50}{\sqrt{29}}\vec{i} - \frac{20}{\sqrt{29}}\vec{j}.$$

The components of $\vec{F}$ in the x and y directions are respectively

$\frac{50}{\sqrt{29}}$ and $-\frac{20}{\sqrt{29}}$ ■

Friction

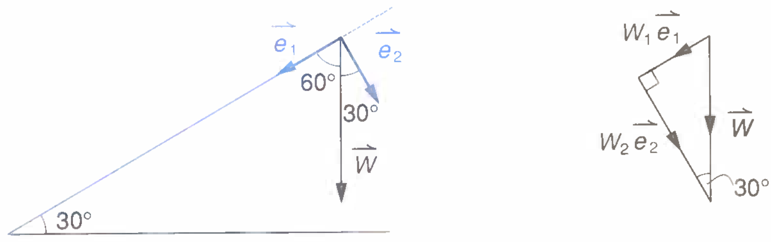
If you place a marble and a book on your desk, then gently incline the surface of your desk, you will notice that the marble rolls off almost immediately, whereas the book will remain motionless. This is because the marble encounters virtually no resistance, and so accelerates down the incline. The book, however, will not slide unless the incline is made a lot steeper. The book is held in equilibrium by a frictional force acting parallel to the surface of your desk.

These two situations will be investigated in Examples 4 and 5.

Example 4 A marble of mass 50 g is on a plane inclined at an angle 30° to the horizontal.

Resolve the force of gravity on the marble into two rectangular components, one of which causes the marble to accelerate.

Solution



space diagram vector diagram

Note: 50 g is the same as $\frac{50}{1000} = 0.05$ kg. The force of gravity on the marble is its weight, $\vec{W}$, whose magnitude is $(0.05)(9.8) = 0.49$ N. The weight acts vertically downwards.

You are asked to resolve $\vec{W}$ in the directions of the unit vectors
$\vec{e}_1$ (parallel to the plane) and
$\vec{e}_2$ (perpendicular to the plane).

Note: The angle between $\vec{W}$ and the horizontal is 90°. Thus the angle θ between $\vec{W}$ and $\vec{e}_1$ is $(180 - 90 - 30)° = 60°$. It follows that the angle between $\vec{W}$ and $\vec{e}_2$ is $(90 - 60)° = 30°$. This is the angle used in the vector diagram.

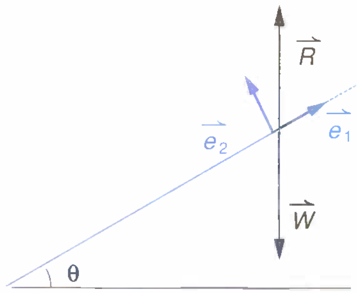
The marble is prevented by the plane from moving in the direction of $\vec{W}$. It can only move parallel to the plane, downhill, along the direction of $\vec{e}_1$.
In this direction, the component of $\vec{W}$ is
$W_1 = |\vec{W}| \sin 30° = (0.49)(1)(0.5\ldots) = 0.245$
In the direction of $\vec{e}_2$, the component of $\vec{W}$ is
$W_2 = |\vec{W}| \cos 30° = (0.49)(1)(0.866\ldots) = 0.424$

The components of the gravitational force on the marble are 0.245 N down the plane, causing the marble to accelerate, and 0.424 N perpendicular to the plane. ■

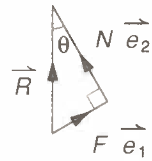
Example 5 A particle of mass m kg is in equilibrium on a rough plane inclined at an angle θ to the horizontal. The equilibrant of the weight is a force called a **reaction**, acting vertically upwards.

Resolve this reaction into a force perpendicular to the plane (called a **normal reaction**), and a force along the plane (called a **frictional force**).

Solution



space diagram



vector diagram

Let the weight be $\vec{W}$, the equilibrant be $\vec{R}$, to be resolved into the normal reaction $N\vec{e}_2$, and the force of friction $F\vec{e}_1$.

The magnitude of the equilibrant, equal to the magnitude of the weight, is mg.

Thus the normal component (the normal reaction) is $N = |\vec{R}|\cos\theta = mg\cos\theta$ and the parallel component (the frictional force) is $F = |\vec{R}|\sin\theta = mg\sin\theta$. ■

The Direction of a Force in 3-space

The following example will show you how to treat a three-dimensional problem on forces.

Example 6 The forces $\vec{P}$ and $\vec{Q}$ are such that $\vec{P} = \overrightarrow{(3,0,2)}$ and $\vec{Q} = \overrightarrow{(-4,6,7)}$. Calculate the magnitude and direction of their resultant $\vec{R}$.

Solution $\vec{R} = \vec{P} + \vec{Q} = \overrightarrow{(3,0,2)} + \overrightarrow{(-4,6,7)} = \overrightarrow{(-1,6,9)}$.

Thus the magnitude of $\vec{R}$, $|\vec{R}| = \sqrt{1^2 + 6^2 + 9^2} = \sqrt{118} \doteq 10.9$.

Now the direction of $\vec{R}$ cannot be specified merely by a 'slope' or 'bearing', since these concepts are meaningful only in 2-space.

A direction in 3-space is specified by the angles α, β and γ that it makes with $\vec{i}$, $\vec{j}$ and $\vec{k}$ respectively.

Now $\vec{R} \cdot \vec{i} = |\vec{R}||\vec{i}| \cos \alpha = |\vec{R}| \cos \alpha$, since $|\vec{i}| = 1$.

Thus $\cos \alpha = \dfrac{\vec{R} \cdot \vec{i}}{|\vec{R}|}$, and similarly $\cos \beta = \dfrac{\vec{R} \cdot \vec{j}}{|\vec{R}|}$ and $\cos \gamma = \dfrac{\vec{R} \cdot \vec{k}}{|\vec{R}|}$

Thus $\cos \alpha = -\dfrac{1}{\sqrt{118}}$, $\cos \beta = \dfrac{6}{\sqrt{118}}$, and $\cos \gamma = \dfrac{9}{\sqrt{118}}$

giving $\alpha \doteq 95°$, $\beta \doteq 56°$, and $\gamma \doteq 34°$. ■

SUMMARY

A vector $\vec{F}$ making an angle of α with $\vec{i}$ is resolved on $\vec{i}$ and $\vec{j}$ as follows.

$$\vec{F} = |\vec{F}| \cos \alpha \ \vec{i} + |\vec{F}| \sin \alpha \ \vec{j}$$

A vector $\vec{F}$ making angles of α, β, and γ with $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively is resolved on $\vec{i}$, $\vec{j}$, and $\vec{k}$ as follows.

$$\vec{F} = |\vec{F}| \cos \alpha \ \vec{i} + |\vec{F}| \cos \beta \ \vec{j} + |\vec{F}| \cos \gamma \ \vec{k}$$

The direction of a vector $\vec{v}$ in $\mathbb{V}_3$ is specified by the angles α, β, and γ that it makes with $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively.

$$\cos \alpha = \frac{\vec{v} \cdot \vec{i}}{|\vec{v}|}, \ \cos \beta = \frac{\vec{v} \cdot \vec{j}}{|\vec{v}|}, \text{ and } \cos \gamma = \frac{\vec{v} \cdot \vec{k}}{|\vec{v}|}$$

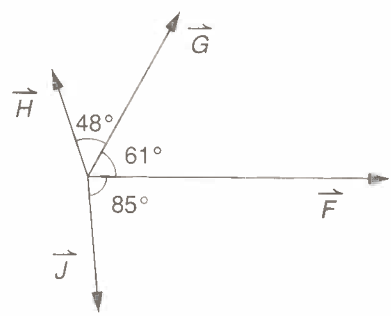
4.2 Exercises

(Unless directed otherwise, give all magnitudes in this exercise in newtons, correct to 3 significant digits, and all angles correct to the nearest degree.)
Use $g \doteq 9.8$ m/s².

1. Two forces $\vec{P}$ and $\vec{Q}$ act on a particle. Calculate the magnitude and the direction of their resultant in the following cases.
   a) $\vec{P} = \overrightarrow{(2,7)}$ and $\vec{Q} = \overrightarrow{(4,1)}$
   b) $\vec{P} = \overrightarrow{(-50,30)}$ and $\vec{Q} = \overrightarrow{(35,60)}$
2. Show that a vector $\vec{F}$ making angles of α, β and γ with $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively is resolved on $\vec{i}$, $\vec{j}$, and $\vec{k}$ as follows.
   $\vec{F} = |\vec{F}| \cos \alpha \, \vec{i} + |\vec{F}| \cos \beta \, \vec{j} + |\vec{F}| \cos \gamma \, \vec{k}$
3. Force $\vec{P} = \overrightarrow{(2,8,-1)}$ and force $\vec{Q} = \overrightarrow{(3,1,5)}$.
   a) Calculate the magnitude of the resultant of $\vec{P}$ and $\vec{Q}$.
   b) Specify the direction of the resultant by finding the angles it makes with $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively.
4. Resolve the force $\vec{F}$ in two orthogonal directions so that one component makes an angle $\theta°$ with $\vec{F}$ in the following cases.
   a) $|\vec{F}| = 20$, $\theta = 30°$
   b) $|\vec{F}| = 100$, $\theta = 10°$
   c) $|\vec{F}| = 100$, $\theta = 80°$
   d) $|\vec{F}| = 50$, $\theta = 130°$

The next two questions are the same as 2 and 6 of 4.1 Exercises. Here, solve them by resolving the forces on $\vec{i}$ and $\vec{j}$.

5. Two forces $\vec{P}$ and $\vec{Q}$ of magnitudes 30 N and 60 N respectively act on a particle. The angle between $\vec{P}$ and $\vec{Q}$ is 75°. Calculate the magnitude and the direction of the resultant of $\vec{P}$ and $\vec{Q}$.
6. The resultant of $\vec{P}$ and $\vec{Q}$ is the force $\vec{F}$, of magnitude 80 N, acting on bearing 070°. $\vec{P}$ has magnitude 25 N and acts due east. Calculate the magnitude and direction of force $\vec{Q}$.
7. Two forces $\vec{P}$ and $\vec{Q}$ act on a particle and make an angle of 115° with each other. If $|\vec{P}| = 16$ N and $|\vec{Q}| = 21$ N, find the magnitude and direction of the resultant force.
8. Four coplanar forces $\vec{F}$, $\vec{G}$, $\vec{H}$, and $\vec{J}$, of magnitudes 40 N, 25 N, 15 N, and 18 N respectively, act on a particle as shown in the figure. Find the magnitude and the direction of the resultant of the four forces.

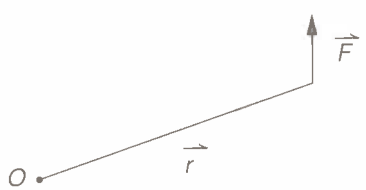


9. Given points $P(4,8)$ and $Q(2,-3)$, a force $\vec{F}$ of 50 N acts in the direction of $\overrightarrow{PQ}$. Resolve $\vec{F}$ on $\vec{i}$ and $\vec{j}$.

10. Given points $P(2,8,-5)$ and $Q(3,-1,2)$, a force $\vec{F}$ of 300 N acts in the direction of $\overrightarrow{PQ}$.
    a) Resolve $\vec{F}$ on $\vec{i}$, $\vec{j}$, and $\vec{k}$.
    b) Find the angles between $\vec{F}$ and each of $\vec{i}, \vec{j}, \vec{k}$.

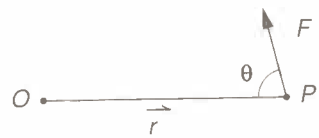
11. A Startrak Space Craft taking off from Earth is being acted upon by three forces. Its main engine is pushing it east with 200 000 N, its vertical jet is pushing it up with 150 000 N, and the north wind is pushing it southward with 30 000 N. By considering a unit vector $\vec{i}$ pointing due south, a unit vector $\vec{j}$ point due east, and a unit vector $\vec{k}$ pointing upward, find the magnitude and direction of the resultant of these three forces on the craft.

12. The moment $\vec{M}$ with respect to the origin O of a force $\vec{F}$ is the cross product $\vec{M} = \vec{r} \times \vec{F}$, where $\vec{r}$ is the position vector of any point on the line along which the force acts. Calculate the following moments.
    a) $\vec{r} = \overrightarrow{(0,3,0)}$; $\vec{F} = \overrightarrow{(2,5,6)}$
    b) $\vec{r} = \overrightarrow{(-2,1,-1)}$; $\vec{F} = \overrightarrow{(4,-3,0)}$



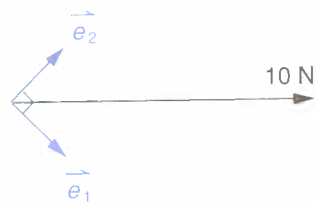
13. The moment of a force about a point is the effectiveness of that force to produce rotation about that point.

    The force $\vec{F}$ acts at the point P, as shown.



    a) What is the angle between $\vec{r}$ and $\vec{F}$?
    b) Write an expression for the moment of force $\vec{F}$ about O, in terms of $|\vec{r}|$, $|\vec{F}|$, and θ. (See question 12.)
    c) For which angle θ is the magnitude of the moment a maximum? a minimum?

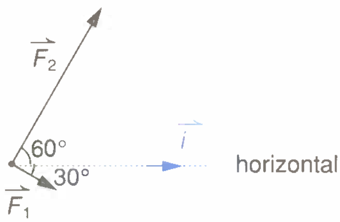
14. Resolve a force of 10 N into two equal rectangular components.



15. Resolve a force of 60 N along two orthogonal directions such that the components are in the ratio 2 : 3. Calculate the angle between the 60 N force and the larger projection.

16. A 150 kg bobsleigh is about to be released on an icy slope inclined at 35° to the horizontal. Calculate the force that must be applied parallel to the slope to keep the bobsleigh stationary.

17. Repeat question 13 of 4.1 Exercises by resolving the forces on $\vec{i}$ and $\vec{j}$.
Two forces $\vec{F_1}$ and $\vec{F_2}$, of magnitudes 7 N and 24 N, act on a particle. $\vec{F_1}$ acts at 30° to the vector $\vec{i}$, and $\vec{F_2}$ acts at 60° to the vector $\vec{i}$, as shown in the figure. Another force $\vec{F_3}$ of magnitude 30 N acts simultaneously on the particle. Calculate the direction of $\vec{F_3}$ so that the resultant of the three forces is in the direction of $\vec{i}$. (Note: The three forces would then cause the particle to move in the direction of $\vec{i}$.)



18. A force $\vec{F} = \overrightarrow{(40,60)}$ is pulling a particle up an inclined plane parallel to the vector $\overrightarrow{(9,2)}$. Calculate the component of the force that is acting parallel to the plane.

19. A particle of mass 20 kg is in equilibrium on a rough plane inclined at 26° to the horizontal.

a) Resolve the equilibrant into a normal reaction and a frictional force.

b) Calculate the number μ, where $|\vec{F}| = \mu|\vec{N}|$. (If the particle is about to slide, μ is called the **coefficient of static friction.**)

c) If the plane were smooth, calculate the *horizontal* force required to stop the particle from sliding down the plane.

20. Repeat question 19 for a mass of m kg on a plane inclined at $\theta°$ to the horizontal.

21. In the following diagrams, the suspended mass is 10 kg.

a) Calculate the tensions in the strings for figure a).

b) Calculate the tension in the string and the pushing force (called a **thrust**) in the strut for figure b).

figure 1

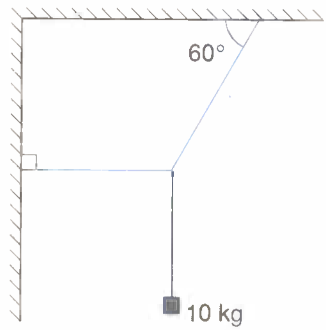
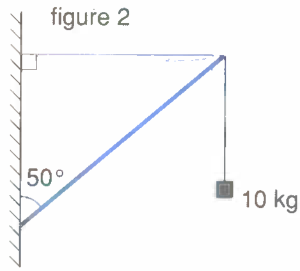
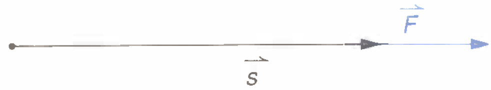


figure 2

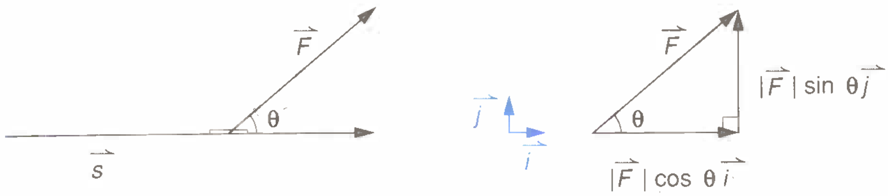


4.3 Work

When a force $\vec{F}$ pulls a particle in its direction along a displacement $\vec{s}$, the 'work' thus done by the force $\vec{F}$ is defined as $|\vec{F}||\vec{s}|$



However, the displacement s is not necessarily in the same direction as the force. For example, if a sled is tugged along a horizontal path, you might have the situation described in this diagram.



In this general case, work is defined as follows.

work done = (magnitude of force *in direction of motion*) (distance moved)

or $\quad W = (|\vec{F}|\cos\theta)(|\vec{s}|)$

or $\quad W = \vec{F}\cdot\vec{s}$

Notice, however, that the sled will only move in the desired direction along the path if θ is acute, that is, if $\theta < 90°$. The cosine of an acute angle is positive, hence *work is a positive scalar*.

If force is measured in newtons, and distance is measured in metres, then work is measured in newton metres (N · m), or joules (J).
1 newton metre equals 1 joule.

Task	Work done (approximate)
Pick up a pencil from the floor	2 J
Push a loaded shopping cart up a short ramp	500 J
Push a car out of a (level) driveway	8000 J

In the above example, you can resolve the force parallel and perpendicular to the path by writing

$$\vec{F} = |\vec{F}|\cos\theta\,\vec{i} + |\vec{F}|\sin\theta\,\vec{j}$$

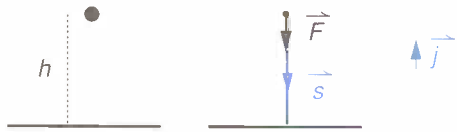
Since the movement is horizontal here, you can say that the parallel force, $|\vec{F}|\cos\theta\,\vec{i}$, does all the work, while the vertical force, $|\vec{F}|\sin\theta\,\vec{j}$, does no work.

In the examples that follow, work will be calculated to the nearest J.

Example 1

A stone of mass m is dropped to the ground from a height h. Calculate the work done by gravity.

Solution



Let the gravitational force be $\vec{F}$ and the displacement be $\vec{s}$.

The magnitude of the gravitational force on the stone is mg.
The stone will travel in the direction of this force, thus the work done
$W = (mg)(h)$
$= mgh$.

Alternatively, using vectors to describe the situation, using $\vec{j}$ as a unit vector pointing directly upwards:
$\vec{F} = -mg\vec{j}$
$\vec{s} = -h\vec{j}$
Work done $= \vec{F} \cdot \vec{s}$

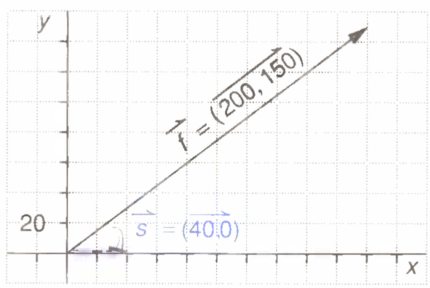
$$\begin{aligned} W &= (-mg\vec{j}) \cdot (-h\vec{j}) \\ &= (-mg)(-h)\vec{j} \cdot \vec{j} \\ &= (mgh)(1) \\ &= mgh. \end{aligned}$$

■

Example 2

A force $\vec{f} = \overrightarrow{(200,150)}$, whose magnitude is measured in newtons, is required to pull a heavy cart along the displacement $\vec{s} = \overrightarrow{(40,0)}$. The magnitude of $\vec{s}$ is measured in metres.
Calculate the magnitude of $\vec{f}$, the magnitude of $\vec{s}$, and the work done by $\vec{f}$.

Solution

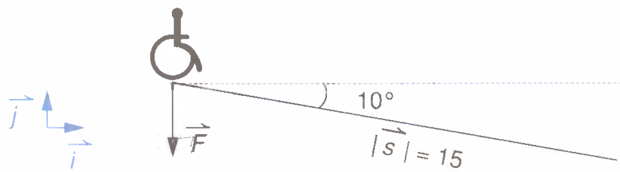


The magnitude of the force, $|\vec{f}| = \sqrt{200^2 + 150^2} = 250$.
The distance, $|\vec{s}| = 40$.
The work done, $W = \vec{f} \cdot \vec{s} = \overrightarrow{(200,150)} \cdot \overrightarrow{(40,0)}$
$= (200)(40) + 0 = 8000$.

Thus the magnitude of the force is 250 N, the distance moved is 40 m, and the work done is 8000 J. ■

Example 3 A man in a wheelchair moves 15 m down a ramp inclined at 10° to the horizontal. The mass of the man and the wheelchair together is 80 kg. Calculate the work done.

Solution Let $\vec{F}$ be the gravitational force on the man and wheelchair. The magnitude of $\vec{F}$ in newtons is $|\vec{F}| = mg = (80)(9.8) = 784$.



Using the horizontal unit vector $\vec{i}$ and the vertical unit vector $\vec{j}$ shown in the diagram, the displacement vector $\vec{s}$ can be written in component form as follows.

$$\vec{s} = \overrightarrow{(15\cos 10°, -15\sin 10°)}$$

and $\vec{F}$ can be written $\vec{F} = \overrightarrow{(0, -784)}$

Thus the work done is $\vec{F} \cdot \vec{s} = 0 + (-784)(-15\sin 10°)$
$= (784)(15)(0.1736\ldots)$
$= 2042.1\ldots$

The work done is about 2042 J. ■

Note: The work done is said to be done *by gravity* when the man moves *down* the ramp. In order to move the same distance *up* the ramp, the man must produce an equal amount of work.

SUMMARY

Work done = (magnitude of force *in direction of motion*)(distance moved)

or $W = |\vec{F}|\cos\theta|\vec{s}|$

or $W = \vec{F} \cdot \vec{s}$

4.3 Exercises

In the following, give answers correct to 3 significant digits where appropriate.

1. Find the work done by $\vec{F}$ as it acts on a particle through a displacement $\vec{s}$ in the following cases.
   a) $\vec{F} = \overrightarrow{(20,40)}$, $\vec{s} = \overrightarrow{(1,3)}$
   b) $\vec{F} = \overrightarrow{(30,-5)}$, $\vec{s} = \overrightarrow{(1,3)}$
   c) $\vec{F} = \overrightarrow{(15,20,-22)}$, $\vec{s} = \overrightarrow{(-4,10,-7)}$

2. A force $\vec{F}$ of 1000 N moves a particle along a path that is at an angle of 42° with the force.
   a) Resolve $\vec{F}$ parallel to and perpendicular to the path.
   b) Calculate the work done if $\vec{F}$ moves the particle 30 m.

3. A tractor is pulling a disabled barge along a canal. The tension in the towrope is 2500 N, and the towrope makes an angle of 25° with the direction of the canal. Calculate the work done in moving the barge 600 m along the canal.

4. A shopper pushes a loaded supermarket cart of mass 20 kg up a ramp inclined at 15° to the horizontal. Calculate the work done if the ramp is 10 m long.

5. A force $\vec{F} = \overrightarrow{(10,10,20)}$ attempts to pull a particle along a displacement $\vec{s} = \overrightarrow{(3,-7,2)}$. Calculate the work done. Explain your answer.

6. A force $\vec{F} = \overrightarrow{(-5,-8,-10)}$ pulls a particle from point $P(25,14,20)$ to point $Q(0,-3,4)$. Calculate the work done.

7. A force $\vec{p}$ of magnitude 50 N acts in the same direction as $\overrightarrow{AB}$ where A has coordinates (5,−6,7) and B has coordinates (8,1,8). Calculate the work done in the following cases.
   a) $\vec{P}$ acts on a particle along the displacement $\overrightarrow{AB}$.
   b) $\vec{P}$ acts on a particle along the displacement $\overrightarrow{MN}$, where M has coordinates (−2,−4,−10) and N has coordinates (4,6,9).

8. The work done by gravity as a book of mass M kg drops to the floor from a height h m is equal to the **kinetic energy** of the book just before it hits the ground. The kinetic energy is given by the formula $\frac{1}{2}Mv^2$, where v is the speed of the book in m/s just before it hits the ground.
   a) It is given that $v = 7$. Calculate h.
   b) It is given that $h = 2$. Calculate v.

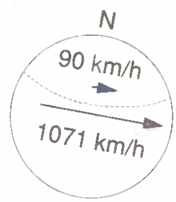
9. a) A pen of mass 100 g is dropped to the floor from a height of 1.5 m. Calculate the work done by gravity.
   b) As the pen falls to the ground, its speed v increases. Just before it hits the ground, the pen has a kinetic energy of $(0.05)v^2$ J, equal to the work done by gravity during the fall. Calculate the speed of the pen at this moment.

4.4 Velocities as Vectors

The word *speed* describes the rate at which *distance* is covered. (Speed is a scalar quantity.) The word *velocity* describes the rate at which *displacement* changes; displacement, unlike distance, includes direction as part of its definition, thus velocity also depends on direction. Like forces, velocities have both magnitude and direction, and furthermore, they can be combined following the laws of vector addition. Hence you can use vectors to represent velocities.

Relative Velocity

It is important to understand that velocities are always 'relative', never 'absolute'. For example, when you say that a truck approaching Winnipeg is travelling due east at 90 km/h, you should realize that this is its velocity *relative to the surface of the earth*. If, for instance, you want to describe its velocity relative to the centre of the earth, you must take into account the velocity of the earth's surface near Winnipeg. That is, at latitude 50° N, the earth's surface moves eastward at 1071 km/h. Thus the velocity of the truck relative to the centre of the earth is 1071 km/h faster than its surface velocity: 1071 + 90 = 1161 km/h, eastward.



The following notation is useful.

$\vec{v}_{TS}$ = velocity of the *Truck* relative to the earth's *Surface*
$\vec{v}_{SC}$ = velocity of the earth's *Surface* (Winnipeg) relative to the earth's *Centre*
$\vec{v}_{TC}$ = velocity of the *Truck* relative to the earth's *Centre*

In this case, the speeds, that is, the magnitudes of the velocities, are such that $|\vec{v}_{TC}| = |\vec{v}_{TS}| + |\vec{v}_{SC}|$, because the vectors involved have the same direction. However, if the vectors are *not* collinear, then the velocity of the truck is obtained by the following *vector addition* of relative velocities.

FORMULA

$$\vec{v}_{TC} = \vec{v}_{TS} + \vec{v}_{SC}$$

In other words, for points T, S, and C moving relative to each other, the velocity of T relative to C is the vector sum of the velocity of T relative to S, and the velocity of S relative to C.

Notice that this formula can be expressed also as a subtraction as follows.

FORMULA

$$\vec{v}_{SC} = \vec{v}_{TC} - \vec{v}_{TS}$$

In other words, the velocity of S relative to C equals the velocity of T relative to some point C, minus the velocity of T relative to S.

Although the verbal descriptions of the relative velocities above may seem difficult to learn, you might appreciate that *the order of letters involved in the notation match exactly those you are familiar with from the triangle law of vector addition, and of vector subtraction.* That is, for example, $\overrightarrow{TC} = \overrightarrow{TS} + \overrightarrow{SC}$.

Similarly, the following observation matches the definition of a negative vector.

Saying "the truck is moving eastward at 90 km/h relative to the surface" is equivalent to saying "the surface is moving westward at 90 km/h relative to the truck". You are motionless inside the truck; the scenery is travelling 'backward'. Thus, if $\vec{v}_{ST}$ is the vector representing this latter velocity, then

$$\vec{v}_{ST} = -\vec{v}_{TS}.$$

Thus the subtraction formula could be written

$$\vec{v}_{SC} = \vec{v}_{CT} - (-\vec{v}_{ST})$$

or

$$\vec{v}_{SC} = \vec{v}_{ST} - \vec{v}_{CT}$$

If a fly is crawling along the truck's dashboard, and you want to find the velocity of the fly relative to the centre of the earth, $\vec{v}_{FC}$, you will require three velocity vectors:

$\vec{v}_{FT}$ = velocity of the fly (relative to the truck)

$\vec{v}_{TS}$ = velocity of the truck (relative to the surface)

$\vec{v}_{SC}$ = velocity of the earth's surface (relative to the centre)

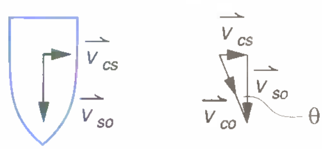
The resultant velocity, $\vec{v}_{FC}$, is obtained by a vector addition of these velocities, that is,

$$\vec{v}_{FC} = \vec{v}_{FT} + \vec{v}_{TS} + \vec{v}_{SC}$$

Example 1 An oil tanker is sailing due south at 60 km/h. The captain, during his daily workout routine, is running perpendicularly across the ship, from starboard (right side) to port (left side), at 25 km/h. What is the captain's velocity relative to the ocean?

Solution $\vec{v}_{SO}$ represents the velocity of the ship relative to the ocean,
$\vec{v}_{CS}$ represents the captain's velocity relative to the ship, and
$\vec{v}_{CO}$ represents the captain's velocity relative to the ocean, then
$\vec{v}_{CO} = \vec{v}_{CS} + \vec{v}_{SO}$, as indicated in the diagram.

space diagram



vector diagram

You can draw your vectors where you want them. The diagram that attempts to portray the position of the ship is called the space diagram. The diagram designed purely for vector algebra is the vector diagram.

By the theorem of Pythagoras,

$|\vec{v}_{CO}|^2 = 25^2 + 60^2 = 4225$

so $|\vec{v}_{CO}| = \sqrt{4225} = 65$,

and $\tan\theta = \dfrac{25}{60} = 0.4166\ldots$, so $\theta \doteq 23°$.

Thus the bearing of $\vec{v}_{CO}$ is $180° - 23° = 157°$.

The captain's velocity relative to the ocean is 65 km/h at a bearing 157°. ■

In the next two examples, alternate solutions that use more traditional notation will also be presented. If you now look back at the introduction to this chapter, you will find that these examples should enable you to solve the airplane pilot's problem.

Note that wind direction always indicates the direction that the wind is blowing *from*, not blowing to.

Example 2

A pilot is heading her plane due north at an airspeed of 160 km/h, while a wind is blowing from the east at 75 km/h. At what speed is she actually travelling, and in what direction, relative to the ground?

Solution

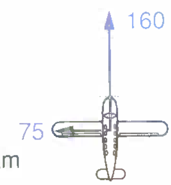
You want

the velocity of the plane relative to the ground, $\vec{v}_{PG}$ (or $\vec{r}$)

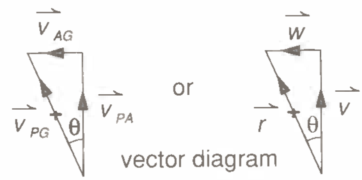
given the plane's velocity relative to the air, $\vec{v}_{PA}$ (or $\vec{v}$)

and the wind velocity, that is,

the velocity of the air relative to the ground, $\vec{v}_{AG}$ (or $\vec{w}$)



space diagram



vector diagram

Observe the diagrams carefully. The plane's groundspeed will be the magnitude of the vector $\vec{v}_{PG}$ (or $\vec{r}$), and the plane will actually fly in the direction θ west of north, *although its nose will always be pointing north.*

The vector sum is thus	or	using traditional notation you can say
$\vec{v}_{PG} = \vec{v}_{PA} + \vec{v}_{AG}$		$\vec{r} = \vec{v} + \vec{w}$
From Pythagoras,		From Pythagoras,
$\lvert\vec{v}_{PG}\rvert^2 = 75^2 + 160^2 = 31225$		$\lvert\vec{r}\rvert^2 = 75^2 + 160^2 = 31225$
$\lvert\vec{v}_{PG}\rvert = 176.70\ldots \doteq 177$		$\lvert\vec{r}\rvert = 176.70\ldots \doteq 177$

Also $\tan\theta = \dfrac{75}{160} = 0.46875 \Rightarrow \theta \doteq 25°$.

Thus her plane is actually travelling at 177 km/h, at a bearing 335°, relative to the ground.

(177 km/h is known as the groundspeed of the plane.) ■

Example 3 A jet is to fly from Toronto to Ottawa. The bearing of Ottawa from Toronto is 055°, and the distance is 370 km. The airspeed of the jet is 500 km/h, and there is a prevailing wind blowing from bearing 350° at 160 km/h. Calculate the following.

a) The direction in which the jet should head, to the nearest degree.

b) The time taken, to the nearest minute, to fly from Toronto to Ottawa.

Solution Let the jet's air velocity be $\vec{v}_{JA}$ (or $\vec{v}$)

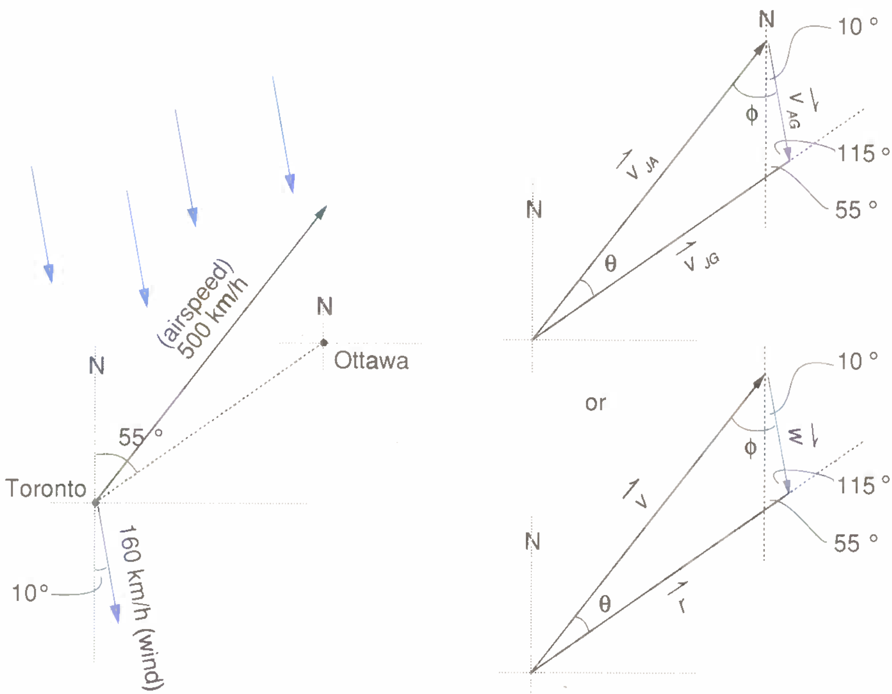
the jet's ground velocity be $\vec{v}_{JG}$ (or $\vec{r}$)

the wind velocity be $\vec{v}_{AG}$ (or $\vec{w}$)

Then $\vec{v}_{JG} = \vec{v}_{JA} + \vec{v}_{AG}$ or $\vec{r} = \vec{v} + \vec{w}$.

You know that $|\vec{v}_{JA}| = |\vec{v}| = 500$, $|\vec{v}_{AG}| = |\vec{w}| = 160$.

Note carefully the angles marked in each diagram. If the wind is from bearing 350°, that means the angle between the wind and the south is 10°. This has been shown in the diagrams. From the space diagram, you can calculate that the angle between the Toronto-Ottawa line and the wind is $(180 - 10 - 55)° = 115°$. This angle is marked in the vector diagrams, where it will be used.



space diagram vector diagram

a) By the sine law in the vector diagram

$$\frac{160}{\sin\theta} = \frac{500}{\sin 115^\circ}$$

$$\sin\theta = \frac{160\sin 115^\circ}{500} = 0.2900\ldots \quad \text{so } \theta = 16.85\ldots^\circ$$

The jet must head about 17° left of the bearing 055°, and thus follow the heading $(055 - 17)^\circ = 038^\circ$.

b) In order to calculate the flight time from Toronto to Ottawa, you need to find the jet's groundspeed, that is, $|\vec{v}_{JB}|$ or $|\vec{r}|$. You must use the sine law again in the vector diagram. First you need the angle ϕ.

$$\phi = 180^\circ - (115^\circ + 16.85\ldots^\circ) = 48.14\ldots^\circ$$

Using the sine law again,

$$\frac{|\vec{v}_{JG}|}{\sin 48.14\ldots^\circ} = \frac{500}{\sin 115^\circ}$$

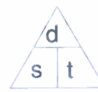
$$|\vec{v}_{JG}| = \frac{500\sin 48.14\ldots^\circ}{\sin 115^\circ} = 410.8\ldots$$

or

$$\frac{|\vec{r}|}{\sin 48.14\ldots^\circ} = \frac{500}{\sin 115^\circ}$$

$$|\vec{r}| = \frac{500\sin 48.14\ldots^\circ}{\sin 115^\circ} = 410.8\ldots$$

Now speed $= \dfrac{\text{distance}}{\text{time}}$, or time $= \dfrac{\text{distance}}{\text{speed}}$



Since the distance to be covered is 370 km, the time taken is

$\dfrac{370}{410.8\ldots} = 0.9004\ldots$, measured in hours.

The time in minutes is thus $(0.9004)(60) = 54.02\ldots$

The flight time from Toronto to Ottawa is about 54 minutes. ■

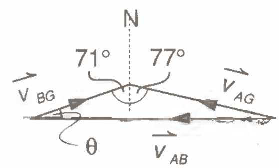
In the final example, the subscript notation should help you to avoid fundamental errors, such as adding the wrong vectors.

Example 4 A small airplane A is flying on bearing 283° at 200 km/h, and another, B, is flying on bearing 071° at 140 km/h. What is the velocity of A relative to B?

Solution

space diagrams

$\vec{v}_{AG}$ represents the velocity of A relative to the ground,
$\vec{v}_{BG}$ represents the velocity of B relative to the ground, and
$\vec{v}_{AB}$ represents the velocity of A relative to B,
then $|\vec{v}_{AG}| = 200$, $|\vec{v}_{BG}| = 140$, and $|\vec{v}_{AB}|$ is unknown.



You can find $\vec{v}_{AB}$ from the vector sum
$\vec{v}_{AB} + \vec{v}_{BG} = \vec{v}_{AG}$,
according to the vector diagram.
As you transcribe the space diagram into the vector diagram, note that the angle between $\vec{v}_{BG}$ and $\vec{v}_{AG}$ is 71° + 77° = 148°.
The cosine law gives

$$|\vec{v}_{AB}|^2 = 140^2 + 200^2 - (2)(140)(200)\cos 148° = 107090.69\ldots$$

so that $|\vec{v}_{AB}| = 327.24\ldots \doteq 327$.
Using the sine law to find the angle θ between $\vec{v}_{AB}$ and $\vec{v}_{BG}$:

$$\frac{200}{\sin\theta} = \frac{327.24\ldots}{\sin 148°}$$

$$\sin\theta = \frac{200\sin 148°}{327.24\ldots} = 0.3238\ldots \Rightarrow \theta \doteq 19°.$$

Thus the angle ϕ between $\vec{v}_{AG}$ and $\vec{v}_{AB}$ is 180° − (148° + 19°) = 13°.
Hence, according to B, A appears to be moving at 327 km/h on a bearing 360° − 77° − 13° = 270°. In other words, B perceives that A is moving away towards the west. ■

SUMMARY

(In the following, given points P and Q moving relative to each other, the notation $\vec{v}_{PQ}$ represents the velocity of P relative to Q.)

Velocity is a vector whose magnitude is called speed.

For any points P and Q moving relative to each other, $\vec{v}_{PQ} = -\vec{v}_{QP}$

Given any points A, B, C moving relative to each other,
$\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$ or $\vec{v}_{BC} = \vec{v}_{AC} - \vec{v}_{AB}$ or $\vec{v}_{BC} = \vec{v}_{BA} - \vec{v}_{CA}$

4.4 Exercises

(Unless directed otherwise, give all speeds in this exercise correct to the nearest km/h, all times correct to the nearest minute, and all angles correct to the nearest degree.)

1. In the following, given points P and Q moving relative to each other, the notation $\vec{v}_{PQ}$ represents the velocity of P relative to Q.

   Find the magnitude and direction of $\vec{v}_{PQ}$ in the following cases. (Speeds are in km/h.)
   a) $|\vec{v}_{PA}| = 40$ pointing due east
   $|\vec{v}_{AQ}| = 25$ pointing due north
   b) $|\vec{v}_{PA}| = 100$ on bearing 280°
   $|\vec{v}_{AQ}| = 35$ on bearing 190°
   c) $|\vec{v}_{PA}| = 550$ on bearing 072°
   $|\vec{v}_{AQ}| = 420$ on bearing 120°
   d) $|\vec{v}_{PA}| = 18$ on bearing 072°
   $|\vec{v}_{AQ}| = 12$ on bearing 300°

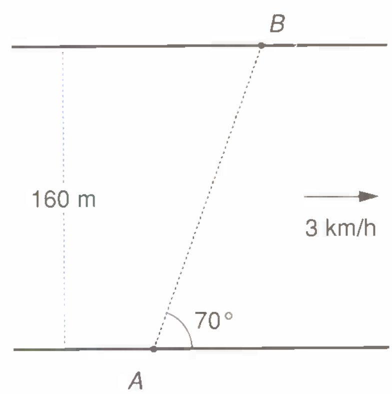
2. The Transcanadian train is travelling westward at 100 km/h. An RCMP officer is on the train. Calculate the magnitude and direction of the RCMP officer's velocity, relative to the ground, in the following cases.
   a) The RCMP officer is walking towards the front of the train at 5 km/h.
   b) The RCMP officer is running towards the back of the train at 15 km/h.
   c) The RCMP officer is running directly across the train, from the north side to the south side, at 12 km/h.

3. A plane is heading due east at an airspeed of 200 km/h, while a wind is blowing from the north at 50 km/h. Find the groundspeed and actual direction of the plane.

4. A is 1000 km due west of B. A plane whose airspeed is 500 km/h flies from A to B, then immediately back to A, on a day when the wind is blowing from the west at 100 km/h. Calculate the total time taken.

5. The Prime Minister asks her pilot to fly her immediately from Ottawa to Toronto on a day when the wind is blowing from the west at 120 km/h. Toronto is 370 km from Ottawa on a bearing 235°. The jet has an airspeed of 900 km/h. Find the following.
   a) the heading the pilot should follow
   b) the flight time

6. A swimmer, whose speed in still water is 3 km/h, wishes to cross a 160 m wide river from A to B as shown in the figure. The river is flowing at 3 km/h.
   a) In what direction should the swimmer head?
   b) How long will it take him to get to B?



7. Repeat question 6 with the following data. The width of the river is w, the speed of the flow is v, the speed of the swimmer in still water is v, and the angle between AB and the direction of the flow is θ.

8. A ship S is sailing on a bearing of 160° at 30 km/h. Another ship T is sailing on a bearing of 250° at 40 km/h.
   a) Calculate the velocity of S relative to T.
   b) Calculate the velocity of T relative to S.

9. A ship is travelling due west at 50 km/h. The smoke emanating from the ship's funnel makes an angle of 25° with the ship's wake. Calculate the speed of the wind if it is known that the wind is blowing from the north.

Mathematics and Aircraft Navigation

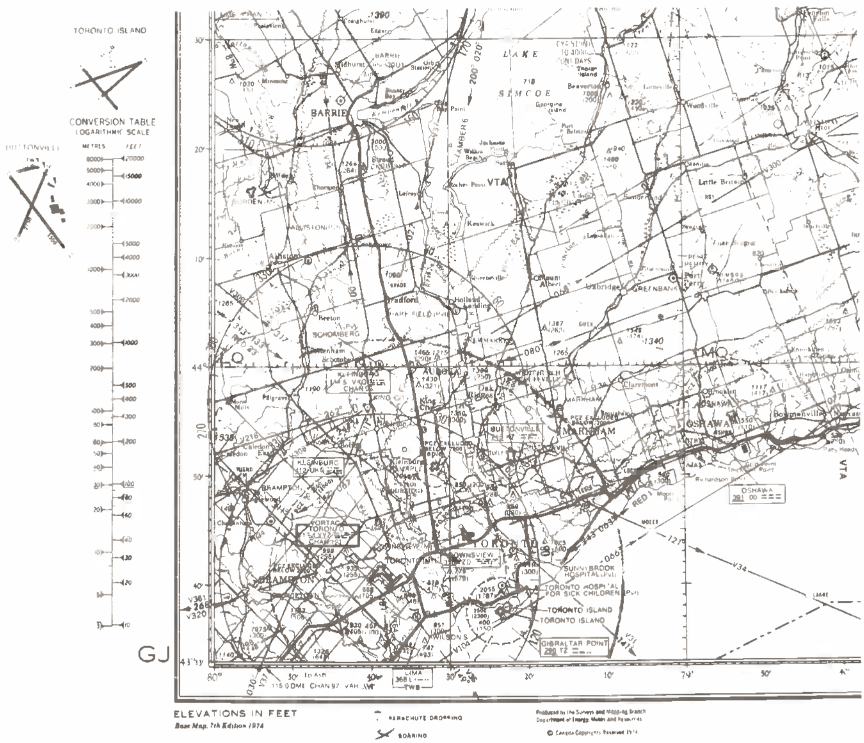
One of the requirements of a private pilot's licence is that you execute at least one long-distance solo flight from your home airport to other airports and return.

Before departing on a cross-country flight, you must prepare a flight plan for review by your instructor and submission to Air Traffic Control. The wind conditions forecast for the flight must be considered. This might involve the following telephone conversation.

"Hello. Toronto Flight Service? Can you give me information for a VFR flight (1) by light aircraft to Barrie this afternoon?"

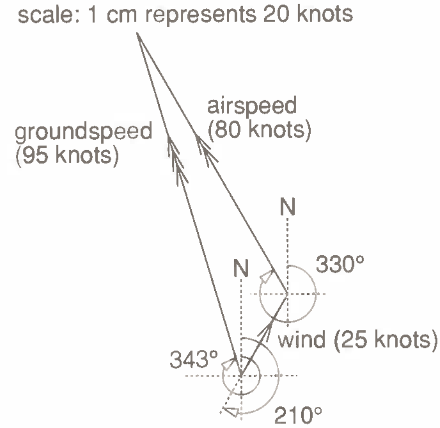
"The weather is clear. The temperature at 4500 ft is 10°C, and the wind is from 210° at 25 knots."

Aircraft altitudes are still measured in feet, and windspeeds are given in knots, or nautical miles per hour. One nautical mile (2) is the distance spanned by one minute of arc, that is, 1/60th of a degree of longitude.



By measurement on the map, you find that the bearing of Barrie airport from Toronto Island is 343°, at a distance of 49 nautical miles (3). You know that your plane has an airspeed of 80 knots.

Your heading and groundspeed are then calculated or measured according to the vector diagram.

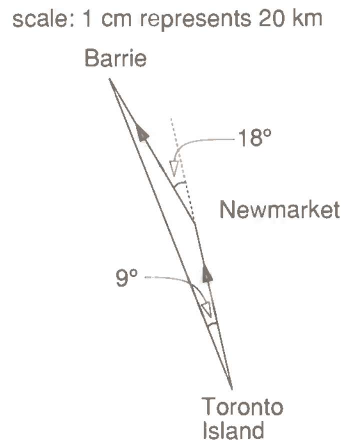


Thus you must fly a "true heading" of 330° to reach Barrie. Your groundspeed will be 95 knots. However, the magnetic north in this part of the world is 10° west of "true north". Thus, you must add 10° to obtain the *magnetic heading*, 330° + 10° = 340°, that you will follow by reference to the magnetic compass in the aircarft.

Note: Your estimated flight time to Barrie is $\frac{\text{distance}}{\text{speed}} = \frac{49}{95} = 0.516$ hours or 31 minutes, to the nearest minute.

HELP!
Due to a wind change, or inaccurate forecast, you find that you are about to fly over Newmarket after about 15 minutes. This is about 9° off course.

A quick sketch
shows that you must alter your
heading by about 2 × 9° = 18°
to the left to reach Barrie.
Thus you now turn to magnetic bearing
340° − 18° = 322°,
and keep your eyes open to find
Barrie airport about 15 minutes later.



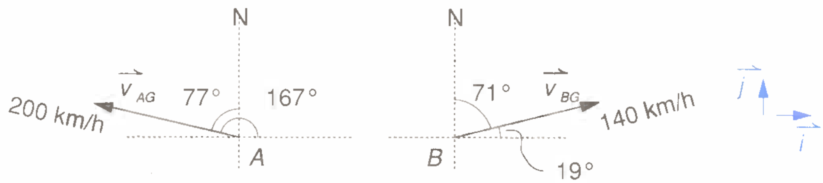
(1) A flight according to Visual Flight Rules.
(2) 1 foot (ft) ≐ 0.3048 m and 1 nautical mile (nm) ≐ 1.852 km.
(3) The usual scale of an air navigation map is 1 : 500 000. You measure 18.2 cm on the map, which represents 91 km, that is 91 × 1.852 ≐ 49 nm. Alternatively, you can measure along a meridian, in minutes of arc, for this distance (degrees and minutes are indicated at the side of the map).

4.5 Velocities Using Components

In this section, you will see that a variety of problems involving velocities can be solved by resolving vectors, or by using components.

The first example is the same as Example 4 of the previous section. This will allow you to compare methods.

Example 1 A small airplane A is flying on bearing 283° at 200 km/h, and another, B, is flying on bearing 071° at 140 km/h. Calculate the direction and magnitude of the velocity of A relative to B.



Solution Resolve the velocities in the eastward and northward directions by using the unit vectors $\vec{i}$ and $\vec{j}$ shown. Note the angles with $\vec{i}$ shown in the diagrams, and recall the resolution formula used in section 4.2, namely: a vector $\vec{v}$ making an angle α with $\vec{i}$ is resolved on $\vec{i}$ and $\vec{j}$ as follows.

$\vec{v} = |\vec{v}| \cos \alpha \; \vec{i} + |\vec{v}| \sin \alpha \; \vec{j}$

Then the velocity of A, $\vec{v}_{AG} = (200) \cos 167° \; \vec{i} + (200) \sin 167° \; \vec{j}$

$= -(194.87\ldots)\vec{i} + (44.99\ldots)\vec{j}$

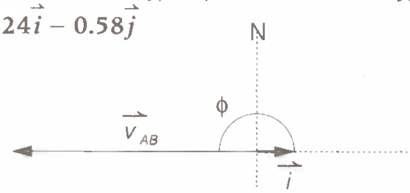
and the velocity of B, $\vec{v}_{BG} = (140) \cos 19° \; \vec{i} + (140) \sin 19° \; \vec{j}$

$= (132.37\ldots)\vec{i} + (45.57\ldots)\vec{j}$

You want the velocity of A relative to B, $\vec{v}_{AB} = \vec{v}_{AG} - \vec{v}_{BG}$

Thus $\vec{v}_{AB} \doteq (-194.87\vec{i} + 44.99\vec{j}) - (132.37\vec{i} + 45.57\vec{j})$

$= -327.24\vec{i} - 0.58\vec{j}$



so $|\vec{v}_{AB}| = \sqrt{327.24^2 + 0.58^2} = 327.24\ldots \doteq 327$

and the angle of $\vec{v}_{AB}$ with $\vec{i}$ is ϕ where

$\vec{v}_{AB} \cdot \vec{i} = |\vec{v}_{AB}||\vec{i}| \cos \phi$

$$\text{so} \quad \cos \phi = \frac{-327.24}{327.24} = -1 \Rightarrow \phi = 180°.$$

Thus, as you found before, B perceives that A is moving at 327 km/h towards the west. ■

Some vector problems on velocities are formulated in component form, and it is convenient to treat them by using components, as in the following example.

Example 2 A ship is travelling with velocity $\vec{v}_{SO} = (20\vec{i} + 30\vec{j})$, and a forklift on board is travelling relative to the ship with velocity $\vec{v}_{FS} = (-5\vec{i} + 6\vec{j})$. An ant is crawling on the engine cover of the forklift, with velocity $\vec{v}_{AF} = (\vec{i} - \vec{j} + 3\vec{k})$ relative to the forklift.
(All velocities in this example are given in km/h.)

Calculate the velocity and the speed of the ant relative to the ocean.

Solution The required velocity is

$$\begin{aligned}\vec{v}_{AO} &= \vec{v}_{AF} + \vec{v}_{FS} + \vec{v}_{SO}\\ &= \overrightarrow{(20,30,0)} + \overrightarrow{(-5,6,0)} + \overrightarrow{(1,-1,3)}\\ &= \overrightarrow{(16,35,3)}.\end{aligned}$$

Thus the speed is

$$|\vec{v}_{AO}| = \sqrt{16^2 + 35^2 + 3^2} = \sqrt{1490} = 38.60\ldots$$

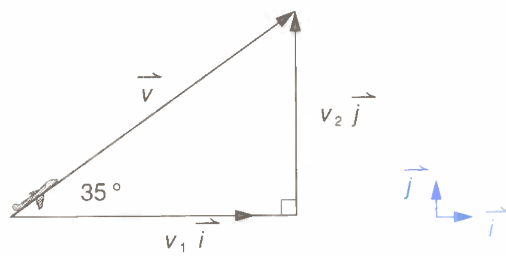
The ant's speed relative to the ocean is about 38.6 km/h. ■

Example 3 The velocity of a jet shortly after takeoff is given by the vector $\vec{v}$ whose magnitude is 500 km/h, and whose direction is 35° from the horizontal, pointing upwards.

a) Calculate the vertical speed of the jet, in m/s.
b) Calculate the speed of the jet's shadow on the ground, if the sun is directly overhead.

Solution The diagram shows that the velocity $\vec{v}$ of the jet can be resolved into a horizontal vector and a vertical vector as follows.

$$\vec{v} = v_1\vec{i} + v_2\vec{j}$$



where v_1 represents the speed of the shadow along the ground, and v_2 represents the vertical speed, or rate of climb, of the jet.

a) $v_2 = |\vec{v}|\sin 35^\circ = (500)(0.5735\ldots) = 286.78\ldots$, in km/h.

To convert to m/s, you must multiply by 1000 and divide by 3600:

vertical speed in m/s is $(286.78\ldots)\left(\frac{1000}{3600}\right) = 79.66$

The rate of climb is about 80 m/s .

b) $v_1 = |\vec{v}|\cos 35^\circ = (500)(0.8191\ldots) = 409.57\ldots$

The horizontal speed of the shadow is about 410 km/h. ■

4.5 Exercises

(Unless directed otherwise, give all magnitudes in this exercise correct to 3 significant digits, and all angles correct to the nearest degree.)
In the following, given points P and Q moving relative to each other, the notation $\vec{v}_{PQ}$ represents the velocity of P relative to Q.

1. Find $\vec{v}_{PQ}$ and $|\vec{v}_{PQ}|$ in the following cases.
   a) $\vec{v}_{PA} = \overrightarrow{(3,1)}$ and $\vec{v}_{AQ} = \overrightarrow{(-2,5)}$
   b) $\vec{v}_{PA} = \overrightarrow{(4,-3,6)}$ and $\vec{v}_{QA} = \overrightarrow{(8,2,-1)}$
   c) $\vec{v}_{AP} = \overrightarrow{(3,2,1)}$ and $\vec{v}_{QA} = \overrightarrow{(9,-6,-5)}$

2. A small airplane P is flying on bearing 165° at 400 km/h, and another airplane, Q, is flying on bearing 220° at 320 km/h. Calculate the magnitude and the direction of the following.
   a) the velocity of P relative to Q
   b) the velocity of Q relative to P

3. A train is travelling with velocity $(36\vec{i} + 48\vec{j})$, and the food trolley is travelling relative to the train with velocity $(6\vec{i} - 4\vec{j})$. A ladybug is crawling on the food trolley with relative velocity $(\vec{i} + \vec{j} - 2\vec{k})$. Calculate the velocity and the speed of the ladybug relative to the ground. (All velocities in this question are given in km/h.)

4. A ship is travelling with velocity $(-25\vec{i} + 50\vec{j})$, and the captain's robot is travelling relative to the ship with velocity $(4\vec{i} - 3\vec{j})$. A mantis is crawling on the robot, with velocity $(-\vec{i} + \vec{j} - 2\vec{k})$ relative to the robot. An ant is crawling on the mantis, with velocity $(2\vec{i} - 2\vec{j} + \vec{k})$ relative to the mantis. (All velocities in this question are given in m/s.)
   Calculate the velocity and the speed relative to the ocean, of
   a) the robot
   b) the mantis
   c) the ant.

5. Repeat question 3 of 4.4 Exercises by using components.

   A plane is heading due east at an airspeed of 200 km/h, while a wind is blowing from the north at 50 km/h.
   Find the groundspeed and actual direction of the plane.

6. Repeat question 5 of 4.4 Exercises by using components.

   The Prime Minister asks her pilot to fly her immediately from Ottawa to Toronto on a day when the wind is blowing from the west at 120 km/h. Toronto is 370 km from Ottawa on a bearing 235°. The jet has an airspeed of 900 km/h.
   Find the following.
   a) the heading the pilot should follow
   b) the flight time

7. A bus is travelling at a steady speed of 20 m/s in the direction of $\vec{i}$ on a rainy day. Raindrops, which are falling vertically (in the direction of $-\vec{j}$), make traces on the side windows of the bus. These traces are inclined at 25° to the horizontal. Calculate the following.
   a) the horizontal component of the drops' velocity with respect to the ground
   b) the horizontal component of the drops' velocity with respect to the bus
   c) the drops' velocity with respect to the ground
   d) the drops' velocity and speed with respect to the bus

8. Repeat question 8 of 4.4 Exercises by using components.

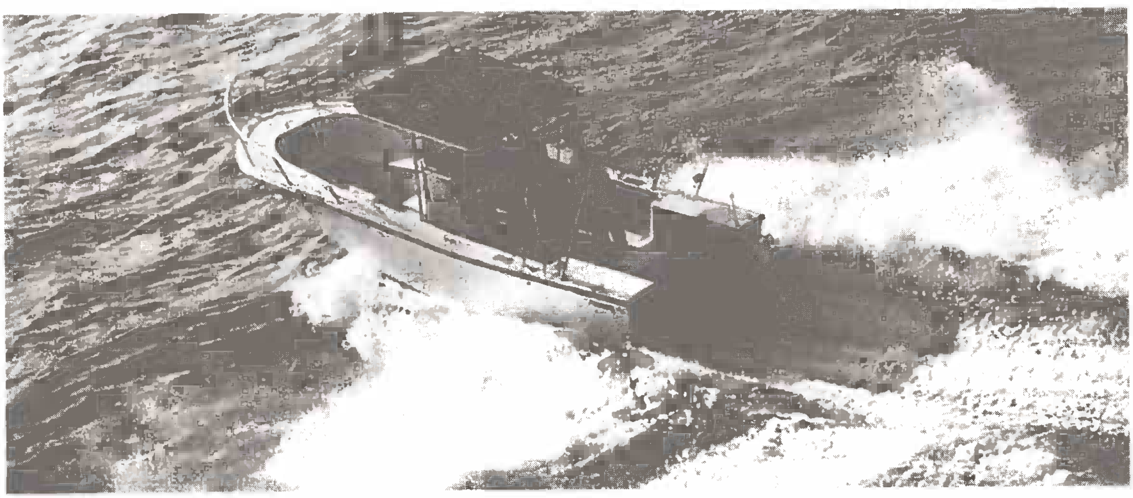
   A ship S is sailing on a bearing of 160° at 30 km/h. Another ship T is sailing on a bearing of 250° at 40 km/h.
   a) Calculate the velocity of S relative to T.
   b) Calculate the velocity of T relative to S.

9. A motor boat is moving at 15 km/h in the direction 048°. The wind is pushing the motorboat 3 km/h westward, and a current is pushing the motorboat 5 km/h northward. Calculate the direction and magnitude of the motorboat due to its own power.

10. A plane P is heading on bearing 060° at 450 km/h, while a plane Q is heading on bearing 340° at 400 km/h, at a time when the air is still.
    a) Calculate the velocity of P relative to Q.
    b) If the wind now starts blowing at 150 km/h from the east, calculate the velocity of P relative to Q.

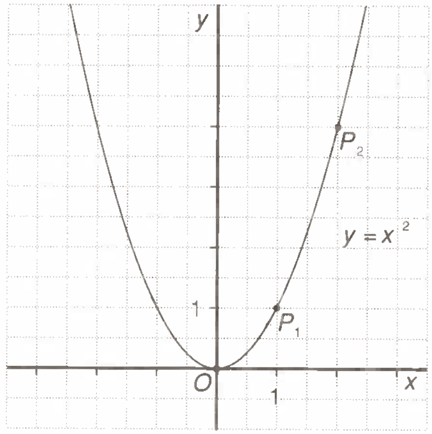
11. A first class passenger on an airliner opens a bottle. The cork pops off and travels at 100 km/h directly across the plane, from the right side to the left side. The plane is heading due east with an airspeed of 500 km/h, and there is a wind blowing from the southwest (that is, bearing 225°) at 120 km/h. Calculate the direction and the magnitude of the velocity of the cork relative to the ground.

12. An airplane pilot heads his plane due east and maintains an airspeed of 170 km/h. After flying for 30 minutes, he finds himself over a village which is 100 km east and 22 km north of his starting point. Find the magnitude and direction of the wind velocity.



In Search of Vector Functions

The points of $\mathbb{R}^2$ whose coordinates (x,y) satisfy the equation $y = x^2$ ① form a parabola.



Notice that if $x = t$ ②

and $y = t^2$ ③

where $t \in \mathbb{R}$ is called a **parameter**, then eliminating t between the equations ② and ③ yields $y = x^2$, the equation ①.

The parabola can be represented
either by the equation ①, called a **Cartesian equation**,
or by the system of equations ② and ③, called **parametric equations**.

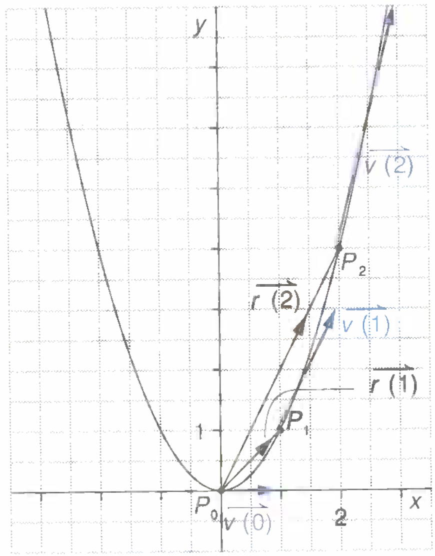
Each value of t gives a specific point on the parabola.
For example,
when $t = 1$, ② gives $x = 1$ and ③ gives $y = 1$.
So $t = 1$ gives the point (1,1).
When $t = 2$, ② gives $x = 2$ and ③ gives $y = 4$.
So $t = 2$ gives the point (2,4).

Now think of $P(x,y)$ as a particle travelling along the curve.
If t is a measure of time, the particle is
at $P_0(0,0)$ at time $t = 0$,
at $P_1(1,1)$ at time $t = 1$,
at $P_2(2,4)$ at time $t = 2$, etc.

Thus the parametric equations can represent the movement of a particle along the parabola, starting at (0,0) and going into the first quadrant.

Now if $\vec{i}$ and $\vec{j}$ are unit vectors along the x and y axes respectively, then the position vector of the particle is $\overrightarrow{OP} = x\vec{i} + y\vec{j}$ or, using ② and ③,

$$\overrightarrow{OP} = t\vec{i} + t^2\vec{j}.$$



$\overrightarrow{OP}$ depends on time, and $\overrightarrow{OP}$ is a vector.

Thus, $\overrightarrow{OP}$ can be called a **vector function** of time,

$\overrightarrow{OP} = \overrightarrow{r(t)} = t\vec{i} + t^2\vec{j}$ ④

The movement of the particle along the parabola can also be described by the vector function ④.

From ④, $\overrightarrow{r(0)} = 0\vec{i} + 0\vec{j} = \vec{0} = \overrightarrow{OP_0}$,
$\overrightarrow{r(1)} = 1\vec{i} + 1^2\vec{j} = \vec{i} + \vec{j} = \overrightarrow{OP_1}$,
$\overrightarrow{r(2)} = 2\vec{i} + 2^2\vec{j} = 2\vec{i} + 4\vec{j} = \overrightarrow{OP_2}$, etc.

The Calculus of Vector Functions

It can be shown that vector functions can be differentiated term by term, just like ordinary functions.

Differentiating ④ yields the velocity vector function

$\overrightarrow{r'(t)} = \overrightarrow{v(t)} = 1\vec{i} + 2t\vec{j} = \vec{i} + 2t\vec{j}$

Observe that $\overrightarrow{v(0)} = \vec{i} + 0\vec{j} = \vec{i}$
$\overrightarrow{v(1)} = \vec{i} + 2\vec{j}$
$\overrightarrow{v(2)} = \vec{i} + 4\vec{j}$

These are the velocity vectors of the particle at times 0, 1, 2 respectively. (These velocity vectors are represented by coloured directed line segments in the figure.)

Differentiating again yields the acceleration vector function.

Example 1

A particle in $\mathbb{R}^2$ moves such that $x = \cos t$ and $y = \sin t$, where t represents time. That is, the position vector of the particle is $\overrightarrow{r(t)} = \cos t\vec{i} + \sin t\vec{j}$.

a) Describe the motion of the particle for $t \geqslant 0$.

b) Find the velocity of the particle at any time, $\overrightarrow{v(t)}$; compare it to $\overrightarrow{r(t)}$.

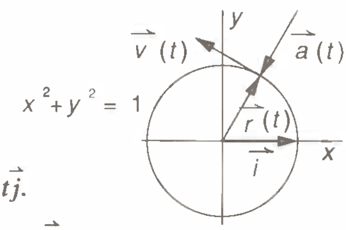
c) Find the acceleration of the particle at any time, $\overrightarrow{a(t)}$; compare it to $\overrightarrow{r(t)}$.

Solution

a) $x = \cos t$ and $y = \sin t$.

Squaring and adding these two equations eliminates t as follows.
$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, or $x^2 + y^2 = 1$.
(See the trigonometrical identities, page 542.)
Thus the particle moves in a circle, centre O, radius 1.

The position vector $\overrightarrow{r(t)} = \cos t\vec{i} + \sin t\vec{j}$ represents a 'moving radius' of this circle.
At the beginning of the motion, when $t = 0$,
$\overrightarrow{r(0)} = \cos 0\vec{i} + \sin 0\vec{j} = 1\vec{i} + 0\vec{j} = \vec{i}$



b) Differentiating $\overrightarrow{r(t)} = \cos t\vec{i} + \sin t\vec{j}$

yields $\overrightarrow{r'(t)} = \overrightarrow{v(t)} = -\sin t\vec{i} + \cos t\vec{j}$.

Note: $\overrightarrow{r(t)} \cdot \overrightarrow{v(t)} = (\cos t\vec{i} + \sin t\vec{j}) \cdot (-\sin t\vec{i} + \cos t\vec{j})$
$= -\cos t \sin t + \sin t \cos t = 0$

Thus, at any given time t, $\overrightarrow{r(t)}$ and $\overrightarrow{v(t)}$ are perpendicular.
This indicates that the velocity $\overrightarrow{v(t)}$ of the particle is *tangential* to the circle, as expected.

c) Differentiating $\overrightarrow{v(t)} = -\sin t\vec{i} + \cos t\vec{j}$

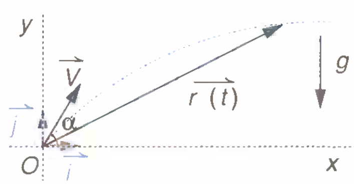
yields $\overrightarrow{v'(t)} = \overrightarrow{a(t)} = -\cos t\vec{i} - \sin t\vec{j}$. ■

Note: $\overrightarrow{a(t)} = -\overrightarrow{r(t)}$.
Thus, the acceleration of a particle travelling steadily around a circle is *along the radius, pointing towards the centre of the circle*. This indicates that the particle, while travelling in a direction perpendicular to the radius at any instant, keeps 'trying' to turn toward the centre. (This is known as **centripetal** acceleration.)

Example 2

A projectile is fired with initial speed V at an angle α to the horizontal. It is subject only to the acceleration due to gravity, g.
Find a vector function describing the motion of the projectile.

Solution



Let $\overrightarrow{r(t)} = x\vec{i} + y\vec{j}$ be the position vector of the projectile, from the point O from which it is fired, at any time t.

Thus x and y are the components of $\overrightarrow{r(t)}$ on $\vec{i}$ and $\vec{j}$, the standard basis vectors, at any time t.

You know that the acceleration acts vertically downwards, so
$\overrightarrow{r''(t)} = -g\vec{j}$.
Integrating (that is, antidifferentiating) gives
① $\overrightarrow{r'(t)} = -gt\vec{j} + \vec{P}$, where $\vec{P}$ is a constant vector.
But you know $\overrightarrow{r'(0)} = V\cos\alpha\vec{i} + V\sin\alpha\vec{j}$, so
$V\cos\alpha\vec{i} + V\sin\alpha\vec{j} = (-g)(0)\vec{j} + \vec{P}$
hence $\vec{P} = V\cos\alpha\vec{i} + V\sin\alpha\vec{j}$.
Thus ① becomes $\overrightarrow{r'(t)} = -gt\vec{j} + V\cos\alpha\vec{i} + V\sin\alpha\vec{j}$
or $\overrightarrow{r'(t)} = V\cos\alpha\vec{i} + (V\sin\alpha - gt)\vec{j}$
Integrating again gives
② $\overrightarrow{r(t)} = Vt\cos\alpha\vec{i} + \left(Vt\sin\alpha - \frac{1}{2}gt^2\right)\vec{j} + \vec{Q}$, where $\vec{Q}$ is a constant vector.

But you know that $\overrightarrow{r(0)} = \vec{0}$, so $\vec{0} = 0\vec{i} + 0\vec{j} + \vec{Q}$, hence $\vec{Q} = \vec{0}$

Thus ② becomes $\overrightarrow{r(t)} = Vt\cos\alpha\vec{i} + \left(Vt\sin\alpha - \frac{1}{2}gt^2\right)\vec{j}$.

The Cartesian equation of this trajectory can be obtained by eliminating t from the two equations

$$\text{and} \quad \begin{cases} x = Vt\cos\alpha \\ y = Vt\sin\alpha - \frac{1}{2}gt^2 \end{cases}$$

This gives

③ $y = \dfrac{-gx^2}{2V^2\cos^2\alpha} + (\tan\alpha)\,x$ which has the form $y = ax^2 + bx + c$. Thus, the path of a projectile is in the shape of a parabola. ■

Activities

1. Verify that the relationship between x and y of Example 2 is the equation ③.
2. Find the range of the projectile by setting $y = 0$.
3. Find the greatest height of the projectile by setting $y' = 0$.
4. A particle in $\mathbb{R}^2$ moves such that $x = a\cos t$ and $y = a\sin t$, where t represents time, and a is a constant.
   a) Describe the motion of the particle for $t \geqslant 0$.
   b) Find the velocity of the particle at any time, $\overrightarrow{v(t)}$, and compare it to $\overrightarrow{r(t)}$.
   c) Find the acceleration of the particle at any time, $\overrightarrow{a(t)}$, and compare it to $\overrightarrow{r(t)}$.
5. Repeat question 4 for $x = a\cos t$ and $y = b\sin t$.

Summary

- The resultant, $\vec{R}$, of a number of forces is the vector sum of those forces. The equilibrant of those forces is $-\vec{R}$.
- A particle is in equilibrium when the vector sum of all forces acting upon it is $\vec{0}$.
- The gravitational force on a particle of mass m kg is mg N, where g is the acceleration due to gravity.
(On earth, $g \doteq 9.8 \text{ m/s}^2$)
- A vector $\vec{F}$ making an angle of α with $\vec{i}$ is resolved on $\vec{i}$ and $\vec{j}$ as follows.
$\vec{F} = |\vec{F}| \cos \alpha \, \vec{i} + |\vec{F}| \sin \alpha \, \vec{j}$
- A vector $\vec{F}$ making angles of α, β and γ with $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively is resolved on $\vec{i}$, $\vec{j}$, and $\vec{k}$ as follows.
$\vec{F} = |\vec{F}| \cos \alpha \, \vec{i} + |\vec{F}| \cos \beta \, \vec{j} + |\vec{F}| \cos \gamma \, \vec{k}$
- The direction of a vector $\vec{v}$ in $\mathbb{V}_3$ is specified by the angles α, β, and γ that it makes with $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively.
$$\cos \alpha = \frac{\vec{v} \cdot \vec{i}}{|\vec{v}|}, \quad \cos \beta = \frac{\vec{v} \cdot \vec{j}}{|\vec{v}|}, \text{ and } \cos \gamma = \frac{\vec{v} \cdot \vec{k}}{|\vec{v}|}$$
- Work done, W, by a force $\vec{F}$ acting along a displacement $\vec{s}$
= (magnitude of force *in direction of motion*) (distance moved) or
$W = \vec{F} \cdot \vec{s}$

(In the following, given points P and Q moving relative to each other, the notation $\vec{v}_{PQ}$ represents the velocity of P relative to Q.)

- Velocity is a vector whose magnitude is called speed.
- For any points P and Q moving relative to each other,
$\vec{v}_{PQ} = -\vec{v}_{QP}$
- Given any points A, B, C moving relative to each other,
$\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$
or $\vec{v}_{BC} = \vec{v}_{AC} - \vec{v}_{AB}$
or $\vec{v}_{BC} = \vec{v}_{BA} - \vec{v}_{CA}$

Inventory

Complete each of the following statements.

1. A force or a velocity can be represented by a ____.
2. An object small enough to be considered as a point is called a ____.
3. Forces are measured in ____.

4. A mass of 1 kg weighs about _____.

5. Two forces or two velocities can be combined in the same way that vectors are _____.

6. If $\vec{P}$ and $\vec{Q}$ are forces, $\vec{R} = \vec{P} + \vec{Q}$ is called the _____ force.

7. If $\vec{R}$ and $\vec{S}$ are forces, and $\vec{R} + \vec{S} = \vec{0}$, then $\vec{S}$ is called the _____ of $\vec{R}$.

8. When the vector sum of all forces acting on a particle is $\vec{0}$, the particle is said to be in _____.

9. A vector diagram helps you to solve a physical problem. The reality of the physical situation is portrayed in a _____ diagram.

10. A vector $\vec{v}$ making an angle of α with $\vec{i}$ is resolved on $\vec{i}$ and $\vec{j}$ as follows. $\vec{v} =$ _____ $\vec{i} +$ _____ $\vec{j}$.

11. A vector $\vec{v}$ making angles of α, β, and γ with $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively is resolved on $\vec{i}$, $\vec{j}$, and $\vec{k}$ as follows.
$\vec{v} =$ _____ $\vec{i} +$ _____ $\vec{j} +$ _____ $\vec{k}$.

12. If $\vec{F} = 3\vec{e}_1 + 5\vec{e}_2$, then the numbers 3 and 5 are called the _____ of $\vec{F}$ on $\vec{e}_1$ and $\vec{e}_2$.

13. An object on a rough inclined plane is kept in equilibrium by a frictional force. This frictional force acts _____ to the plane.

14. The _____ of a vector $\vec{v}$ in $\mathbb{V}_3$ is specified by the angles α, β, and γ that it makes with _____, _____, and _____ respectively.

15. 'Speed' describes the _____ of velocity. 'Distance' describes the magnitude of _____.

16. The work done by a force is a *vector/scalar*. (Delete the incorrect term.)

17. Work is measured in _____. (These are also the units of energy.)

18. The work done by a force is the _____ of the force vector and the displacement vector.

19. The work done by a force acting perpendicularly to the displacement is _____.

20. The symbol $\vec{v}_{AB}$ describes the velocity of A _____ to B.

21. Complete the following relative velocity formulas.
    a) $\vec{v}_{AC} = \vec{v}_{AB} +$ _____
    b) $\vec{v}_{BC} = \vec{v}_{AC} -$ _____
    c) _____ $= \vec{v}_{BA} - \vec{v}_{CA}$.

Review Exercises

Where appropriate, give all angles correct to the nearest degree, all times correct to the nearest minute, and all other numerical answers correct to 3 significant digits.
Given points P and Q moving relative to each other, the notation $\overrightarrow{v_{PQ}}$ represents the velocity of P relative to Q.
Use $g \doteq 9.8$ m/s^2.

1. Two forces $\vec{P}$ and $\vec{Q}$ of magnitudes 80 N and 50 N respectively act on a particle. The angle between $\vec{P}$ and $\vec{Q}$ is 62°. Calculate the magnitude and the direction of the resultant of $\vec{P}$ and $\vec{Q}$.

2. A particle is being pulled by two forces $\vec{F_1}$ and $\vec{F_2}$. $\vec{F_1}$ acts vertically upward and its magnitude is 400 N. $\vec{F_2}$ acts at an angle of 20° to the horizontal, and its magnitude is 500 N.
   a) Find the magnitude and the direction of the resultant force.
   b) Find the direction and magnitude of the equilibrant.

3. Three forces of 50 N, 85 N, and 40 N act simultaneously on a particle, which remains in a state of equilibrium. Calculate the angles between the forces.

4. Is it possible for a particle to remain in equilibrium under the action of three forces whose magnitudes are 60 N, 30 N, and 20 N? Explain.

5. a) Two perpendicular forces $\vec{P}$ and $\vec{Q}$ are such that $2|\vec{P}| = 5|\vec{Q}|$. Given that their resultant has magnitude 90 N, calculate $|\vec{P}|$ and $|\vec{Q}|$.
   b) Two perpendicular forces $\vec{P}$ and $\vec{Q}$ are such that $|\vec{P}| = 2|\vec{Q}|$. If their resultant has magnitude 55 N, calculate $|\vec{P}|$ and $|\vec{Q}|$.

6. Two forces of magnitude 15 N acting on a particle have a resultant of magnitude 5 N. Calculate the angle between the two forces.

7. A particle of mass 8 kg is suspended by cords from two points A and B on a horizontal ceiling such that $AB = 2$ m. The lengths of the cords are 1.6 m and 1.1 m. Calculate the tension in each cord.

8. Force $\vec{P} = \overrightarrow{(50,-26,14)}$ and force $\vec{Q} = \overrightarrow{(16,-10,-45)}$.
   a) Calculate the magnitude of the resultant of $\vec{P}$ and $\vec{Q}$.
   b) Specify the direction of the resultant by finding the angles it makes with $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively.

9. Four coplanar forces $\vec{F}$, $\vec{G}$, $\vec{H}$, and $\vec{J}$ are such that their resultant is 2000 N along bearing 330°. $\vec{F}$ has magnitude 800 N and acts along 030°. $\vec{G}$ has magnitude 600 N and acts along 180°. $\vec{H}$ has magnitude 800 N and acts along 233°. Calculate the direction and magnitude of the force $\vec{J}$.

10. Given points $P(4,8,-3)$ and $Q(1,-2,5)$, a force $\vec{F}$ of 100 N acts in the direction of $\overrightarrow{PQ}$. Resolve $\vec{F}$ on $\vec{i}$, $\vec{j}$, and $\vec{k}$.

11. Resolve a force of 100 N into equal components along three mutually orthogonal directions.

12. A 70 kg luge is about to be released on an icy slope inclined at 50° to the horizontal. Calculate the force that must be applied parallel to the slope to keep the luge stationary.

13. A particle of mass 50 kg is in equilibrium on a rough plane inclined at 32° to the horizontal.
    Resolve the equilibrant into a normal reaction and a frictional force.

14. In the following diagrams, the suspended mass is 10 kg.

a) Calculate the tension in the strings for figure a).

b) Calculate the tension in the string, and the pushing force (called a thrust) in the strut for figure b).

figure a

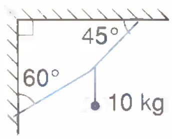
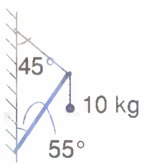


figure b



15. A particle of mass M kg remains stationary on a smooth plane inclined at an angle of $\theta°$ to the horizontal. It is held in equilibrium by a horizontal force. Show that the magnitude of this force in newtons is $Mg \sin \theta \cos \theta$.

16. A force $\vec{P}$ of magnitude 300 N acts in the same direction as $\overrightarrow{AB}$ where A has coordinates (–1,5,8) and B has coordinates (3,4,9). Calculate the work done in the following cases.

a) $\vec{P}$ acts on a particle along the displacement $\overrightarrow{AB}$.

b) $\vec{P}$ acts on a particle along the displacement $\overrightarrow{MN}$, where M has coordinates (4,4,–7) and N has coordinates (2,–1,3).

17. A force $\vec{F}$ of 800 N moves a particle along a path that is at an angle of 82° with the force.

a) Resolve $\vec{F}$ parallel to and perpendicular to the path.

b) Calculate the work done if $\vec{F}$ moves the particle 10 m.

18. A traveller in an airport pushes a loaded luggage cart of mass 45 kg up a ramp inclined at 12° to the horizontal. Calculate the work done if the ramp is 20 m long.

19. A microlight aircraft is heading due west at an airspeed of 90 km/h, while a wind is blowing from the south at 40 km/h. Find the groundspeed and actual direction of the aircraft.

20. An airliner is to take off from Winnipeg to fly to Montréal, 1850 km away on a bearing 094°. The captain prepares her flight plan according to the following data. At the altitude at which the airliner is to fly, the wind is blowing from bearing 330° at 100 km/h. The airspeed of the airliner is 900 km/h.

a) Calculate the heading that the airliner should take.

b) Calculate the flight time.

21. A ship S is sailing on bearing of 005° at 25 km/h. Another ship T is sailing on a bearing of 160° at 32 km/h.

a) Calculate the velocity of S relative to T.

b) Calculate the velocity of T relative to S.

22. A hovercraft is travelling with velocity $(30\vec{i} - 42\vec{j})$, and a trolley is being pushed relative to the hovercraft with velocity $(-5\vec{i} - 6\vec{j})$. A fly is crawling on the trolley with relative velocity

$$\left(-\vec{i} + \frac{1}{2}\vec{j} - \vec{k}\right).$$

Calculate the velocity and the speed of the fly relative to the sea. (All velocities in this question are given in km/h.)

VECTORS, MATRICES
and
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with
International Baccalaureate
questions

Jean-Paul GINESTIER
and
John EGSGARD

CHAPTER FIVE

EQUATIONS OF LINES IN 2- AND 3-SPACE

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CHAPTER FIVE

Equations of Lines in 2- and 3-space

A city has a number of power lines going out underground in straight lines from points *A* and *B* at different depths below the main distribution centre. The map showing these lines has been accidentally destroyed. Your task is to find out which lines coming from point *A* intersect with a line coming from point *B*. If two lines do not intersect, you are required to determine the shortest distance between these lines, and to find the points on the lines where this shortest distance occurs.

You will be able to solve this problem if you can determine the equation of a line in 3-space.

In 3-space, as in 2-space, two lines can intersect or not intersect. In 2-space two lines that do not intersect must be parallel. But, as explained in section 1.2, in 3-space there are lines that are not parallel *and* do not intersect. You can observe this in your classroom by noticing that a line on the front wall parallel to the floor will not intersect a line on the back wall that is perpendicular to the floor, even though the two lines are not parallel.

As you saw in chapter 1, two lines in 3-space that are not parallel and do not intersect are called *skew lines*.

In this chapter you will learn how to find the equations of lines in 3-space. With this and other information from this chapter you will have the mathematics necessary to solve the 'power line' problem given above.

5.1 The Vector Equation of a Line

You are familiar with equations such as $y = -2x + 6$, $4x - 2y = 15$, $x = 5$, and $y = -3$. Each represents a straight line and is called a *linear equation*. What is meant by saying that $y = -2x + 6$ is an equation for a straight line? If you graph every point P whose coordinates (x,y) satisfy this equation, then the points P will all lie in a straight line. For example, the point (1,4) lies on this line because $x = 1$, $y = 4$ substituted in the equation $y = -2x + 6$ gives $4 = -2(1) + 6$, which is true. Because the line is defined in terms of points $P(x,y)$ in a Cartesian coordinate system, this equation is also called the **Cartesian equation of a line in 2-space.**

You will recall that for the line $y = -2x + 6$, the slope of the line is -2 and the y-intercept is 6, that is, the line intersects the y-axis in the point (0,6). Figure 1 shows the line drawn using the axes of a Cartesian coordinate system. The chart gives the coordinates of several points on the line.

x	4	3	2	1	0	−1	−2
y	−2	0	2	4	6	8	10

The line consists of the set of all points $P(x,y)$ such that $y = 2x + 6$. Note that a point whose coordinates does *not* satisfy this equation does *not* lie on the line. The point $O(0,0)$ is clearly not on the line. If $x = 0$, $y = 0$ are substituted in $y = -2x + 6$, the result $0 = -2(0) + 6$ is not true.

The line described by the Cartesian equation $y = -2x + 6$ can also be defined by an equation that contains vectors, called a **vector equation of a line**. Instead of looking at the coordinates (x,y) of a point P on the line, a vector equation describes the position vector $\overrightarrow{OP}$ of a point P on the line. Figure 2 shows position vectors for several points on the line.

figure 1

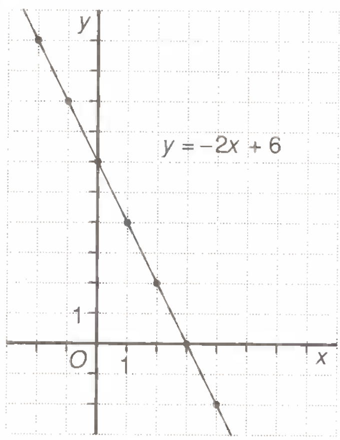
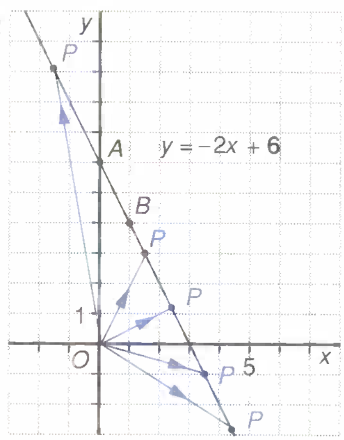
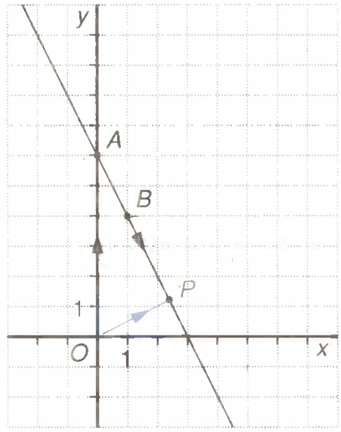


figure 2



Two facts are needed about this line in order to obtain its vector equation. For example, the point $A(0,6)$ and the point $B(1,4)$ are on the line. These points are used to determine a vector parallel to the line, namely $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (1,4) - (0,6) = (1,-2)$.

Let P be any point on the line.
From the figure, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$ ①
But $\overrightarrow{OA} = (0,6)$, and $\overrightarrow{AP}$ is a vector collinear with vector $\overrightarrow{AB}$.
Thus, $\overrightarrow{AP} = k\overrightarrow{AB}$ where $k \in \mathbb{R}$.

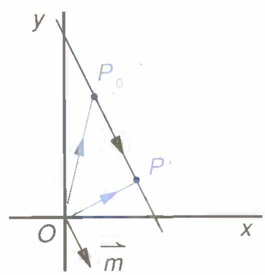


Different values for k will give different positions for the point P on the line.
Substituting in ①
$\overrightarrow{OP} = \overrightarrow{OA} + k\overrightarrow{AB}$
$\overrightarrow{OP} = (0,6) + k\overrightarrow{(1,-2)}$
This is a vector equation for the line through the points A and B.

A vector, such as $\overrightarrow{AB}$, that is collinear with or parallel to a line, is called a **direction vector** of the line.

The above method can be used *to find a vector equation of the line passing through the fixed point P, and having direction vector* $\vec{m}$.

Let P be any point on the line. From the figure,
$\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$.
But $\overrightarrow{P_0P}$ is parallel to $\vec{m}$.
Thus $\overrightarrow{P_0P} = k\vec{m}$, $k \in \mathbb{R}$.
Thus $\overrightarrow{OP} = \overrightarrow{OP_0} + k\vec{m}$.
This is the required vector equation of the line.

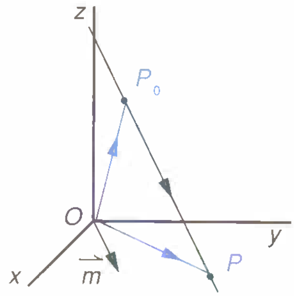


It is customary to use the abbreviations $\vec{r} = \overrightarrow{OP}$ and $\vec{r}_0 = \overrightarrow{OP_0}$

FORMULA

The vector equation of a line is
$\vec{r} = \vec{r}_0 + k\vec{m}$
where
$\vec{r} = \overrightarrow{OP}$, the position vector of *any* point P on the line,
$\vec{r}_0 = \overrightarrow{OP_0}$ the position vector of a *given* point P_0 on the line,
$\vec{m}$ is a vector parallel to the line,
k is any real number and is called a **parameter**.

One of the beautiful mathematical results of vector geometry is that this vector equation of a line in 2-space, namely, $\vec{r} = \vec{r_0} + k\vec{m}$, is also the vector equation of a line in 3-space. The derivation of this equation for 3-space is identical to its derivation in 2-space. You can easily see that this is true by employing the preceding argument using the diagram below.



However, there is one difference.
In 2-space the vectors $\vec{r}$, $\vec{r_0}$, and $\vec{m}$ have two components, while in 3-space the vectors $\vec{r}$, $\vec{r_0}$, and $\vec{m}$ have three components.

Example 1 a) Find a vector equation of the line, through the point $A(2,-1)$, that is parallel to the vector $\overrightarrow{(-5,3)}$.

b) Find a vector equation of the line, through the point $B(2,3,1)$, having $\vec{m} = \overrightarrow{(4,5,6)}$ as direction vector.

Solution For both a) and b) the vector equation of a line is $\vec{r} = \vec{r_0} + k\vec{m}$.

a) Here, $\vec{r_0} = \overrightarrow{OA} = \overrightarrow{(2,-1)}$ and $\vec{m} = \overrightarrow{(-5,3)}$
Thus $\vec{r} = \overrightarrow{(2,-1)} + k\overrightarrow{(-5,3)}$ ①
which is the required vector equation of the line.

b) Here, $\vec{r_0} = \overrightarrow{OB} = \overrightarrow{(2,3,1)}$ and $\vec{m} = \overrightarrow{(4,5,6)}$
Thus $\vec{r} = \overrightarrow{(2,3,1)} + k\overrightarrow{(4,5,6)}$ is a vector equation of the line. ■

Note 1 Any vector parallel to $\overrightarrow{(-5,3)}$ could have been used for $\vec{m}$ in part a). If $\vec{m} = \overrightarrow{(-10,6)}$ the equation is $\vec{r} = \overrightarrow{(2,-1)} + s\overrightarrow{(-10,6)}$. ②

2 Any point on the line besides point A could be used for P_0 and hence for $\overrightarrow{OP_0} = \vec{r_0}$. Therefore, if $k = 1$ in $\vec{r} = \overrightarrow{(2,-1)} + k\overrightarrow{(-5,3)}$, $\vec{r} = \overrightarrow{(2,-1)} + (1)\overrightarrow{(-5,3)} = \overrightarrow{(-3,2)}$. Hence, $\overrightarrow{(-3,2)}$ could have been substituted for $\vec{r_0}$. Thus, another equation for the same line in a) would be $\vec{r} = \overrightarrow{(-3,2)} + t\overrightarrow{(-5,3)}$ ③

3 A comparison of equations ① ② and ③, all of which represent the same line, should help you to realize that it is not always easy to see at a glance whether two equations represent the same line.

Similar remarks could be made about part b) of Example 1.

Example 2 a) Find a vector equation of the line passing through the points $A(3,2)$ and $B(0,-5)$.

b) Find a vector equation of the line passing through the points $C(3,2,-1)$ and $D(0,-5,8)$.

Solution For both a) and b) the vector equation of a line is $\vec{r} = \vec{r_0} + k\vec{m}$.

a) Here, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \overrightarrow{(0,-5)} - \overrightarrow{(3,2)}$

Thus $\overrightarrow{AB} = \overrightarrow{(-3,-7)}$

is a direction vector for the line.

Hence $\vec{m} = \overrightarrow{(-3,-7)}$.

Since both A and B lie on the line, either $\overrightarrow{OA}$ or $\overrightarrow{OB}$ can be used for $\vec{r_0}$.

Using $\vec{r_0} = \overrightarrow{OA} = \overrightarrow{(3,2)}$, a vector equation is

$\vec{r} = \overrightarrow{(3,2)} + k\overrightarrow{(-3,-7)}$.

b) Here, $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$= \overrightarrow{(0,-5,8)} - \overrightarrow{(3,2,-1)}$

Thus $\overrightarrow{CD} = \overrightarrow{(-3,-7,9)}$ is a direction vector for the line.

Hence $\vec{m} = \overrightarrow{(-3,-7,9)}$.

Since both C and D lie on the line, either $\overrightarrow{OC}$ or $\overrightarrow{OD}$ can be use for $\vec{r_0}$.

Using $\vec{r_0} = \overrightarrow{OC} = \overrightarrow{(3,2,-1)}$, a vector equation is

$\vec{r} = \overrightarrow{(3,2,-1)} + k\overrightarrow{(-3,-7,9)}$. ■

Example 3 Find a vector equation of the line passing through the point $E(3,0,2)$ that is perpendicular to vector $\vec{u} = \overrightarrow{(4,-1,2)}$, and also perpendicular to vector $\vec{v} = \overrightarrow{(1,0,-3)}$.

Solution A vector equation of the line is $\vec{r} = \vec{r_0} + k\vec{m}$.

Here, $\vec{r_0} = \overrightarrow{OE} = \overrightarrow{(3,0,2)}$.

Since the line is perpendicular to both $\vec{u}$ and $\vec{v}$, the line is parallel to any vector that is perpendicular to both $\vec{u}$ and $\vec{v}$.

cross product

But $\vec{u} \times \vec{v}$ is a vector that is perpendicular to both $\vec{u}$ and $\vec{v}$.

Hence a direction vector $\vec{m} = \vec{u} \times \vec{v} = \overrightarrow{(4,-1,2)} \times \overrightarrow{(1,0,-3)} = \overrightarrow{(3,14,1)}$.

Substituting in $\vec{r} = \vec{r_0} + k\vec{m}$ gives

$$\vec{r} = \overrightarrow{(3,0,2)} + k\overrightarrow{(3,14,1)}$$

This is a vector equation for the line. ■

5.1 Exercises

1. For each of the following vector equations of lines, state the coordinates of a point on the line and a vector parallel to the line.
   a) $\vec{r} = \overrightarrow{(2,-1)} + k\overrightarrow{(4,2)}$
   b) $\vec{r} = \overrightarrow{(8,-3)} + t\overrightarrow{(5,-4)}$
   c) $\vec{r} = \overrightarrow{(3,-1,4)} + k\overrightarrow{(5,-2,1)}$
   d) $\vec{r} = t\overrightarrow{(1,0,-8)} + \overrightarrow{(-4,7,5)}$

2. For each of the following find a vector equation of the line passing through the point P_0 that has the given vector $\vec{m}$ as direction vector.
   a) $P_0\ (3,7)$ $\quad \vec{m} = \overrightarrow{(1,5)}$
   b) $P_0\ (-2,0)$ $\quad \vec{m} = \overrightarrow{(-9,-2)}$
   c) $P_0\ (6,9)$ $\quad \vec{m} = \overrightarrow{(-2,4)}$
   d) $P_0\ (3,2,7)$ $\quad \vec{m} = \overrightarrow{(1,5,3)}$
   e) $P_0\ (0,-2,0)$ $\quad \vec{m} = \overrightarrow{(-9,-2,5)}$
   f) $P_0\ (2,4,-3)$ $\quad \vec{m} = \overrightarrow{(0,0,6)}$

3. a) Find a vector equation of the line that passes through the points $A(4,-6)$ and $B(2,7)$.
   b) Find a vector equation of the line that passes through the points $A(4,-6,2)$ and $B(-1,2,7)$.

4. a) Find a vector equation of the line that passes through the point $C(-5,2)$ and is parallel to the line through the points $K(1,4)$ and $M(3,7)$.
   b) Find a vector equation of the line that passes through the point $C(3,-5,2)$ and is parallel to the line through the points $K(1,4,-2)$ and $M(3,7,4)$.

5. Find a vector equation of the line that passes
   a) through the point $A(3,-1)$ and is parallel to the line with equation $\vec{r} = \overrightarrow{(0,2)} + k\overrightarrow{(-3,2)}$
   b) through the point $A(3,-1,-5)$ and is parallel to the line with equation $\vec{r} = \overrightarrow{(0,2,0)} + k\overrightarrow{(0,-3,2)}$.

6. Find a vector equation of the line that passes through the point $A(3,-1)$ and is perpendicular to the line with equation $\vec{r} = \overrightarrow{(0,2)} + k\overrightarrow{(-3,2)}$.

7. Find a vector equation of the line passing through the point $A(3,0,2)$ that is perpendicular to vector $\vec{u} = \overrightarrow{(4,-1,2)}$ and is also perpendicular to vector $\vec{v} = \overrightarrow{(1,0,-3)}$.

8. Find a vector equation of the line that passes through the point $D(3,-1,2)$ and is perpendicular to the line with equation $\vec{r} = \overrightarrow{(4,0,2)} + k\overrightarrow{(5,-3,2)}$ and is also perpendicular to the line $\vec{r} = \overrightarrow{(1,1,2)} + s\overrightarrow{(-2,1,3)}$.

9. a) Find the value of t so that the two lines $\vec{r} = \overrightarrow{(1,2)} + k\overrightarrow{(3,-1)}$ and $\vec{r} = \overrightarrow{(4,1)} + s\overrightarrow{(4,t)}$ will be perpendicular.
   b) Find the value of t so that the two lines $\vec{r} = \overrightarrow{(1,8,2)} + k\overrightarrow{(-4,3,-1)}$ and $\vec{r} = \overrightarrow{(4,1,2)} + s\overrightarrow{(4,t,-3)}$ will be perpendicular.

10. Given the points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ with position vector $\vec{r}_1$ and position vector $\vec{r}_2$ respectively.
    a) Show that $\vec{r} = (1-k)\,\vec{r}_1 + k\vec{r}_2$ is a vector equation of the line through P_1 and P_2.
    b) Describe the position of points on the line in part a) for $0 < k < 1$.
    c) Repeat part b) for $k > 1$ and for $k < 0$.

11. a) Find a vector equation of the line, passing through the point $A(3,8)$, that is parallel to the x-axis.
    b) Find a vector equation of the line, passing through the point $A(3,8,1)$, that is parallel to the y-axis.
    c) Find a vector equation of the line, passing through the point $A(3,8,1)$, that is parallel to the z-axis.

5.2 Parametric Equations of a Line

Suppose $P(x,y)$ is any point on the line in 2-space, through the point $P_0(x_0,y_0)$, that is parallel to the vector $\vec{m} = \overrightarrow{(m_1,m_2)}$.

A vector equation for this line is

$$\vec{r} = \vec{r_0} + k\vec{m} \quad (1)$$

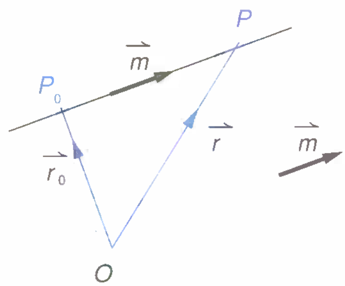
But $\vec{r} = \overrightarrow{OP} = \overrightarrow{(x,y)}$

$\vec{r_0} = \overrightarrow{OP_0} = \overrightarrow{(x_0,y_0)}$

Substituting in (1) gives

$\overrightarrow{(x,y)} = \overrightarrow{(x_0,y_0)} + k\overrightarrow{(m_1,m_2)}$ or,

$\overrightarrow{(x,y)} = \overrightarrow{(x_0 + km_1, y_0 + km_2)}$



By the definition of equal vectors, corresponding components must be equal.

Thus $\begin{cases} x = x_0 + km_1 \\ y = y_0 + km_2 \end{cases}$

These equations for the components are called **parametric equations** of a line, through the point $P_0(x_0,y_0)$, that is parallel to the vector $\vec{m} = \overrightarrow{(m_1,m_2)}$.

The vector $\vec{m}$ determines the direction of the line. Hence m is called a **direction vector** of the line. The components m_1 and m_2, written with the x components first, are called **direction numbers** of the line.

In a similar manner parametric equations can be found for a line in 3-space.

If P and P_0 have 3-space coordinates (x,y,z) and (x_0,y_0,z_0) respectively, and $\vec{m} = \overrightarrow{(m_1,m_2,m_3)}$ then $\overrightarrow{OP} = \overrightarrow{(x,y,z)}$ and $\overrightarrow{OP_0} = \overrightarrow{(x_0,y_0,z_0)}$.

Therefore, equation (1) becomes

$\overrightarrow{(x,y,z)} = \overrightarrow{(x_0,y_0,z_0)} + k\overrightarrow{(m_1,m_2,m_3)}$ or

$\overrightarrow{(x,y,z)} = \overrightarrow{(x_0 + km_1, y_0 + km_2, z_0 + km_3)}$

Thus, parametric equations of the line are

$$\begin{cases} x = x_0 + km_1 \\ y = y_0 + km_2 \\ z = z_0 + km_3 \end{cases}$$

Example 1 a) Find parametric equations of the line, through the point $P_0(2,3)$, having $\vec{m} = \overrightarrow{(4,5)}$ as direction vector.

b) Find parametric equations of the line, through the point $P_0(2,3,1)$, having $\vec{m} = \overrightarrow{(4,5,6)}$ as direction vector.

Solution A vector equation of a line is

a) $$\vec{r} = \vec{r}_0 + k\vec{m}.$$

Here, $\vec{r} = \overrightarrow{(x,y)}$, $\vec{r}_0 = \overrightarrow{(2,3)}$ and $\vec{m} = \overrightarrow{(4,5)}$

Thus $\overrightarrow{(x,y)} = \overrightarrow{(2,3)} + k\overrightarrow{(4,5)}$

$\overrightarrow{(x,y)} = \overrightarrow{(2 + 4k, 3 + 5k)}$ is a vector equation of the line.

Thus $\begin{cases} x = 2 + 4k \\ y = 3 + 5k \end{cases}$

are the required parametric equations.

b) A vector equation of a line is

$$\vec{r} = \vec{r}_0 + k\vec{m}.$$

Here, $\vec{r} = \overrightarrow{(x,y,z)}$, $\vec{r}_0 = \overrightarrow{(2,3,1)}$ and $\vec{m} = \overrightarrow{(4,5,6)}$

Thus $\overrightarrow{(x,y,z)} = \overrightarrow{(2,3,1)} + k\overrightarrow{(4,5,6)}$

$\overrightarrow{(x,y,z)} = \overrightarrow{(2 + 4k, 3 + 5k, 1 + 6k)}$ is a vector equation of the line.

Thus $\begin{cases} x = 2 + 4k \\ y = 3 + 5k \\ z = 1 + 6k \end{cases}$

are the required parametric equations. ■

Note 1 Both part a) and part b) could be done by substituting directly into the formula for the parametric equations of a line. But Example 1 shows that the parametric equation formulas need not be memorized.

2 The multipliers of the parameters, namely the numbers 4 and 5 (in part a) and 4, 5, and 6 (in part b) are the same numbers as the *direction numbers* of the line. *Direction numbers should always be given in the order*
x then y in 2-space and
x then y then z in 3-space.

3 If the vector equation of a line is written in column form as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + k\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

then the parametric equations are more easily recognised.

The following Examples 2 and 3 will be done for lines in 2-space. Similar solutions can be used for lines in 3-space.

Example 2

Given the line L with parametric equations $\begin{cases} x = 2 - 5k \\ y = -1 + 3k \end{cases}$

a) Determine the coordinates of three points on line L.
b) Find the value of k that corresponds to the point $(12,-7)$ that lies on line L.
c) Show that the point $(-3,4)$ does not lie on line L.

Solution

a) Each value of the parameter k gives the position vector of a point on line L. Let $k = 1$, then $x = 2 - 5(1) = -3$ and $y = -1 + 3(1) = 2$.
Hence $(-3,2)$ is a point on line L.
Similarily using, say, the values $k = 2$ and $k = 3$ you will obtain the points $(-8,5)$ and $(-13,8)$ on the line L.

b) Since the point $(12,-7)$ lies on line L, $x = 12$ and $y = -7$ may be substituted in the parametric equations for L.
Thus $12 = 2 - 5k$ and $-7 = -1 + 3k$. Each equation solves to give $k = -2$.

c) If the point $(-3,4)$ lies on line L, $x = -3$ and $y = 4$ may be substituted in the parametric equations for L.

Thus $-3 = 2 - 5k$ and $4 = -1 + 3k$
or $-5 = -5k$ and $5 = 3k$
or $k = 1$ and $k = \frac{5}{3}$, which is impossible.

Hence the point $(-3,4)$ does not lie on line L. ■

Example 3

Explain why the following parametric equations represent the same line.

① $\begin{cases} x = 1 + 3k \\ y = 5 + 2k \end{cases}$ ② $\begin{cases} x = 1 - 6t \\ y = 5 - 4t \end{cases}$

Solution

From ①, a point on the line is $(1,5)$ and a direction vector is $\vec{u} = \overrightarrow{(3,2)}$.
From ②, a point on the line is $(1,5)$ and a direction vector is $\vec{v} = \overrightarrow{(6,4)}$.
Since $\vec{v} = \overrightarrow{(6,4)} = 2\overrightarrow{(3,2)} = 2\vec{u}$, the vectors $\vec{u}$ and $\vec{v}$ are parallel.
Because the two lines have a common point $(1,5)$ and have direction vectors that are parallel, therefore the two lines are the same line. ■

SUMMARY

$\vec{r} = \overrightarrow{OP}$ the position vector of *any* point P on the line,
$\vec{r}_0 = \overrightarrow{OP_0}$ the position vector of a *given* point P_0 on the line,
$\vec{m}$ is a vector parallel to the line,
k is any real number called a *parameter*.

	2-Space	*3-Space*
vector equation	$\vec{r} = \vec{r}_0 + k\vec{m}$	
parametric equations	$\begin{cases} x = x_0 + km_1 \\ y = y_0 + km_2 \end{cases}$	$\begin{cases} x = x_0 + km_1 \\ y = y_0 + km_2 \\ z = z_0 + km_3 \end{cases}$

5.2 Exercises

1. For each of the following parametric equations of lines, state the coordinates of a point on the line and a vector parallel to the line. Indicate whether the line is in 2-space or in 3-space.
   a) $\begin{cases} x = 5 + 2k \\ y = 2 + 4k \end{cases}$
   b) $\begin{cases} x = -3 + 8s \\ y = 1 - 5s \end{cases}$
   c) $\begin{cases} x = -2 + a \\ y = 3a \end{cases}$
   d) $\begin{cases} x = 6k - 4 \\ y = 1 \end{cases}$
   e) $\begin{cases} x = 5 + 2k \\ y = 2 + 4k \\ z = 2 - 5k \end{cases}$
   f) $\begin{cases} x = -3 + 8s \\ y = 1 - 5s \\ z = -2 + 5s \end{cases}$
   g) $\begin{cases} x = -2 + a \\ y = 3a \\ z = 2a + 4 \end{cases}$
   h) $\begin{cases} x = 6k - 4 \\ y = 1 \\ z = -7k \end{cases}$

2. State direction numbers for each of the lines in question 1.

3. a) Write the coordinates of three points on the line whose parametric equations are
   $\begin{cases} x = -6 + 4k \\ y = 3 - 2k \end{cases}$
   b) Write three other sets of parametric equations representing the line in part a).

4. a) Write the coordinates of three points on the line whose parametric equations are
   $\begin{cases} x = -6 + 4k \\ y = 3 - 2k \\ z = 5 - k \end{cases}$
   b) Write three other sets of parametric equations representing the line in part a).

5. Find a vector equation and parametric equations for each of the following lines.
   a) through the point $A(-3,4)$ with direction vector $\overrightarrow{(5,1)}$
   b) through the points $A(-3,4)$ and $B(7,2)$
   c) through the point $A(-3,4)$ with direction numbers 6 and −2
   d) through the point $B(7,2)$ parallel to the line $\vec{r} = \overrightarrow{(4,7)} + k\overrightarrow{(4,-5)}$
   e) through the point $C(8,-3)$ and parallel to the x-axis
   f) through the point $C(8,-3)$ and parallel to the y-axis

6. Find a vector equations and parametric equations for each of the following lines.
   a) through the point $A(5,-3,4)$ with direction vector $\overrightarrow{(-6,2,1)}$
   b) through the points $A(5,-3,4)$ and $B(7,2,-1)$
   c) through the point $A(5,-3,4)$ with direction vector $\overrightarrow{(6,7,-2)}$
   d) through the point $B(7,2,-1)$ parallel to the line $\vec{r} = \overrightarrow{(4,7,0)} + k\overrightarrow{(4,-5,1)}$
   e) through the point $C(8,-3,4)$ and parallel to the x-axis
   f) through the point $C(8,-3,4)$ and parallel to the y-axis
   g) through the point $C(8,-3,4)$ and parallel to the z-axis

7. Given the line with parametric equations
   $\begin{cases} x = 5 - 3t \\ y = 2 + 4t \end{cases}$
   determine whether or not the following points lie on this line.
   $A(2,6)$ $B(-1,3)$ $C(1.5,4)$ $D(8,-2)$ $E(-3,4)$

8. Given the line with parametric equations
   $\begin{cases} x = 5 - 3t \\ y = 2 + 4t \\ z = -1 + 2t \end{cases}$
   determine whether or not the following points lie on this line.
   $A(2,6,1)$ $B(-1,3,3)$ $C(3.5,4,0)$ $D(8,-2,-3)$ $E(-3,4,0)$

9. In each of the following, determine whether or not the given point lies on the line with the given equation.

point	line equation
a) $(-2,-1,-6)$	$\vec{r} = \overrightarrow{(2,1,0)} + k\overrightarrow{(4,2,3)}$
b) $(10,17,2)$	$\vec{r} = \overrightarrow{(1 + 3k,-4 + 7k,5 - k)}$
c) $(2,12,1)$	$\vec{r} = \overrightarrow{(2,3k,5)}$

10. Explain why each of the following equations represents the same line as the equation $\vec{r} = \overrightarrow{(1,2)} + k\overrightarrow{(4,6)}$

a) $\vec{r} = \overrightarrow{(1,2)} + k\overrightarrow{(8,12)}$

b) $\vec{r} = \overrightarrow{(1,2)} + w\overrightarrow{(-2,-3)}$

c) $\vec{r} = \overrightarrow{(5,8)} + t\overrightarrow{(4,6)}$

d) $\vec{r} = \overrightarrow{(-1,-1)} + s\overrightarrow{(2,3)}$

11. Explain why each of the following equations represents the same line as the equation $\vec{r} = \overrightarrow{(1,2,3)} + k\overrightarrow{(4,6,-2)}$

a) $\vec{r} = \overrightarrow{(1,2,3)} + k\overrightarrow{(8,12,-4)}$

b) $\vec{r} = \overrightarrow{(1,2,3)} + w\overrightarrow{(-2,-3,1)}$

c) $\vec{r} = \overrightarrow{(5,8,1)} + t\overrightarrow{(4,6,-2)}$

d) $\vec{r} = \overrightarrow{(-1,-1,4)} + s\overrightarrow{(2,3,-1)}$

12. Show that the equations $\vec{r} = \vec{r_0} + k\overrightarrow{(a,b)}$ and $\vec{r} = \vec{r_0} + t\overrightarrow{(sa,sb)}$ represent the same line, where a, b, k, t, and $s \in \mathbb{R}$.

13. Given the line with Cartesian equation $3x + 2y = 5$,

a) find parametric equations for the line using the parameter k by letting $x = k$ and solving for y in terms of k

b) find parametric equations for the line using the parameter t by letting $x = 3 + 2t$ and solving for y in terms of t.

14. a) Find parametric equations of the line, passing through the point $D(-3,2)$, that is parallel to the vector $\vec{w} = \overrightarrow{(-4,7)}$.

b) For each parametric equation found in part a), express the parameter in terms of x or y.

c) Eliminate the parameter from the equations found in b), hence obtain a Cartesian equation for the line.

15. a) Find parametric equations of the line, passing through the point $D(-3,2,1)$, that is parallel to the vector $\vec{u} = \overrightarrow{(2,-4,7)}$.

b) For each parametric equation found in part a), express the parameter in terms of x, y, or z.

c) Solve each equation found in b) for the parameter. Equate the three values of the parameter to obtain Cartesian equations for the line.

16. Given the points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ with position vector $\vec{r_1}$ and position vector $\vec{r_2}$ respectively. In question 10 of 5.1 Exercises you showed that a vector equation of this line is $\vec{r} = (1 - k)\,\vec{r_1} + k\vec{r_1}$.

Let $\overrightarrow{OQ} = \vec{q}$ be the position vector of the point that divides segment P_1P_2 internally in the ratio $a:b$.

a) Show that the value of k corresponding to point Q is $\dfrac{a}{a+b}$

b) Show that $\vec{q} = \dfrac{b}{a+b}\vec{r_1} + \dfrac{a}{a+b}\vec{r_2}$

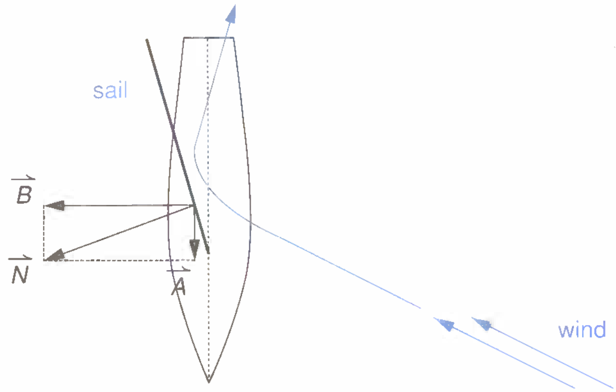
c) Use the results of part b) to find the coordinates of the point dividing the segment $P_1(3,1)$ to $P_2(5,4)$ internally in the ratio 7:2.

17. Show that the lines
$L_1: \vec{r} = \overrightarrow{(2,0,0)} + k\overrightarrow{(0,1,3)}$ and
$L_2: \vec{r} = \overrightarrow{(1,2,3)} + t\overrightarrow{(2,1,0)}$ never intersect.

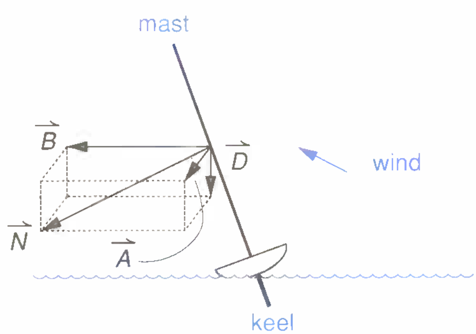
M A K I N G

Sailboards and Sailboats

A sailboat can use the power of the wind to travel 'upwind', as shown in the top-view diagram below. If the sail is properly set, the wind exerts a horizontal force $\vec{N}$ normal to the sail as shown. This force can be resolved into two rectangular components, $\vec{A}$ parallel to the keel (1) of the craft, and $\vec{B}$ perpendicular to it. The force $\vec{A}$ drives the boat forward, while the force $\vec{B}$ is counteracted by the keel.

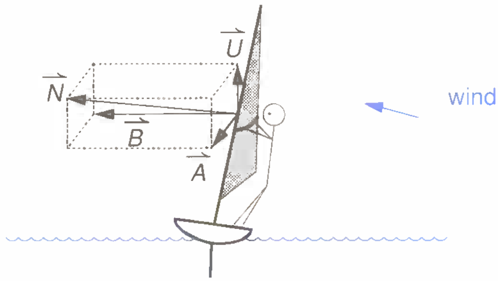


This model works well when the wind is light. However, the stronger the wind, the more another factor must be considered. The wind also causes the boat to lean sideways. Thus, the normal force $\vec{N}$ described above is not actually horizontal, but points partly downward as shown in this front-view diagram. $\vec{N}$ can thus be resolved in three directions, $\vec{A}$ driving the boat forward, $\vec{B}$ opposed by the keel, and $\vec{D}$ pushing the boat downward into the water, thus increasing its weight, and thereby its resistance to forward motion.



A sailboard is a device that has a free sail system. This means that the mast is attached to the board by a universal joint (rather than being fixed rigidly to it). Thus the mast, and the sail, are free to move in any direction.

A sailboard or sailboat acts in the same way in light winds, when the mast is approximately vertical. However, when the wind is strong, riders of sailboards must keep their balance by pulling the mast *toward* them. Thus, the normal force $\vec{N}$ points partly *upward*.



$\vec{N}$ can be resolved into the following forces. $\vec{A}$ drives the board forward. $\vec{B}$ is opposed by the daggerboard (2) and $\vec{U}$ *decreases* the weight of the craft in the water, and thereby its resistance to forward motion.

In fact, $\vec{U}$ can be strong enough to lift the entire sailboard off the surface of the water! When airborne, however, the rider has the problem of keeping both board and sail balanced so as to avoid a crash landing. Good luck if you try it!

(1) The keel is a plate, often weighted, under the boat and along its length, that helps the boat resist being pushed sideways by the wind.
(2) The daggerboard is a removable plate that when positioned beneath the sailboard helps a sailboard resist being pushed sideways by the wind.

5.3 Cartesian Equations of Lines

You are familiar with the Cartesian equation of a line in 2-space. A single equation such as $y = mx + b$ or $Ax + By + C = 0$ is sufficient to determine a line in 2-space. You should wonder what form the Cartesian equation of a line in 3-space will take. In this section you will learn how to find Cartesian equations of lines.

Example 1 Find a Cartesian equation of the line having the vector equation $\vec{r} = \overrightarrow{(2 + 3k, 4 + 5k)}$.

Solution Since a vector equation of the line is $\vec{r} = \overrightarrow{(2 + 3k, 4 + 5k)}$, parametric equations are

$$\begin{cases} x = 2 + 3k \\ y = 4 + 5k \end{cases}$$

Solving each equation for k gives

$$k = \frac{x - 2}{3} \text{ and } k = \frac{y - 4}{5}$$

Equating the two values for k gives the Cartesian equation

$$\frac{x - 2}{3} = \frac{y - 4}{5},$$ where (2,4) is a point on the line and 3,5 are direction numbers of the line. ■

This form of a Cartesian equation is called a **symmetric equation** of the line in 2-space. The word "symmetric" is used because x and y appear symmetrically in the equation.

The equation $\frac{x - 2}{3} = \frac{y - 4}{5}$ can be written $5(x - 2) = 3(y - 4)$ or $5x - 3y + 2 = 0$, the usual form for the Cartesian equation of a line in 2-space.

Example 2 Find Cartesian equations in symmetric form of the line with vector equation $\vec{r} = \overrightarrow{(2,3,1)} + k\overrightarrow{(4,5,6)}$.

Solution The equation $\vec{r} = \overrightarrow{(2,3,1)} + k\overrightarrow{(4,5,6)}$ may be written $\overrightarrow{(x,y,z)} = \overrightarrow{(2 + 4k, 3 + 5k, 1 + 6k)}$, giving the parametric equations

$$\begin{cases} x = 2 + 4k \\ y = 3 + 5k \\ z = 1 + 6k \end{cases}$$

Solving the equations for k gives

$$x - 2 = 4k,\ y - 3 = 5k,\ z - 1 = 6k \text{ or}$$

$$\frac{x - 2}{4} = k,\ \frac{y - 3}{5} = k,\ \frac{z - 1}{6} = k$$

Hence, Cartesian equations of the line in symmetric form are

$$\frac{x - 2}{4} = \frac{y - 3}{5} = \frac{z - 1}{6} \ (=k),$$ where (2,3,1) is a point on the line and 4,5,6 are direction numbers of the line. ■

Note 1 In 2-space, a single symmetric equation is sufficient to determine a line.
2 In 3-space, two symmetric equations are needed to determine a line.
3 Cartesian equations of lines are also called **scalar equations of a line.**
4 If one of the direction numbers equals 0 the symmetric equations take on a different form. For example, if the direction numbers are 3,2,0 and the line passes through the point (4,5,6), the symmetric equations are written

$$\frac{x-4}{3} = \frac{y-5}{2},\ z-6=0 \text{ rather than } \frac{x-4}{3} = \frac{y-5}{2} = \frac{z-6}{0}$$

S U M M A R Y

2-space
For a line in 2-space, through the point $P_0(x_0,y_0)$, having direction numbers m_1, and m_2,

$$\frac{x-x_0}{m_1} = \frac{y-y_0}{m_2} \quad m_1, m_2 \neq 0$$

3-space
For a line in 3-space, through the point $P_0(x_0,y_0,z_0)$, having direction numbers m_1, m_2, and m_3,

$$\frac{x-x_0}{m_1} = \frac{y-y_0}{m_2} = \frac{z-z_0}{m_3} \quad m_1, m_2, m_3 \neq 0$$

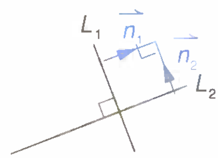
The remainder of this section deals with lines in 2-space and their Cartesian equations. Indeed, the familiar equation $Ax + By + C = 0$ for a 2-space line can be derived by using the fact that two non-zero vectors $\vec{u}$ and $\vec{v}$ are perpendicular if and only if $\vec{u}\cdot\vec{v} = 0$.

Recall that a vector $\vec{n}$ perpendicular to a vector $\vec{a}$ is called a vector *normal to* $\vec{a}$ or a *normal vector* for vector $\vec{a}$.

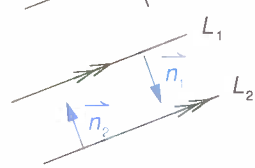
P R O P E R T Y

The following are relationships among two lines in 2-space and the normal vectors to these lines.

1. Two lines are perpendicular if and only if their normal vectors are perpendicular.



2. Two lines are parallel if and only if their normal vectors are parallel.



Example 3 Find a Cartesian equation of the line that passes through the point $P_0(1,2)$ with $\vec{n} = \overrightarrow{(3,4)}$ as a normal vector.

Solution Let $P(x,y)$ be any point on the line.
Since $\vec{n} = \overrightarrow{(3,4)}$ is a normal vector, then $\vec{n}$ is perpendicular to the line.
Thus, $\overrightarrow{P_0P} \perp \vec{n}$. Hence, $\overrightarrow{P_0P}\cdot\vec{n} = 0$.
But $\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0} = \overrightarrow{(x,y)} - \overrightarrow{(1,2)} = \overrightarrow{(x-1,y-2)}$ and $\vec{n} = \overrightarrow{(3,4)}$
Thus, $\overrightarrow{(x-1,y-2)}\cdot\overrightarrow{(3,4)} = 0$
or $(x-1)(3) + (y-2)(4) = 0$, that is, $3x + 4y - 11 = 0$.

Thus, a Cartesian equation of the line is $3x + 4y - 11 = 0$. ■

Notice that the coefficients of x and y in the Cartesian equations are the direction numbers of the normal vector $\vec{n} = \overrightarrow{(3,4)}$. The following shows this to be true for any Cartesian equation of a line written in the form $Ax + By + C = 0$

THEOREM

The line that passes through the point $P_0(x_0,y_0)$, with $\vec{n} = \overrightarrow{(A,B)}$ as normal vector, has Cartesian equation $Ax + By + C = 0$.

Proof: Let $P(x,y)$ be any point on this line.

Since $\vec{n} = \overrightarrow{(A,B)}$ is a normal vector, then $\vec{n}$ is perpendicular to the line.

Thus, $\overrightarrow{P_0P} \perp \vec{n}$

Hence, $\overrightarrow{P_0P} \cdot \vec{n} = 0$.

But $\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0} = \overrightarrow{(x,y)} - \overrightarrow{(x_0,y_0)} = \overrightarrow{(x - x_0, y - y_0)}$

and $\vec{n} = \overrightarrow{(A,B)}$

Thus, $\overrightarrow{(x - x_0, y - y_0)} \cdot \overrightarrow{(A,B)} = 0$

or $(x - x_0)A + (y - y_0)B = 0$

that is $Ax + By + (-Ax_0 - By_0) = 0$.

If the number $-Ax_0 - By_0$ is replaced by the constant C, this equation becomes the Cartesian equation $Ax + By + C = 0$.

PROPERTY

$\vec{n} = \overrightarrow{(A,B)}$ is a normal to the line with Cartesian equation $Ax + By + C = 0$.

Observe that: the slope of $Ax + By + C = 0$ is $-\frac{A}{B}$

the slope of a normal to $Ax + By + C = 0$ is $\frac{B}{A}$

Example 4

Find a Cartesian equation of the line, passing through the point $D(-6,2)$, that is perpendicular to the vector $\vec{u} = \overrightarrow{(5,-4)}$.

Solution

Let the equation be $Ax + By + C = 0$. ①

Since $\vec{u} = \overrightarrow{(5,-4)}$ is perpendicular to the line, a normal vector $\overrightarrow{(A,B)} = \overrightarrow{(5,-4)}$

Substituting for A and B in ① gives

$$5x + (-4)y + C = 0.$$

But the point $D(-6,2)$ lies on the line,

Thus $5(-6) + (-4)(2) + C = 0$ or, $C = 38$.

Thus a Cartesian equation of the line is $5x - 4y + 38 = 0$. ■

Note: This equation could have been obtained from first principles by using the fact that $\overrightarrow{DP} \cdot \overrightarrow{(5,-4)} = 0$, where $\overrightarrow{DP} = \overrightarrow{(x + 6, y - 2)}$.

5.3 Exercises

1. For each of the following equations of a line, state the coordinates of a point on the line and direction numbers for the line.

   a) $\frac{x-3}{5} = \frac{y-2}{6}$

   b) $\frac{x+1}{2} = \frac{y-4}{-7}$

   c) $\frac{x-2}{-8} = \frac{y}{3}$

   d) $\frac{2x-6}{5} = \frac{3-y}{4}$

   e) $\frac{x-2}{3} = \frac{y-4}{8} = \frac{z-1}{5}$

   f) $\frac{x+3}{-3} = \frac{2y-4}{7} = \frac{1-z}{9}$

   g) $\frac{x-2}{3} = \frac{y-4}{5}, z-7=0.$

   h) $\frac{x-1}{-3} = \frac{z+2}{4}, y=2$

2. Find a vector equation for the lines in question 1, parts a) to h).

3. Find, where possible, symmetric equations for each of the lines having the given vector equations.

   a) $\vec{r} = \overrightarrow{(3-2k,4+5k)}$

   b) $\vec{r} = \overrightarrow{(6+4t,-3-t)}$

   c) $\vec{r} = \overrightarrow{(2,1,0)} + k\overrightarrow{(4,2,3)}$

   d) $\vec{r} = \overrightarrow{(-3,4,2)} + k\overrightarrow{(-1,5,0)}$

   e) $\vec{r} = \overrightarrow{(2,3k,5)}$

   f) $\vec{r} = \overrightarrow{(-4k,3k-1)}$

4. Find, where possible, symmetric equations for each of the following lines.

   a) through the point $A(5,-3,4)$, with direction numbers $-6,2,1$

   b) through the point $E(4,-1,0)$, with direction vector $\overrightarrow{(1,0,-3)}$

   c) through the points $A(5,-3,4)$ and $B(7,2,-1)$

   d) through the point $B(7,2,-1)$, parallel to the line $\vec{r} = \overrightarrow{(4,7,0)} + k\overrightarrow{(4,-5,1)}$

   e) through the point $C(8,-3,4)$ and parallel to the x-axis

   f) through the point $C(8,-3,4)$ and parallel to the y-axis

   g) through the point $C(8,-3,4)$ and parallel to the z–axis

5. Find a vector equation for the line with symmetric equation $\frac{x-1}{4} = \frac{y+3}{2} = \frac{z-5}{-3}$

6. Find the value of the variable t so that the line with symmetric equations $\frac{x-1}{6} = \frac{y+2}{5} = \frac{z-3}{-2}$ is perpendicular to the line with vector equation $\vec{r} = \overrightarrow{(0,3,1)} + k\overrightarrow{(2,t,11)}$.

7. Find the value of the variable s so that the line with symmetric equations $\frac{x+3}{2} = y-5 = \frac{z-5}{-8}$ is parallel to the line with vector equation $\vec{r} = \overrightarrow{(9,2,6)} + k\overrightarrow{(-1,s,4)}$.

8. Show that the following lines are the same line.

   L_1: $\frac{x-1}{5} = \frac{y+2}{3} = \frac{z-5}{-1}$ and

   L_2: $\frac{x-11}{10} = \frac{y-4}{6} = \frac{z-3}{-2}$

9. Find the value of the variable t so that the following lines are perpendicular.

$$L_1: \frac{x-4}{2t-1} = \frac{y+1}{3} = \frac{z+2}{4t-1} \text{ and}$$

$$L_2: \frac{x-5}{8} = \frac{y-1}{-2t} = \frac{z+3}{-2}.$$

10. State direction numbers for a normal to each of the following lines.
   a) $3x + 4y + 6 = 0$
   b) $5x - 2y = 3$
   c) $x - 5y = 1$
   d) $2x + 3 = 0$
   e) $y = 5$

11. For each of the following find a scalar equation of the line passing through the given point P_0 and having $\vec{n}$ as a normal vector.
   a) $P_0(3,-1)$ $\quad \vec{n} = \overrightarrow{(-4,1)}$
   b) $P_0(0,-5)$ $\quad \vec{n} = \overrightarrow{(6,4)}$
   c) $P_0(-4,7)$ $\quad \vec{n} = \overrightarrow{(2,-3)}$
   d) $P_0(-2,-5)$ $\quad \vec{n} = \overrightarrow{(-2,0)}$

12. Use normal vectors to decide which pairs of lines are parallel and which pairs of lines are perpendicular.
   a) $3x + 2y = 5$
      $6x + 4y = 1$
   b) $4x - 5y = 7$
      $5x + 4y = 1$
   c) $5x - 3y + 1 = 0$
      $3x + 5y + 2 = 0$
   d) $x - 13y = 18$
      $3x - 39y = 0$
   e) $2x + 1 = 0$
      $x = 10$
   f) $3y + 5 = 0$
      $2x = 3$

13. a) Show that a line with equation $Ax + By + C = 0$ has $\overrightarrow{(B,-A)}$ as a direction vector.
   b) Find a vector equation of the line in 2-space, passing through the point $P_0(3,5)$, that is parallel to the line $2x + 5y + 9 = 0$.

14. Given the line L with equation $4x - 7y + 9 = 0$ and the point $P_0(1,-3)$.
   a) Find a vector equation of the line through P_0 parallel to the line L.
   b) Find a vector equation of the line through P_0 perpendicular to the line L.

15. Find the value of the variable t so that the line with scalar equation $4x + 7y = 3$ is perpendicular to the line with vector equation $\vec{r} = \overrightarrow{(0,1)} + k\overrightarrow{(4,t)}$

16. Find the value of the variable s so that the line with scalar equation $5x + 2y = 8$ is parallel to the line with vector equation $\vec{r} = \overrightarrow{(2,1)} + k\overrightarrow{(3,s)}$

17. Write the symmetric equation $\frac{x-3}{5} = \frac{y-2}{6}$ in the scalar form $Ax + By + C = 0$.

18. Write the Cartesian equation $4x + 7y - 11 = 0$ in symmetric form.

19. a) Show that the vector $\vec{n} = \overrightarrow{(3,-1,2)}$ is a normal of the line $\frac{x-a}{2} = \frac{y-b}{6}, z - c = 0.$
   b) Are all normals to this line parallel to $\vec{n}$?

5.4 Direction Numbers and Direction Cosines of a Line

The direction of a line in 3-space can be given by many direction numbers or vectors. For the line with vector equation $\vec{r} = \overrightarrow{(1 + 2k, 4 - 3k, 5 + 7k)}$, the numbers 2, −3, and 7 are *direction numbers* and $\vec{a} = \overrightarrow{(2,-3,7)}$ is a *direction vector*. Any scalar multiple of $\vec{a}$ is parallel to $\vec{a}$ and thus is also a direction vector; for example, $\overrightarrow{(4,-6,14)}$, $\overrightarrow{(-6,9,-21)}$ and $\overrightarrow{(2t,-3t,7t)}$, $t \in \mathbb{R}$, are direction vectors of the line. Thus the ordered sets of numbers 4,−6,14, and −6,9,−21, and $2t,-3t,7t$ are also direction numbers of the line.

None of the these direction numbers in 3-space specify direction in the same way as *slope* does in 2-space. You need to compare the direction of the line with the directions of the *three coordinate axes*. This is done using a special set of direction numbers called **direction cosines**. These are associated with the cosines of the three angles a direction vector of a line makes with the x-axis, the y-axis, and the z-axis.

Example 1 A line has a direction vector $\vec{a} = \overrightarrow{(2,-3,7)}$. Calculate, to the nearest degree, the angle this line makes with the x-axis.

Solution Let the required angle be α. Then $\vec{a} \cdot \vec{i} = |\vec{a}||\vec{i}| \cos \alpha$

$$\Rightarrow \cos \alpha = \frac{\vec{a} \cdot \vec{i}}{|\vec{a}||\vec{i}|}$$

$$= \frac{\overrightarrow{(2,-3,7)} \cdot \overrightarrow{(1,0,0)}}{\sqrt{2^2 + (-3)^2 + 7^2}\sqrt{1^2 + 0^2 + 0^2}} = \frac{2}{\sqrt{62}\sqrt{1}} = 0.2540002\ldots$$

$$\Rightarrow \quad \alpha = 75.2856\ldots^\circ$$

The angle between the line and the x-axis is 75°. ■

Example 1 can be generalized to find the angle a line makes with the three coordinate axes.

Let α, β and γ be the angles a direction vector $\vec{m} = \overrightarrow{(m_1,m_2,m_3)}$ makes with the x-axis, y-axis, and z-axis respectively.

Since the basis vector $\vec{i}$ lies along the x-axis, the angle between $\vec{m}$ and $\vec{i}$ must be α.

$$\text{Thus } \cos \alpha = \frac{\vec{m} \cdot \vec{i}}{|\vec{m}||\vec{i}|}, = \frac{\overrightarrow{(m_1,m_2,m_3)} \cdot \overrightarrow{(1,0,0)}}{|\vec{m}|} = \frac{m_1}{|\vec{m}|}$$

In a similar manner you can prove that $\cos \beta = \frac{m_2}{|\vec{m}|}$ and $\cos \gamma = \frac{m_3}{|\vec{m}|}$

PROPERTY The direction cosines of a line with direction vector $\vec{m} = \overrightarrow{(m_1,m_2,m_3)}$ are

$$\frac{m_1}{|\vec{m}|}, \frac{m_2}{|\vec{m}|}, \frac{m_3}{|\vec{m}|}$$

Similarily you can show that in 2-space the direction cosines of a line with direction vector $\overrightarrow{(m_1,m_2)}$ are $\frac{m_1}{|\vec{m}|}$, $\frac{m_2}{|\vec{m}|}$

Example 2 Given the line with vector equation $\vec{r} = \overrightarrow{(1,2,3)} + k\overrightarrow{(3,-4,5)}$

a) find the direction cosines of the line, correct to 4 decimal places, and the angles α, β, and γ the line makes with the coordinate axes, correct to the nearest degree

b) calculate $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma$.

Solution a) Direction numbers of the given lines are 3,−4,5.

Hence $m_1 = 3$, $m_2 = -4$, $m_3 = 5$.

Thus, $|m| = \sqrt{3^2 + (-4)^2 + 5^2} = \sqrt{50}$

Therefore, the direction cosines are

$$\cos \alpha = \frac{3}{\sqrt{50}} \quad \cos \beta = -\frac{4}{\sqrt{50}} \quad \cos \gamma = \frac{5}{\sqrt{50}}$$

or, $\cos \alpha \doteq 0.424\,264$ $\quad \cos \beta \doteq -0.565\,854$ $\quad \cos \gamma \doteq 0.707\,106$

Thus, to the nearest degree, $\alpha = 65°$, $\beta = 124°$ and $\gamma = 45°$.
Thus the direction cosines are 0.4243, −0.5659, and 0.7071.
The angles made with the coordinates axes are 65°, 124° and 45°.

Note: Since a line has two 'directions', $\vec{m}$ could have been replaced by $-\vec{m} = \overrightarrow{(-3,4,-5)}$. Thus $\cos \alpha$, $\cos \beta$ and $\cos \gamma$ would have been the negatives of the above. α, β, γ would then have been $180° - 65° = 115°$, $180° - 124° = 56°$ and $180° - 45° = 135°$.

b) $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma$

$$= \left(\frac{3}{\sqrt{50}}\right)^2 + \left(-\frac{4}{\sqrt{50}}\right)^2 + \left(\frac{5}{\sqrt{50}}\right)^2 = \frac{9}{50} + \frac{16}{50} + \frac{25}{50} = \frac{50}{50} \text{ or } 1.$$ ■

You will now see by calculating the length of the vector $\vec{u} = \overrightarrow{(\cos \alpha, \cos \beta, \cos \gamma)}$ that the result of Example 2 part b) is always true.

Now $|\vec{u}|^2 = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma$

$$= \left(\frac{m_1}{|\vec{m}|}\right)^2 + \left(\frac{m_2}{|\vec{m}|}\right)^2 + \left(\frac{m_3}{|\vec{m}|}\right)^2 = \frac{(m_1)^2 + (m_2)^2 + (m_3)^2}{|\vec{m}|^2} = \frac{|\vec{m}|^2}{|\vec{m}|^2} = 1.$$

Therefore, the vector $\overrightarrow{(\cos \alpha, \cos \beta, \cos \gamma)}$ is a *unit vector*. If $\vec{m}$ is a direction vector of the line, you can call the vector $\overrightarrow{(\cos \alpha, \cos \beta, \cos \gamma)} = \vec{e}_m$.

The following property follows directly.

PROPERTY

If α, β, and γ are the angles the direction vector of a line makes with the x-axis, the y-axis, and the z-axis respectively, then $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

5.4 Exercises

1. Given a line with direction numbers 3,2,−4,
   a) state three other sets of direction numbers
   b) find the values of the direction cosines, correct to 4 decimal places
   c) find the angles a direction vector of the line makes with the coordinates axes, correct to the nearest degree.

2. Repeat question 1 parts b) and c) for the following sets of direction numbers
   a) −1,3,5
   b) 0,−5,8
   c) 6,−2,−1
   d) −1,0,2

3. Find direction cosines for each of the lines given by the following.
   a) $\vec{r} = \overrightarrow{(3,-1,4)} + k\overrightarrow{(5,-2,1)}$
   b) $\vec{r} = \overrightarrow{(-4,7,5)} + t\overrightarrow{(1,0,-8)}$
   c) $\begin{cases} x = 5 + 2k \\ y = 2 + 4k \\ z = 2 - 5k \end{cases}$
   d) $\begin{cases} x = -3 + 8s \\ y = 1 - 5s \\ z = -2 + 5s \end{cases}$
   e) $\dfrac{x-2}{3} = \dfrac{y-4}{8} = \dfrac{z-1}{5}$
   f) $\dfrac{x+3}{-3} = \dfrac{2y-4}{7} = \dfrac{1-z}{9}$
   g) $\dfrac{x-1}{-3} = y - 2$

4. Two of the direction cosines of a vector are 0.3 and 0.4. Find the other direction cosine, correct to 4 decimal places.

5. A direction vector of a line makes an angle of 30° with the x-axis and an angle of 70° with the y-axis. Find, correct to the nearest degree, the angle the direction vector makes with the z-axis.

6. A line through the origin of a 3-space coordinate system makes an angle of 45° with the y-axis and an angle of 80° with the z-axis. Find a vector equation of the line.

7. The direction cosines of two vectors are $\dfrac{1}{\sqrt{18}}, \dfrac{1}{\sqrt{18}}, \dfrac{4}{\sqrt{18}}$ and $0, \dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}$. Find the angle between the vectors, correct to the nearest degree.

8. The direction cosines of two intersecting lines are $\dfrac{1}{2}, \dfrac{\sqrt{2}}{2}, \dfrac{1}{2}$ and $-\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{\sqrt{2}}{2}$. Find the angle between the lines, correct to the nearest degree.

9. The line with vector equation $\vec{r} = \overrightarrow{(1,4,-6)} + k\overrightarrow{(-3,1,-2)}$ is perpendicular to the line with symmetric equations $\dfrac{x-2}{4} = \dfrac{y-4}{t} = \dfrac{z-1}{5}$. Find the direction cosines of each line.

10. Vectors $\vec{m}_1$ and $\vec{m}_2$ make angles $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ respectively with the coordinate axes.
    a) If $\vec{m}_1$ is perpendicular to $\vec{m}_2$ show that $\cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2 = 0$.
    b) If θ is the angle between $\vec{m}_1$ and $\vec{m}_2$ show that $\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2$

11. Find the direction cosines of a line perpendicular to both of the lines $\dfrac{x-1}{2} = \dfrac{y-1}{3} = \dfrac{z-1}{6}$ and $\dfrac{x}{3} = \dfrac{y}{8} = \dfrac{z}{12}$

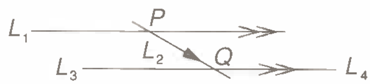
12. a) For a line in 2-space with direction vector $\vec{m} = \overrightarrow{(m_1,m_2)}$ show that the direction cosines are
    $$\cos\alpha = \frac{m_1}{|\vec{m}|},\ \cos\beta = \frac{m_2}{|\vec{m}|}$$
    b) For the line in part a) show that $\cos^2\alpha + \cos^2\beta = 1$.

13. A unit vector makes equal angles with each of the three coordinates axes. Find this unit vector.

5.5 The Intersection of Lines in 2-Space

You have studied intersections of lines in 2-space before, using Cartesian equations of lines. In this section you will investigate the intersection of lines using vector and parametric equations. You will also examine the various relationships that can occur among lines. Your understanding of the different ways lines can intersect in 2-space will help you to understand the intersection of lines in 3-space.

Consider the figure in which three lines L_1, L_2, and L_3 are drawn.
L_1 and L_2 intersect at point P. L_2 and L_3 intersect at point Q. L_1 and L_3 are parallel and distinct hence they do not intersect.
Two parallel lines L_3 and L_4 that have one point in common must have every point in common. The lines are the same line and are sometimes called *coincident* lines.



Situations like these are modelled in the following examples.

Example 1 Given the line L_1 with equation $3x + 2y = 8$. Determine whether or not L_1 is parallel to each of the following lines. If the lines are not parallel, find their point of intersection. If the lines are parallel, determine whether the lines are the same line, or distinct lines.
L_2: $4x - 5y = 3$ L_3: $12x + 8y = 32$ L_4: $6x + 4y = 2$

Solution *L_1 and L_2*

For L_1, $3x + 2y = 8$, so a normal is $\vec{n}_1 = \overrightarrow{(A,B)} = \overrightarrow{(3,2)}$.
For L_2, $4x - 5y = 3$, so a normal is $\vec{n}_2 = \overrightarrow{(4,-5)}$.
Since $\vec{n}_1$ is not a scalar multiple of $\vec{n}_2$, vectors $\vec{n}_1$ and $\vec{n}_2$ are not parallel.
Thus, L_1 and L_2 intersect. To find the point of intersection you must solve the system

$$3x + 2y = 8 \quad ①$$
$$4x - 5y = 3 \quad ②$$

Eliminating x, $4 \times ① - 3 \times ②$ gives $23y = 23$. Thus, $y = 1$.
Substituting $y = 1$ in ① or ② gives $x = 2$.
Hence, the point of intersection of L_1 and L_2 is $(2,1)$.

L_1 and L_3

For L_1, $3x + 2y = 8$, so a normal is $\vec{n}_1 = \overrightarrow{(3,2)}$.
For L_3, $12x + 8y = 32$, so a normal is $\vec{n}_3 = \overrightarrow{(12,8)}$.
Since $\vec{n}_3 = 4\vec{n}_1$, the normals $\vec{n}_3$ and $\vec{n}_1$ are parallel.
But the equation for L_3 is $12x + 8y = 32$ or $4(3x + 2y) = 4(8)$
Dividing by 4 gives $3x + 2y = 8$ which is the equation for L_1. Hence, every point $P(x,y)$ that is a solution for equation ① is also a solution for ③.
Thus, L_1 and L_3 are the same line.

L_1 and L_4

For L_1, $3x + 2y = 8$, so a normal is $\vec{n}_1 = \overrightarrow{(3,2)}$.

For L_4, $6x + 4y = 2$, so a normal is $\vec{n}_4 = \overrightarrow{(6,4)}$.

Since $\vec{n}_4 = 2\vec{n}_1$, the normals $\vec{n}_4$ and $\vec{n}_1$ are parallel.

But the equation for L_4 is $6x + 4y = 2$ or $2(3x + 2y) = 2(1)$

Dividing by 2 gives $3x + 2y = 1$.

Comparing this equation with the equation for L_1: $3x + 2y = 8$, you can see that no point $P(x,y)$ that is a solution for equation $3x + 2y = 8$ can be a solution for equation $3x + 2y = 1$ (nor for equation $6x + 4y = 2$).

Thus, L_1 and L_3 can not have a common point. Lines L_1 and L_3 are parallel and distinct. ■

If two linear equations in two variables have *one or more solutions*, then the system of equations is said to be *consistent*.

If a consistent system has *exactly one solution* the system is *independent*.

If a consistent system has *more than one solution* the system is *dependent*.

If two linear equations in two variables have *no solutions*, then the system of equations is said to be *inconsistent*.

SUMMARY

For a system of two equations in two variables

	number of solutions	*geometric description*
consistent and independent	one	two intersecting lines
consistent and dependent	infinite	same line
inconsistent	none	lines are parallel, distinct

Example 2 Given the line L_1 with vector equation $\vec{r} = \overrightarrow{(5 + 3k,-1 + 4k)}$. Determine whether or not L_1 is parallel to each of the following lines. If the lines are not parallel, find their point of intersection. If the lines are parallel, then determine whether the lines are distinct, or the same line.

L_2: $\vec{r} = \overrightarrow{(4 - 2t,5 + t)}$ L_3: $\vec{r} = \overrightarrow{(-4 - 6s,7 - 8s)}$ L_4: $\vec{r} = \overrightarrow{(8 + 3a,3 + 4a)}$

Solution *L_1 and L_2*

A direction vector for L_1 is $\vec{m}_1 = \overrightarrow{(3,4)}$.

A direction vector for L_2 is $\vec{m}_2 = \overrightarrow{(-2,1)}$.

Since no scalar k exists such that $\vec{m}_1 = k\vec{m}_2$, $\vec{m}_1$ and $\vec{m}_2$ are not parallel.

Hence, L_1 and L_2 are not parallel, so they intersect.

The system is consistent (a solution) and independent (only one solution).

Parametric equations for L_1 and L_2 are

$$\begin{cases} x = 5 + 3k \\ y = -1 + 4k \end{cases} \text{ and } \begin{cases} x = 4 - 2t \\ y = 5 + t \end{cases}$$

At the point of intersection, (x,y) for L_1 equals (x,y) for L_2. Thus,

$$\begin{cases} 5 + 3k = 4 - 2t \\ -1 + 4k = 5 + t \end{cases} \text{ or } \begin{cases} 3k + 2t = -1 & ① \\ 4k - t = 6 & ② \end{cases}$$

Eliminating k: $4 \times ① - 3 \times ②$ gives $11t = -22$, or $t = -2$.
Substitute in ① or ② to obtain $k = 1$.
Substituting $k = 1$ and $t = -2$ in either the parametric equations for L_1, or for L_2, produces $x = 8$ and $y = 3$.
Thus, the point of intersection is $(8,3)$.

L_1 and L_3

A direction vector for L_1 is $\vec{m_1} = \overrightarrow{(3,4)}$.
A direction vector for L_3 is $\vec{m_3} = \overrightarrow{(-6,-8)}$.
Since $\vec{m_3} = -2\vec{m_1}$, the vectors $\vec{m_3}$ and $\vec{m_1}$ are parallel.
Hence, L_1 and L_3 are parallel.
Are L_1 and L_3 the same line or distinct lines? If the lines have any one point in common they must have all points in common, and be the same line.
Test one point from L_1 in L_3. Now $(5,-1)$ is on L_1.
Parametric equations for L_3 are

$$\begin{cases} x = -4 - 6s \\ y = 7 - 8s \end{cases}$$

Substituting 5 for x and -1 for y gives
$5 = -4 - 6s$ and $-1 = 7 - 8s$

$$s = -\frac{9}{6} = -\frac{3}{2} \text{ or } s = \frac{-8}{-8} = 1 \neq -\frac{3}{2}.$$

Since the two values for s are different, $(5,-1)$ does not lie on L_3.
Hence, L_1 and L_3 are parallel and distinct.
The system is inconsistent (no solution).

L_1 and L_4

A direction vector for L_1 is $\vec{m_1} = \overrightarrow{(3,4)}$.
A direction vector for L_4 is $\vec{m_4} = \overrightarrow{(3,4)}$.
Since $\vec{m_4} = \vec{m_4}$ the vectors $\vec{m_4}$ and $\vec{m_1}$ are parallel.
Hence, L_1 and L_4 are parallel.
To determine whether L_1 and L_4 are the same line or distinct lines one point from L_1 should be tested in L_4. Now $(5,-1)$ is on L_1.
Parametric equations for L_4 are

$$\begin{cases} x = 8 + 3a \\ y = 3 + 4a \end{cases}$$

Substituting 5 for x and -1 for y gives
$5 = 8 + 3a$ and $-1 = 3a + 4a$

$$a = -\frac{3}{3} = -1 \text{ or } a = -\frac{4}{4} = -1$$

Since both values of a are the same, $(5,-1)$ does lie on L_4.
Thus, L_1 and L_4 are the same line.
The system is consistent (a solution) and dependent (an infinite number of solutions). ■

5.5 Exercises

1. a) A system of two linear equations in two variables is inconsistent. Explain what this means both geometrically and with respect to the number of solutions of the system.
   b) Repeat part a) for a system that is consistent and dependent.
   c) Repeat part a) for a system that is consistent and independent.

2. For each of the following systems determine whether or not the corresponding lines are parallel. If the lines are parallel, indicate whether they are distinct, or the same line. If the lines are not parallel, find their point of intersection.
   a) $\begin{cases} 4x - 5y = 22 \\ 3x + 2y = 5 \end{cases}$
   b) $\begin{cases} -2x + 6y = -16 \\ 3x - 9y = 15 \end{cases}$
   c) $\begin{cases} 7x + 3y = -30 \\ 21x + 9y = -90 \end{cases}$
   d) $\begin{cases} 8x + 6y = 18 \\ -2x - 9y = -15 \end{cases}$

3. For each of the systems in question 2 indicate which of the following terms apply.
   a) consistent and independent
   b) consistent and dependent
   c) inconsistent

4. For each of the following pairs of lines in 2-space determine whether or not the lines are parallel. If the lines are parallel, then determine whether the lines are distinct or the same line. If the lines are not parallel, find their point of intersection.
   a) $\vec{r} = \overrightarrow{(3 + 2k, 7 + k)}$
      $\vec{r} = \overrightarrow{(-5 + 4t, 8 - 3t)}$
   b) $\vec{r} = \overrightarrow{(19 - 5k, 16 - 6k)}$
      $\vec{r} = \overrightarrow{(2 + s, 4 - 3s)}$
   c) $\vec{r} = \overrightarrow{(1 + 8a, 4 - 6a)}$
      $\vec{r} = \overrightarrow{(2 - 4b, -5 + 3b)}$
   d) $\vec{r} = \overrightarrow{(1 + t, 3 - 2t)}$
      $\vec{r} = \overrightarrow{(-1 + 4k, -7 + 6k)}$

5. Find the point of intersection of the line $\vec{r} = \overrightarrow{(1,-6)} + k\overrightarrow{(-3,5)}$ and the line through the points $A(6,-4)$ and $B(-10,2)$.

6. Find the point of intersection of the line $\vec{r} = \overrightarrow{(-3,1)} + k\overrightarrow{(4,7)}$ and the line $\frac{x - 11}{-2} = \frac{y}{5}$

7. Find the point of intersection of the lines $\frac{x + 3}{2} = \frac{y + 2}{4}$ and $\frac{x - 5}{-3} = \frac{y - 4}{-1}$

8. Given the triangle $A(1,4)$ $B(-5,6)$ $C(3,-2)$
   a) find a vector equation of the median through vertex A
   b) find a vector equation of the median through vertex B
   c) find a vector equation of the median through vertex C
   d) find the point of intersection of the medians through vertices A and B
   e) show that the point of intersection found in part d) also lies on the median through vertex C.

9. Given the parallelogram $O(0,0)$, $Q(5,0)$, $R(8,4)$, $S(3,4)$
   a) find a vector equation of the diagonals OR and QS
   b) find the point of intersection of the diagonals in part a)
   c) use your point in part b) to show that the diagonals of parallelogram $OQRS$ bisect each other.

10. Given the point $P(3,10)$ and the line L, $\vec{r} = \overrightarrow{(1 + 3k, 17 - 4k)}$
    a) find a vector equation of the line through P that is perpendicular to L
    b) find the point A of intersection of L and the line in part a)
    c) find the perpendicular distance from point P to the line L.

11. Find a vector equation of the line through the origin that passes through the point of intersection of the lines $\frac{x - 1}{-1} = \frac{y - 2}{2}$ and $\frac{x + 1}{3} = \frac{y - 1}{2}$

5.6 The Intersection of Lines in 3-Space

In the introduction to this chapter you learned of a 'power line' problem.

Read page 196 again.

Recall that two lines in 3-space can intersect or not intersect. If the two lines do not intersect, then the two lines may be parallel, or not parallel. Two lines in 3-space that are not parallel *and* do not intersect are called *skew lines*.

The following examples will give you the mathematics necessary to solve the 'power line' problem.

Example 1 Given the line L_1 with vector equation $\vec{r} = \overrightarrow{(3 + 2k, 4 - 3k, 5 + k)}$. Determine whether or not L_1 is parallel to each of the following lines. If the lines are not parallel, then determine whether the lines intersect or are skew. If the lines intersect, then find the point of intersection.

L_2: $\vec{r} = \overrightarrow{(-7 + 6t, 5 + 5t, -1 + 4t)}$

L_3: $\vec{r} = \overrightarrow{(-1 + 3a, 3 - a, 7 + 6a)}$

L_4: $\vec{r} = \overrightarrow{(5 - 4w, 2 + 6w, 4 - 2w)}$

Solution (Using elimination)

L_1 and L_2

A direction vector for L_1 is $\vec{m}_1 = \overrightarrow{(2,-3,1)}$.

A direction vector for L_2 is $\vec{m}_2 = \overrightarrow{(6,5,4)}$.

Since no scalar k exists such that $\vec{m}_1 = k\vec{m}_2$, $\vec{m}_1$ and $\vec{m}_2$ are not parallel.
Hence, L_1 and L_2 are not parallel.
To determine if the lines intersect, you will need their parametric equations.
Now the parametric equations for L_1 and L_2 are

$$L_1\begin{cases} x = 3 + 2k \\ y = 4 - 3k \\ z = 5 + k \end{cases} \qquad L_2\begin{cases} x = -7 + 6t \\ y = 5 + 5t \\ z = -1 + 4t \end{cases}$$

At the point of intersection, (x,y,z) for L_1 equals (x,y,z) for L_2. Thus,

$$\begin{cases} 3 + 2k = -7 + 6t \\ 4 - 3k = 5 + 5t \\ 5 + k = -1 + 4t \end{cases} \text{ or } \begin{cases} 2k - 6t = -10 & \text{①} \\ -3k - 5t = 1 & \text{②} \\ k - 4t = -6 & \text{③} \end{cases}$$

Solving ① and ② gives $k = -2$, $t = 1$.
Substituting in ③ you will obtain $(-2) - 4(1) = -6$, which is true.
For $k = -2$ and $t = 1$ substitution in either the parametric equations for L_1 or L_2 gives $x = -1$, $y = 10$, and $z = 3$.
Thus, the point of intersection is $(x,y,z) = (-1,10,3)$.
(Notice that the linear system for L_1 and L_2, consisting of equations ① ② and ③, is consistent and independent.)

L_1 and L_3

A direction vector for L_1 is $\vec{m}_1 = \overrightarrow{(2,-3,1)}$.

A direction vector for L_3 is $\vec{m}_3 = \overrightarrow{(3,-1,6)}$.

Since no scalar k exists such that $\vec{m}_1 = k\vec{m}_3$, $\vec{m}_1$ and $\vec{m}_3$ are not parallel.
Hence, L_1 and L_3 are not parallel.
To determine if the lines intersect, you will need their parametric equations.
Now parametric equations for L_1 and L_3 are

$$L_1 \begin{cases} x = 3 + 2k \\ y = 4 - 3k \\ z = 5 + k \end{cases} \qquad L_3 \begin{cases} x = -1 + 3a \\ y = 3 - a \\ z = 7 + 6a \end{cases}$$

At the point of intersection, (x,y,z) for L_1 equals (x,y,z) for L_3. Thus,

$$\begin{cases} 3 + 2k = -1 + 3a \\ 4 - 3k = 3 - a \\ 5 + k = 7 + 6a \end{cases} \text{ or } \begin{cases} 2k - 3a = -4 & \text{④} \\ -3k + a = -1 & \text{⑤} \\ k - 6a = 2 & \text{⑥} \end{cases}$$

Solving ④ and ⑤ gives $k = 1$, and $a = 2$.
Substituting in ⑥ you will obtain $(1) - 6(2) = 2$ or $-11 = 2$, which is not true.
Thus, the lines L_1 and L_3 do not intersect. Since L_1 and L_3 are also not parallel, L_1 and L_3 are skew lines.
(Notice that the linear system for L_1 and L_3, consisting of equations ④ ⑤ and ⑥, is inconsistent.)

L_1 and L_4

A direction vector for L_1 is $\vec{m}_1 = \overrightarrow{(2,-3,1)}$.

A direction vector for L_4 is $\vec{m}_4 = \overrightarrow{(-4,6,-2)}$.

Since $\vec{m}_4 = -2\vec{m}_1$, therefore $\vec{m}_1$ and $\vec{m}_4$ are parallel.
Hence, L_1 and L_4 are parallel.
Since the lines are parallel, they are either distinct lines or the same line.
To determine whether L_1 and L_4 are the same line or distinct lines, one point from L_1 should be tested in L_4.
Now $(3,4,5)$ is on L_1.
Parametric equations for L_4 are

$$\begin{cases} x = 5 - 4w \\ y = 2 + 6w \\ z = 4 - 2w \end{cases}$$

Substituting 3 for x, 4 for y, and 5 for z gives

$$\begin{cases} 3 = 5 - 4w \\ 4 = 2 + 6w \\ 5 = 4 - 2w \end{cases}$$

Solving each of the equations for w, gives $w = -\frac{1}{2}$, $w = \frac{1}{3}$, and $w = -\frac{1}{2}$.

But w cannot have different values at the same time. Thus, the point $(3,4,5)$ does not lie on line L_4. Hence, the lines can not have a common point. Hence the lines are parallel and distinct.
(Notice that the linear system for L_1 and L_4, obtained by equating components, will be inconsistent.) ■

The systems of equations for the parameters k and t of lines L_1 and L_2 and for k and a of lines L_1 and L_3 in Example 1 can also be solved using matrices as follows.

Alternate Solution for part of Example 1 using Matrices

L_1 and L_2

As in Example 1, the three equations are

$$\begin{cases} 2k - 6t = -10 & ① \\ -3k - 5t = 1 & ② \\ k - 4t = -6 & ③ \end{cases}$$

The augmented matrix for the three equations in the two variables k and t and the reduced form of the matrix follows.

$$\begin{bmatrix} 2 & -6 & -10 \\ -3 & -5 & 1 \\ 1 & -4 & -6 \end{bmatrix} \quad \begin{array}{r} \\ 3 \times \text{row}① + 2 \times \text{row}② \\ \text{row}① - 2 \times \text{row}② \end{array} \begin{bmatrix} 2 & -6 & -10 \\ 0 & -28 & -28 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ \text{row}② + 14 \times \text{row}③ \end{array} \begin{bmatrix} 2 & -6 & -10 \\ 0 & -28 & -28 \\ 0 & 0 & 0 \end{bmatrix}.$$

From row ② of the reduced matrix, $-28t = -28$. Thus, $t = 1$.
From row ① of the reduced matrix, $2k - 6t = -10$.

Substituting $t = 1$ in this equation gives $k = -2$.

L_1 and L_3

As in Example 1, the equations are

$$\begin{cases} 2k - 3a = -4 & ④ \\ -3k + a = -1 & ⑤ \\ k - 6a = 2 & ⑥ \end{cases}$$

The augmented matrix for three equations in the two variables k and a and the reduced form of the matrix follows.

$$\begin{bmatrix} 2 & -3 & -4 \\ -3 & 1 & -1 \\ 1 & -6 & 2 \end{bmatrix} \quad \begin{array}{r} \\ 3 \times \text{row}① + 2 \times \text{row}② \\ \text{row}① - 2 \times \text{row}③ \end{array} \begin{bmatrix} 2 & -3 & -4 \\ 0 & -7 & -14 \\ 0 & 9 & -8 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ 9 \times \text{row}② - 7 \times \text{row}③ \end{array} \begin{bmatrix} 2 & -3 & -4 \\ 0 & -7 & -14 \\ 0 & 0 & -70 \end{bmatrix}$$

From the last line of the reduced matrix, $0a = -70$.

No value of a satisfies this equation so there is no solution to equations ④, ⑤, and ⑥. L_1 and L_3 do not intersect.

Because the direction vectors of L_1 and L_3 are not parallel, lines L_1 and L_3 are skew lines.

L_1 and L_4

The parametric equations for L_1 and L_4 are

$$L_1 \begin{cases} x = 3 + 2k \\ y = 4 - 3k \\ z = 5 + k \end{cases} \qquad L_4 \begin{cases} x = 5 - 4w \\ y = 2 + 6w \\ z = 4 - 2w \end{cases}$$

Equating the components for L_1 and L_4, then simplifying, gives the following system of equations.

$$\begin{aligned} 2k + 4w &= 2 \quad ⑦ \\ -3k - 6w &= -2 \quad ⑧ \\ k + 2w &= -1 \quad ⑨ \end{aligned}$$

The augmented matrix for this system is

$$\begin{bmatrix} 2 & 4 & 2 \\ -3 & -6 & -1 \\ 1 & 2 & -1 \end{bmatrix} \quad \begin{matrix} \\ 3 \times \text{row}① + 2 \times \text{row}② \\ \text{row}① - 2 \times \text{row}③ \end{matrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

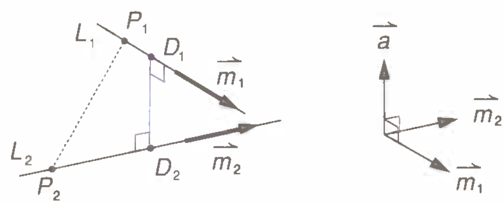
From the last line of the reduced matrix, $0w = 4$.

Since this equation has no solution, the lines L_1 and L_4 do not intersect. Because the direction vectors of L_1 and L_4 are parallel, lines L_1 and L_4 are parallel and distinct. ■

The Shortest Distance between Two Lines

If two lines are skew, they do not intersect. How close do they come to each other? In other words, what is the shortest distance between two skew lines? The following analysis will explain *how to find the shortest distance between two skew lines L_1 and L_3.*

The shortest distance between two skew lines L_1 and L_2 is the distance between the two points D_1 and D_2 on L_1 and L_2 respectively, where $D_1D_2 \perp L_1$ and $D_1D_2 \perp L_2$.



Suppose the vector $\vec{a} = \vec{m}_1 \times \vec{m}_2$, where $\vec{m}_1$ and $\vec{m}_2$ are direction vectors for L_1 and L_2 respectively. Then $\vec{a}$ is perpendicular to $\vec{m}_1$, and to $\vec{m}_2$, and hence to L_1 and to L_2. Hence, $\overrightarrow{D_1,D_2}$ is parallel to $\vec{a}$.

Let d = the shortest distance between L_1 and L_2.

Then $d = |\overrightarrow{D_1D_2}|$. Let P_1 and P_2 be points on L_1 and L_2 respectively.

Thus, $D_1D_2 \perp D_1P_1$ and $D_1D_2 \perp D_2P_2$.

$$\begin{aligned} \text{Thus, } d &= |\text{component of } \overrightarrow{P_1P_2} \text{ along } \overrightarrow{D_1D_2}| \\ &= |\text{component of } \overrightarrow{P_1P_2} \text{ along } \vec{a}| \qquad \vec{a} \parallel \overrightarrow{D_1D_2} \end{aligned}$$

$$\text{Thus } d = \frac{|\overrightarrow{P_1P_2} \cdot \vec{a}|}{|\vec{a}|}$$

FORMULA

The shortest distance d between two skew lines L_1 and L_2 is

$$d = \frac{|\overrightarrow{P_1P_2} \cdot \vec{a}|}{|\vec{a}|}$$

where P_1 is a point on L_1, P_2 is a point on L_2, L_1 has direction vector $\vec{m}_1$, L_2 has direction vector $\vec{m}_2$, and $\vec{a} = \vec{m}_1 \times \vec{m}_2$.

Example 2 a) Find the shortest distance, D_1D_2, between the skew lines L_1: $\vec{r} = \overrightarrow{(-1 + 2k, -2 + 2k, 3k)}$ and L_2: $\vec{r} = \overrightarrow{(9 + 6a, 3 - a, 1 + 2a)}$

b) Find the coordinates of points D_1 and D_2.

Solution a) The shortest distance between the lines is given by the formula

$$d = \frac{|\overrightarrow{P_1P_2} \cdot \vec{a}|}{|\vec{a}|}$$

Here

$P_1 = (-1,-2,0)$, $P_2 = (9,3,1)$ so $\overrightarrow{P_1P_2} = \overrightarrow{(9,3,1)} - \overrightarrow{(-1,-2,0)} = \overrightarrow{(10,5,1)}$

Also, $\vec{a} = \vec{m}_1 \times \vec{m}_2 = \overrightarrow{(2,2,3)} \times \overrightarrow{(6,-1,2)} = \overrightarrow{(7,14,-14)}$

$$|\vec{a}| = \sqrt{7^2 + 14^2 + (-14)^2} = \sqrt{441} = 21$$

Thus $d = \dfrac{|\overrightarrow{(10,5,1)} \cdot \overrightarrow{(7,14,-14)}|}{21} = \dfrac{126}{21}$ or 6

The shortest distance between the lines is 6.

b) You can find the coordinates of points D_1 and D_2 by using the fact that $D_1D_2 \perp L_1$ and $D_1D_2 \perp L_2$

Since $\overrightarrow{OD_1} = \overrightarrow{(-1 + 2k, -2 + 2k, 3k)}$ for some value of k, and

$\overrightarrow{OD_2} = (9 + 6a, 3 - a, 1 + 2a)$ for some value of a,

therefore $\overrightarrow{D_1D_2} = \overrightarrow{OD_2} - \overrightarrow{OD_1}$

$= \overrightarrow{(6a - 2k + 10, -a - 2k + 5, 2a - 3k + 1)}$

For $D_1D_2 \perp L_1$

$\overrightarrow{D_1D_2} \cdot \vec{m}_1 = 0$

$\overrightarrow{(6a - 2k + 10, -a - 2k + 5, 2a - 3k + 1)} \cdot \overrightarrow{(2,2,3)} = 0$

$12a - 4k + 20 - 2a - 4k + 10 + 6a - 9k + 3 = 0$

$16a - 17k = -33$ ①

For $D_1D_2 \perp L_2$

$\overrightarrow{D_1D_2} \cdot \vec{m}_2 = 0$

$\overrightarrow{(6a - 2k + 10, -a - 2k + 5, 2a - 3k + 1)} \cdot \overrightarrow{(6,-1,2)} = 0$

$36a - 12k + 60 + a + 2k - 5 + 4a - 6k + 2 = 0$

$41a - 16k = -57$ ②

Solving ① and ② gives $k = 1$ and $a = -1$.

Hence the coordinates of D_1 are $(-1 + 2[1], -2 + 2[1], 3[1]) = (1,0,3)$ and of D_2 are $(9 + 6[-1], 3 - [-1], 1 + 2[-1] = (3,4,-1))$. ■

5.6 Exercises

1. For each of the following pairs of lines in 3-space determine whether or not the lines are parallel.
If the lines are not parallel, determine whether they intersect, or are skew.
If the lines intersect, find the point of intersection.

a) $\vec{r} = \overrightarrow{(5 + 2k, 8 + k, 13 + 3k)}$
$\vec{r} = \overrightarrow{(3 + 4t, 2 - 3t, 2 - 2t)}$

b) $\vec{r} = \overrightarrow{(10 - 5k, 4 - 6k, 3 + k)}$
$\vec{r} = \overrightarrow{(3 + s, 4 - 3s, -6 + 5s)}$

c) $\vec{r} = \overrightarrow{(1 - 3k, 2 + 5k, 4 + k)}$
$\vec{r} = \overrightarrow{(3 + b, -3b, -2 + 6b)}$

d) $\vec{r} = \overrightarrow{(1 + 8a, 4 - 6a, 2 + 2a)}$
$\vec{r} = \overrightarrow{(2 - 4t, -5 + 3t, -3 - t)}$

e) $\vec{r} = \overrightarrow{(1 + t, 3 - 2t, -14 + 5t)}$
$\vec{r} = \overrightarrow{(-1 + 4k, -7 + 6k, -2 - 2k)}$

f) $\vec{r} = \overrightarrow{(5 + a, -5 - 2a, 3 + 4a)}$
$\vec{r} = \overrightarrow{(3 + 4k, -10 + k, -1 - 2k)}$

2. For each of the pairs of lines in question 1 indicate which of the following terms apply to the corresponding linear system.
a) consistent and independent
b) consistent and dependent
c) inconsistent

3. Find the point of intersection of the line $\vec{r} = \overrightarrow{(-1,4,6)} + k\overrightarrow{(-3,5,1)}$ and the line through the points $A(-4,-3,7)$ and $B(8,1,3)$

4. Find the point of intersection of the line $\vec{r} = \overrightarrow{(-3,4,1)} + k\overrightarrow{(4,-1,7)}$ and the line $\frac{x - 11}{-2} = \frac{y - 5}{-1} = \frac{z}{5}$

5. Find the point of intersection of the lines
L_1: $\frac{x + 2}{5} = \frac{y + 3}{2} = \frac{z + 2}{4}$ and
L_2: $\frac{x + 5}{4} = \frac{y - 5}{-3} = \frac{z - 4}{-1}$

6. Find the distance between the lines in questions 1 c) and 1 f).

7. a) Prove that the lines
L_1: $\vec{r} = \overrightarrow{(1 + 5k, -2 + k, 4 - 3k)}$ and
L_2: $\vec{r} = \overrightarrow{(2 + 3t, 6 - 2t, 7 - 4t)}$ are skew lines.

b) What is the distance between the skew lines L_1 and L_2?

8. Given the lines
L_1: $\vec{r} = \overrightarrow{(4 + 2k, 4 + k, -3 - k)}$ and
L_2: $\vec{r} = \overrightarrow{(-2 + 3s, -7 + 2s, 2 - 3s)}$

a) Prove that the lines are skew.

b) Find the shortest distance between the lines.

c) Find the coordinates of the point P_1 on L_1 and the point P_2 on L_2 such that P_1P_2 is the shortest distance of part b).

9. Given the lines $\vec{r} = \overrightarrow{(3 - k, 4 + 5k, 1 - 2k)}$ and $\vec{r} = \overrightarrow{(8 + 2w, 5 + 3w, -9 - 6w)}$.

a) Prove that the lines intersect.

b) Find the angle between the lines.

10. Prove that the lines
$\vec{r} = \overrightarrow{(-5 + 3k, 2 + 2k, -7 + 6k)}$ and
$\vec{r} = \overrightarrow{(s, -6 - 5s, -3 - s)}$ lie in the same plane.

11. Two power lines L_1 and L_2 from the point A described in the introduction to this chapter have equations
$\vec{r} = \overrightarrow{(1 + 5k, 11k, k)}$ and
$\vec{r} = \overrightarrow{(1 - s, s, -3s)}$ respectively.
A line L_3 from point B has equation
$\vec{r} = \overrightarrow{(3 - t, -4 - 5t, -5 - 2t)}$.
 a) Determine if either of the power lines from point A intersect with the line from point B.
 b) If either of the power lines from point A do not intersect with the line from point B, then find the shortest distance between the lines.
 c) For the lines in part b) find the points on the lines that give this shortest distance.

12. a) Find an equation of the line through the origin that intersects the lines
$\vec{r} = \overrightarrow{(1 - t, 2 + 2t, 3 + t)}$ and
$\vec{r} = \overrightarrow{(-1 + 3k, 1 + 2k, -1 - k)}$
 b) Find the point of intersection of the line found in part a) and the first of the given lines.

13. By definition, a diagonal of a cube joins a vertex of the cube to an opposite vertex not on a same face of the cube. A certain cube has sides of length d. Find the shortest distance between any diagonal of the cube and an edge that is skew to that diagonal.

14. Prove that the lines $\vec{r} = \vec{a} + k\vec{v}$ and $\vec{r} = \vec{b} + t\vec{v}$ lie in the same plane.

15. Determine whether or not the following pairs of equations represent the same line.
 a) $\vec{r} = \overrightarrow{(2,3,-1)} + k\overrightarrow{(-5,0,1)}$ and
 $\vec{r} = \overrightarrow{(-3,3,0)} + t\overrightarrow{(10,0,-2)}$
 b) $\vec{r} = \overrightarrow{(4,5,4)} + k\overrightarrow{(1,2,3)}$ and
 $\vec{r} = \overrightarrow{(4,5,4)} + t\overrightarrow{(3,2,1)}$
 c) $\vec{r} = \overrightarrow{(1,-3,7)} + k\overrightarrow{(1,1,0)}$ and
 $\vec{r} = \overrightarrow{(2,-1,7)} + t\overrightarrow{(1,1,0)}$

16. Given the points $D_1(-1,3,2)$ and $D_2(4,0,1)$.
 a) Find $\overrightarrow{D_1D_2}$.
 b) Prove that the line
 L_1: $\vec{r} = \overrightarrow{(-1,3,2)} + k\overrightarrow{(a,b,1)}$ is perpendicular to D_1D_2 provided $5a - 3b = 1$.
 c) Prove that the line
 L_2: $\vec{r} = \overrightarrow{(4,0,1)} + t\overrightarrow{(p,q,2)}$ is perpendicular to D_1D_2 provided $5p - 3q = 2$.
 d) Find an equation of any one line L_1, and for any one line L_2.
 e) Prove that the shortest distance between the lines L_1 and L_2 of d) is equal to the length of segment D_1D_2.
 f) Prove that the shortest distance between any line L_1 of b) and any line L_2 of c) equals the length of segment D_1D_2.

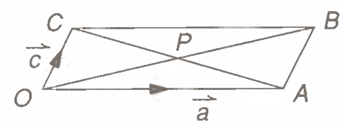
5.7 Geometric Proofs Using Vector Equations

Now that you have learned how to form vector equations of lines, you can apply this knowledge to some geometric proofs. You saw problems of this type already in section 2.6. Some of the examples will be the same as in section 2.6. You can use the method presented here as an alternative.

Recall that the vector equation of a line is $\vec{r} = \vec{r_0} + k\vec{m}$, where $\vec{r}$ is the position vector of any point on the line, $\vec{r_0}$ is the position vector of a given point on the line, $\vec{m}$ is a vector parallel to the line, and $k \in \mathbb{R}$ is a parameter.

Example 1 Prove that the diagonals of a parallelogram bisect each other.

Solution Let the parallelogram be $OABC$.
Let $\overrightarrow{OA} = \vec{a}$, and $\overrightarrow{OC} = \vec{c}$.



In order to find P, you need to pinpoint the intersection of lines OB and AC; thus you will require the equations of lines OB and AC.
Taking the point O as origin, you can use the vectors $\vec{a}$ and $\vec{c}$ to determine these equations as follows.
In general, $\vec{r} = \vec{r_0} + k\vec{m}$, $k \in \mathbb{R}$.

Equation of line OB:
Here, $\vec{r_0} = \vec{0}$, $\vec{m} = \overrightarrow{OB} = (\vec{a} + \vec{c})$.
Using $k \in \mathbb{R}$ as parameter,
$\vec{r} = \vec{0} + k\overrightarrow{OB}$ ①
or $\vec{r} = \vec{0} + k(\vec{a} + \vec{c})$ or $\vec{r} = k(\vec{a} + \vec{c})$

Equation of line AC:
Here, $\vec{r_0} = \overrightarrow{OA} = \vec{a}$, $\vec{m} = \overrightarrow{AC} = (\vec{c} - \vec{a})$.
Using $t \in \mathbb{R}$ as parameter,
$\vec{r} = \overrightarrow{OA} + t\overrightarrow{AC}$ ②
or $\vec{r} = \vec{a} + t(\vec{c} - \vec{a})$

These lines intersect at point P when $\vec{r} = \overrightarrow{OP}$ in equations ① and ②,

so $k(\vec{a} + \vec{c}) = \vec{a} + t(\vec{c} - \vec{a})$

$\Rightarrow$ $k\vec{a} + k\vec{c} = \vec{a} + t\vec{c} - t\vec{a}$

$\Rightarrow (k + t - 1)\,\vec{a} + (k - t)\,\vec{c} = \vec{0}$

But $\vec{a}$ and $\vec{c}$ are linearly independent, so

$k + t - 1 = 0$ and $k - t = 0$, giving $k = \frac{1}{2}$ and $t = \frac{1}{2}$

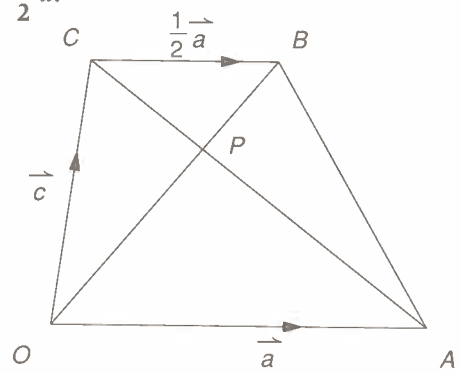
These are the values which give the point P on each line.

Thus, in equation ①, reaching P along line OB requires $\frac{1}{2}\overrightarrow{OB}$, and, in equation ②, reaching P along line AC requires $\frac{1}{2}\overrightarrow{AC}$. More simply,

$\overrightarrow{OP} = \frac{1}{2}\overrightarrow{OB}$ and $\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AC}$; the diagonals do indeed bisect each other. ■

Example 2 Given a trapezoid $OABC$ in which $OA = 2CB$, $OA \parallel CB$, prove that the diagonals each intersect at $\frac{2}{3}$ of their lengths.

Solution To find P, the point of intersection of the diagonals, you need the equations of the lines OB and AC. Using O as origin, choose $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OC} = \vec{c}$, then $\overrightarrow{CB} = \frac{1}{2}\vec{a}$.



Equation of line OB: $\vec{r} = \vec{0} + k\overrightarrow{OB}$ ①

or $\vec{r} = k\left(\vec{c} + \frac{1}{2}\vec{a}\right)$

Equation of line AC: $\vec{r} = \overrightarrow{OA} + t\overrightarrow{AC}$ ②

or $\vec{r} = \vec{a} + t(\vec{c} - \vec{a})$

The lines intersect at P when $\vec{r} = \overrightarrow{OP}$ in both ① and ②.

Thus, $$k\left(\vec{c} + \frac{1}{2}\vec{a}\right) = \vec{a} + t(\vec{c} - \vec{a})$$

$$\Rightarrow \quad k\vec{c} + \frac{1}{2}k\vec{a} = \vec{a} + t\vec{c} - t\vec{a}$$

$$\Rightarrow \quad (k - t)\,\vec{c} + \left(\frac{1}{2}k + t - 1\right)\vec{a} = \vec{0}$$

But $\vec{a}$ and $\vec{c}$ are linearly independent, so $k - t = 0$ and $\frac{1}{2}k + t - 1 = 0$,

which gives $k = \frac{2}{3}$ and $t = \frac{2}{3}$ at the point P.

Thus, in equation ①, reaching P along line OB requires $\frac{2}{3}\overrightarrow{OB}$, that is,

$\overrightarrow{OP} = \frac{2}{3}\overrightarrow{OB}$.

In equation ②, reaching P along line AC requires $\frac{2}{3}\overrightarrow{AC}$, that is,

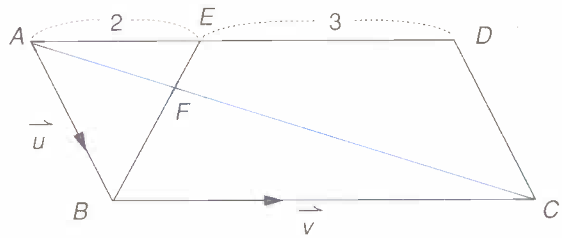
$\overrightarrow{AP} = \frac{2}{3}\overrightarrow{AC}$.

Hence, the required result is confirmed. ■

Example 3 (This is the same as section 2.6, Example 3, on page 104.)

In parallelogram $ABCD$, E divides AD in the ratio 2 : 3. BE and AC intersect at F. Find the ratio into which F divides AC.

Solution



To find the intersection of lines AC and BE, you need the equations of AC and BE.

To obtain these, choose any origin (say point A), and any two independent vectors (say $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{AD} = \overrightarrow{BC} = \vec{v}$). If E divides AD in the ratio 2 : 3, then $\overrightarrow{AE} = \frac{2}{5}\overrightarrow{AD}$, that is, $\overrightarrow{AE} = \frac{2}{5}\vec{v}$.

Equation of line AC: $\vec{r} = \vec{0} + k\overrightarrow{AC}$ ①

or $\vec{r} = k(\vec{v} + \vec{u})$

Equation of line BE: $\vec{r} = \overrightarrow{AB} + t\overrightarrow{BE}$ ②

or $\vec{r} = \vec{u} + t(\overrightarrow{AE} - \overrightarrow{AB})$

or $\vec{r} = \vec{u} + t\left(\frac{2}{5}\vec{v} - \vec{u}\right)$

The lines intersect at point F when $\vec{r} = \overrightarrow{AF}$ in both ① and ②.

Thus, $$k(\vec{v} + \vec{u}) = \vec{u} + t\left(\frac{2}{5}\vec{v} - \vec{u}\right)$$

$$\Rightarrow \quad k\vec{v} + k\vec{u} = \vec{u} + \frac{2t}{5}\vec{v} - t\vec{u}$$

$$\Rightarrow \quad (k + t - 1)\,\vec{u} + \left(k - \frac{2t}{5}\right)\vec{v} = \vec{0}$$

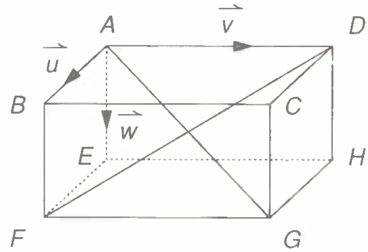
But $\vec{u}$ and $\vec{v}$ and are linearly independent, so $k + t - 1 = 0$ and $k - \frac{2t}{5} = 0$.

These equations give $k = \frac{2}{7}$ and $t = \frac{5}{7}$ at the point F. From equation ①, using $k = \frac{2}{7}$, $\overrightarrow{AF} = \frac{2}{7}\overrightarrow{AC}$, that is F divides AC in the ratio 2 : 5. ■

(Also, as a bonus, you can use $t = \frac{5}{7}$ in equation ② to find that $\overrightarrow{BF} = \frac{5}{7}\overrightarrow{BE}$, so F divides BE in the ratio 5 : 2.)

This method is readily extended to 3-space problems, as in the following example. The only difference is that three independent vectors are required to define the equations of the lines.

Example 4 Consider the rectangular box *ABCDHGFE* shown. Prove that the diagonals *AG* and *FD* intersect, and determine where the point of intersection lies.



Solution First you need the equations of lines *AG* and *FD*.

To obtain these, use any origin (say point *A*), and any three independent vectors (say $\overrightarrow{AB} = \vec{u}$, $\overrightarrow{AD} = \vec{v}$, and $\overrightarrow{AE} = \vec{w}$).

Notice that $\overrightarrow{AB} = \overrightarrow{DC} = \overrightarrow{HG} = \overrightarrow{EF} = \vec{u}$, $\overrightarrow{AD} = \overrightarrow{BC} = \overrightarrow{FG} = \overrightarrow{EH} = \vec{v}$, and $\overrightarrow{AE} = \overrightarrow{BF} = \overrightarrow{CG} = \overrightarrow{DH} = \vec{w}$.

Equation of line *AG*: $\vec{r} = \vec{0} + k\overrightarrow{AG}$

or $\vec{r} = k(\vec{u} + \vec{v} + \vec{w})$ ①

Equation of line *FD*: $\vec{r} = \overrightarrow{AF} + t\overrightarrow{FD}$

or $\vec{r} = (\vec{u} + \vec{w}) + t(\vec{v} - \vec{u} - \vec{w})$ ②

If the lines intersect, then from ① and ②

$$k(\vec{u} + \vec{v} + \vec{w}) = (\vec{u} + \vec{w}) + t(\vec{v} - \vec{u} - \vec{w})$$

$$\Rightarrow \quad k\vec{u} + k\vec{v} + k\vec{w} = \vec{u} + \vec{w} + t\vec{v} - t\vec{u} - t\vec{w}$$

$$\Rightarrow (k + t - 1)\,\vec{u} + (k - t)\,\vec{v} + (k + t - 1)\,\vec{w} = \vec{0}$$

But $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly independent, so

$k + t - 1 = 0$

$k - t = 0$

$k + t - 1 = 0.$

These equations form a consistent system with solution $k = \frac{1}{2}$ and $t = \frac{1}{2}$.

Thus, the lines do intersect. If the point of intersection is *P*, then, from equations ① and ②,

$\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AG}$, and $\overrightarrow{FP} = \frac{1}{2}\overrightarrow{FD}$.

Thus, *P* is the midpoint of each diagonal. That is, the diagonals *AG* and *FD* bisect each other. ■

SUMMARY

You can use vector equations of lines to solve any geometric problem in which you are required to find the intersection of two well-defined lines, by choosing an origin and any known independent vectors to write your equations.

Equating the position vector of a general point on each line will yield the values of the parameters at the point of intersection.

5.7 Exercises

Use vector methods to solve the following problems.

1. $OBCD$ is a parallelogram. E is the midpoint of side OD. Segments OC and BE intersect at point F. Find the ratio into which OC divides BE.

2. $OBCD$ is a parallelogram. E is the point that divides side OD in the ratio $2:5$. Segments OC and BE intersect at point F. Find the ratio into which OC divides BE.

3. a) In $\triangle OAB$, medians AD and BE intersect at point G. Find the ratios into which G divides AD and BE.
   b) Show the medians of a triangle trisect each other.

4. In $\triangle OBC$, E is the midpoint of side OB. Point F is on side OC such that segment EF is parallel to side BC. Into what ratio does F divide side OC?

5. In $\triangle OBC$, E is the point that divides side OB into the ratio $1:2$. Point F is on side OC such that segment EF is parallel to side BC. Into what ratio does F divide side OC?

6. In $\triangle OBC$, E is the point that divides side OB into the ratio $1:k$, $k \neq 0$. Point F is on side OC such that segment EF is parallel to side BC. Into what ratio does F divide side OC?

7. In $\triangle ABC$, D divides AB in the ratio $1:2$ and E divides AC in the ratio $1:4$. BE and CD intersect at point F. Find the ratios into which F divides each of BE and CD.

8. In parallelogram $PQRS$, A divides PQ in the ratio $2:5$, and B divides SR in the ratio $3:2$. Segments PR and AB intersect at C. Find the ratio into which C divides segment PR.

9. $PQRS$ is a trapezoid with PQ parallel to SR. PR and QS intersect at point A. If A divides segment QS in the ratio $2:3$, then find the ratio into which A divides PR.

10. $ABCD$ is a parallelogram. E is the point that divides side AD in the ratio $1:k$, where $k \geq 0$. Segments AC and BE intersect at point F. Find the ratio into which point F divides AC.

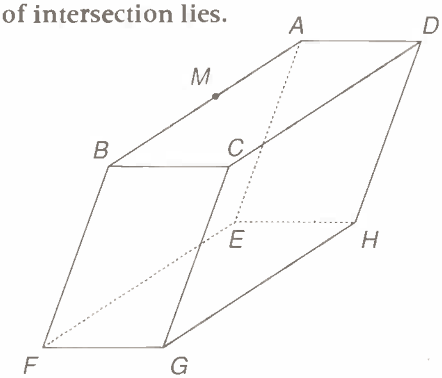
11. Let M be the midpoint of median AD of $\triangle ABC$. BM extended and AC intersect at K. Find the ratio into which K divides AC.

12. $ABCD$ is a trapezoid in which AD is parallel to BC. P and Y divide AB and DC respectively in the same ratio. Q is the point on diagonal AC such that PQ is parallel to BC. Prove that points P, Q, and Y are collinear.

13. In a tetrahedron, prove that the line segments joining a vertex to the centroid of the opposite face intersect at a point that divides the line segments in the ratio $1:3$. (The centroid of a triangle is the point of intersection of the medians. See also question 3.)

14. Show that the point found in question 13 is the same as the point of intersection of the line segments joining the midpoints of opposite edges of a tetrahedron.

15. The box shown, called a parallelepiped, is made up of three pairs of congruent parallelograms. Prove that the diagonals BH and EC intersect, and determine where the point of intersection lies.



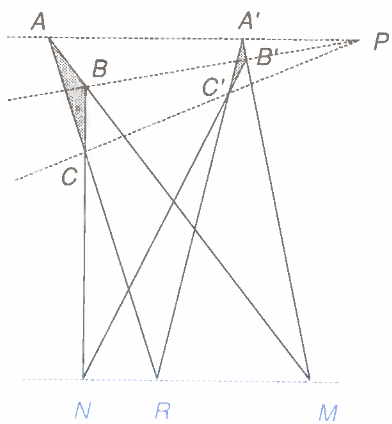
16. In the box shown, let M be the midpoint of AB. Prove that MG and FD do not intersect.

In Search of A Proof of Desargues' Theorem

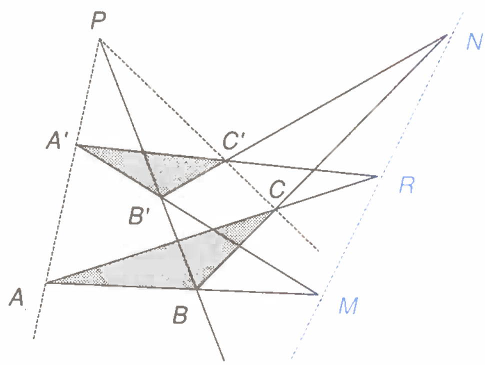
The French geometer Girard Desargues (1591-1661) was one of the first mathematicians to initiate the formal study of projective geometry. Projective geometry studies the properties of geometric configurations that are invariant under a projection. An important theorem that he proved involves the relationship between two triangles whose vertices lie on three concurrent lines. The point of intersection of the lines is called the *centre of the projection* of one triangle on the other. The lines can be in 2-space or 3-space. Desargues' theorem is true in both spaces.

Desargues' theorem

If two triangles have corresponding vertices joined by concurrent lines, then the intersections of corresponding sides are collinear.



coplanar triangles



non-coplanar triangles

Given: $\triangle ABC$ and $\triangle A'B'C'$ such that lines AA', BB', and CC' intersect at point P.
Lines AB and $A'B'$ intersect at M.
Lines BC and $B'C'$ intersect at N.
Lines AC and $A'C'$ intersect at R.

Prove: Points M, N and R are collinear.

Analysis: You can prove M, N, and R are collinear by finding real numbers p and q such that $\overrightarrow{ON} = p\overrightarrow{OM} + q\overrightarrow{OR}$ where $p + q = 1$. ① (See page 98.)

Use the fact that P lies on each of the lines AA', BB', and CC' to determine a relationship among $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{a}'$, $\vec{b}'$, and $\vec{c}'$ where $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, etc.
Use these relationships to find an equation like ①.

Proof: *For Line AA′:*

An equation of line AA' is $\vec{r} = \vec{r_0} + k\vec{m}$ where $\vec{r_0} = \vec{a}$, and $\vec{m} = \overrightarrow{A'A} = \vec{a} - \vec{a'}$.

Thus, an equation for line AA' is $\vec{r} = \vec{a} + k(\vec{a} - \vec{a'})$

Similarly line BB' has equation $\vec{r} = \vec{b} + t(\vec{b} - \vec{b'})$

and line CC' has equation $\vec{r} = \vec{c} + s(\vec{c} - \vec{c'})$

Since point P lies on both of AA' and BB',

$\overrightarrow{OP} = \vec{a} + k(\vec{a} - \vec{a'})$ and $\overrightarrow{OP} = \vec{b} + t(\vec{b} - \vec{b'})$

Therefore, $\vec{a} + k(\vec{a} - \vec{a'}) = \vec{b} + t(\vec{b} - \vec{b'})$

or, $t\vec{b'} - k\vec{a'} = (1 + t)\vec{b} - (1 + k)\vec{a}$ ②

Dividing by $t - k$ gives

$$\frac{t}{t-k}\vec{b'} - \frac{k}{t-k}\vec{a'} = \frac{1+t}{t-k}\vec{b} - \frac{1+k}{t-k}\vec{a}$$

This statement implies there is a point Q such that

$$\overrightarrow{OQ} = \frac{t}{t-k}\vec{b'} - \frac{k}{t-k}\vec{a'}, \text{ where } \frac{t}{t-k} + \frac{-k}{t-k} = \frac{t-k}{t-k} = 1$$

Hence Q is collinear with A' and B'. Also

$$\overrightarrow{OQ} = \frac{1+t}{t-k}\vec{b} - \frac{1+k}{t-k}\vec{a}, \text{ where } \frac{1+t}{t-k} + \frac{-1-k}{t-k} = \frac{t-k}{t-k} = 1$$

Hence Q is collinear with points A and B.

Hence Q is the point M.

Thus, $\overrightarrow{OM} = \frac{t}{t-k}\vec{b'} - \frac{k}{t-k}\vec{a'}$

or $(t - k)\overrightarrow{OM} = t\vec{b'} - k\vec{a'}$ ③

Now the same argument can be repeated for points N and R giving

$(k - s)\overrightarrow{ON} = k\vec{a'} - s\vec{c'}$ ④

$(s - t)\overrightarrow{OR} = s\vec{c'} - t\vec{b'}$ ⑤

Adding equations ③, ④, and ⑤ gives the following.

$(t - k)\overrightarrow{OM} + (k - s)\overrightarrow{ON} + (s - t)\overrightarrow{OR} = \vec{0}$

or, $(k - s)\overrightarrow{ON} = -(t - k)\overrightarrow{OM} - (s - t)\overrightarrow{OR}$

that is, $\overrightarrow{ON} = \frac{-t+k}{k-s}\overrightarrow{OM} + \frac{-s+t}{k-s}\overrightarrow{OR}$

The sum of the coefficients on the *R.S.* is

$$\frac{-t+k}{k-s} + \frac{-s+t}{k-s} = \frac{k-s}{k-s} = 1.$$

Thus, points M, N, and R are collinear, as required.

Activities

1. Is there any situation where you could not divide by any one of $t - k$, $k - s$, or $s - t$ in the proof?

2. The proof assumes that $AB \nparallel A'B'$, $BC \nparallel B'C'$, and $CA \nparallel C'A'$. If $AB \| A'B'$ and $BC \| B'C'$, then prove that $CA \| C'A'$.

Summary

- A vector that is collinear with (or parallel to) a line is called a *direction vector* of the line.
- The components of a direction vector are called *direction numbers.*
- Special direction numbers associated with the angles a direction vector makes with the coordinate axes are the *direction cosines* of a line.
- A vector that is perpendicular to a line is called a *normal vector* of the line.
- In the following table
  $\vec{r} = \overrightarrow{OP}$, the position vector of *any* point P on the line,
  $\vec{r}_0 = \overrightarrow{OP_0}$ the position vector of a *given* point P_0 on the line,
  $\vec{m}$ is a vector parallel to the line,
  k is any real number called a *parameter*.

Lines	*2-Space*	*3-Space*
vector equation	$\vec{r} = \vec{r}_0 + k\vec{m}$	
parametric equations	$\begin{cases} x = x_0 + km_1 \\ y = y_0 + km_2 \end{cases}$	$\begin{cases} x = x_0 + km_1 \\ y = y_0 + km_2 \\ z = z_0 + km_3 \end{cases}$
symmetric equations	$\frac{x - x_0}{m_1} = \frac{y - y_0}{m_2}$, $m_1, m_2, \neq 0$	$\frac{x - x_0}{m_1} = \frac{y - y_0}{m_2} = \frac{z - z_0}{m_3}$, $m_1, m_2, m_3 \neq 0$
direction numbers	m_1, m_2	m_1, m_2, m_3
direction cosines	$\cos\alpha = \frac{m_1}{\lvert\vec{m}\rvert}$, $\cos\beta = \frac{m_2}{\lvert\vec{m}\rvert}$, $\lvert\vec{m}\rvert = \sqrt{m_1^2 + m_2^2}$	$\cos\alpha = \frac{m_1}{\lvert\vec{m}\rvert}$, $\cos\beta = \frac{m_2}{\lvert\vec{m}\rvert}$, $\cos\gamma = \frac{m_3}{\lvert\vec{m}\rvert}$, $\lvert\vec{m}\rvert = \sqrt{m_1^2 + m_2^2 + m_3^2}$
angle relation	$\cos^2\alpha + \cos^2\beta = 1$	$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$
scalar equation	$Ax + By + C = 0$	none exists (applies to planes in 3-space)
slope	$\frac{m_2}{m_1}$ or $-\frac{A}{B}$	does not exist in 3-space
normal vector	$\overrightarrow{(-m_2, m_1)}$ or $\overrightarrow{(A, B)}$	not unique in 3-space

- In 2-space, two lines can be parallel and distinct, or be parallel and identical, or intersect.

- If a system of linear equations has *one or more solutions*, then the system is said to be *consistent*.
  If a consistent system has *exactly one solution* the system is *independent*.
  If a consistent system has *more than one solution* the system is *dependent*.
  If a system of linear equations has *no solutions*, then the system is said to be *inconsistent*.
  For a system of two equations in two variables

	number of solutions	*geometric description*
consistent and independent	one	two intersecting lines
consistent and dependent	infinite	same line
inconsistent	none	lines are parallel, distinct

- In 3-space, two lines can be parallel, or not parallel. If the lines are not parallel, then they may intersect, or not intersect.
  Two lines in 3-space that are not parallel *and* do not intersect are called *skew lines*.

- The shortest distance d between two skew lines
  L_1, with direction vector $\vec{m}_1$, and L_2, with direction vector $\vec{m}_2$, is the distance between the two points D_1 and D_2 on L_1 and L_2 respectively, where $D_1D_2 \perp L_1$ and $D_1D_2 \perp L_2$.

$$d = \frac{|\overrightarrow{P_1P_2} \cdot \vec{a}|}{|\vec{a}|}$$

where P_1 is a point on L_1, P_2 is a point on L_2, and $\vec{a} = \vec{m}_1 \times \vec{m}_2$.

Using Vectors in Euclidean Geometry

A method of solving geometric problems using vector equations of lines can be found on page 229.

Inventory

1. In the vector equation $\vec{r} = \vec{r}_0 + k\vec{m}$, $\vec{r}$ is _____, $\vec{r}_0$ is _____, $\vec{m}$ is _____, and k is _____.

2. In the vector equation $\vec{r} = \overrightarrow{(1 + 3k, 4 + 5k)}$, a point on the line is _____, a direction vector is _____, the parameter is _____.

3. In the vector equation $\vec{r} = \overrightarrow{(5 - 2t, -1 + 3t, -6t)}$, a point on the line is _____, a direction vector is _____, the parameter is _____.

4. In the parametric equations $x = -4 + 3s$, $y = 2 - 5s$, a point on the line is _____, a direction vector is _____, the parameter is _____, the slope is _____.

5. In the parametric equations $x = -6t$, $y = -1 + t$, $z = 4$, a point on the line is _____, a direction vector is _____, the parameter is _____.

6. In the symmetric equation $\frac{x-3}{4} = \frac{y+2}{-3}$, a point on the line is _____, a direction vector is _____.

7. In the symmetric equations $\frac{x+1}{4} = \frac{y}{4} = z - 3$, a point on the line is _____, a direction vector is _____.

8. For the line $4x + 3y + 7 = 0$, the slope is _____ and a normal vector is _____.

9. For direction cosines of a line, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma =$ _____, where α is _____, β is _____, and γ is _____.

10. A vector with direction cosines as components has a length of _____.

11. Two lines in 2-space that do not intersect must be _____.

12. Two lines in 3-space that do not intersect must be either _____ or _____.

13. For a system of two linear equations that is consistent and dependent, there is (are) _____ solution(s) of the system and the two lines have _____ point(s) of intersection.

14. For a system of two linear equations that is consistent and independent, there is (are) _____ solution(s) of the system and the two lines have _____ point(s) of intersection.

15. For a system of two linear equations that is inconsistent there is (are) _____ solution(s) of the system and the two lines have _____ point(s) of intersection.

16. In the formula $d = \frac{|\overrightarrow{P_1P_2} \cdot \vec{a}|}{|a|}$

d is _____, P_1 is _____, P_2 is _____, and $\vec{a}$ is _____.

17. For points $P_1(3, 2, 1)$ and $P_2(5, 6, 8)$, the vector $\overrightarrow{P_1P_2}$ is _____.

Review Exercises

1. Find a vector equation and parametric equations for each of the following lines.
   a) through the point $A(2,-5)$ with direction vector $\overrightarrow{(3,2)}$
   b) through the points $A(-5,-4)$ and $B(1,-6)$
   c) through the point $A(-7,0)$ with direction numbers 2 and -5
   d) through the point $B(2,1)$ parallel to the line $\vec{r} = \overrightarrow{(3,-8)} + k\overrightarrow{(-3,-2)}$
   e) through the point $C(-4,1)$ and parallel to the x-axis
   f) through the point $C(-4,1)$ and parallel to the y-axis

2. Given the line with parametric equations
$$\begin{cases} x = 4 - t \\ y = 3 + 2t \end{cases}$$
determine whether or not the following points lie on this line.
$A(3,5)$ $B(6,-1)$ $C(5,0)$ $D(2.4,4)$

3. Find a Cartesian equation for each of the lines having the given vector equations.
   a) $\vec{r} = \overrightarrow{(1 - 2k, 3 - 4k)}$
   b) $\vec{r} = \overrightarrow{(3 - 2t, t + 1)}$
   c) $\vec{r} = \overrightarrow{(-2 + s, -4 - s)}$

4. For each of the following, find a scalar equation of the line passing through the given point P_0 and having $\vec{n}$ as a normal vector.
   a) $P_0(-1,5)$ $\vec{n} = \overrightarrow{(4,-1)}$
   b) $P_0(4,0)$ $\vec{n} = \overrightarrow{(8,-2)}$
   c) $P_0(11,-5)$ $\vec{n} = \overrightarrow{(0,4)}$

5. Use normal vectors to decide which pairs of lines are parallel and which pairs of lines are perpendicular.
   a) $5x - 2y = 5$
      $2x + 5y = 9$
   b) $4x - 3y + 1 = 0$
      $-8x + 6y = 80$
   c) $2x + 3y = 0$
      $3x = 2y + 1$

6. There are an infinite number of vectors and hence lines that are coplanar with vectors $\vec{a} = \overrightarrow{(2,0,1)}$ and $\vec{b} = \overrightarrow{(-3,4,-5)}$. Find a vector equation of any one of these lines that passes through the point $A(-3,6,1)$.

7. $OBCD$ is a parallelogram with point O at the origin of a 2-space Cartesian coordinate system. $\overrightarrow{OB} = \vec{b}$ lies along the x-axis. $\overrightarrow{OD} = \vec{d}$.
   a) Show that $\overrightarrow{OC} = \vec{b} + \vec{d}$, and $\overrightarrow{BD} = \vec{d} - \vec{b}$.
   b) Find a vector equation for the line through the points O and C in terms of vectors $\vec{b}$ and $\vec{d}$.
   c) Find a vector equation for the line through the points B and D in terms of vectors $\vec{b}$ and $\vec{d}$.

8. a) A line is parallel to the vector $\vec{m} = \overrightarrow{(4,5)}$. Show that the slope of the line is $\frac{5}{4}$
   b) A line is parallel to the vector $\vec{m} = \overrightarrow{(m_1,m_2)}$. Show that the slope of the line is $\frac{m_2}{m_1}$
   c) Find a vector equation of the line passing through the point $A(3,4)$ that has slope equal to $\frac{2}{7}$

9. Given the line L with vector equation $\vec{r} = \overrightarrow{(1,2)} + k\overrightarrow{(-3,-1)}$ and the point $P_0(-4,3)$.
   a) Find a scalar equation of the line through P_0 parallel to the line L.
   b) Find a scalar equation of the line through P_0 perpendicular to the line L.

10. Find the value of the variable t so that the line with scalar equation $2x - 5y = 3$ is perpendicular to the line with vector equation $\vec{r} = \overrightarrow{(4,2)} + k\overrightarrow{(t,3)}$.

11. Find the value of the variable s so that the following lines are perpendicular.
$$\frac{x-6}{4s+5} = \frac{y+1}{1} \text{ and } \frac{x-3}{2} = \frac{y-1}{-4s-2}$$

12. Find a vector equation, parametric equations, and symmetric equations for each of the following lines.

a) through the point $A(2,3,0)$ with direction vector $\overrightarrow{(4,2,-2)}$

b) through the points $A(-2,3,5)$ and $B(3,-1,6)$

c) through the point $A(1,0,10)$ with direction numbers 1, 2, and -4

d) through the point $B(5,6,10)$, parallel to the line $\vec{r} = \overrightarrow{(6,1,1)} + k\overrightarrow{(-2,1,3)}$

e) through the point $C(-4,3,6)$ and parallel to the x-axis

f) through the point $C(-4,3,6)$ and parallel to the z-axis

13. In each of the following, determine whether or not the given point lies on the line with the given equation.

	point	line equation
a)	$(-1,4,9)$	$\vec{r} = \overrightarrow{(1,3,4)} + k\overrightarrow{(-2,1,5)}$
b)	$(-10,-8,4)$	$\vec{r} = \overrightarrow{(0,1,0)} + k\overrightarrow{(5,4,-1)}$
c)	$(11,8,0)$	$\vec{r} = \overrightarrow{(5+2k,-4+4k,3-k)}$

14. a) Find a vector equation of the line that passes through the points $A(3,-2,1)$ and $B(4,0,7)$.

b) Show that the line with equation $\vec{r} = \overrightarrow{(5,2,13)} + t\overrightarrow{(1,2,6)}$ is the same as the line in part a).

15. Find a vector equation of the line that passes through the point $A(-1,-2,4)$ and is parallel to the line with equation $\vec{r} = \overrightarrow{(0,4,9)} + k\overrightarrow{(-3,1,4)}$.

16. Find a vector equation of the line passing through the point $A(4,1,5)$ that is perpendicular to vector $\vec{u} = \overrightarrow{(0,-1,7)}$ and also perpendicular to vector $\vec{v} = \overrightarrow{(2,4,-3)}$.

17. Find a vector equation of the line with symmetric equations

$$\frac{x-3}{4} = \frac{y+1}{5} = \frac{z+3}{-1}$$

18. Find the value of t so that the two lines $\vec{r} = \overrightarrow{(1,7,3)} + k\overrightarrow{(-2,3,5t)}$ and $\vec{r} = \overrightarrow{(4,1,2)} + s\overrightarrow{(t,1,-3)}$ will be perpendicular.

19. Find the value of the variable k so that the line with symmetric equations $\frac{x-3}{2} = \frac{y+1}{4} = \frac{z+3}{k}$ is perpendicular to the line with vector equation $\vec{r} = \overrightarrow{(5+2ks,3-4s,-8-12s)}$

20. Find the value of the variable t so that the following lines are perpendicular.

$$\frac{x-3}{3t+1} = \frac{y+6}{2} = \frac{z+3}{2t};\ \frac{x+7}{3} = \frac{y+8}{-2t} = \frac{z+9}{-3}$$

21. Which of the following triples of numbers can be the direction cosines of a line?

a) $\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}$

b) $\frac{5}{8}, \frac{1}{3}, \frac{1}{2}$

c) $1, -\frac{1}{2}, \frac{1}{2}$

d) $\cos A, \sin A, 0$

22. a) A line passing though the point $(1,-2,3)$ has direction cosines $\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}$ and $\cos\gamma$, where $\cos\gamma$ is positive. Find a vector equation of the line.

b) Repeat part a) but take $\cos\gamma$ to be negative.

23. If vectors $\vec{a}$ and $\vec{b}$ are linearly dependent prove that their direction cosines are the same.

24. Given the vectors $\vec{a} = \overrightarrow{(1,2,3)}$ and $\vec{b} = \overrightarrow{(-2,0,1)}$ find the direction cosines of any vector coplanar with $\vec{a}$ and $\vec{b}$.

25. Find direction cosines for each of the lines given by the following.

a) $\vec{r} = k\overrightarrow{(2,-3,4)}$

b) $\begin{cases} x = 1 - 2a \\ y = 7 + 5a \\ z = 5 + a \end{cases}$

c) $\frac{x}{2} = \frac{y-1}{-1} = \frac{z-2}{3}$

26. A direction vector of a line makes an angle of 40° with the x-axis and an angle of 70° with the z-axis. Find, correct to the nearest degree, the angle the direction vector makes with the y-axis.

27. A line through the origin of a 3-space coordinate system makes an angle of 25° with the x-axis and an angle of 85° with the z-axis. Find a vector equation of the line.

28. For each of the following systems determine whether or not the corresponding lines are parallel. If the lines are not parallel, find their point of intersection.

a) $3x - 2y = 4$
$4x + y = 9$

b) $-x + 6y = -1$
$3x - 18y = 5$

c) $4x - 3y = 5$
$8x - 6y = 10$

29. For each of the systems in the previous question, indicate which of the following terms apply.
consistent and independent
consistent and dependent
inconsistent

30. For each of the following pairs of lines in 2-space, determine whether or not the lines are parallel. If the lines are parallel, then determine whether the lines are distinct or the same line. If the lines are not parallel, find their point of intersection.

a) $\vec{r} = \overrightarrow{(4 + k, 5 - 3k)}$
$\vec{r} = \overrightarrow{(4 + 2t, 3 - 4t)}$

b) $\vec{r} = \overrightarrow{(11 - 6k, -5k)}$
$\vec{r} = \overrightarrow{(2 + 5s, 8 - s)}$

c) $\vec{r} = \overrightarrow{(1 + 3a, 4 - 2a)}$
$\vec{r} = \overrightarrow{(2 - 6b, -5 + 4b)}$

31. a) Find the point of intersection of the line $\vec{r} = \overrightarrow{(-4,2)} + k\overrightarrow{(1,6)}$ and the line $\dfrac{x+5}{3} = \dfrac{y-10}{4}$.

b) Find the angle between the given lines.

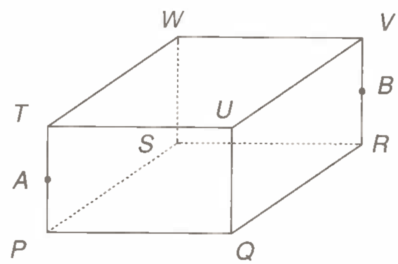
32. $OBCD$ is a parallelogram with point O at the origin of a 2-space Cartesian coordinate system. $\overrightarrow{OB} = \vec{b}$ lies along the x-axis. $\overrightarrow{OD} = \vec{d}$. Find a vector equation for the line through the points B and D in terms of vectors $\vec{b}$ and $\vec{d}$.

33. In the figure, A and B are the midpoints of the sides TP and VR respectively of a rectangular box. The corner S appears to lie on the line though A and B. The coordinates of five of the corners are $P(0,0,0)$, $Q(2,0,0)$, $R(2,4,0)$, $V(2,4,3)$, and $W(0,4,3)$.

a) Find the coordinates of the remaining three corners S, T, and U.

b) Find the coordinates of the points A and B.

c) Prove that the point S does not lie on the line through A and B.



34. For each of the following pairs of lines in 3-space determine whether or not the lines are parallel. If the lines are not parallel, determine whether the lines intersect or are skew. If the lines intersect, find the point of intersection.

a) $\vec{r} = \overrightarrow{(3 + k, 7 - 2k, 4 + 3k)}$
$\vec{r} = \overrightarrow{(-2 + 2t, 1 + 4t, 11 - 5t)}$

b) $\vec{r} = \overrightarrow{(2 + 3k, 1 + 4k, 4 + k)}$
$\vec{r} = \overrightarrow{(3 + b, 1 + 2b, -2 + b)}$

c) $\vec{r} = \overrightarrow{(-1 + a, 4 - 3a, -6 + a)}$
$\vec{r} = \overrightarrow{(2 - 3t, -5 + 9t, -3 - 3t)}$

d) $\vec{r} = \overrightarrow{(6 - 2w, -9 + 4w, 7 + w)}$
$\vec{r} = \overrightarrow{(7 - 5k, -3 + 2k, -3 - 4k)}$

35. For each of the pairs of lines in the previous question, indicate which of the following terms apply to the corresponding linear system.
consistent and independent
consistent and dependent
inconsistent

36. Find the point of intersection of the line $\vec{r} = \overrightarrow{(0,-8,4)} + k\overrightarrow{(2,7,-1)}$ and the line through the points $A(8,4,-1)$ and $B(0,8,5)$.

37. Find the point of intersection of the lines $\frac{x+1}{4} = \frac{y+2}{-2} = \frac{z}{3}$; $\frac{x+7}{5} = \frac{y+10}{3} = \frac{z-5}{-1}$.

38. a) Prove that the lines
$\vec{r} = \overrightarrow{(1+k, 4+3k, 3-2k)}$ and
$\vec{r} = \overrightarrow{(-3+3t, 6+2t, -3t)}$ are skew lines.
b) Find the distance between the given skew lines.

39. Given the lines $\vec{r} = \overrightarrow{(-1+k, 3+2k, 1-3k)}$ and $\vec{r} = \overrightarrow{(3+2w, 10+3w, -6-w)}$.
a) Prove that the lines intersect.
b) Find the angle between the lines.

40. Prove that the following three lines intersect and lie in the same plane.
L_1: $\vec{r} = \overrightarrow{(-1,0,2)} + k\overrightarrow{(3,1,1)}$
L_2: $\vec{r} = \overrightarrow{(-3,-4,1)} + k\overrightarrow{(-4,2,-1)}$
L_3: $\vec{r} = \overrightarrow{(-5,2,1)} + k\overrightarrow{(2,4,1)}$

41. Prove that the following three lines intersect and do not lie in the same plane.
L_1: $\vec{r} = \overrightarrow{(4,1,1)} + k\overrightarrow{(2,1,2)}$
L_2: $\vec{r} = \overrightarrow{(9,1,4)} + k\overrightarrow{(-1,2,1)}$
L_3: $\vec{r} = \overrightarrow{(-1,0,-1)} + k\overrightarrow{(3,1,2)}$

42. a) Find the shortest distance between the skew lines

$$x = 3, \frac{y}{-2} = \frac{z-5}{1} \text{ and}$$

$$\frac{x+2}{3} = \frac{y+1}{2} = \frac{z+1}{-2}$$

b) Find a symmetric equation of the line joining the points on the lines that are at this shortest distance.

43. Find symmetric equations for the following lines.
a) $x - 3y + 6 = 0 = x - 2z - 4$
b) $2x + z = 5 - y = y + z$

Use vector equations of lines to solve problems 44–48.

44. *ABCD* is a parallelogram. *E* is the point that divides side *AD* in the ratio 4:7. Segments *AC* and *BE* intersect at point *F*. Find the ratio into which *AC* divides *BE*.

45. In a parallelogram *ABCD, H* is the midpoint of *AD*, and *E* divides *BC* in the ratio 3:2. If *BH* and *AE* intersect at *M*, find the ratio *AM* : *ME*.

46. In $\triangle ABC$, *E* is the point that divides side *AB* into the ratio 3:2. Point *F* is on side *AC* such that segment *EF* is parallel to side *BC*. Into what ratio does *F* divide side *AC*?

47. In $\triangle ABC$, *D* divides *AB* in the ratio 3:2 and *E* divides *AC* in the ratio 5:4. *BE* and *CD* intersect at point *F*. Find the ratios into which *F* divides each of *BE* and *CF*.

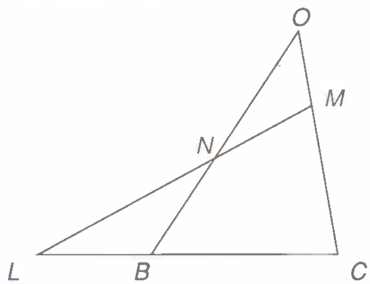
48. In parallelogram *PQRS*, *A* divides *PQ* in the ratio 2:1, and *B* divides *SR* in the ratio 3:4. Segments *PR* and *AB* intersect at *C*. Find the ratio into which *C* divides segment *PR*.

49. The point *P* with coordinates (1,1,3) lies on the line *L* with equation $2x = y + 1 = z - 1$. Find the coordinates of two points Q_1 and Q_2 on *L* such that $PQ_1 = PQ_2 = 6$.
(87 H)

50. Two lines L_1 and L_2 have equations
$\frac{x+3}{3} = \frac{y+4}{2} = \frac{z-6}{-2}$ and
$\frac{x-4}{-3} = \frac{y+7}{4} = \frac{z+3}{-1}$ respectively.
a) Find the coordinates of a point P_1 on the line L_1 and a point P_2 on the line L_2 such that the line (P_1P_2) is perpendicular to both the lines L_1 and L_2.
b) Show that the length of (P_1P_2) is 7.
(84 H)

51 i) The lines l_1 and l_2 have equations
$\vec{r} = \overrightarrow{(2,3,1)} + \lambda\overrightarrow{(1,2,2)}$ and
$\vec{r} = \overrightarrow{(5,-1,-13)} + \mu\overrightarrow{(-2,1,6)}$
respectively. Show that these lines intersect by finding the co-ordinates of their point of intersection.
Find, in the form $ax + by + cz + d$, the equation of the plane which contains both l_1 and l_2.

ii)



In the above diagram $\overrightarrow{OM} = \frac{1}{3}\overrightarrow{OC}$ and $\overrightarrow{ON} = \frac{2}{3}\overrightarrow{OB}$. The line (CB) intersects the line (MN) at L. If $\overrightarrow{LB} = \lambda\overrightarrow{BC}$ find the value of λ.

(82 H)

52 Find the position vector of the point of intersection of the lines with equations

$$\vec{r} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \text{ and}$$

$$\vec{r} = \begin{pmatrix} 5 \\ 6 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

(85 H)

53 The lines l_1 and l_2 have equations
$\frac{x+2}{1} = \frac{y-1}{-2} = \frac{z}{3}$ and
$\frac{x-2}{2} = \frac{y-1}{4} = \frac{z-7}{1}$ respectively. Which one of the following is true?

A. l_1 and l_2 are parallel.
B. l_1 and l_2 are perpendicular.
C. l_1 and l_2 have no common point.
D. l_1 and l_2 intersect at the point with co-ordinates $(-2,1,0)$.
E. l_1 and l_2 intersect at the point with co-ordinates $(0,-3,6)$.

(85 H)

54 Three points A, B and C are given whose coordinates in a rectangular Cartesian system are $(0,9)$, $(6,7)$ and $(8,3)$ respectively.

a) Find the equation of the line l_1 perpendicular to $\overrightarrow{BC}$ and passing through the point A.
b) Find the equation of the line l_2 perpendicular to $\overrightarrow{CA}$ and passing through the point B.
c) Find the coordinates of H, the point of intersection of the lines l_1 and l_2.
d) Verify that $\overrightarrow{HC} \cdot \overrightarrow{AB} = 0$.
e) What geometrical result has now been established for $\triangle ABC$?
f) Find the coordinates of the point S such that $\overrightarrow{SH} = \overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC}$.

(88 S)

VECTORS, MATRICES and COMPLEX NUMBERS

with
International Baccalaureate
questions

Jean-Paul GINESTIER
and
John EGSGARD

CHAPTER SIX

EQUATIONS OF PLANES

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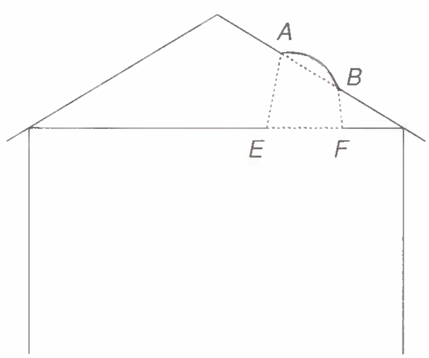
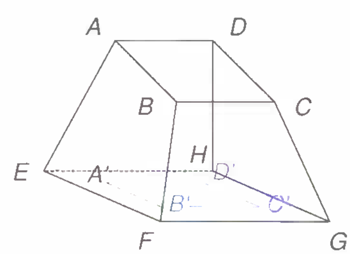
CHAPTER SIX

Equations of Planes

You have a house with a second floor room that does not receive much sunlight. You decide to increase the sunlight by building a skylight in the roof. In order for the light to pass through the skylight into the room, you must also make a hole in the ceiling of the room, as shown in the figure.

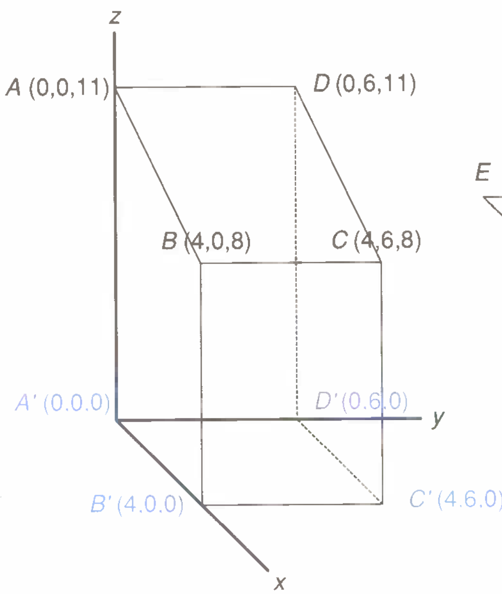


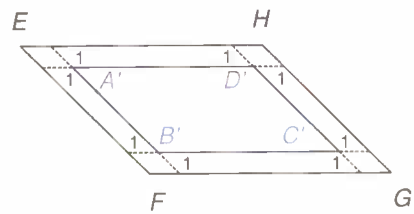


You decide that the hole *ABCD* in the roof and the hole *EFGH* in the ceiling will both be rectangular in shape. For maximum light, you wish the opening *EFGH* in the ceiling to be larger than the projection $A'B'C'D'$ of *ABCD* on the ceiling. Further, you will enclose the opening joining the roof and the ceiling using four flat walls. Can this be done?

You cut a rectangular hole in the roof the size of the skylight that you have bought, mark the projection $A'B'C'D'$ of this hole on the ceiling and then draw on the ceiling a rectangle *EFGH* larger than this projection. Before cutting the hole in the ceiling you check to see if points *ABFE* are coplanar. Unfortunately you discover that they are not. You draw a different rectangle *EFGH* with the same result. You wonder if it is possible to draw a rectangle on the ceiling larger that the projection of *ABCD* so that the sets of points $\{A,B,F,E\}$, $\{A,D,H,E\}$, $\{C,D,H,G\}$, and $\{B,C,G,F\}$ will be coplanar sets.

Suppose the axes of a 3-space coordinate system are placed with origin at point A' as shown in the figure.





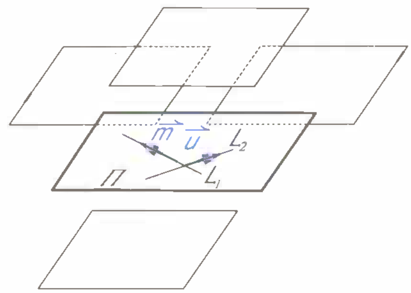
Let the coordinates of the corners of the opening in the roof be $A(0,0,11)$, $B(4,0,8)$, $C(4,6,8)$, and $D(0,6,11)$. Then the corners of the projection of *ABCD* will have coordinates $A'(0,0,0)$, $B'(4,0,0)$, $C'(4,6,0)$, and $D'(0,6,0)$. You try again making an opening in the ceiling by moving each corner one unit parallel to the x-axis and one unit parallel to the y-axis as shown. Thus, the corners will have coordinates $E(-1,-1,0)$, $F(5,-1,0)$, $G(5,7,0)$ and $H(-1,7,0)$. Will the sets of points $\{A,B,F,E\}$, $\{A,D,H,E\}$, $\{C,D,H,G\}$, and $\{B,C,G,F\}$ be coplanar sets?

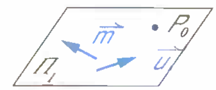
If, instead, you had moved each corner one unit parallel to the x-axis and zero units parallel to the y-axis so that the corners had coordinates $E(-1,0,0)$, $F(5,0,0)$, $G(5,6,0)$, and $H(-1,6,0)$, would this have provided coplanar sets?

In this chapter you will learn how to find vector and Cartesian equations of planes. You will be able to use such equations to answer the above questions about coplanar points.

6.1 Vector and Parametric Equations of a Plane in 3-Space

If two lines L_1 and L_2 in 3-space are not parallel and intersect in a single point, then the two lines determine a unique plane. Denote this plane by the symbol Π.

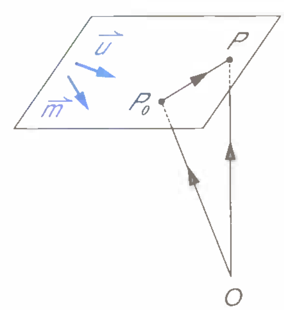




A vector $\vec{m}$ parallel to L_1 can be translated so that the vector lies in the plane Π along L_1. Likewise a vector $\vec{u}$ parallel to L_2 can be translated so that the vector lies in the plane Π along L_2. Also $\vec{m}$ and $\vec{u}$ can be translated together off of plane Π so that they will lie in any plane parallel to plane Π. Thus, a pair of non parallel vectors $\vec{m}$ and $\vec{u}$ determines a family of parallel planes. To obtain a single member Π_1 of this family, you need only give a point P_0 that lies in plane Π_1. You will now use this idea to obtain the vector and parametric equations of a plane.

Suppose a plane passes through the point P_0 and is parallel to the linearly independent (non-parallel) vectors $\vec{m}$ and $\vec{u}$.

Since $\vec{m}$ is not parallel to $\vec{u}$,
then $\vec{m}$ and $\vec{u}$ determine a family of parallel planes.
Let P be any point on the member of this family that passes through the point P_0.



Thus, $\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$
Let $\overrightarrow{OP} = \vec{r}$ and $\overrightarrow{OP_0} = \vec{r}_0$

Since $\overrightarrow{P_0P}$ lies in the plane of $\vec{m}$ and $\vec{u}$, and $\vec{m} \nparallel \vec{u}$, real numbers k and s exist such that

$\overrightarrow{P_0P} = k\vec{m} + s\vec{u}$

Thus, a vector equation of the plane through P_0 is

$$\vec{r} = \vec{r}_0 + k\vec{m} + s\vec{u} \quad ①$$

Suppose the coordinates of P are (x,y,z), and of P_0 are (x_0,y_0,z_0).
Let vector $\vec{m} = \overrightarrow{(m_1,m_2,m_3)}$ and $\vec{u} = \overrightarrow{(u_1,u_2,u_3)}$.
Thus $\overrightarrow{OP} = \overrightarrow{(x,y,z)}$ and $\overrightarrow{OP_0} = \overrightarrow{(x_0,y_0,z_0)}$.
Thus ① becomes

$$\overrightarrow{(x,y,z)} = \overrightarrow{(x_0,y_0,z_0)} + k\overrightarrow{(m_1,m_2,m_3)} + s\overrightarrow{(u_1,u_2,u_3)}$$

or $\overrightarrow{(x,y,z)} = \overrightarrow{(x_0,y_0,z_0)} + \overrightarrow{(km_1,km_2,km_3)} + \overrightarrow{(su_1,su_2,su_3)}$,

or $\overrightarrow{(x,y,z)} = \overrightarrow{(x_0 + km_1 + su_1, y_0 + km_2 + su_2, z_0 + km_3 + su_3)}$

Equating components on left side and right side, you obtain the following parametric equations of a plane.

$$\begin{cases} x = x_0 + km_1 + su_1 \\ y = y_0 + km_2 + su_2 \\ z = y_0 + km_3 + su_3 \end{cases}$$

SUMMARY

For a plane in 3-space

vector equation

$$\vec{r} = \vec{r}_0 + k\vec{m} + s\vec{u}$$

parametric equations

$$\begin{cases} x = x_0 + km_1 + su_1 \\ y = y_0 + km_2 + su_2 \\ z = z_0 + km_3 + su_3 \end{cases}$$

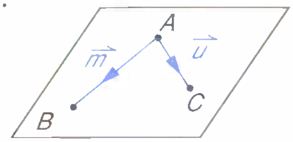
where $\vec{r} = \overrightarrow{OP}$, the position vector of *any* point $P(x,y,z)$ on the plane
$\vec{r}_0 = \overrightarrow{OP_0}$, the position vector of a *fixed* point $P_0(x_0,y_0,z_0)$ on the plane
$\vec{m} = \overrightarrow{(m_1,m_2,m_3)}$ and $\vec{u} = \overrightarrow{(u_1,u_2,u_3)}$ are vectors parallel to the plane such that $\vec{m} \nparallel \vec{u}$,
and k and s are parameters.

Example 1 Find vector and parametric equations of the plane passing through the point $P_0(1,2,3)$, and parallel to the vectors $\vec{m} = \overrightarrow{(3,0,-1)}$ and $\vec{u} = \overrightarrow{(-1,4,5)}$.

Solution The vector equation of a plane is $\vec{r} = \vec{r}_0 + k\vec{m} + s\vec{u}$
Here $\vec{r}_0 = \overrightarrow{OP_0} = \overrightarrow{(1,2,3)}$, $\vec{m} = \overrightarrow{(3,0,-1)}$ and $\vec{u} = \overrightarrow{(-1,4,5)}$.
Substituting gives $\vec{r} = \overrightarrow{(1,2,3)} + k\overrightarrow{(3,0,-1)} + s\overrightarrow{(-1,4,5)}$
or $\vec{r} = \overrightarrow{(1,2,3)} + \overrightarrow{(3k,0,-1k)} + \overrightarrow{(-s,4s,5s)}$
or $\vec{r} = \overrightarrow{(1 + 3k - s, 2 + 4s, 3 - k + 5s)}$ ①
which is a vector equation of the given plane.
To find the parametric equations of the given plane, use the fact that $\vec{r} = \overrightarrow{OP} = \overrightarrow{(x,y,z)}$, then equate components on the left side and right side of the equation.
Thus, equation ① becomes
$\overrightarrow{(x,y,z)} = \overrightarrow{(1 + 3k - s, 2 + 4s, 3 - k + 5s)}$

Thus, the parametric equations of the given plane are $\begin{cases} x = 1 + 3k - s \\ y = 2 + 4s \\ x = 3 - k + 5s \end{cases}$ ■

Example 2 Find a vector equation of the plane that contains the three points $A(1,2,3)$, $B(4,-1,8)$, and $C(0,-2,4)$.

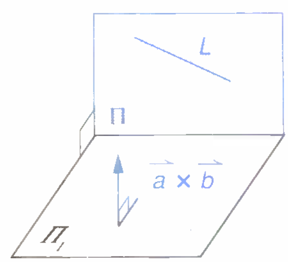


Solution A vector equation of the plane is $\vec{r} = \vec{r_0} + k\vec{m} + s\vec{u}$.
Here, $\vec{r_0}$ can be the position vector of any one of the points A, B, or C.
Take $\vec{r_0} = \overrightarrow{OA} = \overrightarrow{(1,2,3)}$.
Also, $\vec{m}$ and $\vec{u}$ can be any two of the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, or $\overrightarrow{BC}$.
Take $\vec{m} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{(4,-1,8)} - \overrightarrow{(1,2,3)} = \overrightarrow{(3,-3,5)}$, and
$\vec{u} = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \overrightarrow{(0,-2,4)} - \overrightarrow{(1,2,3)} = \overrightarrow{(-1,-4,1)}$.

(Notice that $\vec{m}$ is not a scalar multiple of $\vec{u}$, so these vectors can be used in the equation. What would happen if $\vec{m}$ and $\vec{u}$ were collinear?)

Substituting in $\vec{r} = \vec{r_0} + k\vec{m} + s\vec{u}$ gives
$$\vec{r} = \overrightarrow{(1,2,3)} + k\overrightarrow{(3,-3,5)} + s\overrightarrow{(-1,-4,1)}.$$
Hence a vector equation of the plane through the given points is
$$\vec{r} = \overrightarrow{(1 + 3k - s, 2 - 3k - 4s, 3 + 5k + s)}.$$ ■

Example 3 Find a vector equation of the plane Π that contains the line
L: $\vec{r} = \overrightarrow{(3,1,0)} + t\overrightarrow{(2,1,4)}$
and is perpendicular to the plane
Π_1: $\vec{r} = \overrightarrow{(1,1,1)} + k\overrightarrow{(1,0,5)} + s\overrightarrow{(-4,2,3)}$.



Solution Since L lies in plane Π, the point $P_0(3,1,0)$ lies in Π, and the direction vector $\vec{m} = \overrightarrow{(2,1,4)}$ of L is parallel to Π.
Since $\Pi \perp \Pi_1$, any vector that is *perpendicular* to plane Π_1 is *parallel* to plane Π.
Since $\vec{a} = \overrightarrow{(1,0,5)}$ and $\vec{b} = \overrightarrow{(-4,2,3)}$ lie in plane Π_1, then
$\vec{a} \times \vec{b} = \overrightarrow{(-10,-23,2)}$ is perpendicular to plane Π_1 and must be parallel to plane Π.
Thus, the equation of the plane Π is $\vec{r} = \vec{r_0} + k\vec{m} + s\vec{u}$, where $\vec{r_0} = \overrightarrow{(3,1,0)}$, $\vec{m} = \overrightarrow{(2,1,4)}$, and $\vec{u} = \vec{a} \times \vec{b} = \overrightarrow{(-10,-23,2)}$. (Notice $\vec{m} \neq k\vec{u}$.)

Substituting and simplifying gives the following equation for plane Π.
$$\vec{r} = \overrightarrow{(3 + 2k - 10s, 1 + k - 23s, 4k + 2s)}.$$ ■

6.1 Exercises

1. For each of the following vector equations in 3-space indicate whether the equation defines a line or a plane.

   a) $\vec{r} = \overrightarrow{(3,4,5)} + k\overrightarrow{(0,-4,6)} + s\overrightarrow{(2,1,5)}$
   b) $\vec{r} = \overrightarrow{(0,1,9)} + s\overrightarrow{(2,3,-4)} + t\overrightarrow{(-1,-2,9)}$
   c) $\vec{r} = \overrightarrow{(3,4,5)} + t\overrightarrow{(6,-3,5)}$
   d) $\vec{r} = \overrightarrow{(4 + 3k + 2s, -2 + 4k, -k + 5s)}$

2. For each of the equations in question 1 that represent a plane, state the coordinates of a point that lies on the plane, and the components of two vectors that are parallel to the plane.

3. Find a vector equation and parametric equations for each of the following planes.
   a) the plane through the point $A(3,2,7)$ that is parallel to the vectors $\overrightarrow{(4,6,2)}$ and $\overrightarrow{(0,-2,5)}$
   b) the plane through the two points $B(2,-3,1)$ and $C(-3,0,2)$ that is parallel to the vector $\overrightarrow{(2,3,1)}$
   c) the plane through the three points $A(3,2,-7)$, $B(2,-3,1)$, and $C(-3,0,2)$

4. Given the plane $\Pi: \vec{r} = \overrightarrow{(3,2,1)} + k\overrightarrow{(2,-1,4)} + s\overrightarrow{(0,5,-6)}$
   a) Find the point on the plane corresponding to the parameters $k = 1$ and $s = -2$.
   b) Prove that the point $D(1,13,-15)$ lies on plane Π.
   c) Prove that the point $E(5,6,7)$ does not lie on plane Π.

5. Find a vector equation of the plane that is parallel to the line $L_1: \vec{r} = \overrightarrow{(2 - t, 3 + 2t, -2 + 4t)}$ and also contains the line $L_2: \vec{r} = \overrightarrow{(4 + 3s, 2s, 5 - 6s)}$

6. Find a vector equation of the plane that contains the point $A(3,2,1)$ and is perpendicular to the vector $\vec{u} = \overrightarrow{(3,2,5)}$.

7. Find a vector equation of the plane that contains the line $L: \vec{r} = \overrightarrow{(5,6,7)} + t\overrightarrow{(1,0,5)}$ and is perpendicular to the plane $\Pi_1: \vec{r} = \overrightarrow{(0,0,5)} + k\overrightarrow{(-1,3,-2)} + s\overrightarrow{(3,-4,6)}$.

8. Find a vector equation of the plane parallel to the vector $\overrightarrow{(3,4,-8)}$ that contains the line $\frac{x-2}{5} = \frac{y+3}{-2} = \frac{z-1}{4}$

9. Show that the line $\vec{r} = \overrightarrow{(2 - s, 6 + 2s, 1 + 3s)}$ lies in the plane $\vec{r} = \overrightarrow{(1 + k - 2t, 8 - 2k + 5t, 4 - 3k + 2t)}$

10. Given the plane Π with parametric equations
$$\begin{cases} x = 2 + 3k + 4s \\ y = 5 - 6k - 2s \\ z = 1 + 2k + s \end{cases}$$
   a) Find a vector equation of the plane Π.
   b) Find a vector equation of the plane that is parallel to plane Π and passes through the point $A(1,2,-3)$.

11. Find a vector equation of the plane that has intercepts of 3, −2, 7 on the x-axis, the y-axis, and the z-axis respectively.

12. Find a vector equation of the plane that passes through the points $A(2,3,1)$ and $B(5,-4,2)$ that does *not* intersect the z-axis.

13. a) There is an infinite number of planes passing through the points $C(3,5,8)$ and $D(7,1,-2)$ that are parallel to the vector $\vec{m} = \overrightarrow{(-2,2,5)}$. Explain.
    b) Find a vector equation of any one of the planes in part a).

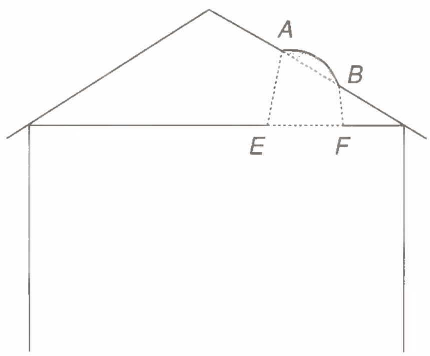
14. Find a vector equation of the plane that passes through the point $A(2,3,1)$ and is perpendicular to the line with parametric equations
$$\begin{cases} x = 2 + 3k \\ y = 5 - 6k \\ z = 1 + 4k \end{cases}$$

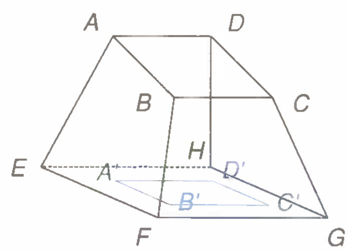
15. Given the four points $A(1,3,2)$, $B(3,4,0)$, $C(5,-1,10)$, and $D(4,3,2)$, prove that the four points are coplanar by finding a vector equation of the plane through any three of the points then proving that the fourth point lies on this plane.

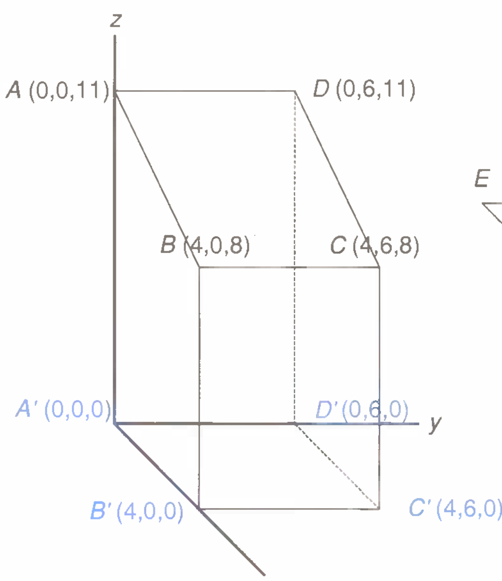
16. In the introduction to this chapter you wondered if the sets of points $\{A,B,F,E\}$, $\{A,D,H,E\}$, $\{C,D,H,G\}$, and $\{B,C,G,F\}$ would be coplanar sets.

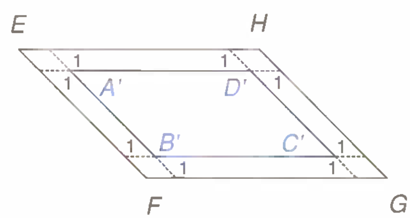
The coordinates of the corners of the roof opening are $A(0,0,11)$, $B(4,0,8)$, $C(4,6,8)$, and $D(0,6,11)$. The corners of the ceiling opening are $E(-1,-1,0)$, $F(5,-1,0)$, $G(5,7,0)$, and $H(-1,7,0)$. Determine which sets of points, if any, are coplanar.

17. If the points E, F, G, and H of question 16 have coordinates $E(-1,0,0)$, $F(5,0,0)$, $G(5,6,0)$, and $H(-1,6,0)$, determine which sets of points, if any, are coplanar.









6.2 The Cartesian Equation of a Plane in 3-Space

In 2-space a Cartesian linear equation such as $3x + 5y + 6 = 0$ represents a straight line. You already know that more than one Cartesian linear equation is needed in 3-space to define a line. For example,

$\frac{x-5}{2} = \frac{y-6}{4} = \frac{z-1}{3}$ represents a straight line through the point (5,6,1) with direction numbers 2, 4, and 3. You will now learn that a single Cartesian linear equation in 3-space defines a plane.

Recall that a normal vector to a line is perpendicular to every vector parallel to that line.

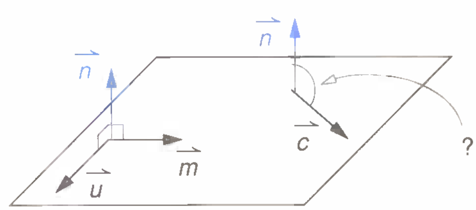
DEFINITION A normal vector to a plane is a vector that is perpendicular to every vector parallel to the plane.

Suppose you place two pencils on a flat surface as a representation of two coplanar vectors. Now hold a third pencil perpendicular to the first two pencils. Any fourth pencil on the surface can be moved parallel to itself to touch the third pencil. Notice the third and fourth pencils are also perpendicular. You will prove this in the following example.

Example 1 $\vec{n}$ is a vector perpendicular to two non-collinear vectors $\vec{m}$ and $\vec{u}$. Prove that $\vec{n}$ is perpendicular to any other vector coplanar with $\vec{m}$ and $\vec{u}$.

Solution Since $\vec{n}$ is perpendicular to both $\vec{m}$ and $\vec{u}$,
$\vec{n} \cdot \vec{m} = 0$, and $\vec{n} \cdot \vec{u} = 0$.
If $\vec{c}$ is coplanar with $\vec{m}$ and $\vec{u}$ then real numbers k and s exist such that
$\vec{c} = k\vec{m} + s\vec{u}$
To prove $\vec{n}$ and $\vec{c}$ are perpendicular you need to prove that $\vec{n} \cdot \vec{c} = 0$.
But

$$\begin{aligned}\vec{n} \cdot \vec{c} &= \vec{n} \cdot (k\vec{m} + s\vec{u}) \\ &= \vec{n} \cdot (k\vec{m}) + \vec{n} \cdot (m\vec{u}) \\ &= k(\vec{n} \cdot \vec{m}) + s(\vec{n} \cdot \vec{u}) \\ &= k(0) + s(0) = 0\end{aligned}$$

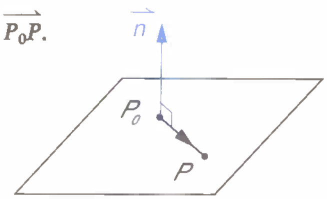


Hence, any vector perpendicular to two non-collinear vectors $\vec{m}$ and $\vec{u}$ is perpendicular to any other vector coplanar with $\vec{m}$ and $\vec{u}$. ■

There is an infinite number of planes having the same normal vector. All of these planes are parallel to each other. To obtain a unique plane with a given normal vector, you only need to find one point on the plane. You will use this idea in the next example to find the Cartesian equation of a plane.

Example 2 Find a Cartesian equation of the plane, passing through the point $P_0(1,2,3)$, that has $\vec{n} = \overrightarrow{(4,5,6)}$ as a normal vector.

Solution Let $P(x,y,z)$ be any point on this plane.
Since $\vec{n} = \overrightarrow{(4,5,6)}$ is a normal vector to the plane, $\vec{n}$ is perpendicular to every vector in the plane.
In particular, $\vec{n}$ is perpendicular to the vector $\overrightarrow{P_0P}$.
Thus, $\vec{n} \cdot \overrightarrow{P_0P} = 0$
Since $\overrightarrow{P_0P} = \overrightarrow{(x-1,y-2,z-3)}$
$\overrightarrow{(4,5,6)} \cdot \overrightarrow{(x-1,y-2,z-3)} = 0$
or $4x - 4 + 5y - 10 + 6z - 18 = 0$



Thus, $4x + 5y + 6z - 32 = 0$ is a Cartesian equation of the given plane. ■

Note: The coefficients 4, 5, and 6 of x, y, and z in the Cartesian equation of the plane are the direction numbers of the normal vector to the plane, $\vec{n} = \overrightarrow{(4,5,6)}$. You will now see that this is true for any Cartesian equation written in the form $Ax + By + Cz + D = 0$.

In the general case, to find a Cartesian equation of the plane passing through the point $P_0(x_0,y_0,z_0)$ that has $\vec{n} = \overrightarrow{(A,B,C)}$ as a normal vector, proceed as follows.

Let $P(x,y,z)$ be any point on this plane.
Since $\vec{n} = \overrightarrow{(A,B,C)}$ is a normal vector to the plane, $\vec{n}$ is perpendicular to every vector in the plane.
In particular, $\vec{n}$ is perpendicular to the vector $\overrightarrow{P_0P}$.
Thus, $\vec{n} \cdot \overrightarrow{P_0P} = 0$
Since $\overrightarrow{P_0P} = \overrightarrow{(x-x_0,y-y_0,z-z_0)}$

$$\overrightarrow{(A,B,C)} \cdot \overrightarrow{(x-x_0,y-y_0,z-z_0)} = 0$$

or $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$

or $Ax + By + Cz + (-Ax_0 - By_0 - Cz_0) = 0$

If the number $(-Ax_0 - By_0 - Cz_0)$ is replaced by the constant D, this equation becomes $Ax + By + Cz + D = 0$.

Thus, $Ax + By + Cz + D = 0$ is a Cartesian equation of a plane.

PROPERTIES

1. The vector $\overrightarrow{(A,B,C)}$ is a normal vector for the plane $Ax + By + Cz + D = 0$.
2. If $\vec{m}$ and $\vec{u}$ are two linearly independent vectors parallel to the plane $Ax + By + Cz + D = 0$, then $\overrightarrow{(A,B,C)}$ is any scalar multiple of the vector $\vec{m} \times \vec{u}$.

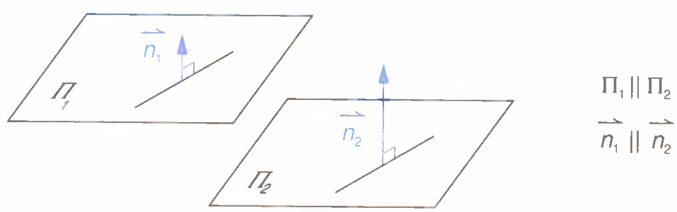
Example 3 Find a Cartesian equation of the plane containing the points $Q(1,2,3)$, $R(-2,3,6)$, and $S(3,0,1)$.

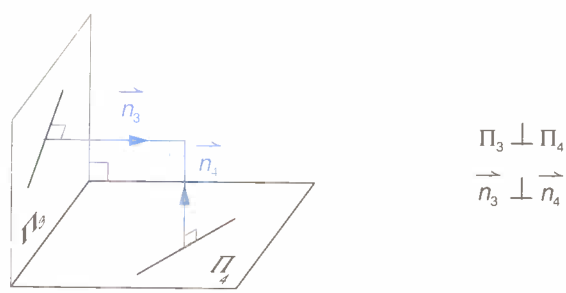
Solution Let the equation of the required plane be $Ax + By + Cz + D = 0$.
The normal vector $\overrightarrow{(A,B,C)}$ can be taken as $\vec{m} \times \vec{u}$ where $\vec{m}$ and $\vec{u}$ are any two of the vectors $\overrightarrow{QR}$, $\overrightarrow{QS}$ or $\overrightarrow{RS}$ provided $\vec{m} \nparallel \vec{u}$.
Take $\vec{m} = \overrightarrow{QR} = \overrightarrow{(-3,1,3)}$ and $\vec{u} = \overrightarrow{QS} = \overrightarrow{(2,-2,-2)}$.
Thus, $\overrightarrow{(A,B,C)} = \overrightarrow{(-3,1,3)} \times \overrightarrow{(2,-2,-2)} = \vec{i}(6-2) - \vec{j}(6-6) + \vec{k}(6-2) = \overrightarrow{(4,0,4)}$
Substituting $A = 4$, $B = 0$, and $C = 4$ into $Ax + By + Cz + D = 0$ gives
$4x + 4z + D = 0$.
To find D it is only necessary to substitute the coordinates of one of the points Q, R, or S into this equation.
Using point Q so that $x = 1$, $y = 2$, and $z = 3$ gives
$4(1) + 4(3) + D = 0$
or $D = -16$

Hence an equation of the plane through the three points Q, R, and S is $4x + 4z - 16 = 0$. This equation can be simplified by dividing each side by 4 to obtain $x + z - 4 = 0$. ■

You should convince yourself that the following facts are true about parallel and perpendicular planes by using pencils as normal vectors and flat pieces of cardboard as planes.

Facts about Parallel and Perpendicular Planes





1. Two planes are *parallel* if and only if their *normal vectors are parallel*.
2. Two planes are *perpendicular* if and only if their *normal vectors are perpendicular*.

Example 4 Find a Cartesian equation of the plane Π that contains the points $M(1,2,3)$ and $T(-2,3,0)$ and is perpendicular to the plane Π_1: $3x + y - 6z = 5$.

Solution Let an equation of plane Π be $Ax + By + Cz + D = 0$.

You will only need to find *three* of the variables A, B, C, and D in terms of the fourth variable. Thus, you must obtain *three* geometric facts that can be translated into *three* algebraic equations.

The first geometric fact, that point M with $x = 1$, $y = 2$, and $z = 3$ lies on the plane, gives the algebraic equation
$A(1) + B(2) + C(3) + D = 0$,
or $\quad A + 2B + 3C + D = 0$ ①

The second geometric fact, that point T with $x = -2$, $y = 3$, and $z = 0$ lies on the plane, gives the algebraic equation
$A(-2) + B(3) + C(0) + D = 0$,
or $\quad -2A + 3B + D = 0$ ②

The third geometric fact, that planes Π and Π_1 are perpendicular, means that the normal vector $\overrightarrow{(A,B,C)}$ of Π is perpendicular to the normal vector $\overrightarrow{(3,1,-6)}$ of Π_1. This gives the algebraic fact
$\overrightarrow{(A,B,C)} \cdot \overrightarrow{(3,1,-6)} = 0$,
or $\quad 3A + B - 6C = 0$ ③

Solving equations ①, ② and ③ for A, B, and C in terms of D produces
$A = -\frac{D}{25}$, $B = -\frac{9D}{25}$, and $C = -\frac{2D}{25}$
To avoid fractions, you should select $D = -25$, giving
$A = 1$, $B = 9$, and $C = 2$.

Hence an equation of the given plane is $x + 9y + 2z - 25 = 0$. ■

Note: The equations ①, ② and ③ of Example 4 can be solved as follows.

Using Matrices

The system is equivalent to the augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ -2 & 3 & 0 & 1 \\ 3 & 1 & -6 & 0 \end{bmatrix}$$

which can be reduced as shown.

$$\begin{matrix} \\ 2 \times \text{row ①} + \text{row ②} \\ 3 \times \text{row ①} - \text{row ③} \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 7 & 6 & 3 \\ 0 & 5 & 15 & 3 \end{bmatrix}$$

$$\begin{matrix} \\ \\ -5 \times \text{row ②} + 7 \times \text{row ③} \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 7 & 6 & 3 \\ 0 & 0 & 75 & 6 \end{bmatrix}$$

From row ③: $75C + 6D = 0$,

thus $$C = -\frac{2D}{25}$$

From row ②: $7B + 6C + 3D = 0$,

or $$7B + 6\left(-\frac{2D}{25}\right) + 3D = 0,$$

thus $$B = -\frac{9D}{25}$$

From row ①: $A + 2B + 3C + D = 0$,

or $$A + 2\left(-\frac{9D}{25}\right) + 3\left(-\frac{2D}{25}\right) + D = 0,$$

thus $$A = -\frac{D}{25}$$

But D can be any real number. To avoid fractions, let $D = -25$ giving $A = 1$, $B = 9$, and $C = 2$.

By Elimination

The three equations are

$$A + 2B + 3C + D = 0 \quad ①$$
$$-2A + 3B \quad + D = 0 \quad ②$$
$$3A + B - 6C \quad = 0 \quad ③$$

Eliminating A from ① and ② using 2 × ① + ② gives

$7B + 6C + 3D = 0$ ④

Eliminating A from ① and ③ using 3 × ① − ③ gives

$5B + 15C + 3D = 0$ ⑤

Eliminating B from ④ and ⑤ using 7 × ⑤ − 5 × ④ gives

$75C + 6D = 0$.

The remainder of the solution is the same as for *Using Matrices*.

6.2 Exercises

1. State the components of a normal vector for each of the following planes.
   a) $3x - 5y + 4z + 3 = 0$
   b) $2x - 5y - 4z = 13$
   c) $x - 5y + 4z = 1$
   d) $x - 2y - 3z = 8$
   e) $2x - 6z + 1 = 0$
   f) $2y + 5z = 4$
   g) $-6x + 10y - 8z = 1$
   h) $2x - 4y - 6z = 16$

2. List the pairs of planes in question 1 that are parallel. Indicate whether the planes are distinct or identical.

3. Prove that the following planes are perpendicular.
   $3x - 4y + 2z + 7 = 0$ and
   $2x + 5y + 7z + 6 = 0$

4. Find a Cartesian equation of the plane through the point $M(1,4,-6)$ having a normal vector $\overrightarrow{(3,6,1)}$.

5. Find a Cartesian equation of the plane through the point $M(1,4,-6)$ that is parallel to the plane $4x - 2y + 7z = 1$.

6. a) A plane Π passes through the point $M(1,4,-6)$. If the Cartesian equation of the plane is $Ax + By + Cz + D = 0$, then state an equation giving a relationship among A, B, C, and D.
   b) The plane Π in part a) also passes through the point $T(3,0,5)$. State another equation giving a relationship among A, B, C, and D.
   c) The plane Π in part a) is also parallel to the line $\vec{r} = \overrightarrow{(1,3,7)} + k\overrightarrow{(3,-4,2)}$. State a third equation giving a relationship among A, B, C, and D.
   d) Use the equations relating A, B, C, and D from parts a), b), and c) to find a Cartesian equation of plane Π.

7. Find a Cartesian equation of the plane through $K(1,4,-1)$, and through the origin, that is parallel to the vector $\vec{u} = \overrightarrow{(-1,0,2)}$.

8. Find a Cartesian equation of the plane through the points $M(1,4,-6)$ and $T(-3,0,5)$ that is perpendicular to the plane $4x - y + 3z = 5$.

9. Find a Cartesian equation of the plane through the points $K(1,2,3)$, $R(1,-1,0)$, and $S(2,-3,-4)$.

10. Find a Cartesian equation of the plane through the points $E(0,1,2)$, $F(3,-4,-8)$, and $G(2,2,4)$.

11. Find Cartesian equations of the three coordinate planes, that is, the xy-plane, the yz-plane, and the zx-plane.

12. Find a Cartesian equation of the plane through the point $E(0,1,2)$ that is parallel to the vectors $\vec{u} = \overrightarrow{(3,2,-6)}$ and $\vec{v} = \overrightarrow{(5,1,1)}$.

13. Find a Cartesian equation of the plane through the point $E(0,1,2)$ that is parallel to the plane
    $\vec{r} = \overrightarrow{(1,1,-1)} + k\overrightarrow{(2,3,-1)} + m\overrightarrow{(4,2,-2)}$.

14. Find a Cartesian equation of the plane with vector equation
    $\vec{r} = \overrightarrow{(5,1,-1)} + k\overrightarrow{(-4,1,0)} + m\overrightarrow{(1,3,-2)}$.

15. Find a vector equation of the plane with Cartesian equation $5x - 2y + 6z + 1 = 0$.

16. Find a Cartesian equation of the plane through the points $E(0,1,2)$, and $F(3,-4,-8)$ that has the same y-intercept as the plane $5x - 2y + z + 10 = 0$.

17. Find a Cartesian equation of the plane through the point $M(1,2,4)$ that contains the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-5}{1}$.

18. Find t so that the point $(1,t,-4)$ lies on the plane $4x + 7y - z = 12$.

19. Prove that the line $\frac{x-1}{2} = \frac{y+3}{3} = \frac{z-2}{4}$ is parallel to the plane $x - 2y + z = 5$.

20. Find the value of C, given that the planes $2x - 4y + Cz = 1$ and $x - 3y + 5z - 2 = 0$ are perpendicular.

21. Find a Cartesian equation of the plane that is perpendicular to the line segment joining the points (2,3,0) and (4,−5,6) at its midpoint.

22. Find the acute angle between the line $\vec{r} = \overrightarrow{(3 - t, 4 + 2t, 1 + 5t)}$ and the plane $3x - 2y + 4z = 5$.

23. Find a Cartesian equation of the plane through the point $T(0,3,-5)$ that has its normal parallel to the line segment joining the points (1,2,3) and (−3,6,11).

24. Find an angle between each of the following pairs of planes.
   a) $3x + 5y - z = 1$ and $4x + 2y - 11z + 2 = 0$
   b) $2x - 2y + z = 8$ and $2x + y + z + 2 = 0$

25. Given the points $P(2,1,5)$ and $Q(1,3,4)$. Find a Cartesian equation of the plane containing the line PQ, that is perpendicular to the plane $x + y + z = 3$.

26. Find a Cartesian equation of the plane containing the line $\frac{x+1}{2} = \frac{y-3}{1} = \frac{z-5}{-4}$, that is parallel to the line $\frac{x+2}{1} = \frac{y+4}{2} = \frac{z+2}{6}$

27. Find a Cartesian equation of the plane parallel to the line $\frac{x-5}{3} = \frac{y+3}{2} = \frac{z-4}{1}$, that passes through $A(1,4,-1)$ and $B(0,2,5)$.

28. Find the values of k and m so that the line $\frac{x+1}{k} = \frac{y-2}{m} = \frac{z+3}{1}$ is perpendicular to the plane through the points $U(1,3,8)$, $W(0,1,1)$ and $V(4,2,0)$.

29. Find a Cartesian equation of the plane that contains the point (1,−1,0) and is perpendicular to each of the planes $3x + 5y + z = 3$ and $2x - 4y + 3z + 1 = 0$.

30. Find the value of t so that the line $\frac{x+1}{3} = \frac{y-2}{6} = \frac{z-3}{4}$ is parallel to the plane $2x + 3y - tz + 7 = 0$.

31. Find a Cartesian equation of the plane that contains the points (3,2,−1) and (2,5,0) and is parallel to the line $\frac{x+4}{3} = \frac{y+2}{2} = \frac{z-5}{-1}$

32. a) If $P(x,y,z)$ is any point on the plane through the points M, R, and S show that an equation of this plane is $\overrightarrow{MP} \cdot \overrightarrow{MR} \times \overrightarrow{MS} = 0$.
   b) Use part a) to find a Cartesian equation of the plane through the points $M(1,3,2)$, $R(0,3,5)$, and $S(-2,1,4)$.

33. Find a Cartesian equation for the plane that contains the z-axis and the point (1,2,3).

34. Find a Cartesian equation of the plane parallel to the x-axis that passes through the points (3,2,−1) and (4,1,3).

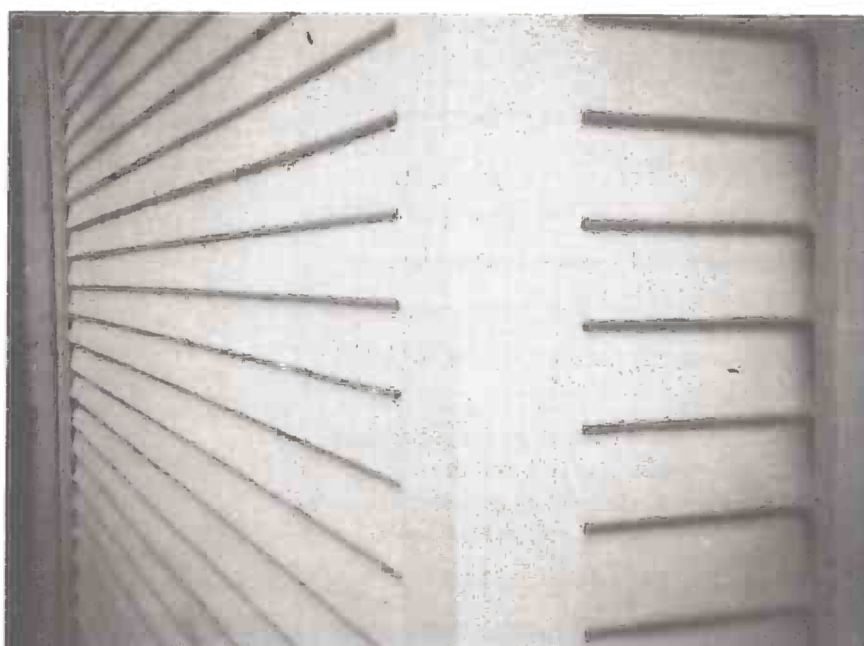
35. Find a Cartesian equation of the plane that intersects its normal through the origin at the point (2,−1,1).

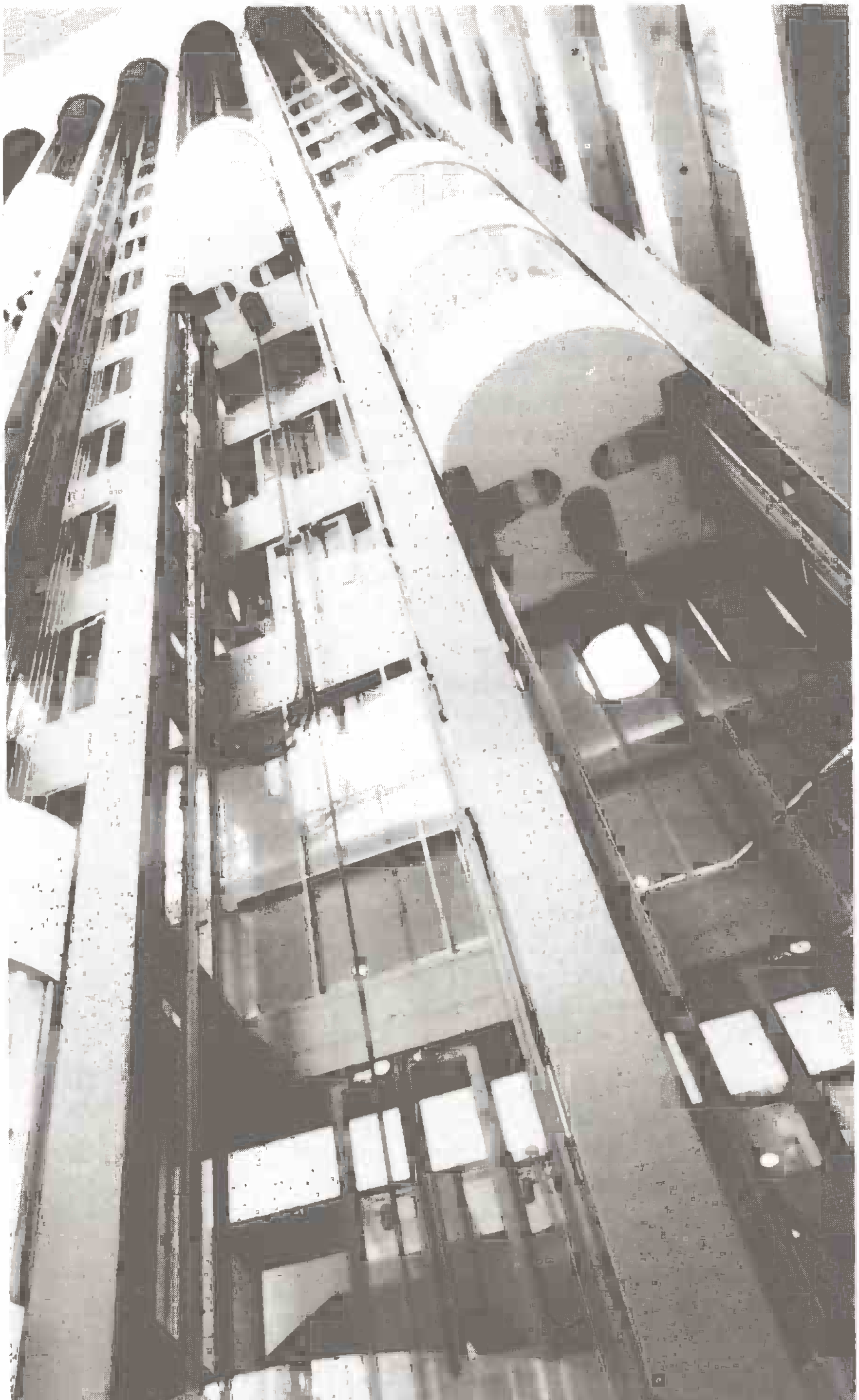
36. For each of the following, find the condition that must exist among A, B, C and D so that the equation $Ax + By + Cz + D = 0$ will represent the given plane.
   a) the plane will have equal intercepts on the x-axis and y-axis
   b) the plane will not intersect the y-axis
   c) The plane will pass through the point that divides the line segment joining (1,2,3) to (−5,6,1) internally in the ratio 2 : 3
   d) the plane will be parallel to the xy-plane
   e) the plane will be perpendicular to the plane $4x - y + 3z + 1 = 0$
   f) the plane will be parallel to the plane $4x - y + 3z + 1 = 0$

37. Given the planes
   $\Pi_1: A_1x + B_1y + C_1z + D_1 = 0$ and
   $\Pi_2: A_2x + B_2y + C_2z + D_2 = 0$
   a) Find a formula for the angle between the planes Π_1 and Π_2.
   b) Prove that the planes Π_1 and Π_2 are perpendicular if and only if $A_1A_2 + B_1B_2 + C_1C_2 = 0$.

M A K I N G

Lines and Planes

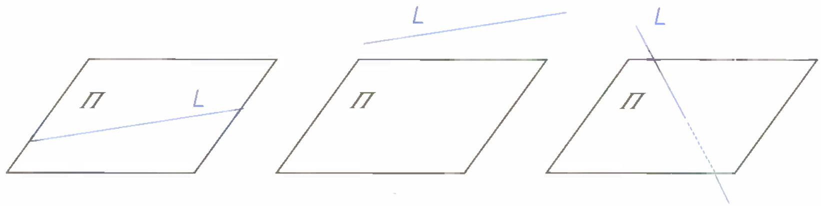




6.3 The Intersection of Lines and Planes

A line L and a plane Π can be related in three ways:

1. line L lies in plane Π
2. line L is parallel to plane Π
3. line L intersects plane Π



In (1), the line L and the plane Π have an infinity of points in common.
In (2), the line L and the plane Π have no points in common.
In (3), the line L and the plane Π have one point in common, called the point of intersection.

You can model these three relationships using a pencil as the line L and the top of your desk as plane Π. For both the pencil and the desk top, only a finite part of the line and plane are being represented.

In this section you will learn to determine the type of intersection, or lack thereof, between a line and a plane.

Example 1 Determine the intersection between the plane Π: $\vec{r} = \overrightarrow{(1 + 2k - s, 3 + k + 2s, -2 - 3k + s)}$ and the line L: $\vec{r} = \overrightarrow{(4 + t, 2 - 2t, 6 + 3t)}$.

Solution At the point of intersection of the plane Π and the line L, some point (x,y,z) on Π must be same as some point (x,y,z) on L.

For plane Π $\qquad$ For line L

$$\begin{cases} x = 1 + 2k - s \\ y = 3 + k + 2s \\ z = -2 - 3k + s \end{cases} \qquad \begin{cases} x = 4 + t \\ y = 2 - 2t \\ z = 6 + 3t \end{cases}$$

Thus,
$$\begin{aligned} 1 + 2k - s &= 4 + t \quad \text{or} \quad 2k - s - t = 3 \quad ① \\ 3 + k + 2s &= 2 - 2t \quad \text{or} \quad k + 2s + 2t = -1 \quad ② \\ -2 - 3k + s &= 6 + 3t \quad \text{or} \quad -3k + s - 3t = 8 \quad ③ \end{aligned}$$

Solving by elimination gives the following.
① − 2 × ②: $-5s - 5t = 5$ ④
3 × ② + ③: $7s + 3t = 5$ ⑤
Eliminate s from ④ and ⑤.
7 × ④ + 5 × ⑤: $-20t = 60$ or $t = -3$
Find x, y, and z by substituting $t = -3$ into the parametric equations of the line L.
$x = 4 + (-3) = 1$, $y = 2 - 2(-3) = 8$, $z = 6 + 3(-3) = -3$.
Thus, plane Π and line L intersect in one point, $(x,y,z) = (1,8,-3)$. ■

Example 2 Determine the intersection of the plane Π_1: $2x + 5y - z = 34$, and the line L: $\vec{r} = \overrightarrow{(4 + t, 2 - 2t, 6 + 3t)}$.

Solution At the point of intersection of the plane Π_1 and the line L, some point (x,y,z) on Π_1 must be the same as some point (x,y,z) on L.
For line L
$x = 4 + t$, $y = 2 - 2t$, and $z = 6 + 3t$
Substituting for x, y, and z in the plane $2x + 5y - z = 34$ gives
$2(4 + t) + 5(2 - 2t) - (6 + 3t) = 34$ or $t = -2$.
To find the point of intersection, substitute $t = -2$ into the parametric equations for line L giving

$$\begin{cases} x = 4 + (-2) = 2 \\ y = 2 - 2(-2) = 6 \\ z = 6 + 3(-2) = 0 \end{cases}$$

Hence the plane Π_1 and line L intersect in the single point $(2,6,0)$. ■

Example 3 Given the line L_1: $\dfrac{x-3}{2} = \dfrac{y+2}{1} = \dfrac{z}{-3}$

a) Show that the line L_1 does not intersect plane Π_2: $2x + 5y + 3z = 1$.
b) Show that the line L_1 lies in the plane Π_3: $2x + 5y + 3z = -4$.

Solution At the point of intersection of the line L_1 and either of the planes Π_2 or Π_3 some point (x,y,z) on the plane must be the same as some point (x,y,z) on L_1.

To find parametric equations for L_1, let $s = \dfrac{x-3}{2} = \dfrac{y+2}{1} = \dfrac{z}{-3}$

Thus
$$\left.\begin{aligned} x &= 2s + 3 \\ y &= s - 2 \\ z &= -3s \end{aligned}\right\} \quad ①$$

a) To find the intersection of L_1 with Π_2 you must substitute the values of x, y, and z from ① into $2x + 5y + 3z = 1$, and solve for s.
$2(2s + 3) + 5(s - 2) + 3(-3s) = 1$
or $4s + 6 + 5s - 10 - 9s = 1$
or $0s = 5$.
Since no value for s satisfies this equation, there can be no point of intersection of line L_1 and plane Π_2.
The line and the plane are parallel.

b) To find the intersection of L_1 with Π_3 you must substitute the values for x, y, and z from ① into $2x + 5y + 3z = -4$, and solve for s.
$2(2s + 3) + 5(s - 2) + 3(-3s) = -4$
or $4s + 6 + 5s - 10 - 9s = -4$
or $0s = 0$.
Since every real value for s satisfies this equation, every point of line L_1 must lie in the plane Π_3. ■

6.3 Exercises

1. Find the intersection of each of the following pairs of a line and a plane. In each case describe the intersection geometrically.
   a) plane: $\vec{r} = (2 + 3k + s, 1 - k + 4s, -1 - k + 2s)$
   line: $\vec{r} = (-2 + t, 4 + 2t, 13 - 3t)$
   b) plane: $\vec{r} = (1 + k + 2s, 1 + 3k - s, -4 - 5k + s)$
   line: $\vec{r} = (2 + t, 4 + 3t, -9 - 5t)$
   c) plane: $x + 11y - 2z = 39$
   line: $\vec{r} = (1 + 2t, 4t, -6 - 3t)$
   d) plane: $4x + 6y + z = 12$
   line: $\vec{r} = (1 + 5t, 1 - 3t, 2 - 2t)$
   e) plane: $x - 2y + 3z + 10 = 0$
   line: $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{-1}$
   f) plane: $2x + y - z = 2$
   line: $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-1}{5}$
   g) plane: $\vec{r} = (3,-2,1) + k(2,1,-3) + s(1,2,1)$
   line: $\frac{x-3}{2} = \frac{y+2}{1} = \frac{z}{-3}$

2. Prove that the line $\frac{x-2}{3} = \frac{y-1}{1} = \frac{z-3}{2}$ is parallel to the plane $2x + 4y - 5z + 6 = 0$.

3. a) Prove that the line $\vec{r} = (-1 + 2t, 3t, 4 + t)$ is perpendicular to the plane $4x + 6y + 2z = 32$.
   b) Find the point of intersection of the line and the plane in part a).
   c) Use a) and b) to find the perpendicular distance from the point $(-1,0,4)$ to the plane $4x + 6y + 2z = 32$.

4. Find a Cartesian equation of the plane through the point $(1,2,1)$ that is perpendicular to the plane $5x + 2y - 4z = 12$ and intersects the line $\vec{r} = (2 + t, 1 + 2t, 6t - 2)$ at the point where $x = 1$.

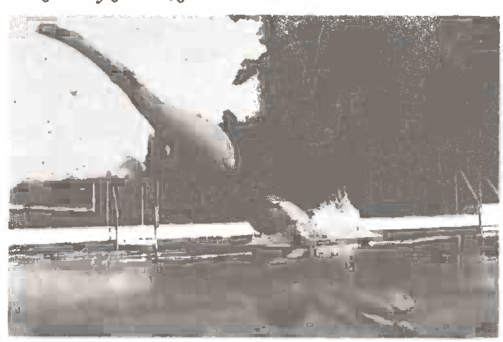
5. Find the value of b so that the line through the point $(2,-4,1)$ and the point $(b,2b,3b)$ does not intersect the plane $x + y + z = 9$.

6. A line L intersects the x-axis in the same point as the plane $3x - 4y + z = 12$ and intersects the xy plane in the same point as the line $\vec{r} = (6 - 2t, 1 + t, t - 1)$. Find a vector equation of line L.

7. Given the point $E(2,3,0)$ and the plane Π: $3x - y + 2z = 17$.
   a) Find parametric equations of the line L_1, through the point E, that is perpendicular to the plane Π.
   b) Find R, the point of intersection of the line L_1 and the plane Π.
   c) Find the perpendicular distance ER from the point E to the plane Π.

8. Find the point of intersection of the line $2y - 6 = 3x + 20$, $z = 1$ with the plane $4x + 2y + 3z + 13 = 0$.

9. Prove that the line $\frac{x - x_0}{m_1} = \frac{y - y_0}{m_2} = \frac{z - z_0}{m_3}$ where $m_1, m_2, m_3 \neq 0$, does not intersect the plane $Ax + By + Cz + D = 0$, if and only if, $Am_1 + Bm_2 + Cm_3 = 0$ and $Ax_0 + By_0 + Cz_0 + D \neq 0$

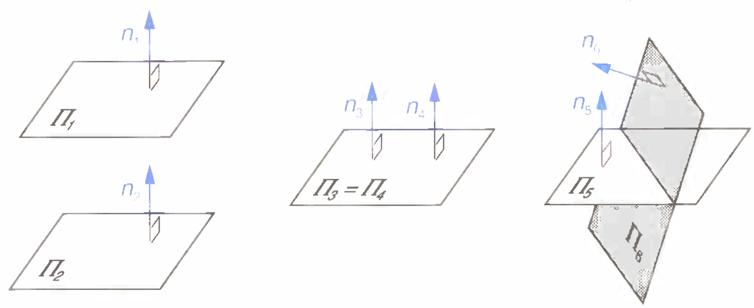


10. A diver is standing on a diving board above a swimming pool at the point $P(2,0,5)$. She leaves the board travelling parallel to the vector $\vec{m} = (0,3,-5)$. If the surface of the pool has equation $z = 0$, then find the coordinates of the point where she enters the pool.

6.4 The Intersection of Two Planes

As you saw in Section 1.2, two planes are either parallel, or they intersect in a straight line. Two parallel planes are either different planes or the same plane. The following diagrams illustrate these situations.

Π_1 and Π_2 are parallel and distinct ($\vec{n}_1 \parallel \vec{n}_2$)	Π_3 and Π_4 are parallel and identical ($\vec{n}_3 \parallel \vec{n}_4$)	Π_5 and Π_6 intersect in a line ($\vec{n}_5 \nparallel \vec{n}_6$)



You should note the relationship between the normal vectors $\vec{n}$ in each of the diagrams. These relationships are summarised as follows.

1. Two planes are parallel if and only if their normal vectors are parallel.
2. Two planes intersect if and only if their normal vectors are not parallel.
3. Two non-parallel planes intersect in a line.

In this section you will learn how to determine algebraically whether or not two planes intersect. If two planes intersect you will find out how to obtain an equation for their line of intersection.

Example 1 Given the plane Π_1: $3x + 2y + 5z = 4$ and the three planes
Π_2: $6x + 4y + 10z = 3$ Π_3: $9x + 6y + 15z = 12$ Π_4: $4x - 3y + z = -1$

a) Prove that planes Π_1 and Π_2 are parallel and distinct.
b) Prove that planes Π_1 and Π_3 are parallel and identical.
c) Prove that planes Π_1 and Π_4 intersect in a line.

Solution a) *planes* Π_1 *and* Π_2

A normal vector for Π_1 is $\vec{n}_1 = \overrightarrow{(3,2,5)}$.

A normal vector for Π_2 is $\vec{n}_2 = \overrightarrow{(6,4,10)}$.

Since $\vec{n}_2 = 2\vec{n}_1$, then $\vec{n}_1$ and $\vec{n}_2$ are parallel. Hence planes Π_1 and Π_2 are parallel.

Now the equation for Π_2 can be written $2(3x + 2y + 5z) = 3$, and the equation for Π_1 can be written $2(3x + 2y + 5z) = 2(4)$. Since $3 \neq 2(4)$, the planes Π_1 and Π_2 are distinct.

b) *planes* Π_1 *and* Π_3

A normal vector for Π_1 is $\vec{n}_1 = \overrightarrow{(3,2,5)}$.

A normal vector for Π_3 is $\vec{n}_3 = \overrightarrow{(9,6,15)}$.

Since $\vec{n}_3 = 3\vec{n}_1$ then $\vec{n}_1$ and $\vec{n}_3$ are parallel. Hence, planes Π_1 and Π_3 are parallel.

Now the equation for Π_3 can be written $3(3x + 2y + 5z) = 3(4)$. Dividing both sides of this equation by 3 gives $3x + 2y + 5z = 4$, which is the equation for plane Π_1. Hence, the planes Π_1 and Π_3 are identical.

c) *planes* Π_1 *and* Π_4

A normal vector for Π_1 is $\vec{n}_1 = \overrightarrow{(3,2,5)}$

A normal vector for Π_4 is $\vec{n}_4 = \overrightarrow{(4,-3,1)}$

Since no real number k exists such that $\vec{n}_4 = k\vec{n}_1$, then $\vec{n}_1$ and $\vec{n}_4$ are not parallel. Hence, planes Π_1 and Π_4 are not parallel. Thus, the planes Π_1 and Π_4 intersect in a line. ■

Example 2

Find parametric equations for the line of intersection of the planes Π_1 and Π_4 of Example 1.

Solution

You must solve the system

$3x + 2y + 5z = 4$ ①

$4x - 3y + z = -1$ ②

Method 1: Using Matrices

The augmented matrix for the system is

$$\begin{bmatrix} 3 & 2 & 5 & 4 \\ 4 & -3 & 1 & -1 \end{bmatrix} \Rightarrow 4 \times \text{row①} - 3 \times \text{row②} \begin{bmatrix} 3 & 2 & 5 & 4 \\ 0 & 17 & 17 & 19 \end{bmatrix}$$

From row ②: $17y + 17z = 19$

Let $z = t$ and solve for y.

$$17y + 17t = 19$$
$$17y = 19 - 17t$$
$$y = \frac{19 - 17t}{17}$$

From row ①: $3x + 2y + 5z = 4$

Substituting for y and z gives

$$3x + 2\left(\frac{19 - 17t}{17}\right) + 5t = 4$$

Multiplying both sides by 17, and removing the bracket, gives

$$51x + 38 - 34t + 85t = 68$$

Thus, $$x = \frac{30 - 51t}{51} \text{ or } x = \frac{10 - 17t}{17}$$

Thus, planes Π_1 and Π_4 intersect in the line with parametric equations

$$x = \frac{10 - 17t}{17}, \quad y = \frac{19 - 17t}{17}, \quad z = t$$

Method 2

Let $z = t$ and substitute into both equations

① $3x + 2y + 5z = 4$ (Π_1) and

② $4x - 3y + z = -1$ (Π_4)

$3x + 2y + 5t = 4$ ③

$4x - 3y + t = -1$ ④

To find parametric equations for the line of intersection, you must eliminate x and y, in turn, from equations ③ and ④, then solve for the remaining variable in terms of the parameter t.

Eliminate x: $4 \times$ ③ $- 3 \times$ ④ gives $17y + 17t = 19$

Thus, $$y = \frac{19 - 17t}{17}$$

Eliminate y: $3 \times$ ③ $+ 2 \times$ ④ gives $17x + 17t = 10$

Thus, $$x = \frac{10 - 17t}{17}$$

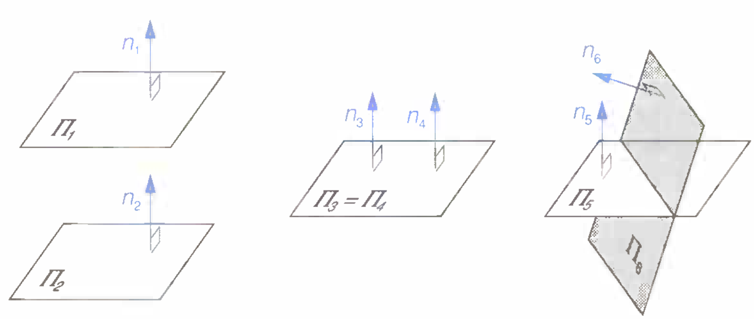
Thus, planes Π_1 and Π_4 intersect in the line with parametric equations

$$x = \frac{10 - 17t}{17}, \quad y = \frac{19 - 17t}{17}, \quad z = t \quad ■$$

Visualising the Intersection of Two Planes

To assist your visualization of the relationship between two planes, take two pieces of rectangular-shaped cardboard and use them to represent parts of two planes. (A plane extends infinitely in all its two-dimensional directions, so that a cardboard can represent only a finite part of a plane.) If you hold the pieces of cardboard so that the cardboards are parallel, then a pencil held perpendicular to one cardboard will be parallel to a pencil held perpendicular to the second cardboard.

If you hold the pieces of cardboard so that the cardboards are not parallel, and a side of one cardboard touches the other cardboard, you can see that non-parallel cardboards intersect in a line segment. You should also notice that a pencil held perpendicular to one cardboard is not parallel to a pencil held perpendicular to the second cardboard.



6.4 Exercises

1. For each of the following planes, use normal vectors to determine whether or not the planes are parallel, or intersect in a line. If the planes are parallel, indicate whether the planes are distinct or identical.
   a) $3x + 5y - z = 10;\quad 2x - y + 4z = 2$
   b) $4x + 6y + 2z = 3;\quad 6x + 9y + 3z = 9$
   c) $x - y + 5z - 2 = 0;\quad -2x + 2y - 10z = 4$
   d) $x - 3y + 5z = 1;\quad 3x - 9y + 15z = 3$
   e) $5x - 3y = 4;\quad 3y + z = 1$
   f) $3x - 2y - 8z + 4 = 0;$
   $4x + 2y + 6z - 3 = 0$

2. For each pair of planes in question 1 that intersect in a line, find parametric equations of their line of intersection.

3. For each pair of planes in question 1 that intersect in a line, find Cartesian equations of their line of intersection.

4. Find an equation of the plane passing through the point $T(-1,3,2)$ that is perpendicular to the line of intersection of the planes
   $2x + y + 3z - 7 = 0$ and $4x - 3y + 4z = -5$.

5. Find an equation of the plane passing through the points $W(0,5,-3)$ and $M(1,-1,3)$ that is parallel to the line of intersection of the planes
   $x + 2y + 3z + 4 = 0$ and $x - y - 3z = 8$.

6. a) Find parametric equations for the line of intersection of the planes
   $2x + y - z + 1 = 0$ and $x - 7y + z = 22$.
   b) Find the three points where the line in part a) intersects the coordinate planes.

7. Given the planes
   $\Pi_1:\ x + y + 2z = 0$
   $\Pi_2:\ 3x - 3y - 2z = 10$
   $\Pi_3:\ x + y - z + 3 = 0$
   $\Pi_4:\ x - y - 2z + 12 = 0$
   Show that the line L_1 of intersection of planes Π_1 and Π_2 is skew with L_2, the line of intersection of planes Π_3 and Π_4.

8. Find a vector equation of the line through the point $Q(-3,0,1)$ that is parallel to the line of intersection of planes Π_1 and Π_2 from question 7.

9. Given the planes Π_1: $kx - y + 2kz = 0$ and Π_2: $kx + y + 4kz = 0$.
   a) Find the value of k so that the line of intersection of planes Π_1 and Π_2 is parallel to the line $\vec{r} = \overrightarrow{(1 + 3t, 4t, 2 - t)}$.
   b) Find the value of k so that the line of intersection of planes Π_1 and Π_2 is perpendicular to the line $\vec{r} = \overrightarrow{(1 + 3t, 4t, 2 - t)}$.

10. Show that the line of intersection of the planes $2x - y + z = 4$ and $x + y + 3z = 1$ is parallel to the line $\dfrac{x+1}{4} = \dfrac{y}{5} = \dfrac{z+2}{-3}$.

11. Show that the line of intersection of the planes $2x + y - 3z = 1$
    and $-2x + 3y + 2z = 4$
    is perpendicular to the
    line $\vec{r} = \overrightarrow{(1,3,2)} + t\overrightarrow{(2,-3,-2)}$.

12. Show that any vector parallel to the line of intersection of the planes $-y + 3z = 1$ and $3x + y + 4 = 0$ is a normal to the plane $3x - 9y - 3z + 1 = 0$.

13. Given the planes $\Pi_1:\ x + y + 5z = -6$
    $\Pi_2:\ x - y - z = 1$
    $\Pi_3:\ kx - 5y - z = 0$

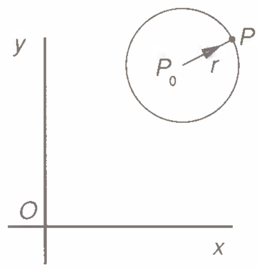
    Find the value of k so that the line of intersection of planes Π_1 and Π_2 lies in the plane Π_3.

14. Given the planes $\Pi_1:\ x - y - z = 2$
    $\Pi_2:\ 4x - 5y - z = -13$
    a) Find parametric equations for L, the line of intersection of Π_1 and Π_2.
    b) Prove that the equation
    $m(x - y - z - 2) + k(4x - 5y - z + 13) = 0$
    represents a plane containing line L, for every pair of real numbers m and k.
    c) Find the plane in b) that contains the point $(1,1,1)$.

In Search of Vector Equations of Circles and Spheres

Circles

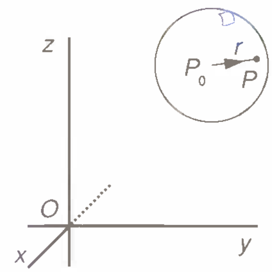
Suppose that a circle has centre P_0 and radius r.
If P is any point on the circle then $|\overrightarrow{P_0P}| = r$.



But $|\overrightarrow{P_0P}| = \sqrt{\overrightarrow{P_0P} \cdot \overrightarrow{P_0P}}$
Hence a vector equation of a circle is
$\sqrt{\overrightarrow{P_0P} \cdot \overrightarrow{P_0P}} = r$, or
$\overrightarrow{P_0P} \cdot \overrightarrow{P_0P} = r^2$.
If P_0 has coordinates (x_0, y_0) and P has coordinates (x, y), then $\overrightarrow{P_0P} = \overrightarrow{(x - x_0, y - y_0)}$.
Hence a Cartesian equation for the circle is
$\overrightarrow{(x - x_0, y - y_0)} \cdot \overrightarrow{(x - x_0, y - y_0)} = r^2$,
or
$(x - x_0)^2 + (y - y_0)^2 = r^2$.

Spheres

Suppose that a sphere has centre P_0 and radius r.
If P is any point on the sphere then $|\overrightarrow{P_0P}| = r$.



But $|\overrightarrow{P_0P}| = \sqrt{\overrightarrow{P_0P} \cdot \overrightarrow{P_0P}}$
Hence a vector equation of a sphere is
$\sqrt{\overrightarrow{P_0P} \cdot \overrightarrow{P_0P}} = r$, or
$\overrightarrow{P_0P} \cdot \overrightarrow{P_0P} = r^2$.
If P_0 has coordinates (x_0, y_0, z_0) and P has coordinates (x, y, z), then $\overrightarrow{P_0P} = \overrightarrow{(x - x_0, y - y_0, z - z_0)}$.
Hence a Cartesian equation for the sphere is
$\overrightarrow{(x - x_0, y - y_0, z - z_0)} \cdot \overrightarrow{(x - x_0, y - y_0, z - z_0)} = r^2$,
or
$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$.

Activities

1. Find a vector equation and a Cartesian equation of the following circles.
   a) centre (0,0), radius 5
   b) centre (2,3), radius 4
   c) centre (−1,5), radius $\sqrt{3}$.

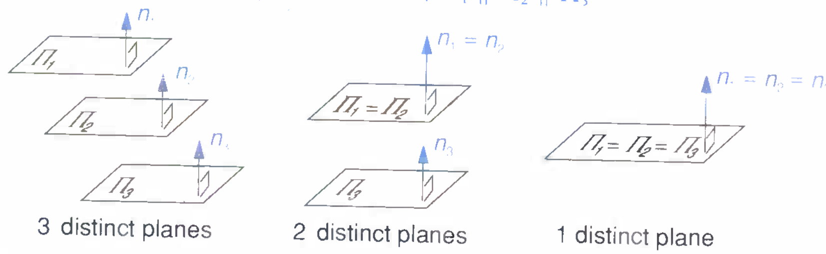
2. Find a vector equation and a Cartesian equation of the following spheres.
   a) centre (0,0,0), radius 5
   b) centre (2,3,4), radius 6
   c) centre (−1,5,−3), radius $\sqrt{7}$.

6.5 The Intersection of Three Planes

There are many ways in which three planes can intersect, as the following demonstrates. You should use three pieces of cardboard to help you to visualize these intersections.

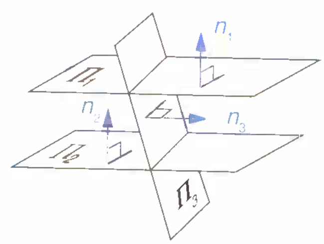
Suppose the planes are called Π_1, Π_2 and Π_3, with normal vectors $\vec{n}_1$, $\vec{n}_2$, $\vec{n}_3$, respectively.

Case 1: The three planes are parallel, that is, $\Pi_1 \parallel \Pi_2 \parallel \Pi_3$



Notice that the three planes may be distinct, or two may be identical, or all three may be identical. In all these situations, the relationship among the normal vectors is $\vec{n}_1 \parallel \vec{n}_2 \parallel \vec{n}_3$.

Case 2: Two planes are parallel but the third plane intersects each parallel plane, that is, $\Pi_1 \parallel \Pi_2 \nparallel \Pi_3$



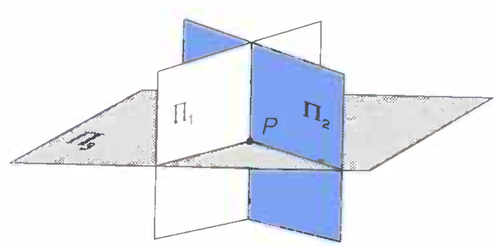
Notice that $\vec{n}_1 \parallel \vec{n}_2 \nparallel \vec{n}_3$.

Case 3: No two of the three planes are parallel

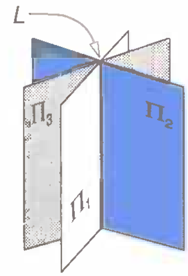
This occurs if no two of the normal vectors $\vec{n}_1$, $\vec{n}_2$, $\vec{n}_3$ are parallel.

The diagrams show that there are three possibilities.

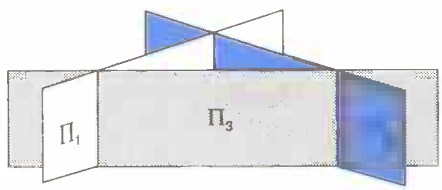
A) The three lines of intersection of the planes have a common point. Hence, the three planes intersect in a single point.



B) The lines of intersection of the three planes are the same line. Hence, the three planes intersect in a line.



C) The lines of intersection of the three planes are distinct and parallel. Hence, the intersection of the three planes forms a triangular prism.



In section 5.5 you learned the meaning of the terms consistent (dependent or independent) or inconsistent, to describe a system of two linear equations in two variables representing lines in 2-space.

The same terms are used in a similar way for a system of three linear equations in three variables in 3-space, as indicated in the following chart.

For a system of three equations in three variables, representing three planes, no two of which are parallel:

	number of solutions	*geometric description*
consistent and independent	one	three planes intersect in a point
consistent and dependent	infinite	three planes intersect in a line
inconsistent	none	three planes form a triangular prism

The following example will show you how to distinguish among these situations algebraically.

Example 1 Prove that the three given planes are parallel and distinct.

$\Pi_1: 6x + 4y + 8z = 5 \quad \Pi_2: 3x + 2y + 4z = 2 \quad \Pi_3: 9x + 6y + 12z = 1$

Solution The normal vector for Π_1 is $\vec{n}_1 = \overrightarrow{(6,4,8)}$
The normal vector for Π_2 is $\vec{n}_2 = \overrightarrow{(3,2,4)}$
The normal vector for Π_3 is $\vec{n}_3 = \overrightarrow{(9,6,12)}$
Since $\vec{n}_1 = 2\vec{n}_2$, and $\vec{n}_3 = 3\vec{n}_2$, the normal vectors $\vec{n}_1$, $\vec{n}_2$, $\vec{n}_3$ are parallel.

Dividing the equation for Π_1 by 2 and the equation for Π_3 by 3 gives

$3x + 2y + 4z = \frac{5}{2}$ ① and $3x + 2y + 4z = \frac{1}{3}$ ② respectively. The left hand side of each of these equations is the same as the left hand side of the equation $3x + 2y + 4z = 2$ ③ of Π_2.

Since the right hand side of each of these three equations is different from that of the others, no triple (x,y,z) can satisfy any two of these equations at the same time. Thus, the three planes are both parallel and distinct. ■

Note: In Example 1, if the right hand side of any of the equations ①, ② or ③ had been the same, the corresponding planes would have been identical.

Example 2 Given the five planes:

$\Pi_1: \quad x + 3y + 2z = 5$ ①
$\Pi_2: \quad 2x - y - 4z = 4$ ②
$\Pi_3: \quad 4x - 3y + z = -3$ ③
$\Pi_4: \quad 5x + y - 6z = 13$ ④
$\Pi_5: \quad 5x + y - 6z = 8$ ⑤

Determine the type of intersection of each of the following sets of three planes.

a) Π_1, Π_2, Π_3 b) Π_1, Π_2, Π_4 c) Π_1, Π_2, Π_5

Solution 1 *Using elimination*

a) The normals to planes Π_1, Π_2, and Π_3 are $\vec{n}_1 = \overrightarrow{(1,3,2)}$, $\vec{n}_2 = \overrightarrow{(2,-1,-4)}$, and $\vec{n}_3 = \overrightarrow{(4,-3,1)}$.

Since none of the normal vectors is a scalar multiple of another, no two normals are parallel. Hence, none of the planes is parallel to another. Each pair of planes will intersect in a line.

The line of intersection of planes Π_1 and Π_2 can be determined, as in Example 2 of section 6.4, to have the following parametric equations.

$$x = \frac{10t + 17}{7}, \; y = \frac{6 - 8t}{7}, \; z = t.$$

The required intersection will be the intersection of this line with plane Π_3. To find this intersection you must substitute the above values for x, y, and z in equation ③ for Π_3 giving

$$4\left(\frac{10t + 17}{7}\right) - 3\left(\frac{6 - 8t}{7}\right) + t = -3$$

Multiplying by 7 to remove fractions, and simplifying, gives

$$40t + 68 - 18 + 24t + 7t = -21, \text{from which } t = -1.$$

Substituting into the parametric equations $x = \frac{10t + 17}{7}$, $y = \frac{6 - 8t}{7}$, $z = t$, gives the point $(1,2,-1)$.
Hence, the three planes intersect in a single point.

b) The normals for the three planes Π_1, Π_2, and Π_4, are $\vec{n_1} = \overrightarrow{(1,3,2)}$, $\vec{n_2} = \overrightarrow{(2,-1,-4)}$, and $\vec{n_4} = \overrightarrow{(5,1,-6)}$.

Since none of these normals is parallel to another, each pair of planes must intersect in a line.
From part a), the line of intersection of Π_1 and Π_2 is
$x = \frac{10t + 17}{7}$, $y = \frac{6 - 8t}{7}$, $z = t$.
Substituting in ④ gives

$$5\left(\frac{10t + 17}{7}\right) + \frac{6 - 8t}{7} - 6t = 13, \text{ or } 50t + 85 + 6 - 8t - 42t = 91, \text{ or } 0t = 0.$$

Hence, t can be any real number. Thus, every point (x,y,z) that lies on the line $x = \frac{10t + 17}{7}$, $y = \frac{6 - 8t}{7}$, $z = t$ also lies on the plane Π_4.

Hence, the three planes intersect in the line $x = \frac{10t + 17}{7}$, $y = \frac{6 - 8t}{7}$, $z = t$.

c) The normals for the three planes Π_1, Π_2, and Π_5 are $\vec{n_1} = \overrightarrow{(1,3,2)}$, $\vec{n_2} = \overrightarrow{(2,-1,-4)}$, and $\vec{n_5} = \overrightarrow{(5,1,-6)}$.

Since none of these normals is parallel to another, each pair of planes must intersect in a line.
From part a) the line of intersection of Π_1 and Π_2 is
$x = \frac{10x + 17}{7}$, $y = \frac{6 - 8t}{7}$, $z = t$.
Substituting in ⑤ gives

$$5\left(\frac{10t + 17}{7}\right) + \frac{6 - 8t}{7} - 6t = 8, \text{ or } 50t + 85 + 6 - 8t - 42t = 56, \text{ or } 0t = -35.$$

But no value of t satisfies this equation. Hence, no point (x,y,z) that lies on the line $x = \frac{10t + 17}{7}$, $y = \frac{6 - 8t}{7}$, $z = t$ also lies on the plane Π_5.
Hence, the three planes have no common intersection. Their three lines of intersection are parallel. The three planes form a triangular prism. ■

Solution 2 *Using matrices*

a) The system of equations for the intersection of planes Π_1, Π_2, and Π_3 is

$$x + 3y + 2z = 5 \quad ①$$
$$2x - y - 4z = 4 \quad ②$$
$$4x - 3y + z = -3 \quad ③$$

The augmented matrix for this system is as follows.

$$\begin{bmatrix} 1 & 3 & 2 & 5 \\ 2 & -1 & -4 & 4 \\ 4 & -3 & 1 & -3 \end{bmatrix}$$

$$\begin{matrix} \\ 2 \times \text{row ①} - \text{row ②} \\ 4 \times \text{row ①} - \text{row ③} \end{matrix} \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 7 & 8 & 6 \\ 0 & 15 & 7 & 23 \end{bmatrix}$$

$$\begin{matrix} \\ \\ 15 \times \text{row ②} - 7 \times \text{row ③} \end{matrix} \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 7 & 8 & 6 \\ 0 & 0 & 71 & -71 \end{bmatrix}$$

From row ③, $71z = -71$

Thus, $z = -1$.

Substituting $z = -1$ in row ② gives

$7y - 8 = 6$ or $y = 2$.

From row ①, using $y = 2$ and $z = -1$,

$x + 6 - 2 = 5$

Thus, $x = 1$.

The three planes intersect in the point $(x,y,z) = (1,2,-1)$

b) The system of equations for the intersection of planes Π_1, Π_2, and Π_4 is

$$x + 3y + 2z = 5 \quad ①$$
$$2x - y - 4z = 4 \quad ②$$
$$5x + y - 6z = 13 \quad ④$$

The augmented matrix for this system is as follows.

$$\begin{bmatrix} 1 & 3 & 2 & 5 \\ 2 & -1 & -4 & 4 \\ 5 & 1 & -6 & 13 \end{bmatrix}$$

$$\begin{matrix} \\ 2 \times \text{row ①} - \text{row ②} \\ 5 \times \text{row ①} - \text{row ④} \end{matrix} \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 7 & 8 & 6 \\ 0 & 14 & 16 & 12 \end{bmatrix}$$

$$\begin{matrix} \\ \\ 2 \times \text{row ②} - \text{row ④} \end{matrix} \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 7 & 8 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From row ④, $0z = 0$.

Thus, z can have any real value, say $z = t$.

Substituting $z = t$ in row ② gives

$7y + 8t = 6$

or $y = \dfrac{6 - 8t}{7}$

From row ①, using $y = \dfrac{6 - 8t}{7}$ and $z = t$,

$x + 3\left(\dfrac{6 - 8t}{7}\right) + 2t = 5,$

or $x = \dfrac{17 + 10t}{7}$

The three planes intersect in the line $x = \dfrac{17 + 10t}{7}$, $y = \dfrac{6 - 8t}{7}$, $z = t$.

c) The system of equations for the intersection of planes Π_1, Π_2, and Π_5 is

$$\begin{aligned} x + 3y + 2z &= 5 \quad ① \\ 2x - y - 4z &= 4 \quad ② \\ 5x + y - 6z &= 8 \quad ⑤ \end{aligned}$$

The augmented matrix for this system is as follows.

$$\begin{bmatrix} 1 & 3 & 2 & 5 \\ 2 & -1 & -4 & 4 \\ 5 & 1 & -6 & 8 \end{bmatrix}$$

$$\begin{matrix} \\ 2 \times \text{row ①} - \text{row ②} \\ 5 \times \text{row ①} - \text{row ⑤} \end{matrix} \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 7 & 8 & 6 \\ 0 & 14 & 16 & 17 \end{bmatrix}$$

$$\begin{matrix} \\ \\ 2 \times \text{row ②} - \text{row ⑤} \end{matrix} \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 7 & 8 & 6 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

From row ⑤, $0z = -5$.

There is no value of z that makes this equation true.

Thus, the three planes have no common intersection. The planes intersect, in pairs, in parallel lines to form a triangular prism. ■

Note for alternate solution:

The type of solution for three non-parallel planes can be deduced from the third row of the reduced matrix.

If row ③ is 0 0 a b where $a \neq 0$, there is one solution: a point.

If row ③ is 0 0 a b where $a = 0$ and $b = 0$, there is an infinity of solutions: a line.

If row ③ is 0 0 a b where $a = 0$ and $b \neq 0$, there is no solution: three planes form a triangular prism.

6.5 Exercises

1. In each of the following, the equations of three planes are given. Describe how the planes in each set are related.

a) $x - 2y + 4z = -4$ ①
$2x - 4y - 8z = 9$ ②
$3x - 6y + 12z = 8$ ③

b) $x + 3y - 5z = -1$ ①
$3x + 9y - 15z = -3$ ②
$5x + 15y - 25z = 5$ ③

c) $3x + 6y - 9z = 6$ ①
$x + 2y - 3z = 2$ ②
$2x + 4y - 6z = 4$ ③

d) $4x + 2y - 6z = 2$ ①
$6x + 3y - 9z = 5$ ②
$-2x - y + 3z = 1$ ③

2. In each of the following, the equations of three non-parallel planes are given. Describe how each set of planes intersects by solving the corresponding linear system.

a) $x - 2y + 4z = -4$
$2x + 4y - z = 9$
$3x + 4y + 2z = 8$

b) $x + 3y - 5z = -1$
$3x - 2y - 3z = 7$
$5x + 4y - 13z = 5$

c) $4x + 2y + 3z = 5$
$x - 6y + z = 14$
$-5x - 14y + 2z = 11$

d) $3x + 5y - 2z = 1$
$-2x - 3y + 4z = 2$
$7x + 11y - 10z = 8$

e) $x + 8y - 2z = 5$
$4x + 7y + 2z = 0$
$5x - 10y - 10z = -25$

f) $7x + y - z = 2$
$6x - 2y + 3z = 1$
$8x + 4y - 5z = 3$

3. For each of the linear systems of three equations in three variables in the previous question, describe the system using the terms consistent (dependent or independent) or inconsistent.

4. Find the value of k so that the following planes will intersect in a point.
$x + 2y + 3z = 5$
$5x + y - kz = 3$
$3x - 4y + kz = -8$

5. Find the value of m so that the following planes will intersect in a line.
$2x - 3y + mz = 7$
$x - my + z = 14$
$5x - 14y + 17z = 28$

6. Find the value of p so that the following planes will form a triangular prism.
$4x + y - 2z = p$
$2x - py + 2z = 1$
$6x + 5y - 6z = 0$

7. There is an infinite number of planes that form a triangular prism with the two planes $x - 2y + 3z = 5$ and $-x + 3y + z = 7$. Find an equation for any one of these planes.

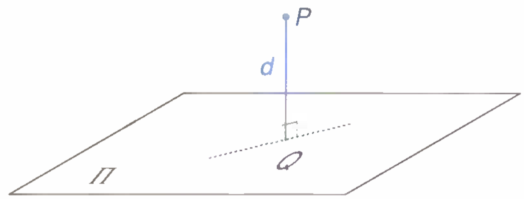
8. In a certain system of three linear equations, the third row of the row-reduced form of the augmented matrix is
$0 \quad 0 \quad a - 7 \quad b - 5$
Describe the intersection of the three planes for the following values of a and b.
a) $a = 7, b = 5$
b) $a = 7, b = 6$
c) $a = 8, b = 5$

9. Given the three planes
$x + y + 2z = 0$
$mx + (m - 1)y + mz = -1$
$mx + (2m - 1)z = 1$
a) For what values of m will the three planes intersect in a point?
b) For what values of m will the three planes intersect in a line?
c) For what values of m will the three planes have no points in common and form a triangular prism?

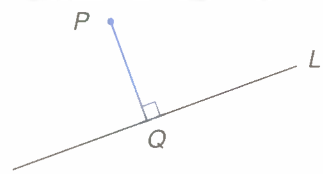
10. a) State conditions on the normal vectors $\vec{n}_1$, $\vec{n}_2$, and $\vec{n}_3$ so that the corresponding planes Π_1, Π_2, and Π_3 will intersect in a point.
b) If the planes Π_1, Π_2, and Π_3 have a common point, state conditions on the normal vectors $\vec{n}_1$, $\vec{n}_2$, and $\vec{n}_3$ so that the planes will intersect in a line.
c) If the planes Π_1, Π_2, and Π_3 do not have a common point, state conditions on the normal vectors $\vec{n}_1$, $\vec{n}_2$, and $\vec{n}_3$ so that the planes will form a triangular prism.

6.6 The Distance from a Point to a Plane or to a Line

The distance from a point P to a plane Π is the length of the line segment PQ where Q is a point on the plane Π and PQ is perpendicular to the plane Π.



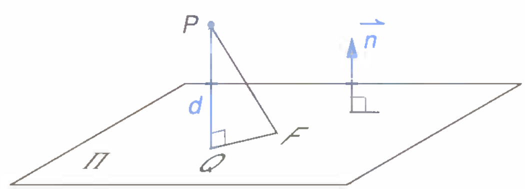
The distance from a point P to a line is the length of the line segment PQ where Q is a point on the line such that PQ is perpendicular to the line.



In this section you will develop formulas for both of these distances.

A formula for the distance from a point P to a plane Π is found as follows.

Let Q be the point on the plane Π such that $PQ \perp$ the plane Π.
Thus, PQ is parallel to the normal $\vec{n}$ of the plane Π.
Let F be any point on the plane Π.



Then the length of $PQ = |\text{the component of } \overrightarrow{PF} \text{ along } \overrightarrow{PQ}|$

$$= |\text{the component of } \overrightarrow{PF} \text{ along } \vec{n}|$$

$$= \left| \frac{\overrightarrow{PF} \cdot \vec{n}}{|\vec{n}|} \right|$$

If the length of $PQ = d$, then you have the following formula.

FORMULA

$$d = \left| \frac{\overrightarrow{PF} \cdot \vec{n}}{|\vec{n}|} \right|$$

where d is the distance from any point P to a plane Π,
F is any point on the plane Π, and $\vec{n}$ is a vector perpendicular to plane Π.

Example 1 Find the distance from the point $P(1,2,3)$ to the plane $4x - 5y + 6z + 8 = 0$.

Solution The required distance d is given by the formula

$$d = \left|\frac{\overrightarrow{PF} \cdot \vec{n}}{|\vec{n}|}\right| \text{ where } P = (1,2,3) \text{ and } \vec{n} = \overrightarrow{(4,-5,6)}.$$

To find a point F on the plane $4x - 5y + 6z + 8 = 0$ ①, you need three numbers x,y,z that make this equation true. There is an infinity of such numbers. To obtain one set, let two of x,y and z have any values. Substitute these in ①, then solve for the remaining variable.

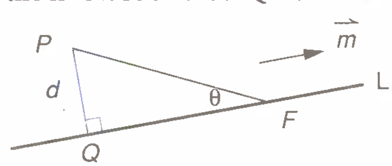
If $x = 1$ and $y = 0$, substituting into ① gives
$4(1) - 5(0) + 6z + 8 = 0$ or $z = -2$.
Thus, use $F = (1,0,-2)$ in the formula.
Therefore, $\overrightarrow{PF} = \overrightarrow{OF} - \overrightarrow{OP} = \overrightarrow{(1,0,-2)} - \overrightarrow{(1,2,3)} = \overrightarrow{(0,-2,-5)}$

$$\text{Thus, } d = \left|\frac{\overrightarrow{(0,-2,-5)} \cdot \overrightarrow{(4,-5,6)}}{\sqrt{4^2 + (-5)^2 + 6^2}}\right| = \frac{|10 - 30|}{\sqrt{77}} \text{ or } \frac{20}{\sqrt{77}}$$

Thus, the distance from the point $P(1,2,3)$ to the plane $4x - 5y + 6z + 8 = 0$ is $\dfrac{20}{\sqrt{77}}$ ■

A formula for the distance from a point P to a line L is found as follows.

Let Q be the point on the line L such that $PQ \perp$ the line L.



Then the required distance is $d = PQ$.
Let $\vec{m}$ be a direction vector of line L, and F any point on line L.
If θ is the angle between line L and PF, then

$d = PQ = |\overrightarrow{PF}|\sin\theta|$ ①

But $|\overrightarrow{PF} \times \vec{m}| = |\overrightarrow{PF}||\vec{m}|\sin\theta|$

$$\text{Thus, } |\sin\theta| = \frac{|\overrightarrow{PF} \times \vec{m}|}{|\overrightarrow{PF}||\vec{m}|}$$

Substituting in ① gives

$$d = \left||\overrightarrow{PF}|\left(\frac{\overrightarrow{PF} \times \vec{m}}{|\overrightarrow{PF}||\vec{m}|}\right)\right|$$

FORMULA

$$d = \left|\frac{\overrightarrow{PF} \times \vec{m}}{|\vec{m}|}\right|$$

where d is the distance from any point P to the line L,
F is any point on the line L, and $\vec{m}$ is a vector parallel to line L.

Example 2 Find the distance from the point $P(1,2,3)$ to the line $L: \vec{r} = \overrightarrow{(4,-1,5)} + t\overrightarrow{(-3,2,6)}$

Solution The required formula is

$$d = \left| \frac{\overrightarrow{PF} \times \vec{m}}{|\vec{m}|} \right|$$

where $P = (1,2,3)$, $\vec{m} = \overrightarrow{(-3,2,6)}$ and F is any point on L. One such point has coordinates $(4,-1,5)$.

Therefore, $\overrightarrow{PF} = \overrightarrow{OF} - \overrightarrow{OP} = \overrightarrow{(4,-1,5)} - \overrightarrow{(1,2,3)} = \overrightarrow{(3,-3,2)}$

Thus, $d = \left| \frac{\overrightarrow{(3,-3,2)} \times \overrightarrow{(-3,2,6)}}{\sqrt{(-3)^2 + 2^2 + 6^2}} \right|$

$$= \frac{\vec{i}(-18 - 4) - \vec{j}(18 + 6) + \vec{k}(6 - 9)}{\sqrt{49}}$$

$$= \frac{|\overrightarrow{(-22,-24,-3)}|}{\sqrt{49}} = \frac{\sqrt{1069}}{7}$$

Thus, the distance from the point $P(1,2,3)$ to the line $L: \vec{r} = \overrightarrow{(4,-1,5)} + t\overrightarrow{(-3,2,6)}$ is $\frac{\sqrt{1069}}{7}$ ■

Note: The above formula can be used in 2-space by expressing each 2-space vector as a 3-space vector with $z = 0$.

Example 3 Find the distance from the point $P(3,1)$ to the line $L: \vec{r} = \overrightarrow{(5,-2)} + t\overrightarrow{(4,7)}$

Solution The point P and the line L are in 2-space.

To use the formula $d = \left| \frac{\overrightarrow{PF} \times \vec{m}}{|\vec{m}|} \right|$

P must be written as the 3-space point $(3,1,0)$ and L must be expressed as the 3-space line $\vec{r} = \overrightarrow{(5,-2,0)} + t\overrightarrow{(4,7,0)}$.

F is any point on L. Such a point is $(5,-2,0)$.

Therefore, $\overrightarrow{PF} = \overrightarrow{OF} - \overrightarrow{OP} = \overrightarrow{(5,-2,0)} - \overrightarrow{(3,1,0)} = \overrightarrow{(2,-3,0)}$

Since $\vec{m} = \overrightarrow{(4,7,0)}$

$$d = \left| \frac{\overrightarrow{(2,-3,0)} \times \overrightarrow{(4,7,0)}}{\sqrt{4^2 + 7^2 + 0^2}} \right| = \vec{i}(0 - 0) - \vec{j}(0 - 0) + \vec{k}(14 + 12)$$

$$= \frac{|\overrightarrow{(0,0,26)}|}{\sqrt{65}} = \frac{26}{\sqrt{65}}$$

Thus, the distance from the point $P(3,1)$ to the line $L: \vec{r} = \overrightarrow{(5,-2)} + t\overrightarrow{(4,7)}$ is $\frac{26}{\sqrt{65}}$ ■

6.6 Exercises

1. Find the distance between the given point and the given plane.
   a) $(2,-1,3)$ $\quad 3x + 5y - z = 5$
   b) $(0,3,-5)$ $\quad 4x - 2y + 3z = 1$
   c) $(9,-2,0)$ $\quad x + 3z = 4$

2. Find the distance between the point and the given plane by first using a cross product to find a normal vector to the plane.
   a) $(-1,9,3)$; $\vec{r} = \overrightarrow{(1 + 3k - 4m, 2 - 4k + 2m, -3 + k - m)}$
   b) $(-4,0,-2)$; $\vec{r} = \overrightarrow{(2k + 3m, 1 + 4k + 3m, 5 - 2k + m)}$
   c) $(5,0,-1)$; $\vec{r} = \overrightarrow{(2 + 5k + 4m, 3 - 2k + 6m, 4k - 2m)}$

3. Find the distance between the given point and the given line.
   a) $(2,-1,3)$ $\quad \vec{r} = \overrightarrow{(3,4,1)} + k\overrightarrow{(1,-2,3)}$
   b) $(5,0,-1)$ $\quad \vec{r} = \overrightarrow{(0,1,4)} + k\overrightarrow{(0,0,-5)}$
   c) $(1,-1,8)$ $\quad \dfrac{x-1}{2} = \dfrac{y+3}{-3} = \dfrac{z-2}{5}$
   d) $(-3,4,8)$ $\quad \dfrac{x-1}{2} = \dfrac{y+3}{-3}, z = 3$

4. Given the parallel planes

   Π_1: $3x + 2y + 4z = 12$, and
   Π_2: $3x + 2y + 4z = 5$

   a) Show that the point $P\,(2,1,1)$ lies on Π_1.
   b) Find the distance between P and plane Π_2.
   c) Explain why the number obtained in b) is the distance between the parallel planes Π_1 and Π_2.

5. Find the distance between the pairs of parallel planes.
   a) $4x + y - 2z = 3$; $4x + y - 2z = 13$
   b) $3x - 2y - 5z = -2$; $3x - 2y - 5z = 5$
   c) $x + 3y - z = 8$; $2x + 6y - 2z = 7$

6. Given the parallel lines
   L_1: $\vec{r} = \overrightarrow{(2 + k, 3 - 4k, 1 - k)}$ and
   L_2: $\vec{r} = \overrightarrow{(k, 1 - 4k, 3 - k)}$
   a) Show that the point $P\,(2,3,1)$ lies on L_1.
   b) Find the distance between P and line L_2.
   c) Explain why the number obtained in b) is the distance between the parallel lines L_1 and L_2.

7. Find the distance between the pairs of parallel lines.
   a) $\vec{r} = \overrightarrow{(1 + 3k, 2 + 5k, 2 - 6k)}$ and $\vec{r} = \overrightarrow{(-1 + 3k, 1 + 5k, 3 - 6k)}$
   b) $\dfrac{x+2}{-1} = \dfrac{y-1}{4} = \dfrac{z+3}{7}$ and $\dfrac{x-5}{-1} = \dfrac{y+2}{4} = \dfrac{z}{7}$

8. Find the value of B so that the distance from the point $(1,-2,-3)$ to the plane $x + By - 2z + 1 = 0$ is 4.

9. Find the value of D so that the distance from the point $(4,0,1)$ to the plane $2x + y - 2z + D = 0$ is 3.

10. Find the distance between the point of intersection of the lines $\vec{r} = \overrightarrow{(1 + t, 3 - 2t, 4t)}$ and $\vec{r} = \overrightarrow{(1 + 2k, 1 - 2k, 7 + k)}$ and the plane through the points $(0,1,0)$, $(-1,2,1)$ and $(3,2,1)$.

11. Find the distance *in 2-space* from the point $P(2,3)$ to the line $\vec{r} = \overrightarrow{(3 - 2t, 1 + t)}$.

12. Find the coordinates of the points on the x-axis that are 4 units from the plane $x + y - 2z = 3$.

13. a) Find the perpendicular distance AN from the point $A(2,3,-5)$ to the plane Π: $4x - y - z + 3 = 0$
    b) Find the coordinates of point N.

14. Given the triangle with vertices $A(1,2,3)$, $B(-4,1,0)$ and $C(0,1,5)$.
  a) Find an equation for the line through the points B and C.
  b) Find the distance from the point A to the line through points B and C.
  c) Find the area of triangle ABC.

15. A tetrahedron is formed from the triangle ABC of the previous question and the point $O(0,0,0)$.
  a) Find an equation of the plane through the points A, B, and C.
  b) Find the distance from point O to the plane in a).
  c) Find the volume of the tetrahedron $OABC$.

16. A geologist discovers that a vein of gold ore completely fills the space between two parallel faces of rock. Her mathematician friend determines that the equations of the rock faces enclosing the ore are $5x - 2y + z = 100$ and $5x - 2y + z = -205$. If the unit of measurement is the metre, find the thickness of the vein of gold ore, correct to 2 decimal places.

17. Prove that the shortest distance between the point $P_0(x_0,y_0,z_0)$ and the plane $Ax + By + Cz + D = 0$ is

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

18. a) Prove that the shortest distance between the point $P_0(x_0,y_0)$ and the line $Ax + By + C = 0$ is

$$d = \left|\frac{\overrightarrow{PF} \times \overrightarrow{m}}{|\overrightarrow{m}|}\right|,$$

for $P_0(x_0,y_0,0)$, $\overrightarrow{m} = \overrightarrow{(B,-A,0)}$, and F any point on the line of intersection of the planes $Ax + By + C = 0$ and $z = 0$.

  b) Prove that the distance

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

Summary

- *For a Plane in 3-space*

vector equation

$\vec{r} = \vec{r}_0 + k\vec{m} + s\vec{u}$

parametric equations

$x = x_0 + km_1 + su_1$

$y = y_0 + km_2 + su_2$

$z = z_0 + km_3 + su_3$

where

$\vec{r} = \overrightarrow{OP}$, the direction vector of *any* point $P(x,y,z)$ on the plane,

$\vec{r}_0 = \overrightarrow{OP_0}$ the direction vector of a *fixed* point $P_0(x_0,y_0,z_0)$ on the plane,

$\vec{m} = \overrightarrow{(m_1,m_2,m_3)}$ and $\vec{u} = \overrightarrow{(u_1,u_2,u_3)}$ are vectors parallel to the plane such that $\vec{m} \nparallel \vec{u}$, and k and s are parameters.

- If $\vec{m}$ and $\vec{u}$ are vectors parallel to plane Π, then $\vec{m} \times \vec{u}$ is perpendicular to the plane Π.

- A normal vector to a plane is a vector that is perpendicular to every vector parallel to the plane.

- $Ax + By + Cz + D = 0$ is a Cartesian equation of a plane, where
  1. The vector $\overrightarrow{(A,B,C)}$ is a normal vector for the plane $Ax + By + Cz + D = 0$.
  2. If $\vec{m}$ and $\vec{u}$ are two linearly independent vectors parallel to the plane $Ax + By + Cz + D = 0$, then $\overrightarrow{(A,B,C)}$ is any scalar multiple of the vector $\vec{m} \times \vec{u}$.

- *Facts about Parallel and Perpendicular Planes*
  1. Two planes are *parallel* if and only if their normal vectors are *parallel*.
  2. Two planes are *perpendicular* if and only if their normal vectors are *perpendicular*.

- A line L and a plane Π can be related in several ways. The line L may be parallel to the plane Π, may lie in the plane Π, or intersect the plane Π.
  In the first situation the line and plane have no point in common.
  In the second situation the line and plane have an infinite number of common points.
  In the third situation plane Π and line L have a single point in common.
  To determine which situation exists, solve the system of equations that represents the planes.

- Two planes may be parallel and distinct, or parallel and identical, or not parallel.
  1. Two planes are parallel if and only if their normal vectors are parallel.
  2. Two planes intersect in a line if and only if their normal vectors are not parallel.
  3. Two non-parallel planes intersect in a line.
- To determine whether or not the planes are parallel, examine their normal vectors. Planes are parallel if their normal vectors are linearly dependent.
- To find the line of intersection of two planes you should solve the system of two equations representing the planes.
- To determine the type of intersection of three planes, no two of which are parallel, solve the linear system of three equations in three variables.
- For a system of three equations in three variables, representing three planes, no two of which are parallel:

	number of solutions	*geometric description*
consistent and independent	one	three planes intersect in a point
consistent and dependent	infinite	three planes intersect in a line
inconsistent	none	three planes form a triangular prism

- The distance d from a point P to a plane Π is the length of the line segment PQ where Q is a point on the plane Π and PQ is perpendicular to the plane Π.

  $$d = \left|\frac{\vec{PF} \cdot \vec{n}}{|\vec{n}|}\right|$$

  where d is the distance from any point P to a plane Π,
  F is any point on the plane Π, and
  $\vec{n}$ is a vector perpendicular to plane Π.

- The distance d from a point P to a line is the length of the line segment PQ where Q is a point on the line such that PQ is perpendicular to the line.

  $$d = \left|\frac{\vec{PF} \times \vec{m}}{|\vec{m}|}\right|$$

  where d is the distance from any point P to the line L,
  F is any point on the line L, and
  $\vec{m}$ is a vector parallel to line L.

Inventory

1. In the vector equation $\vec{r} = \vec{r}_0 + k\vec{m} + s\vec{u}$, $\vec{r}$ is ____, $\vec{r}_0$ is ____, $\vec{m}$ and $\vec{u}$ are ____ where $\vec{m}$ is not ____ $\vec{u}$, and s and k are ____.

2. In the vector equation $\vec{r} = \overrightarrow{(1 + 2s + 3t, -2 + 5s - 4t, 6 - 7s + t)}$, a point on the plane is ____, and two vectors parallel to the plane are ____ and ____.

3. In the parametric equations $x = 3 + 2t - 4s$, $y = 5t + 2s$, $z = -3 + 9t + 2s$, a point on the plane is ____ and two vectors parallel to the plane are ____ and ____.

4. The presence of ____ parameters in a vector equation indicates the equation represents a plane. The presence of one parameter shows the equation represents a ____.

5. For the plane $3x + 4y + 5z + 6 = 0$, a normal vector is ____.

6. If two planes are parallel then their normal vectors are ____.

7. If a system of three equations in three variables has one solution, then the planes they represent intersect ____.

8. If a system of three equations in three variables has an infinity of solutions, then the planes they represent intersect ____.

9. If a system of three equations in three variables has no solution, and the planes represented by the equations are not parallel, then the planes form ____.

10. If the planes $3x + 2y + 4z = 3$ and $Ax + By + Cz + D = 0$ are perpendicular then ____ $= 0$.

11. If a line is not parallel to a plane then the intersection of the line and the plane is ____.

12. If a line is parallel to a plane where the line and the plane have a common point, then the intersection of the line and the plane is ____.

13. In the formula, $d = \left| \dfrac{\overrightarrow{PF} \cdot \vec{n}}{|\vec{n}|} \right|$

    d is ____, F is ____, and $\vec{n}$ is ____.

14. In the formula, $d = \left| \dfrac{\overrightarrow{PF} \times \vec{m}}{|\vec{m}|} \right|$

    d is ____, F is ____, and $\vec{m}$ is ____.

Review Exercises

1. Find a vector equation and parametric equations for each of the following planes.
   a) the plane through the point $A(1,3,6)$, that is parallel to the vectors $\overrightarrow{(0,2,2)}$ and $\overrightarrow{(1,6,5)}$
   b) the plane through the two points $B(5,-1,4)$ and $C(3,-3,0)$, that is parallel to the vector $\overrightarrow{(4,-1,2)}$
   c) the plane through the three points $A(1,2,3)$, $B(1,-1,0)$, and $C(2,-3,-4)$
2. Find a vector equation of the plane parallel to the line L_1: $\vec{r} = \overrightarrow{(2-3t,1+4t,1+2t)}$, that contains the line L_2: $\vec{r} = \overrightarrow{(2s,1-5s,4+s)}$.
3. Find a vector equation of the plane that contains the point $A(-1,0,1)$ and is perpendicular to the vector $\vec{u} = \overrightarrow{(4,1,2)}$.
4. Find a vector equation of the plane that contains the line L: $\vec{r} = \overrightarrow{(0,6,1)} + t\overrightarrow{(1,2,5)}$ and is perpendicular to the plane $3x + 2y - z = 4$.
5. Find a vector equation of the plane parallel to the vector $\overrightarrow{(4,8,-3)}$, that contains the line $\frac{x}{2} = \frac{y-3}{-1} = \frac{z+1}{5}$
6. List the pairs of planes that are parallel. Indicate whether the planes are distinct or identical.
   a) $4x - 5y + 3z + 3 = 0$
   b) $4x - 6y - 2z = 10$
   c) $x - 2y + 3z = 1$
   d) $2x - 3y - z = 5$
   e) $2x - 4y + 5z = 2$
   f) $4x - 5y + 3z = 3$
7. Find a Cartesian equation of the plane through the point $T(1,5,-3)$ having normal vector $\overrightarrow{(1,3,5)}$.
8. Find the point of intersection of the plane $2x + 3y - z = 1$ and the line of intersection of the two planes $x - y + z = 2$ and $x + 2y - 2z = -3$.
9. Find a Cartesian equation of the plane through the point $M(1,1,-1)$ and through the point $T(0,-2,3)$, that is parallel to the vector $\vec{u} = \overrightarrow{(6,-1,2)}$.
10. Find a Cartesian equation of the plane through the points $M(1,1,-1)$ and $T(0,-2,3)$, that is perpendicular to the plane $5x + 2y - 3z = 1$.
11. Find a Cartesian equation of the plane through the points $K(1,2,3)$, $R(1,-1,0)$, and $S(2,-3,4)$.
12. Find a Cartesian equation of the plane through the point $E(1,4,2)$ that contains the x-axis.
13. Find a Cartesian equation of the plane with vector equation $\vec{r} = \overrightarrow{(2,3,-4)} + k\overrightarrow{(-2,0,1)} + m\overrightarrow{(3,1,-1)}$.
14. Find a vector equation of the plane with Cartesian equation $3x - 2y + z = -2$.
15. Find the acute angle between the line $\vec{r} = \overrightarrow{(3-2t,4+t,2+5t)}$ and the plane $4x - 3y + 5z = 1$, correct to the nearest degree.
16. Find the acute angle between the planes $2x - 3y + z = 1$ and $3x - 5y - z - 3 = 0$, correct to the nearest degree.
17. Find a Cartesian equation for the plane that contains the z-axis and is parallel to the line $\vec{r} = \overrightarrow{(1+3t,-4t,-3+2t)}$.
18. Find a Cartesian equation of the plane parallel to the xy-plane, that passes through the point $(4,1,3)$.

19. Find the intersection of each of the following pairs of a line and a plane. In each case describe the intersection geometrically.

a) plane:
$\vec{r} = \overrightarrow{(2 + 4k - m, 1 + k - 4m, 2 - k + 2m)}$
line: $\vec{r} = \overrightarrow{(-2 + t, -t, 8 + 2t)}$

b) plane: $2x + 3y - 2z = 13$
line: $\vec{r} = \overrightarrow{(2 + t, 3 + 2t, 2t)}$

c) plane: $x - 2y + z = 2$
line: $\frac{x-1}{2} = \frac{y}{3} = \frac{z-1}{4}$

20. Find the value of k so that the line through the point $(2,-4,1)$, and also the point $(k,-k,2k)$, is parallel to the plane $x + y + z = 9$.

21. Show the line of intersection of the planes $x - 3y + 6 = 0$ and $x - 2z - 4 = 0$ lies in the plane $x - 9y + 4z + 26 = 0$.

22. a) You are given the 2-space equation $Ax + By + C = 0$ where A and B are *fixed* numbers and C is a *parameter*. Explain why the equation represents a family of parallel lines.

b) You are given the 3-space equation $Ax + By + Cz + D = 0$ where A, B and C are *fixed* numbers and D is a *parameter*. Explain why this equation represents a family of parallel planes.

23. A plane passes through the points $P_1(x_1,y_1,z_1)$, $P_2(x_2,y_2,z_2)$ and $P_3(x_3,y_3,z_3)$. Prove that an equation of the plane is $Ax + By + Cz + D = 0$ where

$$A = \begin{vmatrix} y_2 - y_1 & z_2 - z_1 \\ y_3 - y_1 & z_3 - z_1 \end{vmatrix}$$

$$B = \begin{vmatrix} z_2 - z_1 & x_2 - x_1 \\ z_3 - z_1 & x_3 - x_1 \end{vmatrix}$$

$$C = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \text{ and}$$

$D = -Ax_1 - By_1 - Cz_1$

24. Find a Cartesian equation of the plane containing the line $\frac{x-1}{4} = \frac{y+3}{2} = \frac{z-5}{4}$, that is parallel to the line $\frac{x+9}{3} = \frac{y-4}{2} = \frac{z+1}{2}$

25. Find a Cartesian equation of the plane passing through the origin, that is perpendicular to the plane $x + 2y - z = 5$ and makes an angle of 45° with the x-axis.

26. A line L intersects the x-axis in the same point as the plane $3x - 4y + z = 12$ and intersects the xy-plane in the same point as the line $\vec{r} = \overrightarrow{(6 - 2t, 1 + t, t - 1)}$. Find a vector equation of line L.

27. Find the point of intersection of the line $2x - 3 = \frac{y+2}{2}$; $z = 3$ with the plane $x + 2y - 3z = 11$.

28. For each of the following planes use normal vectors to determine whether or not the planes are parallel or intersect. If the planes are parallel, indicate whether the planes are distinct or identical. If the planes intersect, then find parametric equations of their line of intersection.

a) $6x + 4y - 2z = 10$
$3x + 2y - z = 2$

b) $8x + 6y + 2z = 4$
$12x + 9y + 3z = 6$

c) $x - 3y + 5z = 1$
$3x + 2y + z = 3$

d) $5x - 3y = 4$
$3y + z = 1$

29. Given the planes
Π_1: $2x + 3y + 2z = 0$
Π_2: $6x - 9y - 2z = 10$
Π_3: $2x + 3y - z + 3 = 0$
Π_4: $2x - 3y - 2z + 12 = 0$
Show that the line L_1 of intersection of planes Π_1 and Π_2 is skew with L_2, the line of intersection of planes Π_3 and Π_4.

30. Find a Cartesian equation of the plane through the point $(2,-3,-11)$ that is perpendicular to the plane $2x - y + z = 2$ and intersects the line $\vec{r} = \overrightarrow{(3 + t, 4 + 2t, 2t)}$ at the point where $y = 0$.

31. There is an infinite number of pairs of planes that have the line $\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{3}$ as their line of intersection. Find equations for any two of these planes.

32. Given the planes Π_1: $x+5y-z=1$ and Π_2: $3x+ky+2z=5$.

a) Find the value of k so that the line of intersection of planes Π_1 and Π_2 is parallel to the line $\vec{r}=\overrightarrow{(2-4t,1+t,-3t)}$

b) Find the value of k so that the line of intersection of planes Π_1 and Π_2 is perpendicular to the line $\vec{r}=\overrightarrow{(3s,2-3s,1+4s)}$

33. In each of the following, the equations of three planes are given. Describe how the planes in each set are related.

a) $x-2y+4z=-4$
$x-2y+4z=0$
$x-2y+4z=2$

b) $x+5y-3z=1$
$2x+10y-6z=2$
$3x+15y-9z=3$

34. In each of the following, the equations of three non-parallel planes are given. Describe how each set of planes intersects by solving the corresponding linear system.

a) $x-2y+4z=-4$
$4x-5y+8z=0$
$x-y+z=2$

b) $x+2y-3z=-1$
$3x+y-3z=6$
$5x+5y-9z=4$

c) $x+5y-3z=6$
$2x+2y-z=2$
$3x+7y-4z=3$

d) $3x+y-z=6$
$9x-4y+2z=21$
$x+4y+7z=-18$

35. Describe each system of the previous question as consistent (dependent or independent) or inconsistent.

36. Given the three planes

$$\begin{aligned} x+y+2z&=0\\ (k+2)x+(k+1)y+(k+2)z&=20\\ (k+2)x+(2k+3)z&=1\end{aligned}$$

a) For what values of k will the planes intersect in a point?

b) For what values of k will the planes intersect in a line?

c) For what values of k will the planes form a triangular prism?

37. The following system of equations represents three distinct parallel planes.

$$\begin{aligned} ax+5y+4z&=3\\ 4x+10y+bz&=3\\ 3x+cy+6z&=d\end{aligned}$$

Find the values of a, b, c, and d.

38. Prove that the system

$$\begin{aligned} a_1x+b_1y+c_1z&=d_1\\ a_2x+b_2y+c_2z&=d_2\\ a_3x+b_3y+c_3z&=d_3\end{aligned}$$

represents three distinct parallel planes provided that
$a_1:a_2:a_3=b_1:b_2:b_3=c_1:c_2:c_3$
and $a_1:a_2\neq d_1:d_2$ and $a_1:a_3\neq d_1:d_3$

39. Find the distance between the given point and the given plane.

a) $(0,0,0)$ $\quad x+2y+3z=3$

b) $(3,2,-4)$ $\quad 5x+3y+7z=3$

c) $(0,2,-3)$
$\vec{r}=\overrightarrow{(2+3k+2m,3-2k+4m,1+k+2m)}$

d) $(1,5,1)$
$\vec{r}=\overrightarrow{(2m+k,1+m+4k,3-2k-3m)}$

40. Find the distance between the given point and the given line.

a) $(0,0,0)$ $\quad \vec{r}=\overrightarrow{(8,1,3)}+k\overrightarrow{(5,3,-2)}$

b) $(3,-1,4)$ $\quad \vec{r}=\overrightarrow{(2,3,1)}+k\overrightarrow{(6,0,-1)}$

c) $(1,-1,8)$ $\quad \frac{x-1}{2}=\frac{y+3}{-3}=\frac{z-2}{5}$

d) $(0,0,0)$ $\quad \frac{x+2}{-1}=\frac{y-1}{4}=\frac{z+3}{7}$

e) $(-3,4,8)$ $\quad \frac{x-1}{2}=\frac{y+3}{-3}, z=3$

41. Given the points $P(-1,3,3)$ and $Q(-3,4,2)$ and the plane
Π: $\vec{r}=\overrightarrow{(2,1,4)}+m\overrightarrow{(1,0,1)}+k\overrightarrow{(2,-1,1)}$.
Prove that every point on the line PQ lies in the plane Π.

42. The plane $\vec{r}=\overrightarrow{(22+3s+t,1-s+2t,1+9t)}$ is parallel to the line $\vec{r}=\overrightarrow{(2,1,6)}+k\overrightarrow{(0,a,b)}$. Find the relationship between a and b.

43. Find the intersection of the plane $\vec{r}=\overrightarrow{(2,6,1)}+s\overrightarrow{(1,-1,-1)}+t\overrightarrow{(3,4,2)}$ and the xy-plane.

44. Find the distance between the pairs of parallel lines
$\vec{r} = \overrightarrow{(4 + k, 1 + 3k, 1 - 7k)}$ and
$\vec{r} = \overrightarrow{(-1 + k, 1 + 3k, 3 - 7k)}$

45. Find the value of A so that the distance from the point $(-2,1,-3)$ to the plane $Ax + y - 2z = -1$ is 4.

46. Given the system of equations

$$\begin{aligned} x + y + z &= 1 \\ kx + my + tz &= a \\ (k+1)x + (m+1)y + (t+1)z &= 3 \end{aligned}$$

where k, m, and $t \in \mathbb{R}$

a) For what value of a will this system have an infinity of solutions?
b) For what value of a will this system have no solution?
c) For what value of a will this system have a single solution?

47. Determine the distance between the parallel planes $2x - 2y + z = 1$ and $x + By + Cz = -3$.

48. Find a vector equation for the line through the point $(1,0,1)$ that is parallel to each of the planes $2x + y + 3z = 1$ and $x - 2y + z = 2$.

49. Given the planes
$p = x + y + z$
$q = 2x - y - z$
$w = 3x + 2y + z$
Express each of x, y and z in terms of p, q, and w.

50. Given the planes $x + y + z = 1$ and $2x - 3y + z = -1$

a) Write an equation for a plane that intersects these two planes in the point $(2,1,-2)$.
b) Write an equation for a plane that intersects these two planes in a line.
c) Write an equation for a plane that forms a triangular prism with these two planes.

51. Given the plane Π: $\vec{r} = \overrightarrow{(11,5,1)} + t\overrightarrow{(3,-2,6)} + s\overrightarrow{(0,-3,7)}$. A point A lies on plane Π such that $\overrightarrow{OA}$ is parallel to $\vec{u} = \overrightarrow{(7,3,0)}$. Find the coordinates of point A.

52. Given the line $\vec{r} = \overrightarrow{(2,a,b)} + k\overrightarrow{(-1,1,1)}$ and the plane $\vec{r} = \overrightarrow{(4 - 2s + t, 2 + 6s + t, 5 + 4s)}$. For what values of a and b will the line lie in the plane?

53. Given the plane Π: $\vec{r} = \vec{r}_0 + k\vec{a} + m\vec{b}$ where $\vec{r}_0 = \overrightarrow{OP_0}$. Show that the point P lies in plane Π if and only if $\overrightarrow{P_0P} \cdot (\vec{a} \times \vec{b}) = 0$.

54. Show that a vector equation of the plane through the three points A, B, and C with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively is $\vec{r} = \vec{a} + k\overrightarrow{(b - a)} + s\overrightarrow{(c - a)}$

55. A tetrahedron $ABCD$ is given with vertices at the points $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$ and $D(1,1,1)$.

a) Find the equation of the plane containing the points A, B and C.
b) Find the equation of the plane containing the points B, C and D.
c) Find the value of the acute angle between the planes ABC and BCD, giving your answer to the nearest degree.
d) Find the perpendicular distance from the point D to the plane ABC, leaving your answer in surd form.
e) Calculate the volume of the tetrahedron $ABCD$.

(85 S)

56. The planes p_1 and p_2 are given with equations
p_1: $2x + 3y - z = 4$,
p_2: $3x + 5y - 4z = -1$.
a) Find, in parametric form, the equation of the line l that is the intersection of the planes p_1 and p_2.
b) The plane with equation
$cx + y + 2z = 9$,
where c is a constant, contains the line l. Find the value of the constant c.
c) Find the equation of the plane containing l and passing through the point with coordinates (0,0,1).
(85 S)

57. A line l passes through the points A and B whose position vectors are $9\vec{i} - 5\vec{j} + 2\vec{k}$ and $5\vec{i} - \vec{j}$ respectively. A plane Π has a normal vector $3\vec{i} - 4\vec{j} - \vec{k}$ and it passes through the point whose position vector is $-\vec{i} + 4\vec{k}$.
a) Show that the position vector of the point of intersection of l and Π is $\vec{i} + 3\vec{j} - 2\vec{k}$.
b) Find the position vector of the foot of the perpendicular from A onto the plane Π.
c) The line m is the projection of the line l onto the plane Π. Find, in any form, the equation of the line m.
d) The line n is the reflection of the line l in the plane Π. Show that a vector in the direction of n is $-\vec{i} + 2\vec{j} + 2\vec{k}$
(88 H)

58. Two lines L_1 and L_2 are given whose equations are

$$L_1: \frac{x+3}{2} = \frac{y+1}{3} = \frac{z}{1}$$

$$L_2: \frac{x}{1} = \frac{y-1}{-1} = \frac{z-3}{2}$$

a) Prove that the lines L_1 and L_2 intersect and find the coordinates of P, their point of intersection.
b) Find a vector which is perpendicular at the same time to both of the lines L_1 and L_2 and hence, or otherwise, find the equation of the plane p which contains both the lines L_1 and L_2.
c) Find the perpendicular distance from the origin O on to the plane p, giving your answer correct to two decimal places.
(84 S)

59. The equations of three planes Π_1, Π_2 and Π_3 are
Π_1 $3x - y + z = 3$
Π_2 $2x + y + z = 4$
Π_3 $8x - y + 3z = 10$
a) Find, correct to the nearest degree, the angle between Π_1 and Π_2.
b) Show that the three planes intersect in a line, and give the Cartesian equations of this line.
c) The point A has position vector $17\vec{i} + 2\vec{j} + 8\vec{k}$. The point B is the foot of the perpendicular from A to the plane Π_3. Find the position vector of B.
d) Show that $AB = \sqrt{296}$.
(87 H)

60. a) Using a rectangular Cartesian co-ordinate system $Oxyz$ determine the equation of the plane Π passing through the origin O and the points $A(1,0,2)$ and $B(0,3,-1)$. Write down the components of a vector $\vec{n}$ normal to Π.
b) Derive the parametric form of the equation of the normal to the plane Π through A. Calculate the co-ordinates of the point of intersection H of this normal with the plane whose equation is $x = -5$.
c) The point $P(0,-4,3)$ is joined to the point $Q(0,0,q)$, $q \in \mathbb{R}$. Given that (PQ) is parallel to the plane Π, find the value of q.
(83 S)

VECTORS, MATRICES
and
COMPLEX NUMBERS

with
International Baccalaureate
questions

Jean-Paul GINESTIER
and
John EGSGARD

CHAPTER SEVEN

MATRICES AND LINEAR TRANSFORMATIONS

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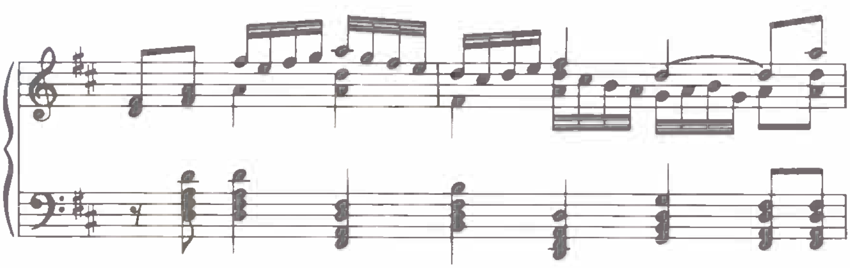
CHAPTER SEVEN

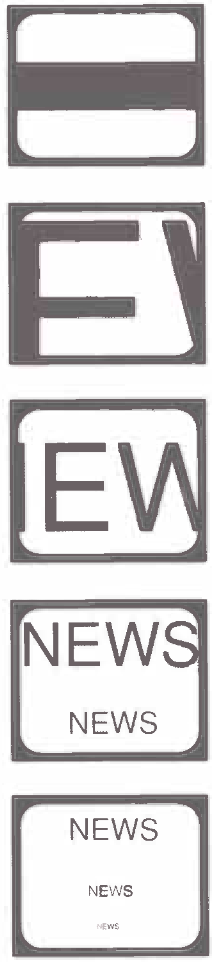
Matrices and Linear Transformations

Many of the introductions to television programs and movies include visual displays of transformations such as rotations and dilatations. Some well-known examples of this are the introductions to "The National" and "The Journal" on CBC television, or the text that appears to move into outer space in the introduction and credits of the "Star Wars" movies.

Transformations are a part of our life in a modern society. Indeed, transformations are involved in any form of representation—be it drawing or painting, sculpture, playing music, etc.



Transformations have an increasingly wide application in design technology. Computer programs are now available to architects as an aid in the design of elaborate structures. Engineers use similar programs to help them develop new projects.



In this chapter, you will be introduced to a new mathematical object called a **matrix** (plural matrices) that can be used as an operator to effect various transformations.

Matrices provide a very compact way of expressing transformations.

Recall that many vector equations can be applied to 2-space, 3-space, and even to spaces of higher dimensions. Matrix equations also have this same universality.

This chapter will provide you with an extensive study of transformations of 2-space, but the principles that you learn will be readily applicable to transformations of 3-space.

7.1 Matrices

In this section, you will be taking an elementary look at matrices, their properties, and some of their algebra. The main purpose of this chapter is the study of matrices as operators that transform vectors. That study will begin in the next section.

DEFINITION A matrix is a rectangular array of numbers.

For example, $A = \begin{bmatrix} 2 & 1 & -1 \\ 5 & 6 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & -4 \\ -1 & 7 \end{bmatrix}$ are matrices.

The numbers composing the matrix are called **elements**. (They are also known as **entries** or **components**.)

The Dimension of a Matrix

A matrix can be described by its **rows** or its **columns** of elements. For example, in A,

the 1st row is 2 1 −1

the 2nd row is 5 6 0

the 1st column is $\begin{matrix} 2 \\ 5 \end{matrix}$; the 2nd column is $\begin{matrix} 1 \\ 6 \end{matrix}$; the 3rd column is $\begin{matrix} -1 \\ 0 \end{matrix}$

Since the matrix A has 2 rows and 3 columns,
A is known as a 2×3 matrix (read "2 by 3 matrix").
Alternatively, A is said to have **dimension** 2×3 (or **order** 2×3, or **shape** 2×3).

DEFINITION A matrix that has m rows and n columns is known as an $m \times n$ matrix.

Example 1 What are the dimensions of the matrices B and C above?

Solution The matrix B above is a 3×1 matrix.
The matrix C above is a 2×2 matrix. ■

Subscript Notation

Given any matrix A, it is often useful to specify its elements in the following way.

The element in the ith row, jth column is represented by a_{ij}.

You can also abbreviate A to $A = [a_{ij}]$.

Thus, if A is a 2×2 matrix, it can be written as

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Equality

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if and only if *all* their corresponding elements are respectively equal.
That is, $a_{ij} = b_{ij}$ for all values of i and j.
(Thus only matrices that have the *same dimensions* can be equal.)

Example 2 Given $M = \begin{bmatrix} 3 & -2 \\ -z & 5 \end{bmatrix}$, $N = \begin{bmatrix} x & 2y \\ 11 & 5 \end{bmatrix}$, and $M = N$, find the values of x, y, and z.

Solution The elements in the 1st row, 1st column, are 3 and x.
Thus $x = 3$.
The elements in the 1st row, 2nd column, are -2 and $2y$.
Thus $y = -1$.
The elements in the 2nd row, 1st column, are $-z$ and 11.
Thus $z = -11$. ■

Square Matrices

DEFINITION

A square matrix of order p is a matrix of dimension $p \times p$.

The properties and operations discussed in the rest of this section will be devoted entirely to 2×2 matrices, that is, square matrices of order 2. This is to prepare you for the 'matrices as operators' that you will be using in the rest of the chapter. Whenever the term "matrix" is used, it will mean "2×2 matrix".

The Algebra of 2×2 Matrices

Addition and Subtraction

Given $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $N = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$, the sum $M + N$ is defined as follows:

$M + N = \begin{bmatrix} a + w & b + x \\ c + y & d + z \end{bmatrix}$ Corresponding elements are added.

The difference is defined as follows:

$M - N = \begin{bmatrix} a - w & b - x \\ c - y & d - z \end{bmatrix}$ Corresponding elements are subtracted.

The Zero Matrix

The 2×2 zero matrix has each element equal to zero. It will be represented by $0_{2\times2}$.

Thus $0_{2\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

The zero matrix is called the **neutral element** for the addition of 2×2 matrices because, for any matrix M,
$M + 0_{2\times2} = M$ and $0_{2\times2} + M = M$.

The zero matrix is also known as the **null matrix**.

Commutativity of Matrix Addition

The addition of matrices is commutative.
That is, for any matrices M and N, $M + N = N + M$.

Proof: Given $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $N = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$,

$$M + N = \begin{bmatrix} a+w & b+x \\ c+y & d+z \end{bmatrix}$$

and

$$N + M = \begin{bmatrix} w+a & x+b \\ y+c & z+d \end{bmatrix}$$

Since the addition of numbers is commutative,
$a + w = w + a$, $b + x = x + b$, $c + y = y + c$, and $d + z = z + d$.
Hence
$M + N = N + M$.

Associativity of Matrix Addition

The addition of matrices is associative. That is, given any matrices L, M and N,
$(L + M) + N = L + (M + N)$.
Thus, brackets are not required when adding matrices. The expression $L + M + N$ can be used to mean $(L + M) + N$.
You will have an opportunity to prove this property in the exercises.

The Negative of a Matrix

Given the matrix $A = [a_{ij}]$, then the matrix $[-a_{ij}] = -A$.
$-A$ is called the **negative** of A.
$-A$ is also called the **additive inverse** of A, because
$A + (-A) = 0_{2\times2} = (-A) + A$.

Multiplication of a Matrix by a Scalar

Since matrix addition is associative, it seems natural to write
$M + M + M = 3M$.
This operation is accepted, and is called
multiplication of a matrix by a scalar.

Given a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and a real number k,

then $$kM = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

Example 3 Given $M = \begin{bmatrix} -4 & 3 \\ -2 & 0 \end{bmatrix}$ and $N = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$, calculate

a) $-1M$ **b)** $4N$ **c)** $2M + 3N$

Solution

a) $-1M = -1\begin{bmatrix} -4 & 3 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -1(-4) & -1(3) \\ -1(-2) & -1(0) \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 2 & 0 \end{bmatrix}$

Notice that the matrix $-1M = -M$.

b) $4N = 4\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4(2) & 4(-1) \\ 4(0) & 4(1) \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ 0 & 4 \end{bmatrix}$

c) $2M + 3N = 2\begin{bmatrix} -4 & 3 \\ -2 & 0 \end{bmatrix} + 3\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -8 & 6 \\ -4 & 0 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -4 & 3 \end{bmatrix}$ ■

Note: A 2×2 matrix is in fact an ordered quadruple of numbers. Thus, it can be considered a "four-dimensional vector". The properties listed above go a long way towards showing that 2×2 matrices, together with the operations of matrix addition, and of multiplication of a matrix by a scalar, form a **vector space**. You will have an opportunity to prove this in the exercises.

Properties of 2×2 Matrices

2×2 Matrices form a vector space $\mathbb{V}_{2\times2}$, that is, the following properties hold.

PROPERTIES

Matrix Addition

A1. $\mathbb{V}_{2\times2}$ is closed under addition: $M, N \in \mathbb{V}_{2\times2}$ implies $M + N \in \mathbb{V}_{2\times2}$

A2. Addition is associative: $L + (M + N) = (L + M) + N$

A3. There is a $0_{2\times2} \in \mathbb{V}_{2\times2}$ such that for all $M \in \mathbb{V}_{2\times2}$, $M + 0_{2\times2} = M$

A4. If $M \in \mathbb{V}_{2\times2}$, then there exists $-M \in \mathbb{V}_{2\times2}$ such that $M + (-M) = 0_{2\times2}$

A5. Addition is commutative: $M + N = N + M$

(These properties mean that $\mathbb{V}_{2\times2}$ is a commutative group with respect to addition.)

PROPERTIES

Multiplication of a Matrix by a Scalar

M1. If $M \in \mathbb{V}_{2\times2}$, $k \in \mathbb{R}$, then $kM \in \mathbb{V}_{2\times2}$

M2. $(kp)M = k(pM)$, $k, p \in \mathbb{R}$

M3. $k(M + N) = kM + kN$

M4. $(k + p)M = kM + pM$

M5. There exists $1 \in \mathbb{R}$ such that $1M = M$

7.1 Exercises

1. Give the dimensions of the following matrices.

$A = \begin{bmatrix} 2 & 6 & 8 & 1 \\ -1 & 0 & 0 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 5 & -1 \\ 6 & 6 \\ 4 & 1 \end{bmatrix}$

$C = \begin{bmatrix} 7 \\ 2 \\ 5 \\ -2 \end{bmatrix}$

2. $A = [a_{ij}] = \begin{bmatrix} 3 & 1 & 0 & 1 \\ -2 & 4 & 5 & 2 \\ -1 & 0 & 8 & -6 \end{bmatrix}$

a) State the values of $a_{11}, a_{14}, a_{23}, a_{33}$.

b) Use the a_{ij} notation to describe the positions of the zero elements in this matrix.

3. Calculate the values of the variables in the following.

a) $\begin{bmatrix} x & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 4 & y \\ z & w \end{bmatrix}$

b) $\begin{bmatrix} 3x & 4y \\ 5 & w \end{bmatrix} = \begin{bmatrix} 9 & -8 \\ w - x & z \end{bmatrix}$

4. Calculate the values of the variables in the following.

a) $\begin{bmatrix} 2 & 3 \\ 5 & z \end{bmatrix} + \begin{bmatrix} x & 6 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & y \\ w & 0 \end{bmatrix}$

b) $\begin{bmatrix} a & 1 \\ 8 & -7 \end{bmatrix} - \begin{bmatrix} 9 & b \\ c & 5 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 8 & d \end{bmatrix}$

c) $2\begin{bmatrix} a & 1 \\ 8 & -7 \end{bmatrix} - 3\begin{bmatrix} 9 & b \\ c & 5 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 8 & d \end{bmatrix}$

d) $\begin{bmatrix} x - y & 4 \\ z & x \end{bmatrix} + \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 3 & x \end{bmatrix}$

5. Given $A = \begin{bmatrix} -5 & 0 \\ 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix}$,

$C = \begin{bmatrix} 7 & 9 \\ -4 & 6 \end{bmatrix}$ calculate the following.

a) $-B$ **d)** $A + B + C$ **g)** $2B + 2C$

b) $3A$ **e)** $\frac{1}{2}A - 2C$ **h)** $3(2A)$

c) $3A - B$ **f)** $2(B + C)$ **i)** $6A$

6. Using the matrices A, B, C given in question 5, calculate the matrix X in the following cases.

a) $2X = A$ **c)** $B - X = A$

b) $X - 3B = 0_{2\times2}$ **d)** $5X + C = 3X - A$

7. Show that no real values of x and y exist such that

$\begin{bmatrix} 3x & x + y \\ 2y & x - y \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 10 & 4 \end{bmatrix}$

8. Using as examples any of the matrices given in question 5, illustrate the following.

a) the commutative property of matrix addition

b) the associative property of matrix addition

For the following questions, in which you will be asked to prove various properties of 2×2 matrices belonging to a vector space, use the matrices

$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $N = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, $L = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$

9. Prove property A1: if M and N are both 2×2 matrices, then $M + N$ is a 2×2 matrix.

10. Prove property A2: matrix addition is associative, that is, $L + (M + N) = (L + M) + N$.

11. Prove property M1: if $k \in \mathbb{R}$, and M is a 2×2 matrix, then kM is a 2×2 matrix.

12. Prove property M2: if $k, p \in \mathbb{R}$, then $(kp)M = k(pM)$. (This was illustrated in parts h) and i) of question 5.)

13. Prove property M3: $k(M + N) = kM + kN$. (This was illustrated in parts f) and g) of question 5.)

14. Prove property M4: $(k + p)M = kM + pM$.

Your solutions to questions 9–14, together with the proofs supplied in the text of section 7.1, show that $\mathbb{V}_{2\times2}$ is a vector space.

7.2 Matrices and Linear Transformations

In this section you will be looking at transformations of vectors in $\mathbb{V}_2$ using 2×2 matrices as operators. It will be necessary to use a different notation when expressing vectors in component form.

The notation $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ will replace the notation $\vec{v} = \overrightarrow{(x,y)}$.

The former notation is called a **column vector** or a 2×1 matrix.

A **transformation** of a vector $\vec{v}$ is a function or mapping that changes $\vec{v}$ into another vector $\vec{v}'$.

For example,

$F\colon \vec{v} \rightarrow \vec{v}'$ where $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{v}' = \begin{bmatrix} 3x \\ -y \end{bmatrix}$ is a transformation.

It can be equally well described in the following ways.

$F\colon \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 3x \\ -y \end{bmatrix}$ or $F\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ -y \end{bmatrix}$ or $\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{F} \begin{bmatrix} 3x \\ -y \end{bmatrix}$ or $F(\vec{v}) = \vec{v}'$

The vector $\vec{v}'$ is known as the image of $\vec{v}$ under F.

You can also say that $\vec{v}$ is mapped onto $\vec{v}'$ by F.

DEFINITION

Linear Transformations

A linear transformation T of a vector space is a transformation that has *both* of the following properties.

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
2. $T(k\vec{v}) = k[T(\vec{v})]$

where $\vec{u}$, $\vec{v}$ are vectors and k is a real number.

Example 1

Check if the transformation F as defined above, namely $F\colon \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 3x \\ -y \end{bmatrix}$, is a linear transformation.

Solution

Let $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} w \\ z \end{bmatrix}$, then $\vec{u} + \vec{v} = \begin{bmatrix} x + w \\ y + z \end{bmatrix}$

$F(\vec{u}) = \begin{bmatrix} 3x \\ -y \end{bmatrix}$ $\quad F(\vec{v}) = \begin{bmatrix} 3w \\ -z \end{bmatrix}$ $\quad F(\vec{u} + \vec{v}) = \begin{bmatrix} 3(x + w) \\ -(y + z) \end{bmatrix} = \begin{bmatrix} 3x + 3w \\ -y - z \end{bmatrix}$

Now $F(\vec{u}) + F(\vec{v}) = \begin{bmatrix} 3x \\ -y \end{bmatrix} + \begin{bmatrix} 3w \\ -z \end{bmatrix} = \begin{bmatrix} 3x + 3w \\ -y - z \end{bmatrix} = F(\vec{u} + \vec{v})$

Thus the first property holds.

$F(k\vec{v}) = F(k\begin{bmatrix} w \\ z \end{bmatrix}) = F\begin{bmatrix} kw \\ kz \end{bmatrix} = \begin{bmatrix} 3kw \\ -kz \end{bmatrix}$

$k[F(\vec{v})] = k\begin{bmatrix} 3w \\ -z \end{bmatrix} = \begin{bmatrix} 3kw \\ -kz \end{bmatrix} = F(k\vec{v})$

Thus the second property also holds.

Hence, F is a linear transformation. ■

Example 2 Check if the following are linear transformations.

a) $G: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x+2 \\ 5x \end{bmatrix}$ b) $H: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x+2y \\ 3x-y \end{bmatrix}$

Solution Let $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} w \\ z \end{bmatrix}$, then $\vec{u} + \vec{v} = \begin{bmatrix} x+w \\ y+z \end{bmatrix}$

a) $G(\vec{u}) = \begin{bmatrix} x+2 \\ 5x \end{bmatrix} \quad G(\vec{v}) = \begin{bmatrix} w+2 \\ 5w \end{bmatrix}$

$G(\vec{u}+\vec{v}) = \begin{bmatrix} (x+w)+2 \\ 5(x+w) \end{bmatrix} = \begin{bmatrix} x+w+2 \\ 5x+5w \end{bmatrix}$

Now $G(\vec{u}) + G(\vec{v}) = \begin{bmatrix} x+2 \\ 5x \end{bmatrix} + \begin{bmatrix} w+2 \\ 5w \end{bmatrix} = \begin{bmatrix} x+w+4 \\ 5x+5w \end{bmatrix} \neq G(\vec{u}+\vec{v})$

Thus the first property does *not* hold. The transformation G is *not* linear.

b) $H(\vec{u}) = \begin{bmatrix} x+2y \\ 3x-y \end{bmatrix} \quad H(\vec{v}) = \begin{bmatrix} w+2z \\ 3w-z \end{bmatrix}$

$H(\vec{u}+\vec{v}) = \begin{bmatrix} (x+w)+2(y+z) \\ 3(x+w)-(y+z) \end{bmatrix} = \begin{bmatrix} x+2y+w+2z \\ 3x-y+3w-z \end{bmatrix}$

$H(\vec{u}) + H(\vec{v}) = \begin{bmatrix} x+2y+w+2z \\ 3x-y+3w-z \end{bmatrix} = H(\vec{u}+\vec{v})$

Thus the first property holds.

$H(k\vec{v}) = H(k\begin{bmatrix} w \\ z \end{bmatrix}) = H\begin{bmatrix} kw \\ kz \end{bmatrix} = \begin{bmatrix} kw+2kz \\ 3kw-kz \end{bmatrix}$

$k[H(\vec{v})] = k\begin{bmatrix} w+2z \\ 3w-z \end{bmatrix} = \begin{bmatrix} kw+2kz \\ 3kw-kz \end{bmatrix} = H(k\vec{v})$

Thus the second property also holds.

Hence, H is a linear transformation. ■

A general linear transformation in $\mathbb{V}_2$ has the following form.

$$T: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad ①$$

where a, b, c and d are real numbers.

You will have an opportunity in the exercises to prove that T thus defined is indeed a linear transformation.

Matrix Notation for Transformations

It is customary to use matrix notation to indicate the relationship between $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$.

DEFINITION $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

This definition of a matrix multiplying a column vector can be remembered in the following way.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

This is called a "row-column" multiplication process. It is the basis of multiplication of matrices in general, as you shall see in the forthcoming sections.

You can also think of the result of this multiplication in the following way.

$$\begin{bmatrix} \text{dot product of the first row with } \begin{bmatrix} x \\ y \end{bmatrix} \\ \text{dot product of the second row with } \begin{bmatrix} x \\ y \end{bmatrix} \end{bmatrix}$$

Also, note that the product (2×2 matrix) $\times$ (2×1 matrix) gives a (2×1 matrix).

Thus the transformation ① can be written in **matrix form**, as follows,

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad ②$$

or even more compactly like this

$$\vec{v} \rightarrow M\vec{v} = \vec{v}', \quad ③$$

where $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\vec{v}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$, and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Here, the matrix M is an operator called the **transformation matrix** of T.

Note: The above discussion implies that a 2×2 matrix will *always* express a linear transformation of $\mathbb{V}_2$.

Example 3 Find the images of each of the following vectors under the transformation matrix $M = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

$\vec{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

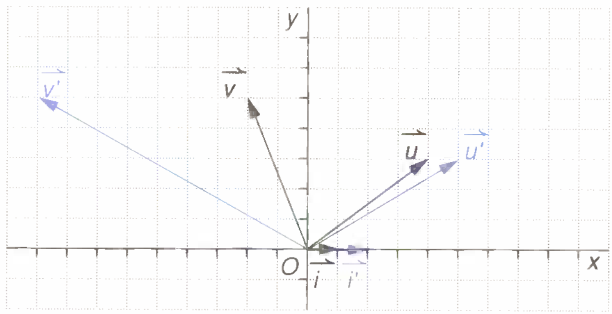
Sketch (as position vectors) $\vec{u}, \vec{v}, \vec{i}$ and their images $\vec{u'}, \vec{v'}, \vec{i'}$.

Solution

$$\vec{u'} = M\vec{u} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} (2)(4) + (-1)(3) \\ (0)(4) + (1)(3) \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\vec{v'} = M\vec{v} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} (2)(-2) + (-1)(5) \\ (0)(-2) + (1)(5) \end{bmatrix} = \begin{bmatrix} -9 \\ 5 \end{bmatrix}$$

$$\vec{i'} = M\vec{i} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (2)(1) + (-1)(0) \\ (0)(1) + (1)(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

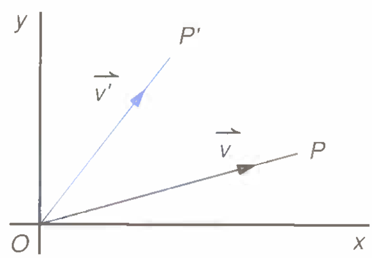


■

Note: The transformation affects the *entire plane*. The vectors $\vec{u}$, $\vec{v}$, and $\vec{i}$ are position vectors of just *some* of the points that are transformed.

Indeed, under a transformation described by matrix M, every vector maps onto a vector $M\vec{v}$.

Suppose $\vec{v}$ is the position vector $\overrightarrow{OP}$ of the point P,
and $M\vec{v}$ is the position vector of the point P'.
Then you can say that
P maps onto point P' under the transformation defined by the matrix M, or
$P \xrightarrow{M} P'$.



The Image of a Line

One of the most important properties of a linear transformation is that it transforms a straight line into another straight line, as the following example will show.

Example 4 Given a linear transformation whose matrix is T, find the image under T of the straight line L whose vector equation is $\vec{r} = \vec{r_0} + k\vec{m}$.

Solution Recall that $\vec{r}$ is the position vector of any point of L.

Let $\vec{r'}$ be the position vector of any point of the *image* of L.

Then $\vec{r'} = T(\vec{r})$

$= T(\vec{r_0} + k\vec{m})$

$= T(\vec{r_0}) + T(k\vec{m})$ — linear transformation, property 1

$= T(\vec{r_0}) + k[T(\vec{m})]$ — linear transformation, property 2

$= \vec{r_0'} + k\vec{m'}$,

where $\vec{r_0'}$ and $\vec{m'}$ are the images of $\vec{r_0}$ and $\vec{m}$ respectively.

Thus the image of L has vector equation $\vec{r'} = \vec{r_0'} + k\vec{m'}$.

This is the equation of a straight line. ■

PROPERTY

Thus, *straight lines are transformed into straight lines by linear transformations.*

The following example uses the above property to show how a diagram can portray the effect of a linear transformation.

Example 5 Consider the unit square S whose vertices are $O(0,0)$, $A(1,0)$, $B(1,1)$, $C(0,1)$. Describe the image of S under the transformation defined by matrix

$$M = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$

Solution Use the following notation.

points	O A B C	images O' A' B' C'
position vectors	$\vec{0}$ $\vec{a}$ $\vec{b}$ $\vec{c}$	images $\vec{0'}$ $\vec{a'}$ $\vec{b'}$ $\vec{c'}$

$$M\vec{0} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0'} = \vec{0}$$

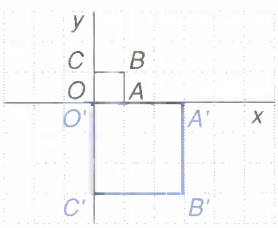
$$M\vec{a} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3+0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \vec{a'} = \overrightarrow{OA'}$$

$$M\vec{b} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3+0 \\ 0-3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \vec{b'} = \overrightarrow{OB'}$$

$$M\vec{c} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 0-3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \vec{c'} = \overrightarrow{OC'}$$

Thus the point O remains at O,
the point $A(1,0)$ has image $A'(3,0)$,
the point $B(1,1)$ has image $B'(3,-3)$,
and the point $C(0,1)$ has image $C'(0,-3)$.

Since straight lines are mapped onto straight lines, you know that OA', $A'B'$, $B'C'$ and $C'O$ are straight lines.
Thus, the transformation is as shown in this diagram.



■

The unit square seems to have been enlarged and reflected in the x-axis. Recall that the transformation affects the entire plane, not just the unit square.

7.2 Exercises

1. Check if the following are linear transformations.

$F: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 3x \\ 4y \end{bmatrix}$ $\quad R: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2 \\ x \end{bmatrix}$

$G: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+y \\ x-y \end{bmatrix}$ $\quad S: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x^2 \\ 2y \end{bmatrix}$

$H: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+1 \\ y \end{bmatrix}$ $\quad T: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 3y \\ -x \end{bmatrix}$

2. State the matrix of each of the linear transformations found in question 1.

3. Prove that the transformation $T: \vec{v} \to \overrightarrow{Mv}$ defined by the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a linear transformation.

4. Calculate the following products.

a) $\begin{bmatrix} -1 & 2 \\ 6 & -4 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

5. Calculate the values of the variables in the following.

a) $\begin{bmatrix} 5 & x \\ 4 & y \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

b) $\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

c) $\begin{bmatrix} 2x & y \\ -y & x \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

d) $\begin{bmatrix} -x & 3 \\ y & z \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \end{bmatrix}$

6. Show that there are no real values of x and y such that $\begin{bmatrix} x & y \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

7. Find the images of the following vectors under the transformation matrix

$M = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$.

$\vec{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ $\quad \vec{w} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Sketch as position vectors $\vec{u}, \vec{v}, \vec{w}$ and their images $\vec{u}', \vec{v}', \vec{w}'$.

8. For $M = \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix}$, $N = \begin{bmatrix} 3 & 5 \\ -7 & 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, find

a) $M\vec{v}$ b) $N\vec{v}$ c) $M(N\vec{v})$ d) $N(M\vec{v})$

Draw conclusions.

9. For the following, use the matrix $M = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

$\vec{0}$ is the zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\vec{i}$ and $\vec{j}$ are the standard basis vectors, that is, $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Calculate the following.

a) $M\vec{0}$ b) $M\vec{i}$ c) $M\vec{j}$

Draw conclusions.

10. The straight line L has vector equation $\vec{r} = \vec{r_0} + k\vec{m}$, where

$\vec{r} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\vec{r_0} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\vec{m} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

a) Find the vector equation of the image of L under the transformation of matrix $A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

b) Graph L and its image on the same set of axes.

11. a) Find the coordinates of two points A and B on the line L whose Cartesian equation is $y = 2x + 1$.

b) Find the image A' of A, and the image B' of B, under the transformation of matrix $M = \begin{bmatrix} -3 & 5 \\ 2 & 0 \end{bmatrix}$.

c) Find the Cartesian equation of the line L' that passes through A' and B'.

d) What is the image of L under M?

12. a) Consider the points $O(0,0)$, $A(3,1)$ and $B(1,4)$. If $OABC$ is a parallelogram, calculate the coordinates of the point C. Sketch $OABC$ on a grid.

b) Find the image $O'A'B'C'$ of the parallelogram $OABC$ under the transformation of matrix $M = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}$

c) Sketch $O'A'B'C'$ on the same grid. Describe the nature of $O'A'B'C'$.

7.3 The Effect of a Linear Transformation

Since any linear transformation T of $\mathbb{V}_2$ can always be represented by its matrix, M, the expression "the transformation M" can clearly replace the expression "the transformation whose matrix is M".

As you have seen, linear transformations are defined for vector spaces. You can observe their effect on points in $\mathbb{R}^2$ as follows. By considering the vectors of $\mathbb{V}_2$ as position vectors of points in $\mathbb{R}^2$, you can examine the effect of a linear transformation of their tips.

Thus, you can speak of linear transformations 'of a plane'.

Some questions of the last exercises illustrated properties of matrices as transformations. These properties will be demonstrated here.

Image of $\vec{0}$ by a Linear Transformation

Consider the general linear transformation $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$M\vec{0} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (a)(0) + (b)(0) \\ (c)(0) + (d)(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

Thus the image of $\vec{0}$ is always $\vec{0}$.

Images of Parallel Lines by a Linear Transformation

Consider the two parallel lines L_1 and L_2 whose vector equations are

L_1: $\vec{r} = \vec{a} + k\vec{m}$

L_2: $\vec{s} = \vec{b} + q\vec{m}$

where L_1 contains a point A whose position vector is $\vec{a}$,

L_2 contains a point B whose position vector is $\vec{b}$,

$\vec{r}$ and $\vec{s}$ are position vectors of a general point on L_1 and L_2 respectively, and k, q are scalars.

(Notice that the lines both have the same direction vector $\vec{m}$. The lines are thus parallel.)

As you saw in Example 4 of the previous section (page 299), the images of L_1 and L_2 under a linear transformation T will be

L_1': $\vec{r'} = \vec{a'} + k\vec{m'}$

L_2': $\vec{s'} = \vec{b'} + q\vec{m'}$.

Thus the image lines will have the same direction $\vec{m'}$, that is, they will be parallel.

These conditions, together with the examples and exercises seen so far in this chapter, lead to the following generalization.

A general linear transformation of a plane may pull, push, turn, stretch, or compress the plane, in any directions, with the two following provisos.

PROPERTIES

1. The origin does not move.
2. Parallelism is preserved.

One thing that a linear transformation *cannot* perform is a *translation*. Indeed, a translation of the plane would violate the first proviso.

Reading and Writing a Matrix

Consider the effect of a general transformation $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ on the standard basis vectors $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$M\vec{i} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \text{ and } M\vec{j} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

Thus, the first column of a matrix M is the image of $\vec{i}$ under M and the second column of a matrix M is the image of $\vec{j}$ under M.

This beautiful property allows you to do *two* very useful things.

1. *WRITING A MATRIX*
   Given a transformation, you can state its matrix as follows.
   Write the first column as the image of $\vec{i}$,
   and the second column as the image of $\vec{j}$.
2. *READING A MATRIX*
   Given a matrix, you can determine the linear transformation it represents as follows.
   Its first column is the image of $\vec{i}$,
   and its second column is the image of $\vec{j}$.

The following examples show how this knowledge can be applied.

Example 1

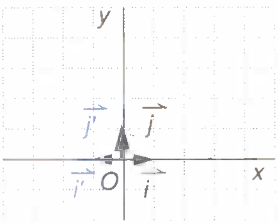
State the matrix M_y of the transformation that reflects the plane in the y-axis.

Solution

If all vectors are reflected in the y-axis then

$\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 \\ 0 \end{bmatrix}$; this is the first column of M_y.

$\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; this is the second column of M_y.



Thus the matrix $M_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ ■

Example 2

Describe the transformation whose matrix is $D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Solution Read the columns of D.

The image of $\vec{i}$ is the first column, $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Thus $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

The image of $\vec{j}$ is the second column, $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Thus $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Thus, both $\vec{i}$ and $\vec{j}$ are doubled in magnitude, or 'enlarged by a factor of 2'.

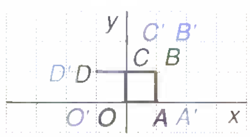
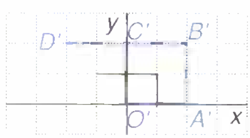
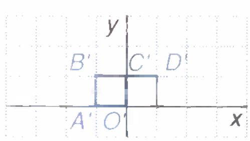
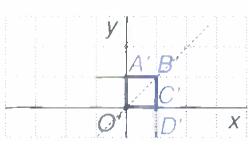
Hence, the transformation D can be described as an enlargement or a dilatation of factor 2. ■

(See a diagram of the effect of "D_2", in the second of the following illustrations.)

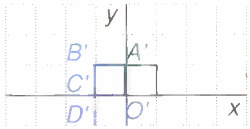
SUMMARY

To summarize this section, you will observe the effects of some common linear transformation matrices, as illustrated on the figure $OABCD$, which is the unit square $OABC$ with BC produced to D so that $BC = CD$. (The use of a non-symmetric figure such as $OABCD$ gives a clearer idea of the transformation in some cases.)

The image figure can be obtained in each case by calculating the image of each of the points O, A, B, C, and D. You will have an opportunity to verify these illustrations in the exercises.

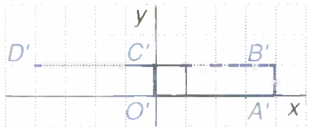
Matrix	Effect	Transformation
$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$		identity
$D_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$		dilatation, of factor 2
$M_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$		reflection in y-axis
$M_{45} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$		reflection in line $y = x$

$$R_{90} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



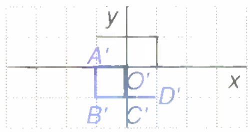
counterclockwise rotation about O, through 90°

$$S_{x4} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$



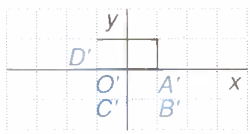
one-way stretch||x-axis, of factor 4

$$D_{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$



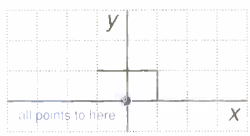
dilatation of factor −1, or reflection in O

$$P_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



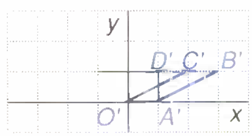
projection onto x-axis

$$0_{2\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$



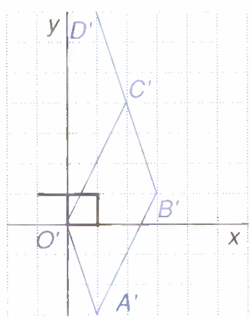
null transformation

$$H_{x2} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$



shear||x-axis, of factor 2

$$G = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$



general linear transformation

When a linear transformation acts on a plane,

1. the origin does not move,
2. parallelism is preserved.

The matrix M of a linear transformation is such that its first column is the image of $\vec{i}$ under M, and its second column is the image of $\vec{j}$ under M.

7.3 Exercises

In the following,
O shall refer to the origin (0,0).

1. Justify the diagrams of pages 304–305, at the end of section 7.3, that show the effect of each of the following transformations.
   I, D_2, M_y,
   M_{45}, R_{90}, S_{x4},
   D_{-1}, P_x, $0_{2\times 2}$,
   H_{x2}, and G.

2. Describe in words the effect of the following transformations, shown on pages 304–305.
   the identity transformation $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
   the null transformation $0_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3. Write the matrix that corresponds to the given transformation by finding the image of $\vec{i}$ and the image of $\vec{j}$.
   a) a dilatation of factor 3
   b) reflection in the x-axis
   c) counterclockwise rotation about O through 270°
   d) counterclockwise rotation about O through 180°
   e) projection onto the y-axis
   f) reflection in the line $y = -x$

4. By finding the images of the points O, $P(1,0)$, $Q(1,1)$, $R(0,1)$ sketch the effect of each of the transformations of question 3 on the unit square.

5. Compare your answer to question 3d) with the matrix $D_{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, listed in section 7.3 as a "dilatation of factor -1" or a "reflection in O".
   Draw conclusions.

6. By reading the columns of the following matrices, describe the associated transformation in each case.
   $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $F = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$
   $B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$ $G = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$
   $C = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ $H = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$
   $D = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
   $E = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ $J = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$

7. By finding the images of the points O, $P(1,0)$, $Q(1,1)$, $R(0,1)$ sketch the effect of each of the transformations of question 6 on the unit square.

8. What is the image of the point O under the transformation matrix $M = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$?

9. The transformation matrix $S = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is known as a horizontal shear of factor 2. Some of the properties of a shear will appear as answers to the following questions.
   a) Find the images of (1,0) and $(a,0)$, where $a \in \mathbb{R}$.
   b) Describe how any point of the x-axis is transformed.
   c) Find the images of (0,1) and $(a,1)$, where $a \in \mathbb{R}$.
   d) Describe how any point on the line $y = 1$ is transformed.
   e) Find the image of the point $(0,b)$, where $b \in \mathbb{R}$.
   f) Describe how any point on the y-axis is transformed.

10. Show the effect of $R = \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix}$ on the unit square.
    Describe the transformation associated with the matrix R.

11. Write the matrix of the following transformations.

a) a stretch parallel to the x-axis of factor 5

b) a dilatation of factor $\frac{1}{4}$

c) a two-way stretch, of factor 2 parallel to the x-axis, and of factor 3 parallel to the y-axis

d) a perpendicular projection onto the line $y = x$

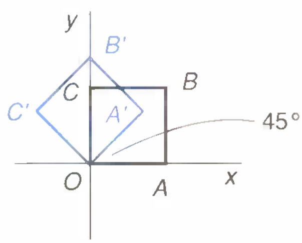
e) a shear parallel to the x-axis of factor -1

f) a shear parallel to the y-axis of factor 5

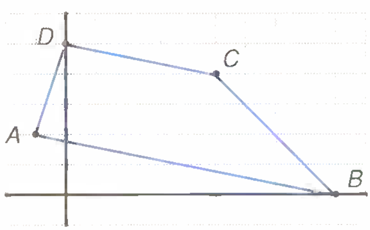
g) a reflection in the y-axis followed by a dilatation of factor 4

12. Sketch the effects of each of the transformations of question 11 on the unit square.

13. Write the matrix of a counterclockwise rotation about the origin, through 45°.



14. a) Verify that the following four points determine the vertices of a trapezoid. $A(-1,2)$, $B(9,0)$, $C(5,4)$, $D(0,5)$



b) Transform this trapezoid with the matrix $M = \begin{bmatrix} 2 & -3 \\ 2 & 5 \end{bmatrix}$

c) Show that the images of the parallel sides of the trapezoid remain parallel under the transformation.

15. Consider the straight lines L_1 and L_2 with the following vector equations.

$$L_1: \vec{r} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + k\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$L_2: \vec{s} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + p\begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

a) Explain why the lines L_1 and L_2 are parallel, and graph them on the same set of axes.

b) Find the vector equations of the images of L_1 and L_2 under the transformation $M = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$

c) Show that these images are also parallel.

16. Given the matrix $S = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, where k is a positive scalar.

a) Find the image under S of the unit square.

b) Calculate the image under S of a general vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

c) Describe the image under S of the entire plane.

17. Repeat question 16 using the matrix $S = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$

18. In question 5 you determined that the following transformations in 2-space are equivalent.

A. a dilatation of factor -1

B. a reflection in O

Consider also another transformation

C. a rotation through 180° about O

Is this transformation equivalent to **A** and to **B**?

Discuss whether or not these transformations are also equivalent in 3-space.

7.4 Rotations and Reflections

In this section you will learn about two important linear transformations, namely rotations, that you will be using in chapter 8, and reflections. You will also make further observations on linear transformations in general. These should help you to solve transformation problems with more assurance.

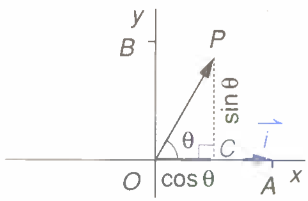
Example 1 Find the matrix R_θ of the transformation that rotates the plane counterclockwise about the origin O, through an angle of θ.

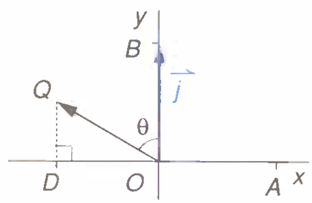
Solution If all vectors are rotated counterclockwise about O, through θ, then this will also be true for $\vec{i}$ and $\vec{j}$.

Let $\vec{i} = \overrightarrow{OA}$, and $\vec{j} = \overrightarrow{OB}$.

Let the image of $\vec{i}$ be $\overrightarrow{OP}$, that is, let $\overrightarrow{OA} \rightarrow \overrightarrow{OP}$. Under a rotation, lengths are invariant, so $|\overrightarrow{OP}| = |\vec{i}| = 1$.

Thus $\overrightarrow{OP} = \begin{bmatrix} OC \\ CP \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$





Similarly, let the image of $\vec{j}$ be $\overrightarrow{OQ}$, that is, let $\overrightarrow{OB} \rightarrow \overrightarrow{OQ}$.

Then the angle between $\overrightarrow{OQ}$ and $\vec{i}$ is $(90° + \theta)$.

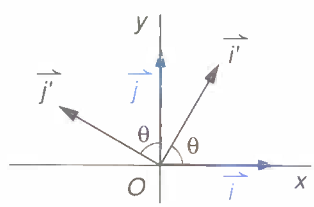
Note also that $|\overrightarrow{OQ}| = |\vec{j}| = 1$.

$$\text{Thus } \overrightarrow{OQ} = \begin{bmatrix} OD \\ DQ \end{bmatrix} = \begin{bmatrix} \cos(90° + \theta) \\ \sin(90° + \theta) \end{bmatrix} = \begin{bmatrix} \sin[90° - (90° + \theta)] \\ \cos[90° - (90° + \theta)] \end{bmatrix}$$
$$= \begin{bmatrix} \sin(-\theta) \\ \cos(-\theta) \end{bmatrix}$$
$$= \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

(The formulas on page 541 have been used.)

Thus, $\vec{i} \rightarrow \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\vec{j} \rightarrow \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

PROPERTY Hence the matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$



This is known as a *positive* rotation through θ. (A counterclockwise rotation is a positive rotation.) ■

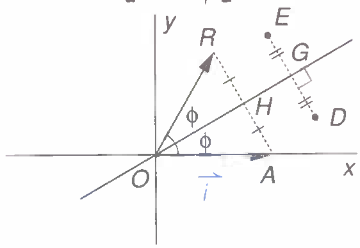
Example 2 Find the matrix M_ϕ of the transformation that reflects the plane in the line L, where L passes through the origin and makes an angle ϕ with the positive x-axis.

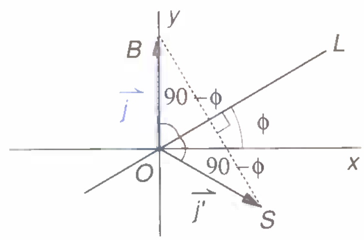
Solution If all vectors are reflected in L, then this will also be true for $\vec{i}$ and $\vec{j}$.
To obtain the image of any point D under a reflection in L, construct the perpendicular DG from D to the line L. Extend this perpendicular on the other side of L so that $DG = GE$. E is then the image of D.

Let the image of $\vec{i}$ be $\overrightarrow{OR}$, that is, let $\overrightarrow{OA} \to \overrightarrow{OR}$. Under a reflection, lengths are invariant, so $|\overrightarrow{OR}| = |\vec{i}| = 1$. Also, the right triangle OAH, where H is the foot of the perpendicular from A to the line L, is congruent to the triangle ORH. Hence, the angle between L and OR is the same as the angle between OA and L, namely ϕ.

Thus, the angle between OR and $\vec{i}$ is 2ϕ.

Hence, $\overrightarrow{OR} = \begin{bmatrix} \cos 2\phi \\ \sin 2\phi \end{bmatrix}$





Similarly, let the image of $\vec{j}$ be $\overrightarrow{OS}$, that is, let $\overrightarrow{OB} \to \overrightarrow{OS}$.
Then the angle between L and OS is equal to the angle between OB and L, namely $90° - \phi$.

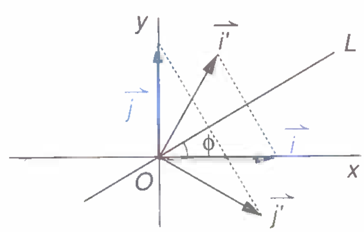
Hence, the angle between OS and $\vec{i}$ is $-([90 - \phi] - \phi) = (2\phi - 90°)$.

$$\text{Thus } \overrightarrow{OS} = \begin{bmatrix} \cos(2\phi - 90°) \\ \sin(2\phi - 90°) \end{bmatrix} = \begin{bmatrix} \cos[-(90° - 2\phi)] \\ \sin[-(90° - 2\phi)] \end{bmatrix} = \begin{bmatrix} \cos[90° - 2\phi] \\ -\sin[90° - 2\phi] \end{bmatrix} = \begin{bmatrix} \sin 2\phi \\ -\cos 2\phi \end{bmatrix}$$

(The formulas on page 541 have been used.)

Thus, $\vec{i} \to \begin{bmatrix} \cos 2\phi \\ \sin 2\phi \end{bmatrix}$ and $\vec{j} \to \begin{bmatrix} \sin 2\phi \\ -\cos 2\phi \end{bmatrix}$

Hence the matrix $M_\phi = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}$



Notice that the slope of the L is $\tan\phi$.
Thus, L has a Cartesian equation $y = (\tan\phi)x$. ■

Note: The matrix $M_\phi = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}$ is very similar to the rotation matrix of Example 1, that is, $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. You must be careful not to confuse them. A reflection matrix, like M_ϕ, is a *symmetric matrix*. Its elements are symmetric about the leading diagonal.

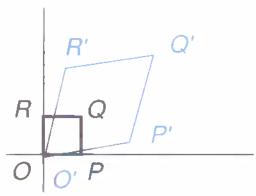
That is, it is of the form $\begin{bmatrix} a & h \\ h & b \end{bmatrix}$ leading diagonal

The observations that follow should help you to make this distinction.

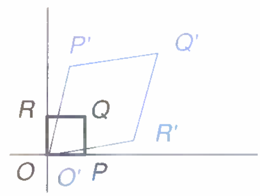
The Size and Orientation of a Transformed Figure

Consider the effect of the general linear transformation $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ on the unit square $OPQR$.

Recall that under M, parallelism is preserved and O does not move. Thus the image of the square $OPQR$ will be a parallelogram $OP'Q'R'$, where P' is the image of P, Q' is the image of Q, and R' is the image of R.



or



$O'P'Q'R'$ has *same* orientation as $OPQR$

$O'P'Q'R'$ has *opposite* orientation to $OPQR$

Reading the columns of M gives $\overrightarrow{OP'} = \begin{bmatrix} a \\ c \end{bmatrix}$, $\overrightarrow{OR'} = \begin{bmatrix} b \\ d \end{bmatrix}$

Recall that the area of a parallelogram whose adjacent sides represent the vectors $\vec{u}$ and $\vec{v}$ is $|\vec{u} \times \vec{v}|$.

However, the cross product is not defined in $\mathbb{V}_2$. In order to calculate the area of the parallelogram $OP'Q'R'$, you must imagine that it lies in the xy-plane of a 3-space coordinate system.

Then the vector $\overrightarrow{OP'} = (a,c,0)$, and the vector $\overrightarrow{OR'} = (b,d,0)$.

Thus the area of $OP'Q'R'$ is

$|\overrightarrow{OP'} \times \overrightarrow{OR'}| = |\overrightarrow{(a,c,0)} \times \overrightarrow{(b,d,0)}| = |ad - bc|$

Since the area of the unit square is 1, the area of the transformed figure is changed by a scale factor $|ad - bc|$.

DEFINITION

The quantity $(ad - bc)$ is known as the **determinant** of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, that is, the determinant of M, or $\det(M)$, or $|M|$.

$\operatorname{Det}(M)$ can also be denoted by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Notice that the original unit square *OPQR* is read *counterclockwise*. Also, from the diagram, the slope of OP' is $\frac{c}{a}$, and the slope of OR' is $\frac{d}{b}$.

If OP' and OR' are in the first quadrant, and $\frac{d}{b} > \frac{c}{a}$, then $OP'Q'R'$ will be read *counterclockwise*.

Figure *OPQR* and figure $O'P'Q'R'$ have the *same orientation*.

Observe that $\frac{d}{b} > \frac{c}{a} \Rightarrow ad > bc \Rightarrow ad - bc > 0 \Rightarrow \det(M) > 0$.

Similarly, you can see that if $\det(M) < 0$, $OP'Q'R'$ will be read *clockwise*.
Figure *OPQR* and figure $O'P'Q'R'$ have *opposite orientation*.

This can be extended to all four quadrants.

Example 3 Calculate the area and describe the orientation of the image of the unit square under each of the following transformations (taken from the examples of the last section).

$$D_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad M_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad P_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution Note that the area of the unit square is 1.

You can find the area scale factors by calculating the determinants.

$$\det(D_2) = (2)(2) - (0)(0) = 4$$

The image is enlarged 4 times.
Since $4 > 0$, the image retains the original orientation.

(D_2 is a dilatation of factor 2.)

$$\det(M_y) = (-1)(1) - (0)(0) = -1.$$

The area of the image is unchanged.

Since $-1 < 0$, the image has the opposite orientation.

(M_y is a reflection in the y-axis.)

$$\det(P_x) = (1)(0) - (0)(0) = 0.$$

The image has zero area. Orientation is not defined for a figure of zero area.

(P_x is a projection onto the x-axis.) ■

Example 4 Calculate the area scale factor and describe the orientation of the transformations described by the following.

a) the rotation matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

b) the reflection matrix $M_\theta = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}$

Solution To simplify the determinants in this question, use the Pythagorean trigonometric identity of page 542.

a) $\det(R_\theta) = \cos^2\theta - (-\sin^2\theta) = \cos^2\theta + \sin^2\theta = 1$. Note that $1 > 0$.
Thus a rotation does not alter the area or orientation of any figure.

b) $\det(M_\phi) = -\cos^2 2\phi - \sin^2 2\phi = -(\cos^2 2\phi + \sin^2 2\phi) = -1$. Note that $-1 < 0$.
Thus a reflection leaves the area invariant, but reverses the orientation of a figure. ■

SUMMARY

These results can be summarized as follows.

If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant, $\det(M) = ad - bc$.

If a figure of area S is transformed by matrix M, the area of the image figure is $|\det(M)|S$.

$|\det(M)|$ is called the area scale factor of matrix M.

If $\det(M) > 0$, the image retains the original orientation;
if $\det(M) < 0$, the image acquires the opposite orientation.

7.4 Exercises

In the following, all rotations are about the origin O, counterclockwise through the indicated angle, unless specified otherwise.

1. Write the matrix of the rotation through the given angle in each case, giving entries correct to 2 decimal places.
   a) 40° c) 90° e) 200°
   b) 80° d) 110° f) 342°

2. Write the matrix of the reflection in the line $y = (\tan\phi)x$ for the following values of ϕ, giving entries correct to 2 decimal places.
   a) 20° b) 45° c) 100°

3. a) Describe the similarities and differences in your answers to 1a) and 2a).
   b) By calculating the determinant of each of these two matrices, show how you can distinguish between a rotation and a reflection.

4. Repeat question 3 by comparing your answers to 1e) and 2c).

5. Write the matrix of the rotation through the given angle in each case, giving exact answers. (Use the trigonometric tables on page 543.)
   a) 45° b) 60° c) 120°

6. Write the matrix of the reflection in the line $y = (\tan\phi)x$ for the following values of ϕ, giving exact answers. (Use the trigonometric tables on page 543.)
   a) 22.5° b) 60° c) 150°

7. Given that R_θ represents a counterclockwise rotation about O of $\theta°$, compare R_{300} and R_{-60}. Explain.

8. Given that M_ϕ represents a reflection in the line $y = (\tan\phi°)x$, compare M_{30} and M_{210}. Explain.

9. Calculate the determinant of each of the following matrices. Hence describe the area scale factor and the orientation of the associated transformations.
   the identity transformation $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
   the null transformation $0_{2\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

10. The following are the matrices whose effect you described in questions 6 and 7 of 7.3 Exercises. Calculate the determinant in each case, to describe the area scale factor and orientation of each transformation.
    $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $F = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$
    $B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$ $G = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$
    $C = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ $H = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$
    $D = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
    $E = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ $J = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$

11. Find k if the determinant of $\begin{bmatrix} 2 & 3 \\ 1 & k \end{bmatrix}$ equals
    a) 5 b) −1 c) 0

12. The transformation matrix $S = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, where k is any real number, is a horizontal shear of factor k.
    Calculate $\det(S)$. Hence, describe the area scale factor and the orientation of a figure transformed by a shear.

13. Given $M = \begin{bmatrix} -k & 1 \\ k^2 & -k \end{bmatrix}$, find det($M$). What is the area of a figure transformed by matrix M?

14. Sketch on a grid the right triangle whose vertices are O, $P(3,0)$, $Q(0,2)$.

a) Calculate the area of the triangle OPQ.

b) Transform O, P, Q into their images O', P', Q' by the matrix $M = \begin{bmatrix} -4 & 3 \\ 5 & -2 \end{bmatrix}$

c) Calculate the area of $O'P'Q'$.

d) Compare the orientation of OPQ and $O'P'Q'$.

15. Use the information of question 14 for the following.

a) By calculating the dot product $\overrightarrow{OP} \cdot \overrightarrow{OQ}$, prove that the angle POQ is 90°.

b) Calculate the angle $P'O'Q'$ by finding the dot product $\overrightarrow{O'P'} \cdot \overrightarrow{O'Q'}$.

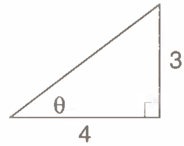
c) Are right angles preserved under linear transformations?

16. a) Show that the *clockwise* rotation through α about O is represented by the matrix

$$C = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

b) Verify that det(C) = 1.

17. In the right triangle shown, $\tan\theta = \frac{3}{4}$.



Give all your answers to the following in fractional form.

a) Calculate $\sin\theta$ and $\cos\theta$.

b) Write the matrix of the rotation through angle θ.

c) Write the matrix of the reflection in the line $y = \left(\tan \frac{1}{2}\theta\right)x$.

18. Describe the following transformations.

$$K = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} \quad L = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$M = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & -\frac{4}{5} \end{bmatrix} \quad N = \begin{bmatrix} -\frac{5}{13} & \frac{12}{13} \\ -\frac{12}{13} & -\frac{5}{13} \end{bmatrix}$$

7.5 Inverse Transformations

So far you have learned how to carry out various linear transformations by using their matrices.

Is it possible to find a transformation that returns the plane to its original status after it has been transformed? That is, if $\vec{v} \xrightarrow{M} \vec{v}'$, can a transformation M^{-1} be found such that $\vec{v}' \xrightarrow{M^{-1}} \vec{v}$? Does a transformation always have an inverse? If an inverse exists, what does its matrix look like?

In this section, these questions will be investigated.

First observe the following examples.

Example 1 Given $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$, try to find a matrix D to reverse the effect of A.

Solution

Consider the vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ and its image under A, $\vec{v}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$

Then $A\vec{v} = \vec{v}'$.

Now if D exists, it will transform $\vec{v}'$ back to $\vec{v}$,

thus $D\vec{v}' = \vec{v}$ or $\vec{v} = D\vec{v}'$.

Thus, if you can find $\begin{bmatrix} x \\ y \end{bmatrix}$ in terms of $\begin{bmatrix} x' \\ y' \end{bmatrix}$,

and express your result as a product $D\begin{bmatrix} x' \\ y' \end{bmatrix}$,

you will have the answer.

Now $A\vec{v} = \vec{v}' \Rightarrow \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\Rightarrow \begin{cases} 3x - y = x' & ① \\ x + y = y' & ② \end{cases}$$

Eliminating y: ① + ② gives $4x = x' + y'$ so $x = \frac{1}{4}x' + \frac{1}{4}y'$

Substituting into ② gives $\frac{1}{4}x' + \frac{1}{4}y' + y = y'$ or $y = -\frac{1}{4}x' + \frac{3}{4}y'$

Thus $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$. Hence the matrix $D = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$ ■

D is called the **inverse** of A and is written A^{-1}.

Notice that $\det(A) = (3)(1) - (-1)(1) = 4$, and that the inverse can also be written $A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$; $\det(A^{-1}) = \frac{1}{4}$.

Observing this form of A^{-1} demonstrates the following facts, that you will prove later.

1. The numbers along the *leading diagonal* $\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$ of A are *exchanged* in A^{-1}.
2. The numbers along the *second diagonal* $\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$ of A are *multiplied by* (-1) in A^{-1}.
3. The *scalar coefficient* of A^{-1} is $\dfrac{1}{\det(A)}$. Also, $\det(A^{-1}) = \dfrac{1}{\det(A)}$

Example 2 Given the matrix $B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$, try to find a matrix B^{-1} that reverses the effect of B.

Solution Proceeding as in Example 1, try to obtain $\begin{bmatrix} x \\ y \end{bmatrix}$ in terms of $\begin{bmatrix} x' \\ y' \end{bmatrix}$

$$B\vec{v} = \vec{v}' \Rightarrow \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2x + 4y = x' & \text{①} \\ x + 2y = y' & \text{②} \end{cases}$$

Eliminating y: ① $- 2 \times$ ② gives $0x = x' - 2y'$.
But x' and y' might be any numbers so that, in general, $x' \neq 2y'$. Hence no value exists for x. Thus no matrix B^{-1} exists. ■

In this case, matrix B is called **non-invertible**, or **singular**.
Its transformation is also called singular.

Note: The determinant of B is $(2)(2) - (4)(1) = 0$, that is $\det(B) = 0$.

FORMULA

The matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse M^{-1} if and only if $\det(M) \neq 0$;

then the inverse of M is $M^{-1} = \dfrac{1}{\det(M)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

If $\det(M) = 0$ the matrix M (and its transformation) are known as singular; no inverse exists.

You will have an opportunity to derive the general formula for M^{-1} in the exercises.

Example 3 Find the inverse, if it exists, of each of the matrices

a) $A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$ b) $B = \begin{bmatrix} \frac{1}{2} & -1 \\ -3 & 6 \end{bmatrix}$

Solution a) $\det(A) = (3)(6) - (5)(4) = -2 \neq 0.$
Thus A has an inverse A^{-1}.

By the formula, $A^{-1} = \frac{1}{-2}\begin{bmatrix} 6 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -3 & \frac{5}{2} \\ 2 & -\frac{3}{2} \end{bmatrix}.$

b) $\det(B) = \left(\frac{1}{2}\right)(6) - (-1)(-3) = 0.$

Thus, B is a singular matrix. No inverse for B exists. ■

Geometric Significance of Singular Transformations

The previous discussion leads to the fact that a matrix M is singular, that is, non-invertible, if its determinant is zero.

Since the area scale factor of a linear transformation of matrix M is $|\det(M)|$, *the area of any figure transformed by a singular matrix is zero.*

Look back at the examples of common transformations in section 7.3 (pages 304–305). The following transformations have zero determinants.

P_x (projection onto the x-axis)

$0_{2\times2}$ (null transformation)

You can see from the diagrams that both these transformations 'squash' the plane onto a single line or a single point. $\mathbb{V}_2$ is said to 'lose some dimension' by a singular transformation. When $\mathbb{V}_2$ loses some dimension by a transformation, that transformation cannot be reversed.

SUMMARY

$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse M^{-1} if and only if $\det(M) \neq 0$;

then the inverse of M is $M^{-1} = \frac{1}{\det(M)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

If $\det(M) = 0$ the matrix M (and its transformation) are known as singular; no inverse exists.

The area of any plane figure transformed by a singular 2×2 matrix is zero.

7.5 Exercises

1. Which of the following are singular?

$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $H = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ $J = \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix}$

$D = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ $K = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{bmatrix}$

$E = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

$F = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ $L = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$

$G = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

2. Find the inverse of each of the invertible matrices of question 1.

3. By finding the images of the points O, $P(1,0)$, $Q(1,1)$, $R(0,1)$, sketch the effect on the unit square of each of the inverses of the transformations of question 1 (when they exist).

4. Calculate the area scale factor and describe the orientation for each of the inverses of the transformations of question 1 (when they exist).

5. a) Which of the matrices in question 1 describe rotations?
   b) Conjecture a formula for the inverse of the rotation matrix
   $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

6. a) Write the matrix R of a rotation, counterclockwise about O, of $\theta°$.
   b) Find R^{-1}.
   c) Explain your answer to b) in terms of a rotation about O of $-\theta°$. (That is, a *clockwise* rotation of $\theta°$.)

7. Find the value of k if the matrix
   $M = \begin{bmatrix} 5 & -2 \\ k & -1 \end{bmatrix}$ is singular.

8. Find the inverse of the identity matrix
   $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Explain your answer.

9. Find the inverse of $S = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
   (the horizontal shear of factor 2). Describe this new transformation.

10. Consider the matrix $M = \begin{bmatrix} 7 & 10 \\ 2 & 3 \end{bmatrix}$
    a) Find the inverse matrix M^{-1}.
    b) Find the image $\vec{v}'$ of the vector $\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ under M.
    c) Verify that $M^{-1}\vec{v}' = \vec{v}$.

11. Repeat question 10 with matrix
    $M = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix}$

12. By following the procedure of Example 1 in the text, page 315, prove that the inverse of the matrix
    $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $M^{-1} = \frac{1}{\det(M)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
    What happens if $\det(M) = 0$?

13. M is an invertible matrix. Calculate $\det(M) \times \det(M^{-1})$.

14. Use $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $\det(M) \neq 0$, to prove the following statement.

    If $\vec{v}$ is any vector, then $M\vec{v} = \vec{0} \Rightarrow \vec{v} = \vec{0}$. (This proves that *only* the zero vector is transformed by an invertible matrix into the zero vector.)

15. The matrix $S = \begin{bmatrix} 3 & -1 \\ 12 & -4 \end{bmatrix}$ maps the entire plane onto a single straight line.
    a) What is the image under S of a general vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$?
    b) What is the Cartesian equation of the image line?
    c) Does S have an inverse? Explain.

7.6 Composition of Transformations and Multiplication of Matrices

To 'compose' two transformations means to make one of the transformations follow the other. You then have a "composite transformation". The same term is used for functions. In this first example, you will observe the effect of the composition of two transformations. Later you will see how this composition is linked to the multiplication of matrices.

Example 1 Consider the two transformations whose matrices are

$P = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

and the general vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

a) Describe each transformation.

b) Calculate $P\vec{v} = \vec{v_1}$, then $Q(P\vec{v}) = \vec{v_2}$.

c) Calculate $Q\vec{v} = \vec{v'}$, then $P(Q\vec{v}) = \overrightarrow{v''}$.

d) Describe the transformations that take $\vec{v}$ to $\vec{v_2}$, and $\vec{v}$ to $\overrightarrow{v''}$.

Solution

a) Read the columns of P: $\vec{i} \rightarrow \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \vec{j} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Thus $\vec{i}$ goes to $-\vec{i}$, and $\vec{j}$ does not move.
Both $\vec{i}$ and $\vec{j}$ are reflected in the y-axis.

P therefore represents a reflection in the y-axis.

Read the columns of Q: $\vec{i} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{j} \rightarrow \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Thus $\vec{i}$ goes to $\vec{j}$, and $\vec{j}$ goes to $-\vec{i}$.
Both $\vec{i}$ and $\vec{j}$ are rotated counterclockwise through 90°.

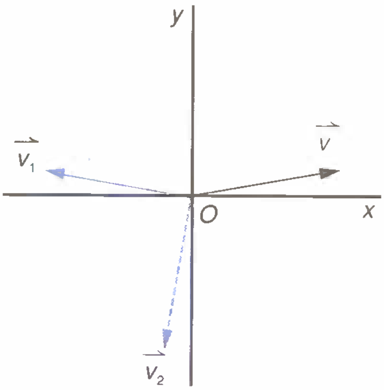
Q therefore represents a counterclockwise rotation about O through 90°. (This will be abbreviated to "rotation of 90°".)

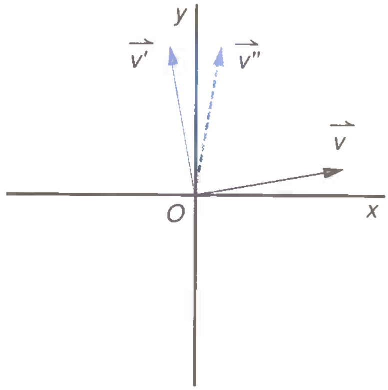
b) $P\vec{v} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} = \vec{v_1}$

$Q(P\vec{v}) = Q(\vec{v_1}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix} = \vec{v_2}$

c) $Q\vec{v} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} = \vec{v'}$

$P(Q\vec{v}) = P(\vec{v'}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} = \overrightarrow{v''}$





d) Observe the figure.

In b), $\vec{v}$ is reflected in the y-axis to $\vec{v}_1$.
$\vec{v}_1$ is then rotated by 90° to $\vec{v}_2$.
Thus, $\vec{v}$ is reflected in the y-axis THEN rotated by 90° to $\vec{v}_2$.
In c), $\vec{v}$ is rotated by 90° to $\vec{v}'$.
$\vec{v}'$ is then reflected in the y-axis to $\vec{v}''$.
Thus, $\vec{v}$ is rotated by 90° THEN reflected in the y-axis to $\vec{v}''$. ■

Note 1 Doing P first, then Q, gives a different result from doing Q first, then P.

2 Writing $Q(P\vec{v})$ means that P acts first, then Q.

This example shows that the composition of transformations is non-commutative. (Beware! "Non-commutative" does not mean "never commutative". There are examples of transformations that do commute, as you shall see later.)

If you now compute a 'matrix product' QP by the same dot product 'row-column' process you used to multiply a 2 × 2 matrix and a 2 × 1 matrix, you have

$$QP = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \overset{(\text{row }1)\cdot(\text{col }1)}{(0)(-1)+(-1)(0)} & \overset{(\text{row }1)\cdot(\text{col }2)}{(0)(0)+(-1)(1)} \\ \underset{(\text{row }2)\cdot(\text{col }1)}{(1)(-1)+(0)(0)} & \underset{(\text{row }2)\cdot(\text{col }2)}{(1)(0)+(0)(1)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = R$$

You obtain a 2 × 2 matrix R that will transform $\vec{v}$ *directly* to $\vec{v}_2$, as follows.

$$R\vec{v} = (QP)\vec{v} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix} = \vec{v}_2$$

Note: The calculations $Q(P\vec{v})$ and $(QP)\vec{v}$ lead to the same result.
That is, $Q(P\vec{v}) = (QP)\vec{v}$

This is one of the manifestations of the associativity of matrix multiplication, which will be discussed later in this section.

Similarly, PQ can be computed.

$$PQ = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = S$$

S will transform $\vec{v}$ *directly* to $\overrightarrow{v''}$, as follows.

$$S\vec{v} = (PQ)\vec{v} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} = \overrightarrow{v''}$$

Recall that $\vec{v_2} \neq \overrightarrow{v''}$, that is, the image by QP was different from the image by PQ.
Hence it is not surprising that the matrix $QP \neq$ matrix PQ.

The Multiplication of 2 × 2 Matrices

DEFINITION

Given a matrix $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and a matrix $Q = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$, then the product PQ is defined as follows.

$$PQ = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

(row 1 of P)·(column 1 of Q) → $aw + by$
(row 2 of P)·(column 1 of Q) → $ax + bz$
(row 1 of P)·(column 2 of Q) → $cw + dy$
(row 2 of P)·(column 2 of Q) → $cx + dz$

$$= \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

The matrix PQ represents a transformation that is the result of doing *Q first, then P*.

Matrix multiplication is non-commutative.

However, there are cases of matrices that commute.

For example, recall that $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix.

The transformation I leaves the plane unchanged.
Thus, if A is any matrix, then $AI = IA = A$.
Hence, I commutes with any matrix.
Also, since any invertible matrix M has an inverse M^{-1} that 'undoes' the effect of M,
$MM^{-1} = M^{-1}M = I$.

Hence, a matrix commutes with its inverse.

You will have an opportunity to verify these properties in the exercises.

Example 2 Consider the matrices $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

a) Describe the transformations whose matrices are M and D.
b) Calculate the product MD, and describe its transformation.
c) Calculate the product DM, and describe its transformation.
d) State whether or not the transformations M and D commute.

Solution

a) For $M: \vec{i} \to \vec{j}$ and $\vec{j} \to \vec{i}$.

Thus M is a reflection in the line $y = x$.

For D: $\vec{i} \to 2\vec{i}$ and $\vec{j} \to 2\vec{j}$.

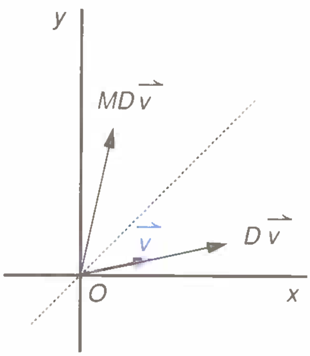
Thus D is a dilatation of factor 2.

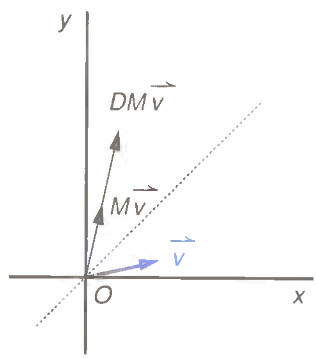
b) $MD = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

The plane is *dilated first, then reflected.*

c) $DM = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

The plane is *reflected first, then dilated.*





d) The order in which the transformations M and D are performed does *not* alter the final image. These transformations, as well as their matrices, *do* commute. ■

The Associativity of Matrix Multiplication

The associative property for matrix multiplication holds. That is, given any 2×2 matrices A, B, and C,
$(AB)C = A(BC)$.

You will be asked to prove this property in the exercises.

Multiplication of Matrices of Different Dimension

Recall that for any 2×2 matrices P and Q, and any column vector (that is, a 2×1 matrix) $\vec{v}$,

PROPERTY

$Q(P\vec{v}) = (QP)\vec{v}$.

This seems to indicate that matrix multiplication is associative even in cases when the matrices are not all of equal dimension.

Indeed, it is valid to 'multiply' any two matrices for which the dot product 'row-column' process is possible. This will be true whenever the number of elements in the *rows* of the first matrix is equal to the number of elements in the *columns* of the second matrix. This leads to the following general results, which will not be proven.

PROPERTY

The product AB of two matrices is defined only if A and B have dimension as follows.
A has dimension $m \times n$, B has dimension $n \times p$.
Then the product AB has dimension $m \times p$.

Whenever the product of matrices is defined, the associative property holds.

The Determinant of a Composite Transformation

PROPERTY

The determinant of a product of matrices is equal to the product of the determinants, that is, given matrices P and Q,
$\det(PQ) = \det(P) \times \det(Q)$

In the exercises, you will have an opportunity to prove this property. An important consequence of this property is demonstrated in the following example.

Example 3 Calculate the area scale factor of AB where
$A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 3 \\ 5 & 1 \end{bmatrix}$

Solution Note that $\det(A) = (2)(-2) - (1)(-4) = 0$.
Hence, $\det(A) \times \det(B) = 0$, or $\det(AB) = 0$.
The transformation AB is singular.
Thus the area scale factor is zero. ■

Example 3 shows that
a singular transformation composed with any other transformation gives a singular transformation.

SUMMARY

Given a matrix $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and a matrix $Q = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$, then the product PQ is defined as follows.

$$PQ = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

The matrix PQ represents a transformation that is the result of doing *Q first, then P.*
(This composition of transformations is non-commutative.)

In general, $QP \neq PQ$ (Matrix multiplication is non-commutative.)

$(PQ)R = P(QR)$ (Matrix multiplication is associative.)

$\det(PQ) = \det(P) \times \det(Q)$

7.6 Exercises

1. Given

$L = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, M = \begin{bmatrix} 4 & -1 \\ 5 & 7 \end{bmatrix}, N = \begin{bmatrix} 6 & 5 \\ -2 & 0 \end{bmatrix},$

calculate

LM, ML, LN, NL, MN, NM.

The following matrices are to be used in answering questions 2–10.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$E = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

$G = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$J = \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix}$

$K = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix}$

2. List which of the transformations defined by the above matrices fit the following.
   a) singular
   b) identity
   c) dilatation
   d) shear
   e) rotation
   f) area scale factor = 1

3. a) Calculate the products *IB, IC, IK.*
   b) Explain your results.

4. a) Calculate the products *CE* and *EC.*
   b) Show the effect of *CE* and of *EC* on the unit square, and describe each of these composite transformations.
   c) Calculate the area scale factor of *CE* and of *EC.*

5. a) Calculate the products *BJ* and *JB.*
   b) Describe each composite transformation. What is special about these?

6. a) Calculate the product *JK* and *KJ.*
   b) Describe each composite transformation. What is special about these?

7. a) Calculate the products *AK* and *GK.*
   b) Calculate the area scale factor in each case. What is special about these composite transformations?

8. Given a matrix *M*, the notation
   $M^n = \underbrace{M \times M \times M \times \ldots\ldots \times M}_{n \text{ times}}$
   a) Calculate I^2 and A^2.
   b) Explain your results.

9. a) Calculate B^2 and E^2.
   b) Show the effect of B^2 and E^2 on the unit square, and describe each composite transformation.

10. a) Write *J* using exact values (see page 543). Calculate J^2, J^3, J^6.
    b) Explain your results.

11. $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the 2 × 2 identity matrix, or unit matrix.
    a) Show that for any vector $\vec{v}$, $I\vec{v} = \vec{v}$.
    b) Show that, for any 2 × 2 matrix *A*, $AI = IA = A$.

12. Consider the matrix $M = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$
    a) Find the inverse matrix M^{-1}.
    b) Calculate MM^{-1} and $M^{-1}M$.
    c) Draw conclusions.

13. Given $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and its inverse

    $M^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$,

    show that MM^{-1} and $M^{-1}M = I$.

    (*I* is the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)

14. Consider the matrices $A = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix}$ and

$B = \frac{1}{17}\begin{bmatrix} 2 & 1 \\ -5 & 6 \end{bmatrix}$

By calculating AB and BA, show that A and B are inverse matrices.

15. Repeat question 14 for $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$

and $B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$

16. Given three matrices

$K = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $L = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, $M = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$

a) Calculate the following products.
KL LM $(KL)M$ $K(LM)$

b) Hence show that matrix multiplication is associative.

17. a) Using the matrices of question 16, calculate the following.
det(K); det(L); det(KL).

b) Hence show that
det(KL) = det(K) × det(L)

18. Using the matrix K of question 16, where det(K) ≠ 0,

a) calculate K^{-1},

b) show that $KK^{-1} = I$ and $K^{-1}K = I$.

c) Do the matrices K and K^{-1} commute?

19. A transformation has matrix $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

a) Show the effect of M on the unit square, and describe M.

b) According to your description, what would happen if M operated twice? three times? four times?

c) Confirm your answer to b) by calculating M^2, M^3, and M^4.

20. Repeat question 19 with the matrix

$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

21. The matrix $T = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

represents a reflection in the line $y = (\tan \theta)x$.

a) What would be the effect of applying T twice?

b) Confirm your answer to a) by finding T^2.

22. The matrix $R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

represents a counterclockwise rotation of angle θ about O.

a) What would be the effect of applying R twice?

b) By calculating R^2, obtain expressions for $\cos 2\theta$ and $\sin 2\theta$ in terms of $\cos \theta$ and $\sin \theta$.

23. Given the matrices

$P = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$ and

$Q = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

a) Describe each transformation, and hence describe the composite transformation PQ.

b) Confirm your answer by calculating the product PQ.

c) What is the inverse of matrix P? of matrix Q?

24. $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

a) Calculate AB and BA.

b) Calculate the inverse matrices A^{-1} and B^{-1}.

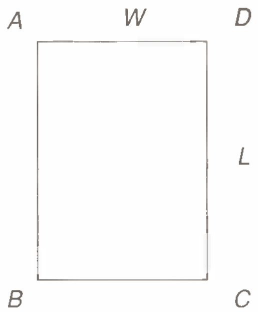
c) Calculate $A^{-1}B^{-1}$.

d) Calculate $(A^{-1}B^{-1})(AB)$ and $(A^{-1}B^{-1})(BA)$.

e) What is the inverse of $(A^{-1}B^{-1})$?

Similarity and Folding

The *golden ratio*, or *golden section*, is a well-known ratio that occurs in more than one branch of mathematics. It has also been used traditionally in architecture as the 'perfect rectangle'.



A rectangle $ABCD$ is said to be 'golden' if its sides form the golden ratio, that is, if

$$\frac{AB}{AD} = \frac{AD}{AB - AD}$$

or $\frac{L}{W} = \frac{W}{L - W}$, where the length $AB = L$, and the width $AD = W$.

Thus, $L(L - W) = W^2$ or $L^2 - LW - W^2 = 0$,

which leads by the quadratic formula to $\frac{L}{W} = \frac{1 + \sqrt{5}}{2} = 1.618\ldots$

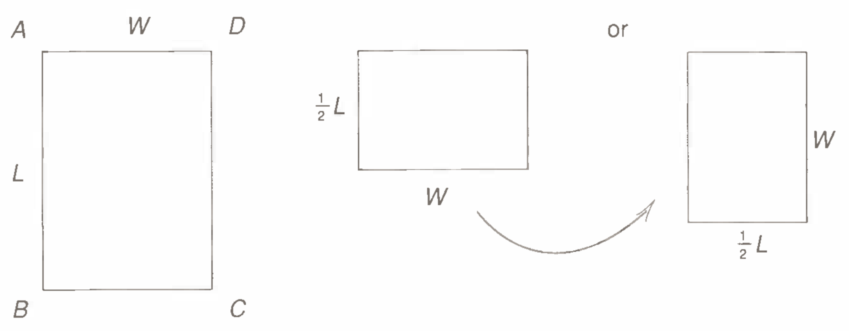
A less well known shape is the following, called the *folding section*.

Consider a sheet of paper of length L and width W.

If the sheet is folded once, its new dimensions will be $\frac{L}{2}$ and W.

If the half-sheet is to have the same shape as the original, then

$\frac{L}{W} = \frac{W}{\frac{1}{2}L}$ or $L^2 = 2W^2$ or $\frac{L}{W} = \sqrt{2} = 1.414\ldots$



Notice that when the sheet is folded again and again, the same shape is always retained. This idea was adopted to create the 'A' series of international paper sheet formats, as follows.

The base sheet size, called A0, has area 1 m^2 or 10 000 cm^2.

Thus $(W)(W\sqrt{2}) = 10\,000$

$$\Rightarrow \qquad W^2 = \frac{10\,000}{\sqrt{2}}$$

$$\Rightarrow \qquad W = \frac{100}{\sqrt[4]{2}} \doteq 84.089\ldots \text{ cm}$$

This leads to the following 'An' sizes in cm, where n represents the number of folds.

A0	84.089	×	118.921
A1	59.46	×	84.089
A2	42.04	×	59.46
A3	29.7	×	42.0
A4	21.0	×	29.7
A5	14.8	×	21.0
A6	10.5	×	14.8
A7	7.4	×	10.5
	etc. . .		

This series of formats has been adopted by most countries.

The most frequently used paper size is A4. Note that any subsequent size can be obtained merely by folding.

Thus, if envelopes are manufactured in sizes marginally larger than these, any paper size can be put into any envelope merely by folding it the required number of times.

7.7 Properties of Matrix Multiplication and Matrix Equations

At this point, a multiplicative algebra of matrices has been established alongside the additive algebra described in section 7.1.
The set S of **invertible** 2×2 matrices together with the operation of matrix multiplication is said to form a **non-commutative group**.

The following properties hold.
M1. S is closed under multiplication: $M, N \in S \Rightarrow MN \in S$
M2. Multiplication is associative: $L(MN) = (LM)N$
M3. There exists $I \in S$ such that for all $M \in S$, $IM = MI = M$
M4. If $M \in S$, then there exists $M^{-1} \in S$ such that $MM^{-1} = M^{-1}M = I$

There remains only one property to be proven to allow you to solve matrix equations in a manner similar to that which you use to solve ordinary algebraic equations in $\mathbb{R}$.

The Distributivity of Matrix Multiplication over Matrix Addition

PROPERTY

Given any 2×2 matrices L, M and N, then
$L(M + N) = LM + LN$
and $(M + N)L = ML + NL$
You will have an opportunity to prove this property in the exercises.

Notice that the order of the letters is crucial. Since matrix multiplication is not commutative, it is *not valid* to replace, say, LM by ML.

You must continually be aware of the non-commutativity of matrix multiplication when working with matrices. With this proviso, you can solve matrix equations. In the first example, you will find the inverse of a composite transformation.

Example 1 Given matrices A and B, find the inverse of the product AB.

Solution Let the image of $\vec{v}$ under AB be $\vec{v'}$.
To find $(AB)^{-1}$, you must express $\vec{v}$ in terms of $\vec{v'}$.

Now	$(AB)\vec{v} = \vec{v'}$	
thus	$(A^{-1})(AB)\vec{v} = (A^{-1})\vec{v'}$	
therefore	$(A^{-1}A)(B)\vec{v} = A^{-1}\vec{v'}$	property M2
or	$(I)(B)\vec{v} = A^{-1}\vec{v'}$	property M4
or	$B\vec{v} = A^{-1}\vec{v'}$	property M3
thus	$(B^{-1})(B\vec{v}) = (B^{-1})(A^{-1})\vec{v'}$	
therefore	$(B^{-1}B)(\vec{v}) = (B^{-1}A^{-1})\vec{v'}$	property M2
so	$I\vec{v} = (B^{-1}A^{-1})\vec{v'}$	property M4
and finally	$\vec{v} = (B^{-1}A^{-1})\vec{v'}$	

PROPERTY

Hence, the inverse of (AB) is $(B^{-1}A^{-1})$. ■

Using Matrix Equations to Solve Linear Systems

If M is an invertible matrix, $M\vec{v} = \vec{u}$

$$M^{-1}(M\vec{v}) = M^{-1}\vec{u}$$
$$(M^{-1}M)\vec{v} = M^{-1}\vec{u}$$
$$I\vec{v} = M^{-1}\vec{u}$$
$$\vec{v} = M^{-1}\vec{u}$$

Hence, solving $M\vec{v} = \vec{u}$ yields $\vec{v} = M^{-1}\vec{u}$.

Thus, if $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\vec{u} = \begin{bmatrix} p \\ q \end{bmatrix}$,

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$ or $\begin{cases} ax + by = p \\ cx + dy = q \end{cases}$ which yields $\begin{bmatrix} x \\ y \end{bmatrix} = \dfrac{1}{\det(M)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix}$ or

FORMULAS

$$x = \frac{dp - bq}{ad - bc} \text{ and } y = \frac{-cp + aq}{ad - bc}, \quad \text{or} \quad x = \frac{pd - bq}{ad - bc} \text{ and } y = \frac{aq - pc}{ad - bc}$$

These results are known as **Cramer's rule** for the solution of a system of two first-degree equations in two unknowns.

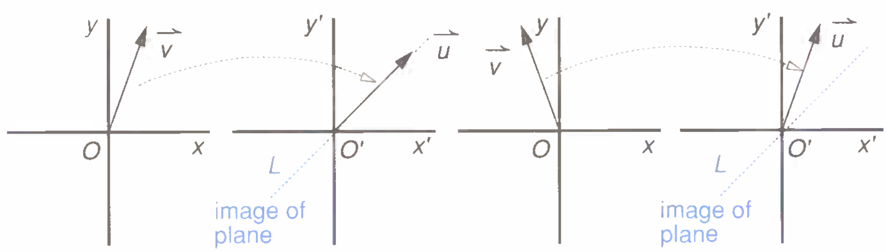
Example 2 Solve the system $2x - 3y = 7$
$4x - y = 1$

Solution Substituting directly into the formulas,

$$x = \frac{(7)(-1) - (-3)(1)}{(2)(-1) - (-3)(4)} = -0.4, \quad y = \frac{(2)(1) - (7)(4)}{(2)(-1) - (-3)(4)} = -2.6 \quad ■$$

However, if M is *not* invertible, then Cramer's rule cannot be used. Recall that the plane will lose some dimension under the effect of a singular transformation. The plane will be 'squashed' into at most a single line, say L. Since the origin O never moves under a linear transformation, L will contain O.

Thus, if M is singular, there will be two possibilities when $M\vec{v} = \vec{u}$.



EITHER $\vec{u}$ is the position vector of a point of L. In that case, solutions for v' must exist.

OR no point of L has u' as position vector. In that case, no solutions for $\vec{v}$ are possible.

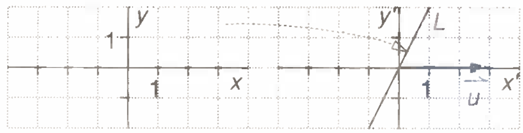
Example 3 Given $M = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, find the vectors $\vec{v}$ that satisfy the equation $M\vec{v} = \vec{u}$ if

a) $\vec{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ b) $\vec{u} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

Solution Notice that $\det(M) = (1)(4) - (2)(2) = 0$, so M is singular. The effect of M on a general vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ is

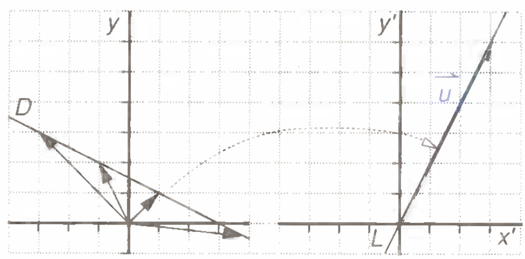
$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + 4y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2(x + 2y) \end{bmatrix} = \begin{bmatrix} a \\ 2a \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ where } a = x + 2y.$$

Thus the entire plane is squashed onto a line L, through the origin, of direction vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.



a) Since no a exists such that $a\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, the point whose position vector is $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ is not on L. Thus the transformation $M\vec{v} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ is impossible. No vector $\vec{v}$ can be found.

b) Since $a\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ for $a = 3$, the point whose position vector is $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ is on L. Thus, solutions exist for $\vec{v}$ given $M\vec{v} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$. Since $a = x + 2y$ and $a = 3$, then $x + 2y = 3$. Hence all the position vectors of points of the line D with equation $x + 2y = 3$ are mapped onto $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$



A vector equation of the line D can be found by introducing a parameter k, as follows. Let $x = k$. Then, from $x + 2y = 3$, $y = \dfrac{-k + 3}{2}$

$$\text{Thus } \vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ \dfrac{-k + 3}{2} \end{bmatrix} = \begin{bmatrix} 0 + k \\ \dfrac{3}{2} - \dfrac{k}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{3}{2} \end{bmatrix} + k\begin{bmatrix} 1 \\ -\dfrac{1}{2} \end{bmatrix}$$

Or, with the notation of chapter 5, $\overrightarrow{(x,y)} = \overrightarrow{\left(0, \frac{3}{2}\right)} + k\overrightarrow{\left(1, -\frac{1}{2}\right)}$ which represents the line through the point $\left(0, \frac{3}{2}\right)$, with direction vector $\overrightarrow{\left(1, -\frac{1}{2}\right)}$.

7.7 Exercises

1. $L = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, M = \begin{bmatrix} 5 & -4 \\ -1 & 1 \end{bmatrix}$

   I is the identity matrix.

   a) Calculate L^{-1} and M^{-1}.
   b) Calculate LM, $L^{-1}M^{-1}$ and $M^{-1}L^{-1}$.
   c) What is the inverse of LM?
   d) Verify that $(LM)(M^{-1}L^{-1}) = I$ and $(LM)(L^{-1}M^{-1}) \neq I$.

2. Given $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
   Show that $B^{-1}A^{-1} = A^{-1}B^{-1}$ in this case. Explain.

3. Solve the following systems by writing each system as a matrix equation.

   a) $5x - 6y = -1$
      $-3x + 4y = 2$
   b) $2x - 7y = 4$
      $4x - 14y = -6$
   c) $2x - 7y = 4$
      $4x - 14y = 8$
   d) $x + 2y = 5$
      $3x + 6y = 8$
   e) $-x + 4y = 7$
      $2x - 8y = -14$
   f) $2x + 3y = 6$
      $5x - y = -1$

4. Solve the systems of question 3 by using Cramer's rule, where possible.

5. Given $M = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$, find the vectors $\vec{v}$ that satisfy the equation $M\vec{v} = \vec{u}$ in the following cases.

   a) $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
   b) $\vec{u} = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$
   c) $\vec{u} = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$
   d) $\vec{u} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

6. Show that the matrix $M = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ maps all the points of the line $2x + y - 2 = 0$ to the point $P(2,4)$.

7. Given three matrices
   $K = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, L = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, M = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$
   a) Calculate
      $L + M$, KL, KM, $K(L + M)$,
      LK, MK, $(L + M)K$.
   b) Hence show that matrix multiplication is distributive over matrix addition. That is, show that $K(L + M) = KL + KM$ and $(L + M)K = LK + MK$.

8. a) Calculate the product $\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix}$
   b) If A and B are two matrices such that $AB = 0_{2\times2}$, is it necessarily true that $A = 0_{2\times2}$ or $B = 0_{2\times2}$? Explain.

9. Three non-zero matrices P, Q and R are such that $PQ = RQ$. Is it necessarily true that $P = R$? Discuss this in the two cases
   a) Q is invertible   b) Q is singular.

10. Show that, for any 2×2 matrices A and B, $(A + B)^2 = A^2 + AB + BA + B^2$.
    Is it possible to simplify the expression on the right side?

11. The matrix T is said to be a square root of the matrix A, if $T^2 = A$.
    Find two different square roots of
    $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

12. Matrix A is such that $A - A^2 = I$, where I is the unit matrix.
    a) Prove that A is invertible.
    b) Prove that $A^3 = -I$.
    c) If X is a matrix such that $AX = I + A$, find the real numbers p and q such that $X = pI + qA$.

13. a) If $P = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$ and $Q = \begin{bmatrix} a & b \\ 4b & 3b + a \end{bmatrix}$,
    $a, b \in \mathbb{R}$, prove that $PQ = QP$.
    b) It is also known that for non-zero vectors $\vec{u}$ and $\vec{v}$, $P\vec{u} = \vec{u}$ and $P\vec{v} = 6\vec{v}$. Find the matrix Q such that $PQ = QP$, $Q\vec{u} = -\vec{u}$, and $Q\vec{v} = 4\vec{v}$.

14. **a)** State the matrix R that rotates the plane counterclockwise through an angle θ, and the matrix M that reflects the plane in the line $y = x$.

b) Solve for θ the equation $RM = MR$, $0° \leqslant \theta < 360°$.

15. An **orthogonal matrix** is one whose columns represent perpendicular unit vectors. Show that the following are orthogonal.

a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b) $\begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

c) the matrix of counterclockwise rotation through an angle θ

d) the matrix of reflection in the line $y = (\tan \alpha)x$.

16. Given $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $M = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}$, show that the following are orthogonal matrices. (See question 15.)

a) R^2

b) M^2

c) RM

(Use the formulas on page 542.)

17. $S = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ is known as a **symmetric matrix.**

a) Verify that $\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$ is a symmetric matrix.

b) Calculate S^2, and show that it is symmetric also.

c) Calculate S^3, and show that it is symmetric also.

18. $A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 5 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 8 & -19 \\ 1 & -4 & 10 \\ 1 & -3 & 7 \end{bmatrix}$

a) Show that $DA = AD = I$, where

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(I is the 3×3 identity matrix. Thus, you have shown here that A and D are inverse 3×3 matrices.)

b) By writing the following system of equations as a matrix equation $A\vec{v} = \vec{b}$, solve the system for x, y and z.

$$\begin{aligned} 2x + y + 4z &= 2 \\ 3x + 5y + z &= 1 \\ x + 2y &= -5 \end{aligned}$$

19. To find the invariant lines of a transformation M, you look for vectors $\vec{v}$ whose images under M are *collinear* with $\vec{v}$. That is, you look for real numbers k and non-zero vectors $\vec{v}$ that satisfy $M\vec{v} = k\vec{v}$.

a) Given $M = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, show that $M\vec{v} = k\vec{v}$ will have non-zero solutions for x and y if and only if $k = 4$ or $k = -1$.

(4 and -1 are known as the **characteristic values** of the matrix M.)

b) If $k = 4$, show that $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ satisfies the equation $M\vec{v} = k\vec{u}$ and if $k = -1$, show that $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ satisfies the equation $M\vec{v} = k\vec{u}$.

(These are known as **characteristic vectors** of the matrix M.)

c) Hence find the Cartesian equations of two lines through O that are invariant under the matrix M.

In Search of Invariant Lines and Characteristic Vectors

Invariant Lines

The algebraic properties of matrices will also allow you to discover whether or not a transformation has any invariant lines.

Recall that when a matrix transforms a plane, the origin does not move. For some transformations, there may be entire lines that do not move. Such a line, called an invariant line, must therefore contain the origin.

A line L through the origin is completely determined by any direction vector $k\vec{v}$, where $\vec{v} \neq 0$ is any vector parallel to this line.

Now a line through (0,0) with direction vector $\vec{v}$ is invariant under a transformation M if $\vec{v}$ maps into some vector parallel to itself. In that case, $M\vec{v} = k\vec{v}$ for some $k \in \mathbb{R}$.

An example should help you to understand the general case.

Example 1 Find the invariant lines of the transformation $M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Solution If a line of direction vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ is invariant, the image of $\vec{v}$ is parallel to $\vec{v}$. Hence $M\vec{v} = k\vec{v}$ for some $k \in \mathbb{R}$.

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = k\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2x + y \\ x + 2y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$$

$$\begin{cases} (2 - k)x + y = 0 \\ x + (2 - k)y = 0 \end{cases}$$

$$\begin{bmatrix} 2 - k & 1 \\ 1 & 2 - k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad ①$$ in matrix form

or $C\vec{v} = \vec{0}$.

An invertible matrix sends only $\vec{0}$ to $\vec{0}$. Thus, matrix C must be singular. That is, $\det(C) = 0$.

$$\det(C) = (2 - k)(2 - k) - (1)(1) = 0$$
$$4 - 4k + k^2 - 1 = 0$$
$$k^2 - 4k + 3 = 0$$
$$(k - 1)(k - 3) = 0$$
$$k = 3 \text{ or } k = 1.$$

Thus, the direction vectors $\vec{v}$ of invariant lines are obtained for $k = 3$ or $k = 1$ (the numbers 3 and 1 are called the **characteristic values** of M).

When $k = 3$ $\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -x + y = 0 \\ x - y = 0 \end{cases}$$

and both these equations lead to $y = x$.
Thus one invariant line has equation $y = x$.

When $k = 1$ $\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x + y = 0 \\ x + y = 0 \end{cases}$$

and these equations are equivalent to $y = -x$.
Thus the other invariant line has equation $y = -x$. ■

Invariant Lines in the General Case—Characteristic Vectors

$\vec{v}$ will be a direction vector of a line that is invariant under the transformation M if $\vec{v}$ is mapped into a vector parallel to $\vec{v}$.

Hence $M\vec{v} = k\vec{v}$ for some $k \in \mathbb{R}$.

Thus $M\vec{v} = k(I\vec{v})$ since $I\vec{v} = \vec{v}$.

$M\vec{v} - (kI)\vec{v} = \vec{0}$

$(M - kI)\vec{v} = \vec{0}$ ②

Since $(M - kI)$ sends a non-zero vector to $\vec{0}$, this matrix *must be singular*. Thus, $\det(M - kI) = 0$.

Now $M - kI = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} a-k & b \\ c & d-k \end{bmatrix}$

Thus $\det(M - kI) = (a - k)(d - k) - bc = 0$ ③

Equation ③, a quadratic equation in k, is called the **characteristic equation** of matrix M.
The values of k that are the roots of equation ③ are the **characteristic values** of matrix M.

If the characteristic values are real, the direction vectors of the invariant lines can be obtained by substituting the characteristic values into equation ②. These vectors, defining invariant lines for M, are the **characteristic vectors** of matrix M.

If the characteristic values are *not* real, then the transformation has no invariant lines.

Note: Characteristic values and characteristic vectors are sometimes known as **eigenvalues** and **eigenvectors** (from the German "eigen" meaning "proper").

Example 2 Find the invariant lines of the transformation $R = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

Solution The characteristic equation of the matrix R is $(a - k)(d - k) - bc = 0$, where $a = 3, b = -4, c = 4, d = 3$.

Therefore $(3 - k)(3 - k) - (-4)(4) = 0$

$$9 - 6k + k^2 + 16 = 0$$

$$k^2 - 6k + 25 = 0.$$

The discriminant for this quadratic equation is $(-6)^2 - 4(1)(25) = -64 < 0$. Hence, the roots of this equation are not real. Thus, the transformation R has no invariant lines. ■

Note: R could be written $5\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

Thus, R represents a rotation through θ, where $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{4}{5}}{\frac{3}{5}} = \dfrac{4}{3}$, together with a dilatation of factor 5. Since every line in the plane is rotated by θ (approximately 53°), no line can be invariant.

Activities

1. a) Find the characteristic values of $M = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$
   b) Find the characteristic vectors of M (if they exist).
   c) Find the equations of the invariant lines of transformation M (if they exist).
2. Repeat question 1 for the following: $M = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$, and $M = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$
3. a) Show that matrix $M = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ has the characteristic equation $k^2 - 3k - 4 = 0$.
   b) Show that the matrix M satisfies its own characteristic equation, that is, show that $M^2 - 3M - 4I = 0_{2\times2}$.
   (This is known as the **Cayley-Hamilton theorem**.)
4. $S = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ is known as a *symmetric matrix.*
   a) Prove that the characteristic values of a symmetric matrix are always real.
   b) Find a condition on a, b, h so that the characteristic values of S are equal.
5. A symmetric matrix S (see activity 4) for which $a \neq b$ always has two distinct real characteristic values p and q, associated with characteristic vectors $\vec{u}$ and $\vec{v}$ as follows.

$$S\vec{u} = p\vec{u} \qquad \text{and} \qquad S\vec{v} = q\vec{v}$$

   a) Show that $(S\vec{u})\cdot\vec{v} = (S\vec{v})\cdot\vec{u}$
   b) Hence show that $(p\vec{u})\cdot\vec{v} = (q\vec{v})\cdot\vec{u}$
   c) Hence show that the characteristic vectors of a symmetric matrix must be orthogonal.

Summary

- A matrix with m rows and n columns is said to have dimension $m \times n$.
- The element in the ith row, jth column is represented by a_{ij}.
- Two matrices are equal if and only if all their corresponding elements are respectively equal.
- Two matrices *that have the same dimensions* are added by adding their corresponding elements.

(In the following, all matrices are 2×2 unless specified otherwise.)

- Matrices form a vector space $\mathbb{V}_{2\times2}$, that is, the following properties hold.

Matrix Addition

A1. $V_{2\times2}$ is closed under addition: $M, N \in \mathbb{V}_{2\times2}$ implies $M + N \in \mathbb{V}_{2\times2}$
A2. Addition is associative: $L + (M + N) = (L + M) + N$
A3. There is a $0_{2\times2} \in \mathbb{V}_{2\times2}$ such that for all $M \in \mathbb{V}_{2\times2}$, $M + 0_{2\times2} = M$
A4. If $M \in \mathbb{V}_{2\times2}$, then there exists $-M \in \mathbb{V}_{2\times2}$ such that $M + (-M) = 0_{2\times2}$
A5. Addition is commutative: $M + N = N + M$

(These properties mean that $\mathbb{V}_{2\times2}$ is a commutative group with respect to addition.)

Multiplication of a Matrix by a Scalar

M1. If $M \in \mathbb{V}_{2\times2}$, $k \in \mathbb{R}$, then $kM \in \mathbb{V}_{2\times2}$
M2. $(kp)M = k(pM)$, $k, p \in \mathbb{R}$
M3. $k(M + N) = kM + kN$
M4. $(k + p)M = kM + pM$
M5. There exists $1 \in \mathbb{R}$ such that $1M = M$

Linear Transformations

- A linear transformation T of a vector space $\mathbb{V}_2$ is such that

$$\left.\begin{array}{ll} 1. & T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \\ 2. & T(k\vec{v}) = k[T(\vec{v})] \end{array}\right\} \text{ where } \vec{u}, \vec{v} \in \mathbb{V}_2 \text{ and } k \in \mathbb{R}$$

- A general linear transformation of $\mathbb{V}_2$ has the following form.

$$T: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

or $\vec{v} \rightarrow M\vec{v} = \vec{v'}$,

where $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\vec{v'} = \begin{bmatrix} x' \\ y' \end{bmatrix}$, and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- When a linear transformation acts on a plane,
  1. the origin does not move,
  2. parallelism is preserved.
- The matrix M of a linear transformation is such that its first column is the image of $\vec{i}$ by M and its second column is the image of $\vec{j}$ by M.
- The matrix of a counterclockwise rotation about O through an angle θ is

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- For various types of linear transformations, see the summary of section 7.3, pages 304–305.

Determinants

- If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant, $\det(M) = ad - bc$.
- If a figure of area S is transformed by matrix M, the area of the image figure is $|\det(M)|\, S$.
- If $\det(M) > 0$, the image retains the orientation of the original figure; if $\det(M) < 0$, the image acquires the opposite orientation.

Inverse Matrices

- $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $\det(M) \neq 0$; then the inverse of M is

$$M^{-1} = \frac{1}{\det(M)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- If $\det(M) = 0$ the matrix M (and its transformation) are known as singular; no inverse exists.
- $\mathbb{V}_2$ 'loses some dimension' by a singular transformation.

Multiplication of Matrices

- Given a matrix $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and a matrix $Q = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$, then the product PQ is defined as follows.

$$PQ = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

- The matrix PQ represents a transformation that is the result of doing *Q first, then P*.
- In general, $QP \neq PQ$. (Matrix multiplication is non-commutative.)
- $(PQ)R = P(QR)$. (Matrix multiplication is associative.)
- Matrix multiplication is distributive over addition: $L(M + N) = LM + LN$ and $(M + N)L = ML + NL$
- The inverse of (AB) is $(B^{-1}A^{-1})$.
- A singular transformation composed with any other transformation gives a singular transformation.

General Matrix Products

- The product AB of two matrices is defined only if A and B have dimension as follows.
  A has dimension $m \times n$, B has dimension $n \times p$.
  The product AB then has dimension $m \times p$.
- Whenever the product of matrices is defined, the product has the associative property.

Cramer's rule

- Cramer's rule for the solution of $\begin{cases} ax + by = p \\ cx + dy = q \end{cases}$

$$x = \frac{pd - bq}{ad - bc} \quad \text{and} \quad y = \frac{aq - pc}{ad - bc} \qquad (ad - bc \neq 0)$$

Inventory

1. In the matrix $A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 4 \end{bmatrix}$, the element a_{23} is ______.

2. A matrix that has p rows and q columns has dimension ______.

3. A square matrix of order n has ______ rows and ______ columns.

4. $\begin{bmatrix} 1 & a \\ b & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 3 & c \end{bmatrix} \Rightarrow a =$ ______, $b =$ ______, $c =$ ______.

5. If T is a linear transformation of $\mathbb{V}_2$, then $T(\vec{u} + \vec{v}) =$ ______ and $T(k\vec{v}) =$ ______.

6. Under a linear transformation, the origin moves to ______.

7. ______ is always preserved under a linear transformation.

8. The first column of the matrix M of a linear transformation is the image of ______ by ______.
   The second column is the ______.

9. θ is an angle such that $\cos\theta = 0.6$ and $\sin\theta = 0.8$. The matrix of the counterclockwise rotation about O through the angle θ is ______.

10. The matrix product AB represents a transformation that is the result of doing ______ first, then ______.

11. The composition of transformations is in general *non-*______.

12. If A has dimension $p \times q$, and B has dimension $r \times s$, then the matrix product AB is defined if and only if ______ = ______. In that case, the dimension of AB is ______.

13. The inverse of the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ exists if and only if ______.
    Then $M^{-1} =$ ______.

14. A matrix M represents a singular transformation if ______.

15. A singular transformation cannot be ______.

16. A singular transformation composed with any other transformation gives a ______.

17. The area scale factor of matrix M is ______.

18. A figure is transformed into a figure of the same orientation if ______; and into a figure of the opposite orientation if ______.

Review Exercises

1. Given

$A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 0 \\ -2 & -3 \end{bmatrix}$, $C = \begin{bmatrix} 5 & -1 \\ -4 & 7 \end{bmatrix}$

calculate the following.

a) $-1A$ d) $A + B + C$ g) $-B + 6C$

b) $3B$ e) $A - 4C$ h) $-4(2A)$

c) $5A - B$ f) $\frac{1}{2}(B + C)$ i) $-8A$

2. Using the matrices A, B, C given in question 1, calculate the matrix X in the following cases.

a) $3X = B$ c) $C + X = A$

b) $X - 2A = 0_{2\times2}$ d) $X + 2B = 5X - A$

3. Calculate the values of the variables in the following.

a) $\begin{bmatrix} 2 & x \\ 1 & y \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -8 \end{bmatrix}$

b) $\begin{bmatrix} -x & 2y \\ y & x \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

4. Find the images of the following vectors under $M = \begin{bmatrix} 1 & -2 \\ 5 & 2 \end{bmatrix}$

$\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$ $\vec{w} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Sketch as position vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ and their images $\vec{u'}$, $\vec{v'}$, $\vec{w'}$.

5. Write the matrix that corresponds to the given transformation in each case.

a) a dilatation of factor $\frac{1}{4}$

b) reflection in the y-axis

c) rotation through $-90°$

d) projection onto the x-axis

e) reflection in the line $y = x$

6. By finding the images of the points O, $P(1,0)$, $Q(1,1)$, $R(0,1)$ sketch the effect of each of the transformations of question 5 on the unit square.

7. The straight line L has vector equation $\vec{r} = \vec{r_0} + k\vec{m}$, where

$\vec{r} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\vec{r_0} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$, $\vec{m} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

a) Find the vector equation of the image of L under the transformation of matrix

$A = \begin{bmatrix} 2 & 5 \\ -1 & 0 \end{bmatrix}$

b) Graph L and its image on the same set of axes.

8. By reading the columns of the following matrices, describe the associated transformation in each case.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $D = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}$ $E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$C = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ $F = \begin{bmatrix} \cos 65° & -\sin 65° \\ \sin 65° & \cos 65° \end{bmatrix}$

9. By finding the images of the points O, $P(1,0)$, $Q(1,1)$, $R(0,1)$ sketch the effect of each of the transformations of question 8 on the unit square.

10. Use the matrices of question 8 to answer the following.

a) For each transformation, describe the orientation and the area of a transformed figure.

b) Which matrices are singular?

c) Find the inverse of each of the invertible matrices.

11. Use the matrices of question 8 to calculate the following.

a) AB c) DE

b) CD d) E^2

12. Write the matrix of the rotation counterclockwise about the origin through the given angle. In each case, give entries correct to 2 decimal places.

a) 50° b) 125° c) 180°

13. Write the matrix of the reflection in the line $y = (\tan \phi)x$ for the following values of ϕ, giving entries correct to 2 decimal places.
a) $70°$ **b)** $132°$ **c)** $90°$

14. Given that R_θ represents a counterclockwise rotation (about the origin) of $\theta°$, compare R_{225} and R_{-135}. Explain.

15. Show that the counterclockwise rotation (about the origin) of $(180° - \alpha)$ is represented by the matrix
$$\begin{bmatrix} -\cos\alpha & -\sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$$

16. Consider the straight lines L_1 and L_2 with the following vector equations.
$$L_1: \vec{r} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + k\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad L_2: \vec{s} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} + p\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
a) Explain why the lines L_1 and L_2 are perpendicular, and graph them on the same set of axes.
b) Find the vector equations of the images of L_1 and L_2 under the transformation
$$M = \begin{bmatrix} 1 & 0 \\ 6 & -5 \end{bmatrix}$$
c) Are the images also perpendicular?

17. Describe the following transformations, where $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, and $x^2 + y^2 = 1$.
$$M = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \qquad N = \begin{bmatrix} x & y \\ y & -x \end{bmatrix}$$

18. **a)** Write the matrix M_θ of the reflection in the line $y = (\tan\theta)x$.
b) Find M_θ^{-1}.
c) Describe the transformation M_θ^{-1}.

19. Given $M = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$, describe the vectors $\vec{v}$ such that $M\vec{v} = \vec{0}$.

20. **a)** Find the inverse of each of the following matrices.
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$C = \begin{bmatrix} \cos 80° & \sin 80° \\ \sin 80° & -\cos 80° \end{bmatrix}$$
b) Explain your answers by describing the transformations involved.

21. Let M^{-1} be the inverse of a matrix M.
a) What is the effect of applying M then M^{-1}?
b) What is $M\,M^{-1}$?

22. $A = \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 7 & 3 \end{bmatrix}$
a) Calculate AB and BA.
b) Calculate $\det(A)$, $\det(B)$, $\det(AB)$ and $\det(AB)$.
c) Verify that $\det(AB) = \det(A)\det(B)$.

23. Given the 2×2 matrices A and B, discuss the statement $AB = 0_{2\times2} \Rightarrow BA = 0_{2\times2}$.

24. Consider a matrix M, and a non-zero vector $\vec{u}$. Discuss whether or not a vector $\vec{v}$ such that $M\vec{v} = \vec{u}$ can always be found in the following cases.
a) M is invertible.
b) M is singular.

25. Consider the matrix $M = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, where none of the elements a, b, or c are zero. It is required to find a non-zero matrix X such that $MX + XM = 0_{2\times2}$.

Find the general solution in terms of a parameter t.

26. Prove that the set S of invertible 2×2 matrices forms a non-commutative group by verifying each of the following properties.

M1. S is closed under multiplication:
$M, N \in S \Rightarrow MN \in S$

M2. Multiplication is associative:
$L(MN) = (LM)N$

M3. There exists $I \in S$ such that for all $M \in S$,
$IM = MI = M$

M4. If $M \in S$, then there exists $M^{-1} \in S$ such that $MM^{-1} = M^{-1}M = I$

27. The matrix M and the vectors $\vec{v}$ and $\vec{w}$ are given by

$$M = \begin{bmatrix} 2 & p \\ 4 & 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} q \\ 5 \end{bmatrix}, \text{ where}$$

$p, q \in \mathbb{R}$.

a) Given that the matrix N is the inverse of M,
   i) write down N, and
   ii) state the value of p if N is singular.

b) Prove that $M(\vec{v} - \vec{w}) = M(\vec{v}) - M(\vec{w})$.

c) Given $M(\vec{v} - \vec{w}) = \vec{v} + \vec{w}$, calculate the values of p and q.

d) Given $\det(M^2) = 64$, calculate the values of p.

(86 SMS)

28. $\vec{u}$, $\vec{v}$ and $\vec{w}$ are three vectors in a two-dimensional rectangular Cartesian coordinate system with origin O. $\vec{i}$ and $\vec{j}$ are unit vectors in the direction of the coordinate axes. A, B and C are three points such that

$$\overrightarrow{OA} = \vec{u} = 2\vec{i} - \vec{j}$$
$$\overrightarrow{OB} = \vec{v} = \vec{i} + 3\vec{j}$$
$$\overrightarrow{OC} = \vec{w} = 7\vec{i} + \frac{9}{2}\vec{j}$$

a) Show A, B and C on a diagram.

b) Find the values of λ and μ such that $\vec{w} = \lambda\vec{u} + \mu\vec{v}$.

c) Prove that $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are perpendicular.

d) Find the coordinates of D so that $ABCD$ is a rectangle and determine the magnitudes of the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

e) The rectangle $ABCD$ is transformed to the quadrilateral $A'B'C'D'$ under the transformation with matrix $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.
   i) Calculate the coordinates of the points A', B', C' and D'.
   ii) Prove that $A'B'C'D'$ is a parallelogram.
   iii) Calculate the value of α, $\alpha \in \mathbb{R}$, such that

$$\overrightarrow{BB'} = \alpha\overrightarrow{BC'}.$$

(83 SMS)

29. Given that P^T denotes the transpose of the matrix P,* which one of the following statements concerning 2×2 matrices and their determinants may be false?

A. $\det 2P = 4 \det P$
B. $\det(PQ) = \det P \det Q$
C. $\det(P + Q) = \det P + \det Q$
D. $\det P^T = \det P$
E. $(PQ)^T = Q^T P^T$.

(83 H)

*see page 362

30. A linear transformation L is such that $L\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $L\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Find $L\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

(85 H)

31. Oxy is a 2-dimensional rectangular Cartesian coordinate system. Points of the system $P(x,y)$ are mapped onto points $P'(x',y')$ by a linear transformation T represented by the (2×2) matrix M, in such a way that the coordinates obey the relation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = M\begin{bmatrix} x \\ y \end{bmatrix},$$

where

$$M = \begin{bmatrix} 1.4 & -0.2 \\ 0.8 & 0.6 \end{bmatrix}.$$

a) Find the images under T of the points O, A, B, C whose coordinates are (0,0), (1,2), (3,1) and (2,−1) respectively.

b) Draw a sketch on squared paper showing the figure $OABC$ and its image $O'A'B'C'$.

c) Prove that all points on the line with equation

$$y = 2x$$

are invariant under T.

d) Describe fully the geometrical effect of the linear transformation T.

e) Determine the images of the points A and C under the linear transformation T^{-1}.

(88 S)

VECTORS, MATRICES
and
COMPLEX NUMBERS

with
International Baccalaureate
questions

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CHAPTER EIGHT

TRANSFORMATIONS OF CONICS

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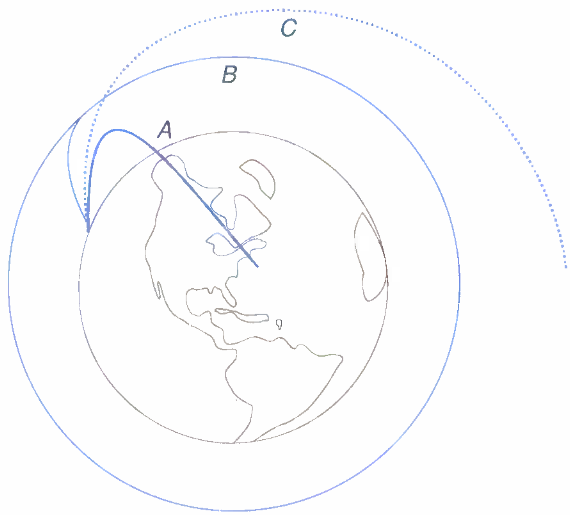
CHAPTER EIGHT

Transformations of Conics

Conics are all around you—the path of a ball thrown through the air; the shape of a cross section of the reflector used in a TV satellite dish or in an automobile headlight; the arch of a bridge; the path of the planets around the sun; the path of a space vehicle, etc.

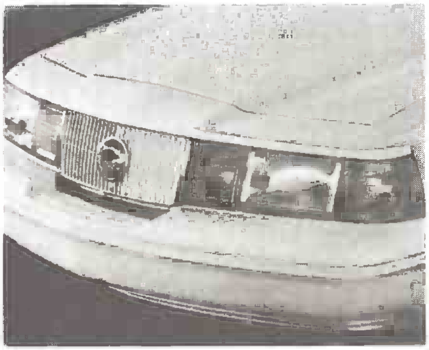
The diagram shows possible conical paths of a space vehicle launched from the surface of the earth.

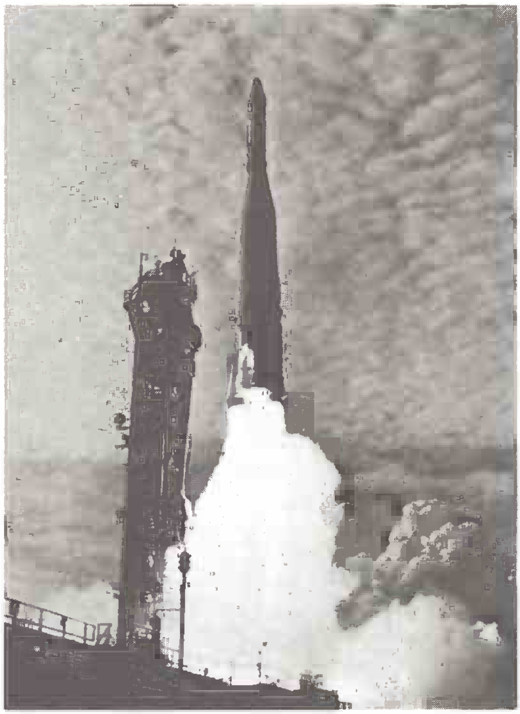


The parabolic path A is taken by a space vehicle that falls back to earth. The circular path B is followed by a space vehicle sent into orbit around the earth. Path C, which may be an ellipse, a hyperbola or a parabola, is that of a space vehicle with sufficient velocity to escape the gravity of the earth and go into orbit around the sun.

In actual space travel, any one of the conical paths A, B, or C may need to be *transformed* into another conical path by a translation, a rotation, a dilatation, or some combination of these transformations.

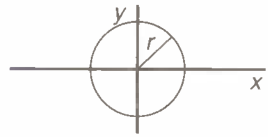
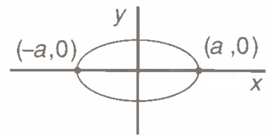
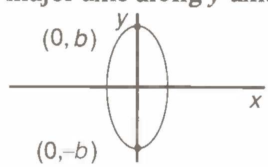
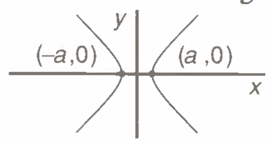
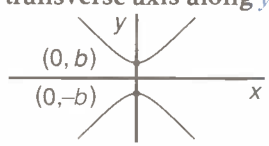
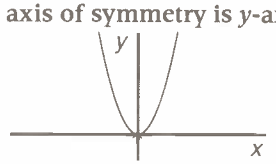
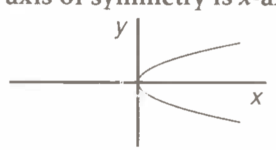
In this chapter you will study the transformation of conics under translations and rotations.





8.1 Conics in Standard Position

In previous grades you met the conic sections, namely, the circle, the ellipse, the hyperbola, and the parabola.
The following chart will remind you of some of the information that you learned about the conics.

conic	equation	centre	vertices	graph
circle	$x^2 + y^2 = r^2$	(0,0)	none	radius $= r$, $r > 0$ 
ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $a > b > 0$	(0,0)	$(a,0)(-a,0)$	major axis along x-axis 
	$b > a > 0$	(0,0)	$(0,b)(0,-b)$	major axis along y-axis 
hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $a > 0, b > 0$	(0,0)	$(a,0)(-a,0)$	transverse axis along x-axis 
	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ $a > 0, b > 0$	(0,0)	$(0,b)(0,-b)$	transverse axis along y-axis 
parabola	$y = kx^2$ $k > 0$: opens up $k < 0$: opens down	none	(0,0)	axis of symmetry is y-axis 
	$x = ky^2$ $k > 0$: opens right $k < 0$: opens left	none	(0,0)	axis of symmetry is x-axis 

The circle, ellipses, and hyperbolas with equations given in the chart have their centres at the origin (0,0) and their major axis or transverse axis along the x-axis or y-axis. These conics are called **central conics**. The central conics and the parabolas with equations given in the chart (vertex at the origin (0,0) and axis of symmetry along the y-axis or x-axis) are said to be in **standard position**, and their equations in **standard form**.

Equations of conics do not have to be in standard form. They can also be in general form.

An equation of a conic written $ax^2 + by^2 + 2gx + 2fy + c = 0$ is in **general form**.

Some examples of standard form and general form follow.

conic	standard form	general form	a	b	g	f	c
ellipse	$\frac{x^2}{4} + \frac{y^2}{9} = 1$	$9x^2 + 4y^2 - 36 = 0$	9	4	0	0	−36
hyperbola	$\frac{x^2}{3} - \frac{y^2}{5} = 1$	$5x^2 - 3y^2 - 15 = 0$	5	−3	0	0	−15
ellipse	$\frac{x^2}{16} + \frac{y^2}{9} = 1$	$9x^2 + 16y^2 - 144 = 0$	9	16	0	0	−144
hyperbola	$\frac{x^2}{4} - \frac{y^2}{9} = -1$	$9x^2 - 4y^2 + 36 = 0$	9	−4	0	0	36
circle	$x^2 + y^2 = \frac{9}{4}$	$4x^2 + 4y^2 - 9 = 0$	4	4	0	0	−9
parabola	$y = 4x^2$	$4x^2 - y = 0$	4	0	0	$-\frac{1}{2}$	0
parabola	$x = -3y^2$	$3y^2 + x = 0$	0	3	$\frac{1}{2}$	0	0

Example

Describe the graph of each of the following conics.

a) $9x^2 + 9y^2 = 16$

b) $16x^2 + 9y^2 = 144$

c) $25x^2 - 16y^2 = 400$

d) $4x^2 - y = 0$

Solution

a) $9x^2 + 9y^2 = 16$ or $x^2 + y^2 = \frac{16}{9}$ is an equation of a circle with centre (0,0) and radius $= \sqrt{\frac{16}{9}} = \frac{4}{3}$.

b) $16x^2 + 9y^2 = 144$ ①
can be divided by 144 to obtain

$$\frac{x^2}{9} + \frac{y^2}{16} = 1.$$

The conic is an ellipse with $a^2 = 9$, $a = 3$ and $b^2 = 16$, $b = 4$. Since $b > a$, the major axis is on the y-axis.

Alternatively, you can find the x-intercepts and y-intercepts and then sketch the graph of the ellipse.

x-intercepts: let $y = 0$ in ①	*y-intercepts*: let $x = 0$ in ①
$16x^2 = 144$	$9y^2 = 144$
$x^2 = 9$	$y^2 = 16$
$x = 3$ or $x = -3$	$y = 4$ or $y = -4$

The graph of $16x^2 + 9y^2 = 144$ is as shown below.

c) $25x^2 - 16y^2 = 400$ ②

Each side can be divided by 400 to obtain

$$\frac{x^2}{16} - \frac{y^2}{25} = 1.$$

The conic is a hyperbola with $a^2 = 16$, $a = 4$ and $b^2 = 25$, $b = 5$. Since this is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the transverse axis is on the x-axis.

Alternatively, you can find the x-intercepts and y-intercepts and then sketch the graph of the hyperbola.

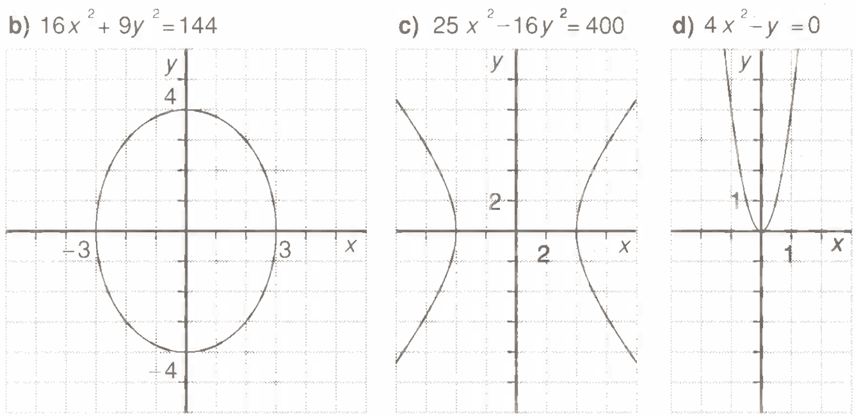
x-intercepts: let $y = 0$ in ②	*y-intercepts*: let $x = 0$ in ②
$25x^2 = 400$	$-16y^2 = 400$
$x^2 = 16$	Since $y^2 > 0$ for $y \in \mathbb{R}$, no real value of
$x = 4$ or $x = -4$	y satisfies this equation.

Hence, the x-intercepts are 4 and -4 but the hyperbola does not intersect the y-axis.

The graph of $25x^2 - 16y^2 = 400$ is as shown below.

d) $4x^2 - y = 0$ can be written $y = 4x^2$, which is an equation for a parabola with vertex at (0,0), axis of symmetry the y-axis, and opening upward.

The graph of $4x^2 - y = 0$ is as shown below.



Note 1 Sketching a conic in this context means to find the x-intercepts and y-intercepts, if they exist, and to show the direction of opening for a hyperbola or a parabola.

2 If more points are desired, obtain points near the vertices. In part c) above, $x = 5$ gives $y = \pm 3.75$, and $x = -5$ also gives $y = \pm 3.75$, which produce the four points (5,3.75), (5,−3.75), (−5,3.75), and (−5,−3.75).

The general form of the equation of a conic is $ax^2 + by^2 + 2gx + 2fy + c = 0$.

(Do not confuse these a's and b's with those of, say, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The two sets are different.)

By comparing the standard form equation of a conic with the general form you can obtain some useful information.

circle

The equation $9x^2 + 9y^2 = 16$ from Example 1 can be rewritten

$$9x^2 + 9y^2 - 16 = 0.$$

Each term of the equation can be multiplied by any non-zero number k to obtain $9kx^2 + 9ky^2 - 16k = 0$.

Comparing this equation with $ax^2 + by^2 + 2gx + 2fy + c = 0$

gives $a = b = 9k$, $g = f = 0$, $c \neq 0$.

Note that $(a)(b) = (9k)(9k) = 81k^2$, a positive number.

You will learn that the important fact to remember about a circle is $ab > 0$, $a = b$.

ellipse

The equation $16x^2 + 9y^2 = 144$ from Example 1 can be rewritten

$$16x^2 + 9y^2 - 144 = 0.$$

Comparing this equation with $ax^2 + by^2 + 2gx + 2fy + c = 0$

gives $a = 16$, $b = 9$, $g = f = 0$, $c \neq 0$.

Note that $(a)(b) = (16)(9) = 144$, a positive number.

You will learn that the important fact to remember about an ellipse is $ab > 0$, $a \neq b$

hyperbola

The equation $25x^2 - 16y^2 = 400$ from Example 1 can be rewritten

$$25x^2 - 16y^2 - 400 = 0.$$

Comparing this equation with $ax^2 + by^2 + 2gx + 2fy + c = 0$

gives $a = 25$, $b = -16$, $g = f = 0$, $c \neq 0$.

Note that $(a)(b) = (25)(-16) = -400$, a negative number.

You will learn that the important fact to remember about a hyperbola is $ab < 0$.

parabola

The equation $4x^2 - y = 0$ from Example 1 is in general form.

Comparing this equation with $ax^2 + by^2 + 2gx + 2fy + c = 0$

gives $a = 4$, $b = 0$, $g = c = 0$, $f = -\frac{1}{2}$. Note that $(a)(b) = (4)(0) = 0$.

You will learn that the important fact to remember about a parabola is $ab = 0$.

You should check the truth of the following summary as you do Exercises 8.1 and 8.2.

SUMMARY

conic	$ax^2 + by^2 + 2gx + 2fy + c = 0$
circle	$ab > 0$, $a = b$
ellipse	$ab > 0$, $a \neq b$
hyperbola	$ab < 0$
parabola	$ab = 0$.

8.1 Exercises

1. Sketch the graph of each of the following conics.
   a) $x^2 + y^2 = 4$
   b) $x^2 + 9y^2 = 9$
   c) $4x^2 - y^2 = 4$
   d) $y = 4x^2$
   e) $x = 2y^2$

2. Sketch the graph of each of the following circles.
   a) $x^2 + y^2 - 16 = 0$
   b) $x^2 + y^2 - 9 = 0$
   c) $4x^2 + 4y^2 - 16 = 0$
   d) $3x^2 + 3y^2 - 15 = 0$
   e) $-2x^2 - 2y^2 + 8 = 0$
   f) $-5x^2 - 5y^2 + 10 = 0$

   Indicate the value of a, b, c, g, and f in the general equation for a conic $ax^2 + by^2 + 2gx + 2fy + c = 0$. In each case verify that $ab > 0$, $a = b$.

3. Sketch the graph of each of the following ellipses.
   a) $9x^2 + 16y^2 - 144 = 0$
   b) $25x^2 + 4y^2 - 100 = 0$
   c) $4x^2 + 25y^2 - 100 = 0$
   d) $-4x^2 - 9y^2 + 36 = 0$
   e) $-32x^2 - 18y^2 + 144 = 0$
   f) $-2x^2 - 3y^2 + 18 = 0$

   Indicate the value of a, b, c, g, and f in the general equation for a conic $ax^2 + by^2 + 2gx + 2fy + c = 0$. In each case verify that $ab > 0$, $a \neq b$.

4. Sketch the graph of each of the following hyperbolas.
   a) $9x^2 - 16y^2 - 144 = 0$
   b) $9x^2 - 16y^2 + 144 = 0$
   c) $4x^2 - y^2 + 4 = 0$
   d) $-8x^2 + 2y^2 + 8 = 0$
   e) $-x^2 + y^2 + 9 = 0$
   f) $3x^2 - 5y^2 - 30 = 0$

   Indicate the value of a, b, c, g, and f in the general equation for a conic $ax^2 + by^2 + 2gx + 2fy + c = 0$. In each case verify that $ab < 0$.

5. Sketch the graph of each of the following parabolas.
   a) $x^2 - y = 0$
   b) $4x^2 - y = 0$
   c) $3x^2 + y = 0$
   d) $-x^2 + 2y = 0$
   e) $4y^2 + x = 0$
   f) $5y^2 - x = 0$

   Indicate the value of a, b, c, g, and f in the general equation for a conic $ax^2 + by^2 + 2gx + 2fy + c = 0$. In each case verify that $ab = 0$.

6. Identify each of the following equations as representing a circle, an ellipse, a hyperbola or a parabola.
   a) $x^2 + y^2 = 25$
   b) $\frac{x^2}{9} - \frac{y^2}{4} = 1$
   c) $y = 9x^2$
   d) $4x^2 - 25y^2 + 100 = 0$
   e) $x^2 + y^2 - 16 = 0$
   f) $\frac{x^2}{9} + \frac{y^2}{4} = 1$
   g) $16x^2 - 9y^2 = 144$
   h) $3y^2 + x = 0$

   For each circle, state the coordinates of the centre and the length of the radius.
   For each ellipse, state the coordinates of the centre, the coordinates of the vertices and the name of the major axis.
   For each hyperbola, state the coordinates of the centre, the coordinates of the vertices and the name of the transverse axis.
   For each parabola, state the coordinates of the vertex and name the axis of symmetry.

7. For each of the following equations of conics, compare the equation with the general equation
   $ax^2 + by^2 + 2gx + 2fy + c = 0$
   and determine the values of a and b. Calculate the value of ab and use this value to determine the type of conic.
   a) $x^2 + 7y^2 - 8 = 0$
   b) $3x^2 - 5y^2 - 1 = 0$
   c) $4x^2 + 5y = 0$
   d) $8x^2 + 8y^2 - 5 = 0$
   e) $3y^2 - 4x = 0$
   f) $3x^2 - 16y^2 + 1 = 0$
   g) $5x^2 + 6y^2 - 2 = 0$
   h) $tx^2 + ty^2 - 14 = 0$, $t \neq 0$
   i) $tx^2 + my^2 - 14 = 0$, $t > 0$, $m < 0$
   j) $tx^2 + my^2 - 14 = 0$, $t \neq m$, $t > 0$, $m > 0$

8.2 Translations of Conics

In section 1.1 you observed the relationship between vectors and translations. Each vector $\vec{a} = \overrightarrow{(h,k)}$ defines the *translation* that maps each point P with coordinates (x,y) into the point P' with coordinates $(x + h, y + k)$, that is,

$$\text{point } P(x,y) \rightarrow \text{point } P'(x + h, y + k).$$

In earlier grades you learned that a translation is an **isometry**, that is, every line segment maps into a congruent line segment, which means that any figure is congruent to its image figure. A special property of a translation is that a line L and its image line L' are parallel.

In this section you will learn how the equation of a conic changes when a conic is translated from its standard position.

Example 1

Given the points $P(2,3)$, $Q(-1,5)$, and $R(0,-2)$

a) find the image of $\triangle PQR$ under the translation defined by the vector $\vec{a} = \overrightarrow{(4,-2)}$

b) sketch $\triangle PQR$ and its image $\triangle P'Q'R'$.

Solution

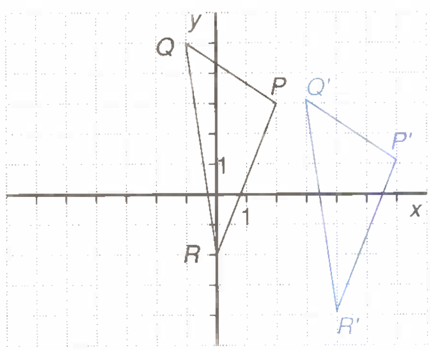
For the translation defined by vector $\vec{a} = \overrightarrow{(4,-2)}$, the point (x,y) maps into the point $(x + 4, y - 2)$, that is,

$$(x,y) \rightarrow (x + 4, y - 2).$$

Thus,

$$P(2,3) \rightarrow P'(2 + 4, 3 - 2) = (6,1)$$
$$Q(-1,5) \rightarrow Q'(-1 + 4, 5 - 2) = (3,3)$$
$$R(0,-2) \rightarrow R'(0 + 4, -2 - 2) = (4,-4)$$

b) $\triangle PQR$ and $\triangle P'Q'R'$ are graphed as shown.



■

Note: $\triangle PQR \cong \triangle P'Q'R'$,
$PQ \parallel P'Q'$,
$PR \parallel P'R'$,
$QR \parallel Q'R'$.

Example 2

Given the ellipse E: $9x^2 + 4y^2 = 36$.

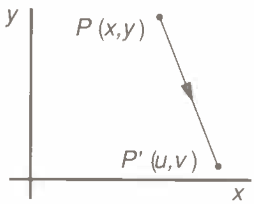
a) Find, in general form, an equation for E', the image of E, under the translation $(x,y) \rightarrow (x + 2, y - 5)$.

b) Sketch a graph of the ellipse E and its image ellipse E'.

c) By comparing the given equation with the general equation $ax^2 + by^2 + 2gx + 2fy + c = 0$, find the values of a, b, c, g, f. Calculate ab for both the equation of E and the equation of E', and note the sign in each case.

Solution

a) To avoid confusing the coordinates of a point P on the ellipse E and a point P' on the image ellipse E', a point P on E will be called (x,y) while a point P' on E' will be denoted (u,v).



Thus, under the translation, $P(x,y) \rightarrow P'(u,v)$

But under this translation $(x,y) \rightarrow (x + 2, y - 5) = (u,v)$

Therefore, $u = x + 2$ and $v = y - 5$ ①

To find the relationship between u and v on E' you must solve ① for x and y in terms of u and v, then substitute these values into the equation for E, namely, into $9x^2 + 4y^2 = 36$ ②

From ①, $x = u - 2$, and $y = v + 5$

Substituting into ② gives

$$9(u - 2)^2 + 4(v + 5)^2 = 36$$

or $9(u^2 - 4u + 4) + 4(v^2 + 10v + 25) - 36 = 0$

or $$E': 9u^2 + 4v^2 - 36u + 40v + 100 = 0.$$

Because it customary to write the equation of a conic using x and y you should replace u by x and v by y to obtain the following equation for E'

$$E': 9x^2 + 4y^2 - 36x + 40y + 100 = 0.$$

b) To sketch image ellipse E' you should find the images of the centre, and of the points of intersection of E: $9x^2 + 4y^2 = 36$ with the x-axis and y-axis.

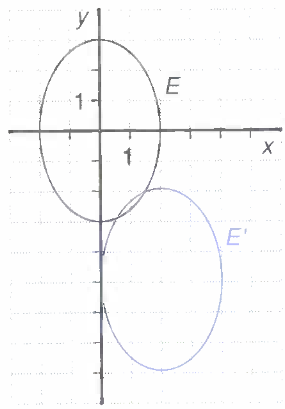
x-intercepts: let $y = 0$
$9x^2 + 0 = 36$
$x^2 = 4$
$x = 2$ or $x = -2$

y-intercepts: let $x = 0$
$0 + 4y^2 = 36$
$y^2 = 9$
$y = 3$ or $y = -3$

Under the translation $(x,y) \rightarrow (x + 2, y - 5)$

ellipse E		image ellipse E'
centre (0,0)	→	centre (2,−5)
(2,0)	→	(4,−5)
(−2,0)	→	(0,−5)
(0,3)	→	(2,−2)
(0,−3)	→	(2,−8)

The graphs of E and E' are as shown.



c) Comparing the general equation $ax^2 + by^2 + 2gx + 2fy + c = 0$ with each equation gives

for E: $9x^2 + 4y^2 - 36 = 0$

$a = 9$, $b = 4$,
$g = f = 0$,
$c = -36$
$ab = 36$ is positive

for E': $9x^2 + 4y^2 - 36x + 40y + 100 = 0$

$a = 9$, $b = 4$,
$2g = -36$, so $g = -18$; $2f = 40$, so $f = 20$,
$c = 100$
$ab = 36$ is positive ■

Example 3

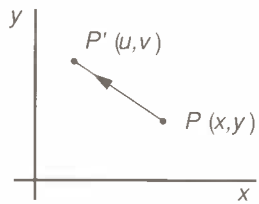
a) Given the hyperbola H: $x^2 - 16y^2 = 16$, find, in general form, an equation for H', the image of H, under the translation $(x,y) \rightarrow (x - 3, y + 2)$.

b) Sketch a graph of the hyperbola H and its image hyperbola H'.

c) By comparing the given equation with the general equation $ax^2 + by^2 + 2gx + 2fy + c = 0$ find the values of a, b, c, g, f. Calculate ab for both the equation of H and the equation of H', and note the sign in each case.

Solution

a) As in Example 2, a point P on H will be called (x,y) while a point P' on H' will be denoted (u,v).



Thus, under the translation, $P(x,y) \rightarrow P'(u,v)$
But under this translation $(x,y) \rightarrow (x - 3, y + 2) = (u,v)$
Therefore, $u = x - 3$ and $v = y + 2$ ③

As in Example 2, solve ③ for x and y, then substitute into
$x^2 - 16y^2 = 16$ ④
From ③, $x = u + 3$, and $y = v - 2$
Substituting into ④ gives
$(u + 3)^2 - 16(v - 2)^2 = 16$
or $u^2 + 6u + 9 - 16(v^2 - 4v + 4) - 16 = 0$

or H': $u^2 - 16v^2 + 6u + 64v - 71 = 0$.

As in Example 2, replace u by x and v by y to obtain the following equation for H'.

H': $x^2 - 16y^2 + 6x + 64y - 71 = 0$.

b) To sketch image hyperbola H' you should find the images of the centre of H: $x^2 - 16y^2 = 16$ and of the points where H intersects the x-axis and the y-axis.

x-intercepts: let $y = 0$
$x^2 + 0 = 16$
$x = 4$ or $x = -4$

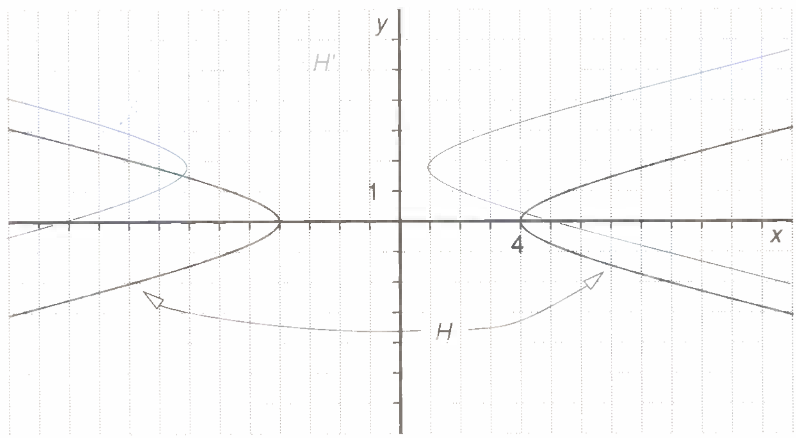
y-intercepts: let $x = 0$
$0 - 16y^2 = 16$
$-16y^2 = 16$ which has no solution in real numbers.

The x-intercepts are 4 and -4. The hyperbola does not intersect the y-axis. Hence, the hyperbola opens along the positive x-axis and the negative x-axis.

Under the translation $(x,y) \rightarrow (x - 3, y + 2)$

hyperbola H		image hyperbola H'
centre $(0,0)$	$\rightarrow$	centre $(-3,2)$
$(4,0)$	$\rightarrow$	$(1,2)$
$(-4,0)$	$\rightarrow$	$(-7,2)$

The graphs of H and H' are as shown.



c) Comparing each equation with the general equation $ax^2 + by^2 + 2gx + 2fy + c = 0$ gives

for H: $x^2 - 16y^2 - 16 = 0$

$a = 1, b = -16,$
$g = f = 0,$
$c = -16$
$ab = -16$ is negative

for H': $x^2 - 16y^2 + 6x + 64y - 71 = 0$

$a = 1, b = -16,$
$2g = 6$, so $g = 3$; $2f = 64$, so $f = 32$,
$c = 71$
$ab = -16$ is negative ■

From Example 2 and Example 3 you can see that, under a translation, the image of an ellipse or hyperbola in standard position has an equation of the form

$$ax^2 + by^2 + 2gx + 2fy + c = 0$$

Note: As you saw in Section 8.1, for an ellipse $ab > 0$ with $a \neq b$, and for a hyperbola $ab < 0$. In 8.2 Exercises, you will find $ab > 0$ with $a = b$ for a circle, and $ab = 0$ for a parabola.

8.2 Exercises

1–10

1. Find the image of the given point under the given translation.
   a) point (3,5) translation $(x,y) \rightarrow (x + 3, y + 6)$
   b) point (−2,4) translation $(x,y) \rightarrow (x - 1, y + 7)$
   c) point (8,0) translation $(x,y) \rightarrow (x - 3, y - 2)$
   d) point (−4,−5) translation $(x,y) \rightarrow (x + 4, y + 5)$

2. Given the ellipse E: $25x^2 + 9y^2 = 225$.
   a) Find, in general form, an equation for E', the image of E, under the translation $(x,y) \rightarrow (x + 3, y + 1)$.
   b) Sketch a graph of the ellipse E and its image ellipse E'.
   c) Find the values of a, b, c, g, f, and the sign of ab for both the equation of E and the equation of E'.

3. Repeat the previous question for the following ellipses and translations.
   a) ellipse $9x^2 + 16y^2 = 144$ translation $(x,y) \rightarrow (x + 3, y + 6)$
   b) ellipse $x^2 + 4y^2 = 16$ translation $(x,y) \rightarrow (x - 1, y + 7)$
   c) ellipse $8x^2 + 200y^2 = 1600$ translation $(x,y) \rightarrow (x - 3, y - 2)$
   d) ellipse $4x^2 + 9y^2 = 36$ translation $(x,y) \rightarrow (x + 4, y + 5)$

4. a) Given the hyperbola H: $x^2 - 16y^2 = 16$, find, in general form, an equation for H', the image of H, under the translation $(x,y) \rightarrow (x - 3, y + 2)$.
   b) Sketch a graph of the hyperbola H and its image hyperbola H'.
   c) Find the values of a, b, c, g, f, and the sign of ab for both the equation of H and the equation of H'.

5. Repeat the previous question for the following hyperbolas and translations.
   a) hyperbola $9x^2 - 16y^2 = 144$ translation $(x,y) \rightarrow (x - 3, y + 5)$
   b) hyperbola $x^2 - 4y^2 = -16$ translation $(x,y) \rightarrow (x + 1, y + 7)$
   c) hyperbola $8x^2 - 50y^2 = 800$ translation $(x,y) \rightarrow (x - 3, y - 2)$
   d) hyperbola $x^2 - y^2 = 36$ translation $(x,y) \rightarrow (x + 4, y - 3)$

6. Given the parabola P: $4x^2 - y = 0$.
   a) Find, in general form, an equation for P', the image of P, under the translation $(x,y) \rightarrow (x + 3, y + 1)$.
   b) Sketch a graph of the parabola P and its image parabola P'.
   c) Find the values of a, b, c, g, f, and the value of ab for both the equation of P and the equation of P'.

7. Repeat the previous question for the following parabolas and translations.
   a) parabola $x^2 - y = 0$ translation $(x,y) \rightarrow (x - 1, y + 6)$
   b) parabola $4y^2 - x = 0$ translation $(x,y) \rightarrow (x - 1, y - 3)$
   c) parabola $8y^2 + x = 0$ translation $(x,y) \rightarrow (x + 2, y - 2)$
   d) parabola $4x^2 + y = 0$ translation $(x,y) \rightarrow (x - 1, y + 3)$

8. Given the circle C: $x^2 + y^2 = 25$.
   a) Find, in general form, an equation for C', the image of C, under the translation $(x,y) \rightarrow (x - 3, y + 1)$.
   b) Sketch a graph of the circle C and its image circle C'.
   c) Find the values of a, b, c, g, f, and the sign of ab for both the equation of C and the equation of C'.

9. Repeat the previous question for the following circles and translations.
   a) circle $x^2 + y^2 = 144$ translation $(x,y) \rightarrow (x + 3, y - 1)$
   b) circle $x^2 + y^2 = 25$ translation $(x,y) \rightarrow (x - 5, y - 2)$
   c) circle $8x^2 + 8y^2 = 16$ translation $(x,y) \rightarrow (x + 3, y - 2)$
   d) circle $-4x^2 - 4y^2 = -16$ translation $(x,y) \rightarrow (x + 4, y + 5)$

10. Examine your results of questions 2–9 to show that for any particular conic the value of ab and hence the sign of ab is invariant under a translation. That is, observe the value of ab before and after translation for each conic to see the following are true.
for an ellipse, $ab > 0$ and $a \neq b$
for a hyperbola, $ab < 0$
for a parabola, $ab = 0$
for a circle, $ab > 0$ and $a = b$

11. Given the curve C: $4x^2 - y + 3 = 0$ and the translation $(x,y) \rightarrow (x,y - 3)$.
   a) Find an equation of its image under the given translation.
   b) Name the type of curve.
   c) Sketch the *image* curve C'. Then sketch the given curve C by using the inverse translation to the given translation.

12. Repeat the previous question for the following curves and translations.
   a) curve $x^2 + y^2 + 4x + 6y - 12 = 0$ translation $(x,y) \rightarrow (x + 2,y + 3)$
   b) curve $2x^2 + 8x + y = 0$ translation $(x,y) \rightarrow (x + 2,y - 8)$
   c) curve $9x^2 + 4y^2 + 72x + 24y + 36 = 0$ translation $(x,y) \rightarrow (x + 4,y + 3)$
   d) curve $4x^2 - y^2 + 16x - 6y + 3 = 0$ translation $(x,y) \rightarrow (x + 2,y + 3)$

13. a) Show by factoring, that the conic $4x^2 - y^2 = 0$ is a **degenerate conic** representing a pair of straight line.
   b) Find an equation of the image of this degenerate conic under the translation $(x,y) \rightarrow (x + 5,y + 2)$
   c) Graph the degenerate conic and its image under the translation.
   d) Why is this conic called a degenerate conic?

14. a) Find the equation of the image of the conic $ax^2 + by^2 + t = 0$ under the translation $(x,y) \rightarrow (x + h,y + k)$.
   b) Use your results of part a) to show that the sign of ab is invariant under a translation.

15. Find the translation under which the curve $x^2 + y^2 + 4x + 6y - 3 = 0$ maps into the circle $x^2 + y^2 = 16$.

16. Given the conic C: $ax^2 + by^2 + c = 0$, where $a, b \neq 0$;
and the translation $(x,y) \rightarrow (x + h,y + k)$,

show that the image of C under the translation is the conic
C': $Ax^2 + By^2 + 2Gx + 2Fy + C = 0$, where
$A = a, B = b, G = -2ah, F = -2bk,$
$C = ah^2 + bk^2 + c.$

17. a) The conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is translated under the translation $(x,y) \rightarrow (x + h,y + k)$.

   Show that the image conic has the equation

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1.$$

   b) Give the coordinates of the centre of the image curve.
   c) State the lengths of the major axis and the minor axis for the image conic.

18. a) The conic $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is translated under the translation $(x,y) \rightarrow (x + h,y + k)$. State an equation for the image curve in a form similar to that of question 17.
   b) Repeat part a) for the curve $y = ax^2$.

19. You will see in a later section that the equation $ax^2 + 2hxy + by^2 + c = 0$ defines an ellipse or a hyperbola for $a \neq 0$, $b \neq 0$, and $a \neq b$. Suppose the conic maps into $Ax^2 + 2Hxy + By^2 + C = 0$ under a translation. Prove the following
   a) $a + b = A + B$
   b) $b^2 - 4ac = B^2 - 4AC$

20. Prove that a translation is an isometry by showing that the length of the line segment joining $P_1(x_1,y_1)$ to $P_2(x_2,y_2)$ equals the length of the line segment joining the images of P_1 and P_2.

8.3 Translating Conics into Standard Position

The equation $ax^2 + by^2 + 2gx + 2fy + c = 0$ represents some conic.

If $g = 0$ and $f = 0$, the conic is a circle, an ellipse, or a hyperbola, *in standard position*: for example, $x^2 + y^2 - 9 = 0$ represents a circle, $4x^2 + 9y^2 - 36 = 0$ an ellipse, or $16x^2 - 25y^2 - 400 = 0$ a hyperbola.

If $a = f = c = 0$, or $b = g = c = 0$, the conic is a parabola *in standard position*: for example, $4y^2 + x = 0$ or $9x^2 - y = 0$.

For conics in standard position, graphs can be drawn as in section 8.1.

If the coefficients indicate that the conic is *not* in standard position, then the conic is graphed by first translating it into standard position, then using the *inverse* translation on the standard position graph.

You will remember from section 8.1 that terms in x and y in non-standard equations arise from perfect squares such as $(x + 3)^2 = x^2 + 6x + 9$. To reverse the process of translating back into standard position you will need to be able to *complete a square*.

Recall that $x^2 + mx$ becomes a perfect square by the addition of $\left(\frac{m}{2}\right)^2$.

Then the following is true.

$$x^2 + mx + \left(\frac{m}{2}\right)^2 = \left(x + \frac{m}{2}\right)^2$$

For example:

$$x^2 + 8x + \left(\frac{8}{2}\right)^2 = \left(x + \frac{8}{2}\right)^2 = (x + 4)^2$$

$$x^2 - 12x + \left(\frac{-12}{2}\right)^2 = \left(x + \frac{-12}{2}\right)^2 = (x - 6)^2$$

$$x^2 + 7x + \left(\frac{7}{2}\right)^2 = \left(x + \frac{7}{2}\right)^2$$

Recall that, in standard position or not, the following is the relationship among a, b, and the conic.

The Graph of $ax^2 + by^2 + 2gx + 2fy + c = 0$

circle	$ab > 0$, $a = b$
ellipse	$ab > 0$, $a \neq b$
hyperbola	$ab < 0$
parabola	$ab = 0$

The following example will demonstrate a method of graphing a conic with equation $ax^2 + by^2 + 2gx + 2fy + c = 0$, that is, when the conic is not in standard position.

Example 1 Given the conic C: $9x^2 + 4y^2 + 36x - 8y + 4 = 0$

a) Name the type of conic.

b) Determine the translation that changes the equation into standard form.

c) State an equation for the image curve C'.

d) Graph the image conic C' and given conic C.

Solution

a) Here $a = 9$, $b = 4$ and $ab = 36$.
Hence $ab > 0$ and $a \neq b$, and so the conic is an ellipse.

b) $g \neq 0$, $f \neq 0$, so the conic is not in standard position. The translation can be obtained by completing the squares of the terms in x and also of the terms in y.

$9x^2 + 4y^2 + 36x - 8y + 4 = 0$ can be written

$9(x^2 + 4x) + 4(y^2 - 2y) = -4$, or

$9(x^2 + 4x + 4) + 4(y^2 - 2y + 1) = -4 + 36 + 4$ add $9(\frac{4}{2})^2$ and $4(\frac{-2}{2})^2$ to both sides of equation

or $9(x + 2)^2 + 4(y - 1)^2 = 36$

Replacing $x + 2$ by u and $y - 1$ by v (to avoid confusing a point on C and a point on C') gives an equation in standard form, namely

$9u^2 + 4v^2 = 36$

Thus, the translation moving $9x^2 + 4y^2 + 36x - 8y + 4 = 0$ into standard position is

$(x,y) \to (u,v)$ or

$(x,y) \to (x + 2, y - 1) = (u,v)$

c) The equation $9u^2 + 4v^2 = 36$ of the image conic C' should be rewritten using x and y, as $9x^2 + 4y^2 = 36$.

d) The curve $9x^2 + 4y^2 = 36$ intersects the x-axis at $A'(2,0)$ and $B'(-2,0)$ and the y-axis at $P'(0,3)$ and $Q'(0,-3)$. The graph of C' is shown below.

To obtain the equation of C', you have applied the translation $(x,y) \to (x + 2, y - 1)$. Hence, to graph C, you must apply the *inverse* translation $(x,y) \to (x - 2, y + 1)$ to the points A', B', P' and Q' on the image ellipse C'.

Under this inverse translation

$(x,y) \to (x - 2, y + 1)$

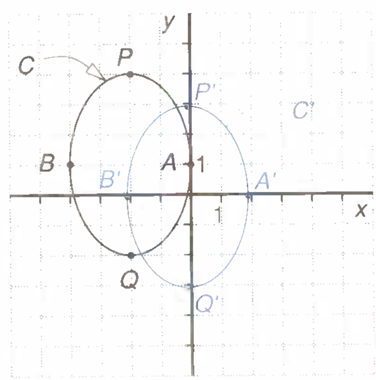
$A'(2,0) \to A(0,1)$

$B'(-2,0) \to B(-4,1)$

$P'(0,3) \to P(-2,4)$

$Q'(0,-3) \to Q(-2,-2)$

The graph of C is shown on the same axes as the graph of C'.



■

Alternate Solution for parts b) and c) of Example 1

b) Let the required translation be $(x,y) \to (x+h, y+k)$.

If (x,y) is a point on C and (u,v) is the image point on C' then
$u = x + h \quad v = y + k$
Hence,
$x = u - h \quad y = v - k$
But $9x^2 + 4y^2 + 36x - 8y + 4 = 0$.
Substituting for x and y gives
$9(u-h)^2 + 4(v-k)^2 + 36(u-h) - 8(v-k) + 4 = 0$
or $9(u^2 - 2hu + h^2) + 4(v^2 - 2kv + k^2) + 36u - 36h - 8v + 8k + 4 = 0$
or $9u^2 + 4v^2 + u(-18h + 36) + v(-8k - 8) + 9h^2 + 4k^2 - 36h + 8k + 4 = 0$*

This equation will be in standard form if the coefficients of u and v are 0.
Hence, $-18h + 36 = 0$ and $-8k - 8 = 0$,
that is, $h = 2$ and $k = -1$.

c) To obtain the equation of E' you should substitute $h = 2$, $k = -1$ into equation* giving
$9u^2 + 4v^2 + u(0) + v(0) + 9(2)^2 + 4(-1)^2 - 36(2) + 8(-1) + 4 = 0$
or $9u^2 + 4v^2 - 36 = 0$.

Example 2

Given the conic C: $4x^2 - y^2 - 8x - 6y - 9 = 0$.

a) Name the type of conic.
b) Determine the translation that changes the equation into standard form.
c) State an equation for the image curve C'.
d) Graph the image conic C' and given conic C.

Solution

a) Here $a = 4$, $b = -1$ and $ab = -4$.
Hence $ab < 0$, so the conic is a hyperbola.

b) The translation can be obtained by completing the squares of the terms in x and also of the terms in y.
$4x^2 - y^2 - 8x - 6y - 9 = 0$ can be written
$4(x^2 - 2x) - (y^2 + 6y) = 9$ or
$4(x^2 - 2x + 1) - (y^2 + 6y + 9) = 9 + 4 - 9$ add $4(\frac{-2}{2})^2$ and $-(\frac{6}{2})^2$ to both sides of the equation
or $4(x-1)^2 - (y+3)^2 = 4$

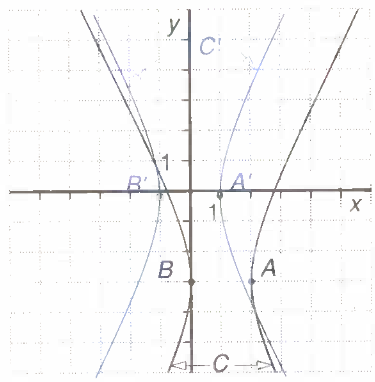
Replacing $x - 1$ by u and $y + 3$ by v (to avoid confusing a point on C and a point on C') gives an equation in standard form, namely
$4u^2 - v^2 = 4$
Thus, the translation moving $4x^2 - y^2 - 8x + 6y - 9 = 0$ into standard position is
$(x,y) \to (u,v)$ or
$(x,y) \to (x - 1, y + 3)$

c) The equation $4u^2 - v^2 = 4$ of the image conic C' should be rewritten using x and y, as $4x^2 - y^2 = 4$.

d) The curve $4x^2 - y^2 = 4$ intersects the x-axis at $A'(1,0)$ and $B'(-1,0)$ and does not intersect the y-axis. Since the hyperbola C' is in standard position and intersects the x-axis, the conic opens right along the positive x-axis and left along the negative y-axis. The graph of C' is as shown below.

To graph C, you must apply the *inverse* translation $(x,y) \rightarrow (x + 1, y - 3)$ to the points A' and B' on the image hyperbola C'.

Under this inverse translation
$A'(1,0) \rightarrow A(2,-3)$
$B'(-1,0) \rightarrow B(0,-3)$
The graph of C is shown on the same axes as the graph of C'.



Example 3

Given the conic C: $3x^2 - 24x - y + 46 = 0$.
a) Name the type of conic.
b) Determine the translation that changes the equation into standard form.

Solution

a) Here $a = 3$, $b = 0$.
Since $ab = 0$, the conic is a parabola.
b) Note that there are no y^2 terms, so that the translation can be obtained by completing the square for the terms in x.

$3x^2 - 24x - y + 46 = 0$
can be written
$3(x^2 - 8x) = y - 46$
$3(x^2 - 8x + 16) = y - 46 + 48$ add $3(\frac{8}{2})^2$ to both sides of the equation
$3(x - 4)^2 = y + 2$
Replacing $x - 4$ by u and $y + 2$ by v gives an equation in standard form, namely $3u^2 = v$.
Thus the translation changing $3x^2 - 24x^2 - y + 46 = 0$ into standard form is
$(x,y) \rightarrow (u,v)$ or
$(x,y) \rightarrow (x - 4, y + 2)$ ■

8.3 Exercises

1. For each of the following conics, indicate whether the conic is an ellipse, a circle, a hyperbola or a parabola.
   a) $3x^2 + 2y^2 - 4x - 7y - 10 = 0$
   b) $5x^2 - 2y^2 - 3x - 2y - 3 = 0$
   c) $4x^2 - 3x - 7y - 1 = 0$
   d) $4x^2 + 4y^2 - 8x + 7y + 3 = 0$
   e) $x^2 + 4y^2 - 3x - 8y - 10 = 0$
   f) $x^2 - y^2 - 2x - 3 = 0$

2. Given the conic $C\colon 4x^2 + y^2 + 8x - 6y + 9 = 0$.
   a) Name the type of conic.
   b) Determine the translation that changes the equation into standard form.
   c) Find an equation for the image curve C'.
   d) Graph the image conic C' and given conic C.

3. Repeat the previous question for the following conics.
   a) $4x^2 - 9y^2 - 8x - 18y - 41 = 0$
   b) $x^2 + y^2 - 6x + 10y - 2 = 0$
   c) $4x^2 + 25y^2 - 48x + 200y + 444 = 0$
   d) $x^2 + 8x - 16y + 32 = 0$
   e) $9x^2 + y^2 + 18x - 6y + 9 = 0$
   f) $4x^2 - 4y^2 + 8x - 16y + 13 = 0$
   g) $2x^2 + 2y^2 + 8x + 12y + 1 = 0$
   h) $y^2 + 4x - 12y + 4 = 0$

4. Given the conic $C\colon 4x^2 - y^2 - 8x - 4y = 0$.
   a) Determine the translation that changes the equation into standard form.
   b) By factoring the image equation show that the image curve C' consists of two lines that intersect at the origin.
   c) Find equations for the two lines that make up the conic C.
   d) Explain why C is called a degenerate hyperbola.

5. Show that the sign of ab is invariant when the conic $ax^2 + by^2 + 2gx + 2fy + c = 0$ is translated under the translation $(x,y) \to (x + h, y + k)$.

6. a) Use translations to show that the conic $x^2 + y^2 + 2gx + 2fy + c = 0$ is a circle with centre $(-g,-f)$ and radius $\sqrt{g^2 + f^2 - c}$
   b) Under what conditions on g, f, and c will the circle have a non-real radius? Such a circle is called an *imaginary circle*.
   c) Under what conditions on g, f, and c will the circle have a radius that is zero? Such a circle is called a *point circle*.

7. Given the conic $C\colon ax^2 + by^2 + 2gx + 2fy + c = 0$.
   a) Show that C maps into $ax^2 + by^2 = k$ under the translation
   $$(x,y) \to \left(x + \frac{g}{a}, y + \frac{f}{a}\right)$$
   b) Show that C is an ellipse if $\frac{g^2}{a} + \frac{f^2}{b} - c > 0$ and $ab > 0$.
   c) Show that C is a hyperbola if $\frac{g^2}{a} + \frac{f^2}{b} - c \neq 0$ and $ab < 0$.

8. Given the conic $C\colon ax^2 + by^2 + 2gx + 2fy + c = 0$ where $b = 0$.
   a) What type of conic is C?
   b) Show that the conic is translated into standard position by the translation
   $$(x,y) \to \left(x + \frac{g}{a}, y + \frac{c}{2f} - \frac{g^2}{2af}\right)$$

9. Show that the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, $a > b$, determines an ellipse with centre (h,k) and having major axis $y = k$ and minor axis $x = h$.

10. Show that the equation $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1$ determines an hyperbola with centre (h,k) and transverse axis $y = k$.

11. Show that the equation $x - h = m(y - k)^2$ defines a parabola with vertex (h,k) and axis of symmetry $y = k$.

8.4 Matrices, their Transposes and the Central Conics

So far you have studied the effect of a translation on conics. Next you will consider what happens to the equation of a central conic (an ellipse or hyperbola) under a rotation.

In section 7.4 you discovered that a rotation transformation can be described using a matrix. One way to study the effect of a rotation transformation on an ellipse or a hyperbola will be to write such conic equations using matrices.

Consider the equation of ellipse $4x^2 + 9y^2 = 36$, and the matrices

$T = [x \quad y]$, $M = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$, $V = \begin{bmatrix} x \\ y \end{bmatrix}$, and $K = [36]$

Because matrix multiplication is associative, the matrix product TMV can be performed either as $(TM)V$ or $T(MV)$. Using the latter,

$$\begin{aligned} TMV &= [x \quad y]\left(\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right) \\ &= [x \quad y]\begin{bmatrix} 4x + 0 \\ 0 + 9y \end{bmatrix} \\ &= [4x^2 + 9y^2] \end{aligned}$$

Hence, the conic equation $4x^2 + 9y^2 = 36$ is equivalent to the matrix equation $[4x^2 + 9y^2] = [36]$ which is written $TMV = K$.

When a central conic in standard position is rotated about the point (0,0) you will see (in section 8.5) that the conic has an equation with an xy term. The form of such an equation is $ax^2 + 2hxy + y^2 = k$. You will now learn how to express such an equation using matrices.

Example 1 Show that the quadratic expression $ax^2 + 2hxy + by^2$ is equivalent to the matrix product TMV where $T = [x \quad y]$, $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$, $V = \begin{bmatrix} x \\ y \end{bmatrix}$

Solution The product $TMV = T(MV)$

$$\begin{aligned} &= [x \quad y]\left(\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right) \\ &= [x \quad y]\begin{bmatrix} ax + hy \\ hx + by \end{bmatrix} \\ &= [ax^2 + 2hxy + by^2] \end{aligned}$$

Hence, the quadratic expression $ax^2 + 2hxy + by^2$ is equivalent to the matrix product $TMV = [x \quad y]\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ ■

Note: The determinant of M, det(M), equals $ab - h^2$. You will learn that the value of this determinant is significant in determining the type of central conic represented by the matrix equation $TMV = K$.

If $K = [k]$, then Example 1 shows you that the quadratic equation $ax^2 + 2hxy + by^2 = k$ is equivalent to the matrix equation

$$TMV = K, \text{ or}$$

$$[x \quad y]\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [k]$$

Observe that the matrix T and the matrix V contain the same elements but the row in T is a column in V.
T is called the *transpose* of matrix V and is denoted V^t.

PROPERTY

The quadratic expression $ax^2 + 2hxy + by^2 = k$ is equivalent to the matrix equation $V^tMV = K$ where $V = \begin{bmatrix} x \\ y \end{bmatrix}$, $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$, $K = [k]$, and $V^t = [x \quad y]$

DEFINITION

The transpose A^t of an $m \times n$ matrix A is the $n \times m$ matrix formed by interchanging the elements in rows and columns, that is,
$A = [a_{ij}] \Rightarrow A^t = [a_{ji}]$

The following are some examples of a matrix and its transpose.

matrix A $\quad \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \quad \begin{bmatrix} m & k \\ k & c \end{bmatrix} \quad \begin{bmatrix} 5 & 4 & 1 \\ 3 & 2 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \begin{bmatrix} 6 & 5 & 4 \\ 7 & 8 & 9 \end{bmatrix}$

transpose A^t $\quad \begin{bmatrix} 5 & 3 \\ 4 & 2 \end{bmatrix} \quad \begin{bmatrix} m & k \\ k & c \end{bmatrix} \quad \begin{bmatrix} 5 & 3 & 7 \\ 4 & 2 & 8 \\ 1 & 6 & 9 \end{bmatrix} \quad \begin{bmatrix} 6 & 7 \\ 5 & 8 \\ 4 & 9 \end{bmatrix}$

Note: The symmetric matrix $\begin{bmatrix} m & k \\ k & c \end{bmatrix}$ equals its transpose.

One important property of the transpose of a matrix is the following.

PROPERTY

If A and B are matrices such that AB exists then $(AB)^t = B^tA^t$.
The proof of this property for 2×2 matrices follows.

Proof: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} m & t \\ r & s \end{bmatrix}$

Then $A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and $B^t = \begin{bmatrix} m & r \\ t & s \end{bmatrix}$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} m & t \\ r & s \end{bmatrix} = \begin{bmatrix} am + br & at + bs \\ cm + dr & ct + ds \end{bmatrix}$$

$$B^tA^t = \begin{bmatrix} m & r \\ t & s \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} am + br & cm + dr \\ at + bs & ct + ds \end{bmatrix} = (AB)^t$$

Hence, $(AB)^t = B^tA^t$.

In the next section you will be performing rotations on conics about the point (0,0) using the rotation matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

In 8.4 Exercises you will be asked to prove the following property of the rotation matrix R and its transpose R^t.

PROPERTY

If R is the rotation matrix and I is the 2×2 unit matrix, then $RR^t = R^tR = I$.

This property means that R^{-1}, the inverse of R, and R^t, the transpose of R, are equal matrices.

Example 2

a) Write the Cartesian form of the matrix equation $V^tMV = K$ where $V = \begin{bmatrix} x \\ y \end{bmatrix}$, $M = \begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}$ and $K = [7]$.

b) Write the matrix equation for the Cartesian equation $3x^2 - 5xy - 7y^2 = 10$.

Solution

a) *Method 1*

From the first property on page 362, the matrix form of the equation $ax^2 + 2hxy + by^2 = k$ is the matrix equation $V^tMV = k$ where $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$, $V = \begin{bmatrix} x \\ y \end{bmatrix}$, and $K = [k]$.

But $M = \begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}$ so $a = 2$, $h = -3$ and $b = 6$. Also, $K = [7]$, so $k = 7$.

Hence, the matrix equation $V^tMV = K$ becomes the Cartesian equation $ax^2 + 2hxy + by^2 = k$ or $2x^2 - 6xy + 6y^2 = 7$.

Method 2

The matrix equation $V^tMV = K$ becomes

$$[x \quad y]\begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [7]$$

$$\left([x \quad y]\begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = [7]$$

$$[2x - 3y \quad -3x + 6y]\begin{bmatrix} x \\ y \end{bmatrix} = [7]$$

$$[2x^2 - 3xy - 3xy + 6y^2] = [7]$$

$$[2x^2 - 6xy + 6y^2] = [7]$$

Hence, the Cartesian equation is $2x^2 - 6xy + 6y^2 = 7$.

b) From the first property on page 362, the matrix M for the Cartesian equation $ax^2 + 2hxy + y^2 = k$ is $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$, and $K = [k]$.

But for $3x^2 - 5xy - 7y^2 = 10$, $a = 3$, $h = -\frac{5}{2}$ or -2.5, $b = -7$, $k = 10$.

Thus the Cartesian equation $3x^2 - 5xy - 7y^2 = 10$ becomes the matrix equation $V^tMV = K$. That is, $[x \quad y]\begin{bmatrix} 3 & -2.5 \\ -2.5 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [10]$. ■

8.4 Exercises

1. Calculate the matrix product.

   a) $[7 \quad 8]\begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

2. For each of the following matrices M, write the Cartesian equation of the conic whose matrix equation is $V^tMV = K$.

   a) $\begin{bmatrix} 7 & 8 \\ 8 & 9 \end{bmatrix}$ d) $\begin{bmatrix} -2 & -4 \\ -4 & 3 \end{bmatrix}$

   b) $\begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}$ e) $\begin{bmatrix} 12 & 10 \\ 10 & 2 \end{bmatrix}$

   c) $\begin{bmatrix} 4 & -5 \\ -5 & 2 \end{bmatrix}$ f) $\begin{bmatrix} -2 & 9 \\ 9 & 1 \end{bmatrix}$

3. For each of the following conics $ax^2 + 2hxy + by^2 = k$, write the matrix M and the matrix K for its matrix equation $V^tMV = K$.

   a) $4x^2 + 6xy + 5y^2 = 3$
   b) $7x^2 - 8xy + 3y^2 = 1$
   c) $-2x^2 + y^2 = 9$
   d) $5x^2 - 3xy - 7y^2 = 8$
   e) $9x^2 - 11y^2 = -5$
   f) $x^2 - xy + 5y^2 = 3$

4. Write the matrix equation $V^tMV = K$ for each of the conics in question 3.

5. Write the transpose of each of the following matrices.

   a) $\begin{bmatrix} 7 & 8 \\ 8 & 9 \end{bmatrix}$ d) $\begin{bmatrix} -2 & -4 \\ -4 & 3 \end{bmatrix}$

   b) $\begin{bmatrix} 6 & 7 \\ 0 & -1 \end{bmatrix}$ e) $\begin{bmatrix} 12 & 0 \\ 10 & 2 \end{bmatrix}$

   c) $\begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix}$ f) $\begin{bmatrix} 1 & 4 & 3 \\ 7 & 1 & 2 \end{bmatrix}$

6. Given $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

   a) Calculate the matrix product AB.
   b) Write $(AB)^t$.
   c) Write the matrix A^t and the matrix B^t and then calculate B^tA^t.
   d) Use your results of parts b) and c) to verify that $(AB)^t = B^tA^t$.

7. Repeat the previous question using the following pairs of matrices

   a) $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix}$

   b) $A = \begin{bmatrix} -4 & 3 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}$

   c) $A = \begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 4 & 1 \end{bmatrix}$

8. For each of the following values of θ, write the rotation matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for the rotation of θ about (0,0). Find the value of each element, correct to 2 decimal places.

   a) 30° b) 20° c) 140° d) 90°

9. For each of the following values of θ, write the rotation matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for the rotation of θ about (0,0). Find the exact value of each element using the table on page 543.

   a) 30° b) 45° c) 60°

10. Suppose that R is the rotation matrix corresponding to a rotation of $\theta = 40°$ about the point (0,0), and $M = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$

    Calculate the matrix product RM. State the value of each element of RM, correct to 2 decimal places.

11. Repeat the previous question for the following values of M and θ.

    a) $M = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}$ and $\theta = 30°$

    b) $M = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ and $\theta = 60°$

12. Repeat question 11 but find exact values for each element of RM.

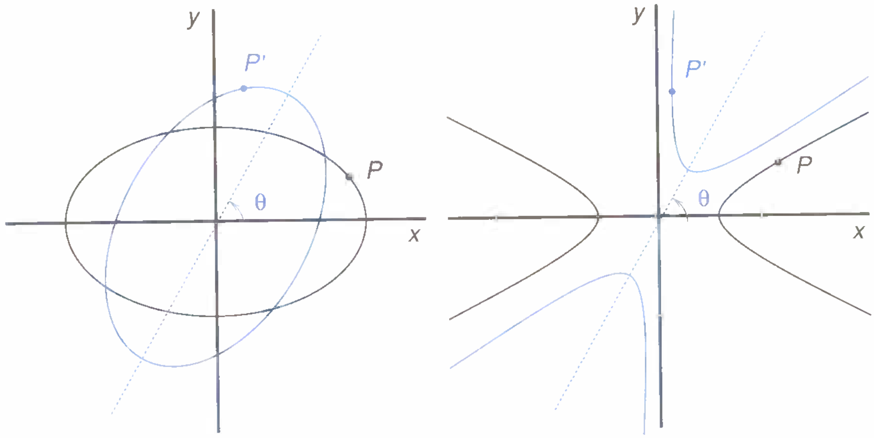
13. Given that R is the rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

    and I is the 2×2 unit matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

    a) Prove that $R^tR = RR^t = I$.
    b) Explain why you can conclude that $R^t = R^{-1}$.

8.5 Rotations of Central Conics

Suppose a central conic C (an ellipse or a hyperbola) in standard position, with its axes of symmetry along the x-axis and y-axis, is rotated through an angle θ about the origin (0,0) as in the figure.



Then a point P on the conic C maps into a point P' on the image conic C'.
If $\overrightarrow{OP} = V$ and $\overrightarrow{OP'} = U$, then under the rotation matrix R, $V \to U$, where

$RV = U$ ①

$R^t(RV) = R^tU$	(multiplying *on the left side* both sides by R^t)
$(R^tR)V = R^tU$	(matrix multiplication is associative)
Thus, $IV = R^tU$	(from section 8.4, $R^tR = I$)
Or, $V = R^tU$	(for any 2×2 matrix W, $WI = IW = W$)
Also, $V^t = (R^tU)^t$	(taking the transpose of the matrices on each side)
or $V^t = U^tR$	($(AB)^t = B^tA^t$, and the transpose of A^t is A)

Thus, under a rotation θ about the point (0,0),
V becomes R^tU, and V^t becomes U^tR.

Hence, upon applying the rotation transformation,
the matrix equation $V^tMV = K$ becomes
the matrix equation $(U^tR)\ M\ (R^tU) = K$ or $U^t(RMR^t)U = K$.

Once this equation has been obtained it is customary to replace U by V and U^t by V^t. Hence, you now have the following property.

PROPERTY

Under the rotation defined by the rotation matrix R
the conic $V^tMV = K$ maps into the conic $V^tM'V = K$,
where $M' = RMR^t$, and

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},\ V = \begin{bmatrix} x \\ y \end{bmatrix},\ \text{and } M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$$

Example 1 Given the ellipse E: $4x^2 + 9y^2 = 36$.

a) Write the corresponding matrix equation $V^tMV = K$.

b) Find the matrix equation for E', the image of E under a rotation of 20° about the point (0,0). Round off numbers to 1 decimal place.

c) Write the Cartesian equation for the image of E.

d) Sketch a graph of E and E'.

Solution

a) For the curve $ax^2 + 2hxy + by^2 = k$, $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$

For E: $4x^2 + 9y^2 = 36$, $a = 4$, $b = 9$ and $h = 0$. Thus, $M = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$

Therefore the matrix equation for E is

$$V^tMV = K \text{ or}$$

$$[x \quad y]\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [36]$$

b) The matrix equation for E' is $V^tM'V = K$, where $M' = RMR^t$. But the rotation angle about (0,0) is 20°, thus

the rotation matrix $R = \begin{bmatrix} \cos 20° & -\sin 20° \\ \sin 20° & \cos 20° \end{bmatrix}$ and,

$$R^t = \begin{bmatrix} \cos 20° & \sin 20° \\ -\sin 20° & \cos 20° \end{bmatrix}$$

Thus, $M' = RMR^t$

$$= \begin{bmatrix} \cos 20° & -\sin 20° \\ \sin 20° & \cos 20° \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} \cos 20° & \sin 20° \\ -\sin 20° & \cos 20° \end{bmatrix}$$

$$= \left(\begin{bmatrix} \cos 20° & -\sin 20° \\ \sin 20° & \cos 20° \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}\right)\begin{bmatrix} \cos 20° & \sin 20° \\ -\sin 20° & \cos 20° \end{bmatrix}$$

$$= \begin{bmatrix} 4\cos 20° & -9\sin 20° \\ 4\sin 20° & 9\cos 20° \end{bmatrix}\begin{bmatrix} \cos 20° & \sin 20° \\ -\sin 20° & \cos 20° \end{bmatrix}$$

$$= \begin{bmatrix} 4\cos^2 20° + 9\sin^2 20° & 4\cos 20° \sin 20° - 9\sin 20° \cos 20° \\ 4\sin 20° \cos 20° - 9\cos 20° \sin 20° & 4\sin^2 20° + 9\cos^2 20° \end{bmatrix}$$

$$= \begin{bmatrix} 4.6 & -1.6 \\ -1.6 & 8.4 \end{bmatrix} \text{ correct to 1 decimal place.}$$

Thus, the matrix equation for E' is

$$V^tM'V = K, \text{ or } [x \quad y]\begin{bmatrix} 4.6 & -1.6 \\ -1.6 & 8.4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [36].$$

c) For the image ellipse E',

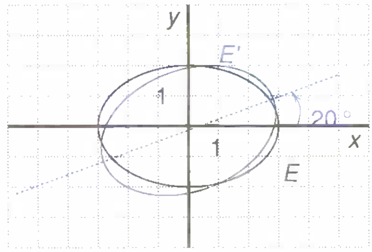
$M' = \begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix}$, thus $a' = 4.6$, $b' = 8.4$ and $h' = -1.6$

Since the equation of E' is $a'x^2 + 2h'xy + b'y^2 = k$, the Cartesian equation for the image ellipse E' is

$$4.6x^2 - 3.2xy + 8.4y^2 = 36$$

d) For $4x^2 + 9y^2 = 36$: the x-intercepts are 3 and −3, while the y-intercepts are 2 and −2. The graph of E is shown in the diagram.

The graph of E' is the graph of E rotated 20° counterclockwise about (0,0), shown on the same axes as that of E.



■

Note: Once you have had some practice using the method of Example 1, the equation of the rotated conic can be found simply by calculating the matrix $M' = RMR^t = \begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix}$ and then substituting into the image equation $a'x^2 + 2h'xy + b'y^2 = k$.

PROPERTY

Under a rotation of θ about the point (0,0) with rotation matrix R, the conic $ax^2 + 2hxy + by^2 = k$ maps into the conic $a'x^2 + 2h'xy + b'y^2 = k$, where $\begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix} = RMR^t$.

Example 2

The hyperbola H: $x^2 - 16y^2 = 5$ is rotated through an angle of 30°. Find an equation of H', the image of H after rotation. Use exact values for the sine and cosine of 30°.

Solution

The image H' has equation $a'x^2 + 2h'xy + b'y^2 = k$, where $\begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix} = RMR^t$ and $k = 5$.

For H: $x^2 - 16y^2 = 5$, so $a = 1$, $h = 0$, $b = -16$. Also $\theta = 30°$. Thus,

$M = \begin{bmatrix} 1 & 0 \\ 0 & -16 \end{bmatrix}$, $R = \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix}$ and $R^t = \begin{bmatrix} \cos 30° & \sin 30° \\ -\sin 30° & \cos 30° \end{bmatrix}$

Thus $M' = \left(\begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -16 \end{bmatrix}\right)\begin{bmatrix} \cos 30° & \sin 30° \\ -\sin 30° & \cos 30° \end{bmatrix}$

$= \begin{bmatrix} \cos 30° & 16 \sin 30° \\ \sin 30° & -16 \cos 30° \end{bmatrix}\begin{bmatrix} \cos 30° & \sin 30° \\ -\sin 30° & \cos 30° \end{bmatrix}$

$= \begin{bmatrix} \cos^2 30° - 16 \sin^2 30° & \cos 30° \sin 30° + 16 \sin 30° \cos 30° \\ \sin 30° \cos 30° + 16 \cos 30° \sin 30° & \sin^2 30° - 16 \cos^2 30° \end{bmatrix}$

$= \begin{bmatrix} -\frac{13}{4} & \frac{17\sqrt{3}}{4} \\ \frac{17\sqrt{3}}{4} & -\frac{47}{4} \end{bmatrix}$ using the table on page 543

Thus, $a' = -\frac{13}{4}$, $b' = -\frac{47}{4}$ and $h' = \frac{17\sqrt{3}}{4}$

Hence, the equation of H' is $-\frac{13}{4}x^2 + \frac{34\sqrt{3}}{4}xy - \frac{47}{4}y^2 = 5$ or

$13x^2 - 34\sqrt{3}xy + 47y^2 = -20$. ■

In Examples 1 and 2 above, examine $ab - h^2$, the value of the determinant M, for the original curves E and H. Compare these values with the value of the determinant of $M' = a'b' - h'^2$ for the image curves E' and H'.

for E: $ab - h^2 = (4)(9) - 0^2 = 36$;
for E': $a'b' - h'^2 = (4.6)(8.4) - (\frac{3.2}{2})^2 \doteq 36$
Thus $ab - h^2$ and $a'b' - h'^2$ are *both positive in value.*
for H: $ab - h^2 = (1)(-16) - 0^2 = -16$;
for H': $a'b' - h'^2 = (-\frac{13}{4})(-\frac{47}{4}) - (\frac{17\sqrt{3}}{4})^2 = -\frac{256}{16} = -16$.
Thus $ab - h^2 = a'b' - h'^2$, and *this value is negative.*
This demonstrates, but does not prove, that the value, and hence the sign, of $ab - h^2$ is invariant for rotations. For an ellipse, $ab - h^2 > 0$. For a hyperbola, $ab - h^2 < 0$.

Even though the value of $ab - h^2$ is invariant (provided neither conic equation has been multiplied by a constant), all that you need to remember is that the *sign of $ab - h^2$ does not change under a rotation.*

In 8.5 Exercises you will be given the opportunity to check that the sign of $ab - h^2$ is invariant under a rotation. You should observe that this invariant sign is the same as for translations, that is, the sign of $ab - h^2$ when $h = 0$.

In Search of a Proof that $ab - h^2$ is Invariant Under a Rotation

Consider the conic $ax^2 + 2hxy + by^2 = k$. This conic has the matrix equation $V^tMV = K$, where $V = \begin{bmatrix} x \\ y \end{bmatrix}$, $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$, and $K = [k]$.
The determinant of matrix M is $ab - h^2$, that is $\det(M) = ab - h^2$.
Under a rotation of θ about the point (0,0), this conic maps into the conic $V^tM'V = K$ where $M' = RMR^t$.
If this new conic has equation $a'x^2 + 2h'xy + b'y^2 = k$, then the matrix $M' = RMR^t$ would equal $\begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix}$. Hence, $\det(M') = a'b' - h'^2$.
To prove that the value, and therefore the sign, of $ab - h^2$ is invariant, you need to prove that $\det(M) = \det(M')$

$$\text{Now } \det(R) = \det\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
$$= \cos^2\theta - (-\sin^2\theta) = \cos^2\theta + \sin^2\theta = 1 \text{ (see page 542)}$$

Similarly, $\det(R^t) = 1$
From chapter 7, for matrices A, B, and C and the matrix product ABC, $\det(ABC) = \det(A) \times \det(B) \times \det(C)$.

$$\text{Thus, } \det(M') = \det(RMR^t) = \det(R) \times \det(M) \times \det(R^t)$$
$$= 1 \times \det(M) \times 1$$

Thus, $\det(M') = \det(M)$.
Thus the value of $ab - h^2$, and hence its sign, is invariant under a rotation.

8.5 Exercises

Round off all numbers to 1 decimal place where appropriate.

1. Given the ellipse E: $x^2 + 4y^2 = 4$.
   a) Write the corresponding matrix equation $V^tM'V = K$.
   b) Use the fact that $M' = RMR^t$ to find the matrix equation $V^tMV = K$, for E', the image of E under a rotation of 30° about the point (0,0).
   c) Write a Cartesian equation for the image of E.
   d) Sketch a graph of E and E'.

2. a) The ellipse E: $4x^2 + y^2 = 16$ is rotated through an angle of 45° about the point (0,0). Find an equation of E', the image of E after rotation.
   b) Sketch a graph of E and E'.

3. Repeat question 2 for the following ellipses and rotation angles θ.
   a) $x^2 + 9y^2 = 9$, $\theta = 120°$
   b) $4x^2 + 25y^2 = 100$, $\theta = 250°$
   c) $25x^2 + 9y^2 = 200$, $\theta = -40°$
   d) $3x^2 + y^2 = 4$, $\theta = 60°$

4. For each of the ellipses in questions 2 and 3 check to see that the value of $ab - h^2$ is invariant under the given rotation and that its sign is positive.

5. Repeat question 2 for the following ellipses and rotation angles θ. Use exact values for the sines and cosines of θ, from the table on page 543.
   a) $4x^2 + y^2 = 16$, $\theta = 45°$
   b) $9x^2 + y^2 = 9$, $\theta = 120°$
   c) $3x^2 + y^2 = 4$, $\theta = 60°$

6. a) The hyperbola H: $9x^2 - y^2 = 9$ is rotated through an angle of 30° about the point (0,0). Find an equation of H', the image of H after rotation.
   b) Sketch a graph of H and H'.

7. Repeat question 6 for the following hyperbolas and rotation angles θ.
   a) $x^2 - 4y^2 = -4$, $\theta = 30°$
   b) $16x^2 - 25y^2 = 400$, $\theta = 120°$
   c) $x^2 - y^2 = -1$, $\theta = -40°$
   d) $x^2 - 3y^2 = -3$, $\theta = 60°$

8. For each of the hyperbolas in questions 6 and 7 check to see that the value of $ab - h^2$ is invariant under the given rotation and that its sign is negative.

9. Repeat question 6 for the following hyperbolas and rotation angles θ. Use exact values for the sines and cosines of θ, from the table on page 543.
   a) $9x^2 - y^2 = 9$, $\theta = 30°$
   b) $16x^2 - 25y^2 = 400$, $\theta = 120°$
   c) $x^2 - 3y^2 = -3$, $\theta = 60°$.

10. Find an equation of each of the following after rotation of 90° about the point (0,0).
    a) $4x^2 + 9y^2 = 36$
    b) $x^2 - 9y^2 = 9$
    c) $4x^2 + y^2 = 16$

11. a) A rotation of 90° about (0,0) is equivalent to interchanging the x-axis and the y-axis. Explain.
    b) Make use of part a) to write the image equations for the previous question, *without making any calculations.*

12. Given the curve C: $17x^2 + 16xy + 17y^2 = 225$ and the rotation of 45° about the point (0,0).
    a) Name the type of curve.
    b) Find an equation of C', the image of C under the given rotation.
    c) Sketch the *image* curve C'. Then sketch the given curve by using the inverse rotation to the given rotation.

13. Repeat the previous question for the curve $x^2 + 3.5xy - y^2 = 1$ and a rotation of 60°.

14. The method of this section can not be applied to a parabola in standard position. Why not?

Earth Satellites

Each flight of a satellite about the earth is achieved by launching a rocket vertically. Eventually the trajetory of the rocket must be tilted. This is necessary so that flight of the rocket is parallel to the surface of the earth at the time that the orbital velocity at the desired altitude is reached. The space vehicle attached to the final stage of the rocket is then in free fall about the earth.

Communication satellites and meterological satellites work best if they remain fixed above one place on the surface of the earth. This will occur when the time for the satellite to move about the earth is equal to the time for the earth to make one complete rotation about its axis. In this case the satellite appears to be stationary in the sky. Such an orbit is called *geostationary* or *geosynchronous*.

What is the height h km above the earth of a satellite in a geosynchronous orbit?

Newton's first law of motion indicates that a body will continue in its state of rest or uniform motion in a straight line unless acted upon by some external force. Thus a rocket launched in a straight line would continue forever along that line, unless some force caused it to change its direction. The force that is continually changing the direction of the rocket and the satellite, causing them to make a circular orbit about the earth, is the gravitational pull of the earth on the rocket and satellite.

Now the force required continually to change the direction of a satellite of mass m (causing the satellite to move in a circular orbit) is called *centripetal* force. For a satellite moving along a circle of radius r with a tangential velocity v, this force equals $\frac{mv^2}{r}$.

But Newton showed that the gravitational pull of the earth that provides this centripetal force equals $\frac{kmE}{r^2}$

where E is the mass of the earth, r is the distance between the centre of the earth and the centre of the satellite, k is a constant number whose value depends only on the units chosen for E, r and m.

Thus, centripetal force = gravitational force, or $\frac{mv^2}{r} = \frac{kmE}{r^2}$, or $v^2r = kE$.

Since k and E do not change, v^2r = constant.

This formula is true for all earth satellites. Since the value of v and r are known for the moon, which is an earth satellite, the value of this constant can be calculated.

Now the moon rotates about the earth every 27.322 days, along a circle of approximate radius 3.844×10^5 km.

Thus the tangential velocity of the satellite

$$= \frac{\text{distance travelled}}{\text{time taken to travel this distance}}$$

$$= \frac{\text{circumference of the orbit of the moon}}{\text{time for one orbit of the moon}}$$

$$= \frac{2\pi \times 3.844 \times 10^5}{27.322 \times 24} \text{ km/h or } 3.683\ 32 \times 10^3 \text{ km/h}$$

Thus, for the moon, $v^2r = (3.683\ 32 \times 10^3)^2 \times 3.844 \times 10^5$
$= 5.2151 \times 10^{12}$

Thus, for any earth satellite, and in particular, a geosynchronous satellite, $v^2r = 5.2151 \times 10^{12}$ ①

But for a geosynchronous satellite the speed is

$$v = \frac{\text{circumference of orbit}}{24} \text{ km/h or } \frac{2\pi r}{24} \text{ km/h}$$

For a satellite at a height h km above the earth, the distance from the centre of the satellite to the centre of the earth equals
r = the mean radius of the earth + h, so $r = 6371 + h$.

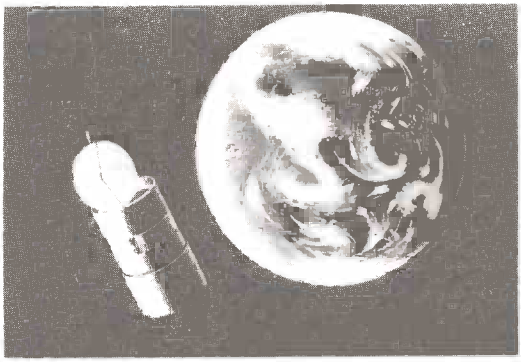
Substituting into ① gives

$$\left[\frac{2\pi(6731 + h)}{24}\right]^2 \times (6371 + h) = 5.2151 \times 10^{12} \text{ or}$$

$$6731 + h = \left(\frac{5.2151 \times 10^{12} \times 24^2}{(2\pi)^2}\right)^{\frac{1}{3}}$$

Thus, $6371 + h = 42\ 375$, or $h = 36\ 004$ km.

Thus, a geosynchronous satellite must be about 36 000 km above the surface of the earth.



8.6 Rotations that Eliminate *xy* Terms

In section 8.5 and 8.5 Exercises you saw that the rotation of an ellipse or hyperbola in standard position changed the equation from the form $ax^2 + by^2 = k$ to $a'x^2 + 2h'xy + b'y^2 = k$ where $h' \neq 0$, thus introducing a term in xy.

In this section you will determine the rotation that you must apply to $a'x^2 + 2h'xy + b'y^2 = k$ to return the equation to the form $ax^2 + by^2 = k$, that is, the form $ax^2 + 2hxy + by^2 = k$, where $h = 0$.

If the conic C' with equation $a'x^2 + 2h'xy + b'y^2 = k$, with $h' \neq 0$, maps into the conic C with equation $ax^2 + 2hxy + by^2 = k$, with $h = 0$, then the matrix equation $V^tM'V = K$ where $M' = \begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix}$ with $h' \neq 0$ must become the matrix equation $V^tMV = K$ where $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ with $h = 0$.

If C' maps into C under the rotation defined by matrix R, then C will have matrix equation $V^tMV = K$ where $M = RM'R^t$.

To determine the rotation that will eliminate the xy term in $ax^2 + 2hxy + by^2 = k$ you only need find the element h in the first row and second column of M and decide what value of θ will make that term zero.

$$\text{Now } M = RM'R^t = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a'\cos\theta - h'\sin\theta & h'\cos\theta - b'\sin\theta \\ a'\sin\theta + h'\cos\theta & h'\sin\theta + b'\cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\text{must become } \begin{bmatrix} a & h \\ h & b \end{bmatrix}, \text{ where } h = 0.$$

The element h in the first row and second column of this product is the dot product of the elements in the first row of the left hand matrix with the elements in the second column of the right hand matrix.

$$\begin{aligned} \text{That is, } h &= (a'\cos\theta - h'\sin\theta,\ h'\cos\theta - b'\sin\theta)\cdot(\sin\theta, \cos\theta) \\ &= a'\cos\theta\sin\theta - h'\sin^2\theta + h'\cos^2\theta - b'\sin\theta\cos\theta \\ &= (a' - b')\sin\theta\cos\theta + h'(\cos^2\theta - \sin^2\theta) \end{aligned}$$

But $\sin\theta\cos\theta = \dfrac{\sin 2\theta}{2}$, and $\cos^2\theta - \sin^2\theta = \cos 2\theta$

Thus, $h = (a' - b')\dfrac{\sin 2\theta}{2} + h'\cos 2\theta$

$h = 0$ means that

$(a' - b')\dfrac{\sin 2\theta}{2} + h'\cos 2\theta = 0$, that is, $(a' - b')\dfrac{\sin 2\theta}{2} = -h'\cos 2\theta$

Thus $\dfrac{\sin 2\theta}{\cos 2\theta} = \dfrac{-2h'}{a' - b'}$

or $\tan 2\theta = \dfrac{2h'}{b' - a'}$ $\qquad \tan\phi = \dfrac{\sin\phi}{\cos\phi}$ and $(a' - b') = -(b' - a')$

This formula assumes that $b' \neq a'$. If $b' = a'$, then $\tan 2\theta$ is undefined, so that 2θ is 90° or 270° (or any angle having the same initial arm and terminal arm with either one). This means that θ is 45° or 135°.
In using this formula it is customary to ignore the primes on a', b' and h', obtaining the following.

FORMULA

To remove the xy term from the conic equation $ax^2 + 2hxy + by^2 = k$, rotate the conic through an angle θ about the point (0,0) where

if $a \neq b$, then $\tan 2\theta = \dfrac{2h}{b-a}$ and if $a = b$, then $\theta = 45°$ or $135°$.

When $a \neq b$, θ can be found using your calculator by keying $\dfrac{2h}{b-a}$ then using the inverse tangent function key. This keying will give 2θ, so you must divide this number by 2. If $\dfrac{2h}{b-a} > 0$, then $\theta > 0$. If $\dfrac{2h}{b-a} < 0$, then $\theta < 0$.
These are not the only rotation angles that will eliminate the xy term. Any one of the infinite number of solutions of $\tan 2\theta = \dfrac{2h}{b-a}$ can be divided by 2 to give a suitable angle of rotation. In particular, if your calculator gives you $m°$ for 2θ, another value for 2θ is $180° + m°$.
One rotation angle brings the major or transverse axis to the x-axis while the other brings it to the y-axis.

Example 1 Determine, correct to 1 decimal place, an angle of rotation about the point (0,0) needed to remove the xy term from each of the following.

a) $9x^2 - 4xy + 6y^2 = 35$ **b)** $x^2 + xy + y^2 = 6$

Solution **a)** For the conic $9x^2 - 4xy + 6y^2 = 35$, $a = 9$, $h = \dfrac{-4}{2} = -2$, $b = 6$.

Thus, $\tan 2\theta = \dfrac{2h}{b-a} = \dfrac{2(-2)}{6-9} = \dfrac{4}{3}$
Using your calculator you obtain $2\theta = 53.1301°$.
Hence, $\theta = 26.565° = 26.6°$, correct to 1 decimal place. Hence, an angle of rotation of 26.6° about the point (0,0) will eliminate the xy term.

b) For the conic $x^2 + xy + y^2 = 6$, $a = 1$, $h = \dfrac{1}{2}$, $b = 1$.

Since $a = b$, a rotation about the point (0,0) of $\theta = 45°$ or $\theta = 135°$ will eliminate the xy term. ■

Note: In part a), 2θ could be $180° + 53.1301° = 233.1301°$. Thus another angle of rotation would be 116.6°.

Before you eliminate an xy term it will be helpful to know the form of the standard equation of the conic to help you to detect errors in your calculations. Hence, you should make use of the following information that you verified in section 8.5 and in 8.5 Exercises.

The Graph of $ax^2 + 2hxy + by^2 = k$

ellipse: $ab - h^2 > 0$ hyperbola: $ab - h^2 < 0$

Example 2 Given the conic $C: -5x^2 + 8xy + y^2 = 21$.

a) Determine the type of conic.

b) Find an angle of rotation about the point (0,0) to eliminate the xy term.

c) Find an equation of the image curve C'.

d) Sketch the graph of the image curve C', then of the original curve C.

Solution

a) Here $a = -5$, $h = \frac{8}{2} = 4$, and $b = 1$.

Thus, $ab - h^2 = (-5)(1) - 16 = -21 < 0$. The conic is a hyperbola.

b) Since $a \neq b$, the angle of rotation about (0,0) that will eliminate the xy term is a solution of $\tan 2\theta = \frac{2h}{b-a} = \frac{(2)(4)}{1-(-5)} = \frac{8}{6}$ or $\frac{4}{3}$

Using your calculator, $2\theta = 53.130\ 102\ldots^\circ$ Thus, $\theta = 26.565\ 051\ldots^\circ$

c) The image C' has equation $a'x^2 + 2h'xy + b'y^2 = k$ where

$$M' = \begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix} = RMR^t, \text{ and } k = 21$$

But here, $M = \begin{bmatrix} -5 & 4 \\ 4 & 1 \end{bmatrix}$ and $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Thus $M' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} -5 & 4 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} -5\cos\theta - 4\sin\theta & 4\cos\theta - \sin\theta \\ -5\sin\theta + 4\cos\theta & 4\sin\theta + \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} -5\cos^2\theta - 8\sin\theta\cos\theta + \sin^2\theta & -4\sin^2\theta + 4\cos^2\theta - 6\sin\theta\cos\theta \\ -4\sin^2\theta + 4\cos^2\theta - 6\sin\theta\cos\theta & -5\sin^2\theta + 8\sin\theta\cos\theta + \cos^2\theta \end{bmatrix}$$

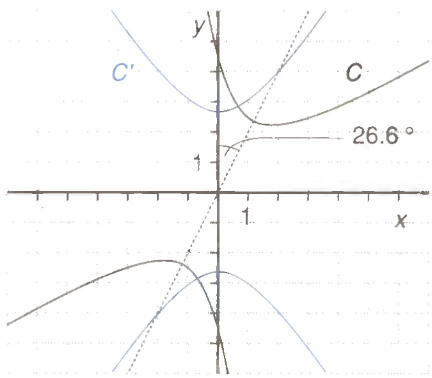
$$= \begin{bmatrix} -7 & 0 \\ 0 & 3 \end{bmatrix}$$

use the memory key for θ

Hence the equation of C' is $-7x^2 + 3y^2 = 21$ or $\frac{x^2}{3} - \frac{y^2}{7} = -1$.

d) The image curve C' is a hyperbola with centre at (0,0) that intersects the y-axis at $(0, \sqrt{7})$ and $(0, -\sqrt{7})$.

Since C maps into C' under a *counterclockwise* rotation of approximately 26.6° about (0,0), C' will map back into C under a *clockwise* rotation of approximately 26.6° about (0,0).



Note: The angle $\theta = 26.6^\circ$ rotates the transverse axis of C onto the y-axis. The other solution, $2\theta = 180^\circ + 53.13 = 233.13^\circ$ gives $\theta \doteq 116.6^\circ$. This rotation maps the transverse axis of C onto the x-axis. ■

In Search of a Method of Eliminating the *xy* Term using Characteristic Values

On page 333, when looking for invariant lines, you learned about characteristic values of a matrix. Recall that the characteristic values k of the matrix M are obtained by solving $\det(M - kI) = 0$.

You also proved the following facts in the activities on page 335.

1. A symmetric matrix always has real characteristic values.
2. The characteristic vectors corresponding to each characteristic value of a symmetric matrix are orthogonal.

Using these facts, you will now prove a theorem about diagonalizing a symmetric matrix, that is, changing the elements of the matrix, so that all *non-zero* elements are along the leading diagonal. An application of this theorem will allow you to eliminate the xy term in a quadratic form, remarkably quickly.

THEOREM

Given the symmetric matrix $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ that has characteristic values c and d.

Prove that this matrix can be diagonalized as $\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$

Proof: Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be a unit characteristic vector associated with the value c. (That is, $\vec{u}$ is such that $M\vec{u} = c\vec{u}$.)

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be a unit characteristic vector associated with the value d. (That is, $\vec{v}$ is such that $M\vec{v} = d\vec{v}$.)

Let $H = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$. Then $H^t = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$.

Matrix M can be diagonalised by computing H^tMH.

You will show that $H^tMH = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$

First note that $M\vec{u} = c\vec{u} \Rightarrow \begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$$\Rightarrow au_1 + hu_2 = cu_1 \quad ①$$
$$\text{and } hu_1 + bu_2 = cu_2 \quad ②$$

Also, $M\vec{v} = d\vec{v}$, so $\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = d\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$\Rightarrow av_1 + hv_2 = dv_1 \quad ③$$
$$\text{and } hv_1 + bv_2 = dv_2 \quad ④$$

$$\text{Now } H^tMH = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

$$= \begin{bmatrix} au_1 + hu_2 & hu_1 + bu_2 \\ av_1 + hv_2 & hv_1 + bv_2 \end{bmatrix}\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

$$= \begin{bmatrix} cu_1 & cu_2 \\ dv_1 & dv_2 \end{bmatrix}\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \qquad \text{using ①, ②, ③ and ④}$$

$$= \begin{bmatrix} cu_1^2 + cu_2^2 & cu_1v_1 + cu_2v_2 \\ du_1v_1 + du_2v_2 & dv_1^2 + dv_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} c|\vec{u}|^2 & c(\vec{u}\cdot\vec{v}) \\ d(\vec{u}\cdot\vec{v}) & d|\vec{v}|^2 \end{bmatrix}$$

$$= \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$$

since $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors (because M is symmetric).

Application to the Rotation of a Conic

In section 8.6 you learned that the conic $ax^2 + 2hxy + by^2 = k$ can have its xy term eliminated ($h = 0$) under a rotation of θ about the origin O, where

$$\tan 2\theta = \frac{2h}{b - a}$$

If the equation of the rotated conic is $cx^2 + dy^2 = k$, then

the matrix RMR^t equals $\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$,

where R is the rotation matrix for a rotation θ about the origin O.

But from the theorem, $H^tMH = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$

This implies that the matrix H and the matrix R^t are the same matrix.

$$\text{Thus, } H = R^t = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

(since $\sin(-\theta) = -\sin\theta$, and $\cos(-\theta) = \cos\theta$)

Hence H corresponds to a rotation of $-\theta$ about the origin O, where

$$\tan 2\theta = \frac{2h}{b - a}$$

Example

a) Eliminate the xy term from $41x^2 - 24xy + 34y^2 = 100$, and describe the conic.

b) Find the rotation that maps the conic to its new position.

Solution

a) The defining matrix is $M = \begin{bmatrix} 41 & -12 \\ -12 & 34 \end{bmatrix}$

Thus the characteristic values are the solutions of $\begin{vmatrix} 41 - c & -12 \\ -12 & 34 - c \end{vmatrix} = 0$

$$(41 - c)(34 - c) - (-12)^2 = 0$$
$$c^2 - 75c + 1250 = 0$$
$$(c - 25)(c - 50) = 0$$
$$c = 25 \text{ or } c = 50.$$

Hence the conic can be rotated so that its equation without the xy term is exactly $25x^2 + 50y^2 = 100$

or $x^2 + 2y^2 = 4$

The rotated conic is an ellipse which intersects the axes at $(\pm 2,0)$, $(0,\pm\sqrt{2})$.

b) To find the rotation, you need unit characteristic vectors of the matrix M.

$$M\vec{u} = 25\vec{u} \Rightarrow \begin{bmatrix} 41 & -12 \\ -12 & 34 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 25\begin{bmatrix} x \\ y \end{bmatrix}$$
$$\Rightarrow \; 41x - 12y = 25x$$
$$\text{and } -12x + 34y = 25y$$

that is, $4x - 3y = 0$ so $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\vec{e_u} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$

$$M\vec{v} = 50\vec{v} \Rightarrow \begin{bmatrix} 41 & -12 \\ -12 & 34 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 50\begin{bmatrix} x \\ y \end{bmatrix}$$
$$\Rightarrow \; 41x - 12y = 50x$$
$$\text{and } -12x + 34y = 50y,$$

that is, $3x + 4y = 0$ so $\vec{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ and $\vec{e_v} = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$

The columns of the rotation matrix R are $\vec{e_u}$ and $\vec{e_v}$, that is $R = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$.

(Notice that if the columns were interchanged, you would not have a rotation matrix.)

This is the matrix of a rotation through $-\theta$,

where $\sin(-\theta) = \frac{4}{5}$ and $\cos(-\theta) = \frac{3}{5}$

Therefore $-\theta \doteq 53.1°$, so that $\theta \doteq -53.1°$ ■

Note: Using the formula $\tan 2\theta = \frac{2h}{b-a} = \frac{-24}{-7}$ gives $\theta \doteq 36.9°$ or $\theta \doteq -53.1°$.

$-53.1°$ rotates the major axis of the ellipse onto the x-axis, while $36.9°$ rotates the major axis onto the y-axis.

8.6 Exercises

Round off all numbers to 1 decimal place where appropriate.

1. Given the conic $x^2 - 2xy + 3y^2 = 1$.
- **a)** Name the type of conic.
- **b)** Determine an angle of rotation about (0,0) needed to remove the xy term.

2. Repeat question 1 for the following conics.
- **a)** $4x^2 - 3xy - 2y^2 = 8$
- **b)** $x^2 - xy + y^2 = 1$
- **c)** $3x^2 - 4xy + 4y^2 = 3$
- **d)** $2x^2 + 9xy + y^2 = 4$
- **e)** $2x^2 - 2\sqrt{6}xy + y^2 = 4$
- **f)** $4x^2 + 3xy + 5y^2 = 8$

3. Given the conic C: $5x^2 + 6xy + 5y^2 = 8$.
- **a)** Determine the type of conic.
- **b)** Find an angle of rotation about (0,0) that will eliminate the xy term.
- **c)** Find an equation of the image curve C'.
- **d)** Sketch the graph of the image curve C', then the graph of C, the original curve.

4. Repeat question 3 for the following conics.
- **a)** $23x^2 - 16xy + 42y^2 = 180$
- **b)** $1175x^2 - 1472xy + 325y^2 = -500$
- **c)** $170x^2 - 1970xy - 170y^2 = -1000$
- **d)** $225x^2 - 135xy + 65y^2 = 1000$
- **e)** $200x^2 - 346xy = 300$
- **f)** $184x^2 - 158xy + 156y^2 = 2000$

5. Given the conic C: $17x^2 + 16xy + 17y^2 = 225$.
- **a)** Determine the type of conic.
- **b)** Find an angle of rotation about (0,0) that will eliminate the xy term.
- **c)** Using exact values for sines and cosines from the table on page 543, find an equation of the image curve C'.
- **d)** Sketch the graph of the image curve C', then the graph of C, the original curve.

6. Repeat question 5 for the following conics.
- **a)** $x^2 + 2\sqrt{3}xy - y^2 = -2$
- **b)** $13x^2 + 10xy + 13y^2 = 8$
- **c)** $5x^2 + 2\sqrt{3}xy + 3y^2 = 36$

7. In section 8.6 you showed that the conic $a'x^2 + 2h'xy + b'y^2 = k$ maps into the conic $ax^2 + 2hxy + by^2 = k$, with $h = 0$, under a rotation θ about (0,0) where

$$h = (a' - b')\frac{\sin 2\theta}{2} + h' \cos 2\theta.$$

- **a)** Show that

$$a = \left(\frac{a' + b'}{2}\right) + \left(\frac{a' - b'}{2}\right) \cos 2\theta - h' \sin 2\theta$$

and that

$$b = \left(\frac{a' + b'}{2}\right) - \left(\frac{a' - b'}{2}\right) \cos 2\theta + h' \sin 2\theta$$

- **b)** Use your results of part a) to verify that $a'b' - h'^2 = ab - h^2$.

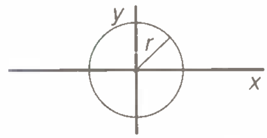
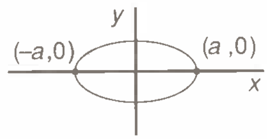
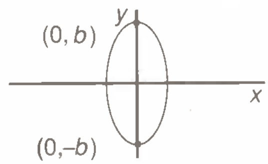
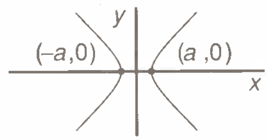
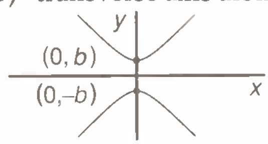
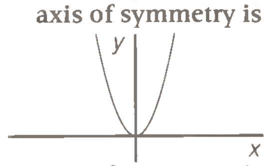
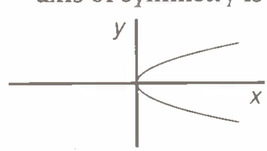
8. Given the conic C:
$130x^2 + 100xy + 130y^2 + 481x + 28y = 0$.
- **a)** Determine the type of conic.
- **b)** Find an angle of rotation about (0,0) that will eliminate the xy term.
- **c)** Find an equation of the image curve C' under this rotation.
- **d)** Translate C' so that the equation of the image C'' of C' is in standard form.
- **e)** Sketch the graph of the final image curve C''.
- **f)** Use the inverse translation to that in part d) to graph C'.
- **g)** Use the inverse rotation to that in part b) to graph C.

9. Repeat the previous question for the following conic.
$5x^2 - 6xy + 5y^2 + 5.7x - 17y + 8 = 0$

10. Repeat question 8 for the conic
$5x^2 - 6xy + 5y^2 + 4\sqrt{2}x - 12\sqrt{2}y + 8 = 0$,
using the exact values for the sines and cosines from the table on page 543.

11. Write a computer program to transform a conic C: $ax^2 + 2hxy + by^2 = k$ into the standard conic C': $a'x^2 + b'y^2 = k$. You may wish to use the results of question 7.

Summary

conic	equation	centre	vertices	graph
circle	$x^2 + y^2 = r^2$	(0,0)	none	radius = r, $r > 0$ 
ellipse	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $a > b > 0$	(0,0)	$(a,0)(-a,0)$	major axis along x-axis 
	$b > a > 0$	(0,0)	$(0,b)(0,-b)$	major axis along y-axis 
hyperbola	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ $a > 0, b > 0$	(0,0)	$(a,0)(-a,0)$	transverse axis along x-axis 
	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$ $a > 0, b > 0$	(0,0)	$(0,b)(0,-b)$	transverse axis along y-axis 
parabola	$y = kx^2$ $k > 0$: opens up $k < 0$: opens down	none	(0,0)	axis of symmetry is y-axis 
	$x = ky^2$ $k > 0$: opens right $k < 0$: opens left	none	(0,0)	axis of symmetry is x-axis 

- The circles, ellipses, and hyperbolas with equations given in the chart have their centres at the origin (0,0) and their major axis or transverse axis along the x-axis or y-axis. Also, the parabolas with equations given in the chart have their vertices at the origin (0,0) and their axes of symmetry along the y-axis or x-axis. Such conics are said to be in *standard position*, and their equations in *standard form*.
- An equation written $ax^2 + by^2 + 2gx + 2fy + c = 0$ is in *general form*.
- Any vector $\vec{a} = \overrightarrow{(h,k)}$ defines the *translation* that maps each point P with coordinates (x,y) into the point P' with coordinates $(x + h, y + k)$, that is, point $P(x,y) \to$ point $P'(x + h, y + k)$.

The Graph of $ax^2 + by^2 + 2gx + 2fy + c = 0$

ellipse	$ab > 0,\ a \neq b$
circle	$ab > 0,\ a = b$
hyperbola	$ab < 0$
parabola	$ab = 0$

- A conic that has an equation $ax^2 + by^2 + 2gx + 2fy + c = 0$ with at least one of g and f not zero is not in standard position. To translate the conic into standard position you should complete the square of the terms in x and y.
- The expression $x^2 + mx$ becomes a perfect square by the addition of $\left(\frac{m}{2}\right)^2$, then $x^2 + mx + \left(\frac{m}{2}\right)^2 = \left(x + \frac{m}{2}\right)^2$
- The *transpose* A^t of an $m \times n$ matrix A is the $n \times m$ matrix formed by interchanging the elements in rows and columns, that is, $A = [a_{ij}]_{m \times n} \Rightarrow A^t = [a_{ji}]_{n \times m}$
- If A and B are matrices such that AB exists, then $(AB)^t = B^tA^t$.
- The conic $ax^2 + 2hxy + by^2 = k$ has matrix equation $V^tMV = K$ where $V = \begin{bmatrix} x \\ y \end{bmatrix}$, $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ and $K = [k]$.
- Under the rotation defined by the rotation matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ the conic $V^tMV = K$ maps into the conic $V^t(M')V = K$, where $M' = RMR^t$.
- Under a rotation of θ about the point (0,0) with rotation matrix R, the conic $ax^2 + 2hxy + by^2 = k$ having $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ maps into the conic $a'x^2 + 2h'xy + b'y^2 = k$ where $\begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix} = M' = RMR^t$.
- If R is a rotation matrix and I is the 2×2 unit matrix, then $RR^t = R^tR = I$.

The Graph of $ax^2 + 2hxy + by^2 = k$

ellipse: $ab - h^2 > 0$ hyperbola: $ab - h^2 < 0$

- To remove the xy term from the conic equation $ax^2 + 2hxy + by^2 = k$, rotate the conic through an angle θ about the point (0,0) where if $a \neq b$, $\tan 2\theta = \dfrac{2h}{b - a}$ and if $a = b$, $\theta = 45^\circ$ or 135°.

Inventory

1. The conic $x^2 + y^2 = 16$ is a(n) _____ with _____ at (0,0) and _____ equal to 4.

2. The conic $9x^2 + 4y^2 = 36$ is a(n) _____ with _____ at (0,0) and x-intercepts _____, and y-intercepts _____.

3. The conic $9x^2 - 4y^2 = 36$ is a(n) _____ with _____ at (0,0) and x-intercepts _____, and y-intercepts _____.

4. The conic $9x^2 - 4y^2 = -36$ is a(n) _____ with _____ at (0,0) and x-intercepts _____, and y-intercepts _____.

5. The conic $9x^2 = y$ is a(n) _____ with _____ at (0,0) and axis of symmetry along _____.

6. The conic $4y^2 = x$ is a(n) _____ with _____ at (0,0) and axis of symmetry along _____.

7. To transform the conic with equation $4x^2 + 9y^2 + 6x + 10y + 1 = 0$ into standard position you must perform a transformation that is a _____. This transformation can be obtained by completing the _____ of the terms $4x^2 +$ _____ and also of the terms _____ $+ 10y$.

8. If $x^2 + 6x + w = (x + p)^2$, then $p =$ _____ and $w =$ _____.

9. If the conic C maps into the conic C' under the translation $(x,y) \to (x + 2, y + 3)$, then the conic C' maps back into the conic C under the translation $(x,y) \to$ _____.

10. If the conic $ax^2 + 2hxy + by^2 = k$ is an ellipse, then _____ is _____ 0. If the conic is a hyperbola, then _____ is _____ 0.

11. To transform the conic with equation $4x^2 + 6xy + 5y^2 = 10$ into standard position you must perform a transformation that is a _____. This transformation will eliminate the _____ term.

12. A rotation of θ about (0,0) that will eliminate the xy term from the equation $ax^2 + 2hxy + by^2 = k$ is given by the formula tan_____ = _____, if a_____b. If a_____b, then $\theta = 45°$ or _____°.

13. The rotation matrix for a rotation of θ about the point (0,0) is $R =$ _____. The transpose of R, that is $R^t =$ _____.

14. For the conic $4x^2 + 6xy + 5y^2 = 7$, $a =$ _____, $b =$ _____, and h _____. The matrix equation is $V^tMV = K$ where V_____, M_____, and K_____.

15. Under a rotation of 20° about the point (0,0) the conic $5x^2 + 6xy + 7y^2 = 2$ maps into the conic $a'x^2 + 2h'xy + b'y^2 = k$ where $\begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix} = RMR^t$. For this conic, $R =$ _____, and $M =$ _____.

Review Exercises

Round off all numbers to 1 decimal place where appropriate.

1. Sketch the graph of each of the following conics.
   a) $x^2 + y^2 = 4$
   b) $4x^2 - 25y^2 = -100$
   c) $9x^2 - y^2 = 36$
   d) $4x^2 - y^2 = 4$
   e) $16x^2 + y^2 = 16$
   f) $y = 8x^2$
   g) $9x^2 + 9y^2 = 25$
   h) $x = -2y^2$

2. Given the ellipse E: $25x^2 + y^2 = 25$.
   a) Find, in general form, an equation for E', the image of E, under the translation $(x,y) \rightarrow (x + 3, y + 1)$.
   b) Sketch a graph of the ellipse E and its image ellipse E'.

3. Repeat the previous question for the following conics and translations.

	conic	translation
a)	$9x^2 - 16y^2 = 144$	$(x,y) \rightarrow (x - 3, y - 1)$
b)	$x^2 + y^2 = 16$	$(x,y) \rightarrow (x - 2, y + 3)$
c)	$y = 4x^2$	$(x,y) \rightarrow (x - 3, y - 2)$

4. For each of the following, indicate whether the conic is an ellipse, a circle, a hyperbola or a parabola.
   a) $2x^2 - 3y^2 - 5x - 7y - 10 = 0$
   b) $5x^2 + 5y^2 + 3x - 2y - 3 = 0$
   c) $8x^2 + 3x - 7y - 1 = 0$
   d) $4x^2 + 9y^2 - 3x + 7y + 3 = 0$
   e) $8x^2 - 4y^2 - 6x + 8y - 0 = 0$

5. Given the conic C: $x^2 + 8y^2 + 8x - 16y + 9 = 0$.
   a) Name the type of conic.
   b) Determine the translation that changes the equation into standard form.
   c) State an equation for C', the image of C.
   d) Graph the image conic C' and then the given conic C.

6. Repeat the previous question for the following conics.
   a) $x^2 + y^2 + 6x + 12y - 2 = 0$
   b) $x^2 - 12x + 4y + 4 = 0$
   c) $9x^2 - 4y^2 - 18x - 8y - 31 = 0$
   d) $4x^2 + y^2 + 8x - 10y + 13 = 0$
   e) $4x^2 - 4y^2 + 16x - 8y + 13 = 0$
   f) $x^2 + 25y^2 - 4x - 200y + 4 = 0$
   g) $y^2 - 8x + 4y - 36 = 0$
   h) $x^2 + y^2 - 8x + 6y - 1 = 0$

7. For each of the following conics $ax^2 + 2hxy + by^2 = k$, write the matrix M and the matrix K for its matrix equation $V^tMV = K$.
   a) $5x^2 - 6xy + 4y^2 = 7$
   b) $3x^2 - 6xy - y^2 = 1$
   c) $2x^2 + 3xy - y^2 = 9$
   d) $3x^2 + 2xy - 5y^2 = 8$
   e) $4x^2 - 12y^2 = 15$
   f) $5x^2 - 4xy + 5y^2 = 13$

8. Write the matrix equation $V^tMV = K$ for each of the conics in question 7.

9. For each of the following matrices M, write the Cartesian equation of the conic whose matrix equation is $V^tMV = K$.
   a) $\begin{bmatrix} 4 & -3 \\ -3 & 9 \end{bmatrix}$
   b) $\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$
   c) $\begin{bmatrix} -7 & -4 \\ -4 & 3 \end{bmatrix}$
   d) $\begin{bmatrix} 17 & 10 \\ 10 & 7 \end{bmatrix}$

10. Write the transpose of each of the following matrices.
    a) $\begin{bmatrix} 4 & 5 \\ -3 & 9 \end{bmatrix}$
    b) $[-4 \quad 3]$

11. Given $A = \begin{bmatrix} 2 & x \\ 0 & y \end{bmatrix}$,
    a) write A^t
    b) find values of x and y so that $A = A^t$.

12. Given the ellipse E: $9x^2 + 4y^2 = 36$.
   a) Write the corresponding matrix equation $V^tMV = K$.
   b) Find the matrix RMR^t for E', the image of E under a rotation of 30° about the point (0,0).
   c) Write a Cartesian equation for the image of E.
   d) Sketch a graph of E and E'.

13. Repeat the previous question for the following conics and rotation angles θ.
   a) $x^2 + 4y^2 = 9, \quad \theta = 50°$
   b) $4x^2 - 9y^2 = 36, \theta = 100°$
   c) $x^2 + 9y^2 = 36, \quad \theta = -20°$
   d) $3x^2 - y^2 = 4, \quad \theta = 60°$

14. a) Given the ellipse E: $9x^2 + 4y^2 = 36$, find the image of E under a rotation of 30° about (0,0). Use exact values, from the table on page 543.
   b) Sketch the graph of E and E'.

15. a) A circle $x^2 + y^2 = 25$ is rotated about the point (0,0). Explain why an xy term will not be introduced into the image equation.
   b) Will an xy term be introduced in the image equation if the circle $x^2 + y^2 - 4x + 8y - 1 = 0$ is rotated about the point (0,0)? Explain.

16. For the rotation matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

and $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ show that

$$RMR^t = \begin{bmatrix} a + (b-a)\sin^2\theta & \frac{a-b}{2}\sin 2\theta \\ \frac{a-b}{2}\sin 2\theta & a + (b-a)\cos^2\theta \end{bmatrix}$$

17. Given the conic $x^2 + 2xy + 5y^2 = 4$.
   a) Name the type of conic.
   b) Determine an angle of rotation about (0,0) needed to remove the xy term.

18. Repeat the previous question for the following conics.
   a) $5x^2 + 2xy - 3y^2 = 4$
   b) $3x^2 - 2xy + y^2 = 2$
   c) $x^2 - 3xy + 4y^2 = 4$
   d) $3x^2 + 4xy + 3y^2 = 1$
   e) $3x^2 - 16xy + 5y^2 = 4$
   f) $6x^2 - 3xy + 2y^2 = 8$

19. Given the conic C: $17x^2 - 15xy + 17y^2 = 32$.
   a) Determine the type of conic.
   b) Find an angle of rotation about the point (0,0) that will eliminate the xy term.
   c) Find an equation of the image curve C'.
   d) Sketch the graph of the image curve C', then the graph of C, the original curve.

20. Repeat question 19 for the following conics.
   a) $x^2 - xy + y^2 = 2$
   b) $100x^2 - 173xy = 600$
   c) $20x^2 - 69xy - 20y^2 = 40$
   d) $30x^2 - 17.3xy + 50y^2 = 360$
   e) $29x^2 + 6xy + 21y^2 = 120$
   f) $22x^2 - 12xy + 17y^2 = 65$
   g) $5x^2 + 18xy + 5y^2 = 26$
   h) $9x^2 + 16xy + 21y^2 = 50$
   i) $47x^2 - 24xy + 57y^2 = 195$
   j) $x^2 - 16xy - 11y^2 = -30$
   k) $x^2 - 24xy - 6y^2 = -30$
   l) $4x^2 - 2xy + 4y^2 = 15$

21. Repeat question 19 for the following conics. Use exact values of the sines and cosines of the rotation angles, from the table on page 543.
   a) $17x^2 - 15xy + 17y^2 = 32$
   b) $13x^2 + 34\sqrt{3}xy + 47y^2 = 20$

22. Given the conic C: $10x^2 + 20xy + 10y^2 - 28x + 28y = 0$.
   a) Find an angle of rotation about (0,0) that will eliminate the xy term.
   b) Find an equation of the image curve C'.
   c) Determine the type of conic.
   d) Sketch the graph of the image curve C', then the graph of C, the original curve.

VECTORS, MATRICES and COMPLEX NUMBERS

with
International Baccalaureate
questions

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and
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CHAPTER NINE

MATHEMATICAL INDUCTION

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CHAPTER NINE

Mathematical Induction

There is a story told of a prince who had been disobedient. The prince was taken to a large room with a lock on the door. Inside that room was a second room and a key that opened the lock on the door of the second room. Inside this second room was a third room and a key that opened the lock on the door of the third room. Inside the third room was a fourth room and a key that opened the lock on the door of the fourth room. These rooms continued forever, with each room containing the key to open the lock on the door of the next room. His punishment was to open lock after lock and open door after door forever. Would the prince be able to continue in this way if he lived forever?

Another story is told of a queen who enjoyed playing with dominoes. She would place one domino on its end, then place a second on its end beside it, then a third beside the second and so on. Each domino was positioned so that it would knock over the next domino if it fell. The queen would push over the first domino which would push over the second which would push over the third, which would push over the fourth, and so on, until all the dominoes were knocked over. She continually added more and more dominoes in order to see them fall over. Then the queen wondered if she could continue to add dominoes forever so that her push on the first would cause all of the others to fall over. The queen offered a prize of 100 gold pieces to whomever could answer this question.

A third story is told of a cow who wanted to reach the moon using a very long ladder. She observed that she could get on the first rung of the ladder. She also realized that once she was on any rung she knew how to climb to the next rung. She wondered if this was enough to ensure that she could climb the ladder to the moon, and perhaps beyond the moon forever.

The topic of mathematical induction which you will study in this chapter will help you to solve the problems introduced in these stories.

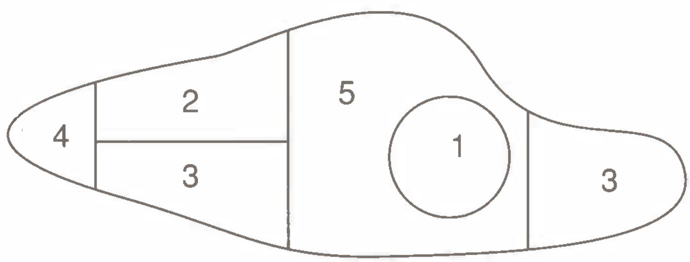
9.1 Making Conjectures

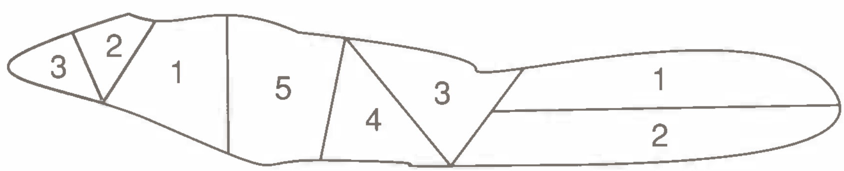
Proof is a very important part of mathematics. But, in the order of time, proof is generally the final event in a mathematical discovery. Mathematicians spend much of their energy trying to discover new mathematical truths. They make guesses or conjectures about what seems to be true then try to prove or disprove their conjectures. Some of these proofs or disproofs come immediately after the conjecture. Other conjectures are shown to be true or false a long time after their discovery. Some conjectures are still awaiting proof or disproof. Some examples of such conjectures are presented in the following paragraphs.

The Four Colour Problem

What is the minimum number of colours needed to colour the map of countries on a surface so that no two countries with the same boundary will be coloured with the same colour?

The following diagrams show some maps that have been coloured with five colours. Each of the five colours is indicated with a different number. Can you colour either map using fewer than five colours?





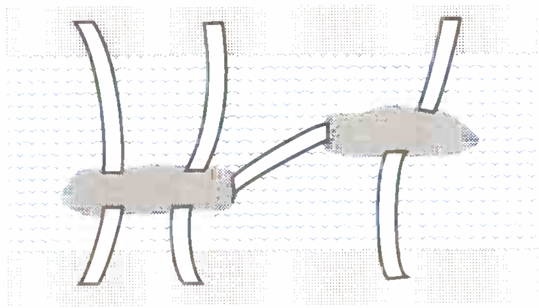
The Four Colour Problem goes back to October 23, 1852 when Francis Gutherie posed it to his teacher, De Morgan, who wrote to W.R. Hamilton. [reference: Stein: *The Man-Made Universe*, page 220] In 1890, P.J. Heawood proved that five colours were sufficient to colour any map. Most mathematicians conjectured that four colours were enough. Indeed, no one was able to draw a map that needed more than four colours. Nevertheless, it was not until 1976 that Kenneth Appel and Wolfgang Haken of the United States proved, using a computer, that four colours are sufficient.

The Koenigsberg Bridge Problem

In 1735 the Swiss mathematician Leonhard Euler described this problem as follows.

In the town of Koenigsberg there is an island called Kneiphof, with two branches of the river Pregel flowing around it. There are seven bridges crossing the two branches. The question is whether a person can walk in such a way that he will cross these bridges once but not more than once.

Here is a diagram of the seven bridges of Koenigsberg. Can a person plan a walk that will take the person across each bridge exactly once?



The problem had been around a long time before Euler. The townspeople used to spend their Sunday afternoons on such a walk, wondering if they could cross all of the bridges without repeating any bridges. They never succeeded in doing so. Euler showed in the same year that such a walk was impossible.

Fermat's Last Theorem

You know that the Pythagorean theorem states that for any right triangle with hypotenuse c and other two sides a, and b, that $a^2 + b^2 = c^2$.

Mathematicians wondered if a similar fact were true for any other power. For example, they tried to discover natural numbers a, b, and c such that $a^3 + b^3 = c^3$. The French mathematician Pierre de Fermat (1601-1665) wrote about this problem in the margin of a book he was reading. He said that he had discovered a truly wonderful proof that the equation $a^n + b^n = c^n$ does not have a solution in integer values of a, b and c for $n \geqslant 3$. He wrote that the margin of the book he was reading was too small to contain the proof. Since that time mathematicians have tried to rediscover Fermat's proof. But *no one* has been able to prove or disprove Fermat's statement!

Note how the simplicity of the statement of these problems gives no clue as to their difficulty.

In this section you will be given the opportunity to make your own conjectures about various mathematical ideas. You will not prove your guesses in this section. You will learn how to prove some of them in the next section 9.2.

Example 1 Conjecture a formula for the sum of the first n odd numbers, that is, for the sum $S_n = 1 + 3 + 5 + 7 + \ldots + (2n-1)$.

Solution To help find a pattern you should list and examine the values of S_1, S_2, S_3, S_4.

$S_1 = 1 = 1$

$S_2 = 1 + 3 = 4$

$S_3 = 1 + 3 + 5 = 9$

$S_4 = 1 + 3 + 5 + 7 = 16$

Observe that $S_1 = 1^2$

$S_2 = 2^2$

$S_3 = 3^2$

$S_4 = 4^2$

One conjecture or guess would be $S_n = n^2$

The conjecture should now be checked for other values of n.

Let $n = 6$.

$S_6 = 1 + 3 + 5 + 7 + 9 + 16 = 36 = 6^2$

as it should be, according to the conjecture. ■

Note: It is important for you to realize that the fact that a formula checks for particular values of n does *not* mean that the formula is true for all values of n.

For the series $1 + 2 + 3 + 4 + 5 + \ldots$

$S_1 = 1, S_2 = 3, S_3 = 6, S_4 = 10.$

The formula $S_n = \dfrac{n(n+1)}{2} + (n-1)(n-2)(n-3)$

produces the following values

$S_1 = 1, S_2 = 3, S_3 = 6, S_4 = 16.$

The formula checks for $n = 1, 2,$ and 3 but *not* for $n = 4$.

Example 2 Guess a formula for the sum S_n of the series

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \ldots + \frac{1}{n(n+1)}$$

Solution To try to guess this sum you might list the partial sums as follows.

$$S_1 = \frac{1}{1 \times 2} = \frac{1}{2}$$

$$S_2 = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

$$S_3 = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} = \frac{2}{3} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4}$$

$$S_4 = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} = \frac{3}{4} + \frac{1}{20} = \frac{16}{20} = \frac{4}{5}$$

These values for S_n suggest the formula $S_n = \frac{n}{n+1}$

Checking the conjecture for $n = 5$, S_n should equal $\frac{5}{5+1}$ or $\frac{5}{6}$.

$$S_5 = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \frac{1}{5 \times 6} = \frac{4}{5} + \frac{1}{30} = \frac{25}{30} = \frac{5}{6},$$

as required. ■

Example 3 Make a conjecture about the values of $n \in \mathbb{N}$, for which $2^n < n!$, where $n! = n(n-1)(n-2)(n-3)\ldots(3)(2)(1)$ and $0! = 1$ ($n!$ is read "n factorial").

Solution Try values of $n \in \mathbb{N}$ beginning with $n = 1$.

Let $n = 1$: $L.S. = 2^1 = 2$, and $R.S. = 1! = 1$
Since $L.S. > R.S.$, the statement is false.

Let $n = 2$: $L.S. = 2^2 = 4$, and $R.S. = 2! = (2)(1) = 2$
Since $L.S. > R.S.$, the statement is false.

Let $n = 3$: $L.S. = 2^3 = 8$, and $R.S. = 3! = (3)(2)(1) = 6$
Since $L.S. > R.S.$, the statement is false.

Let $n = 4$: $L.S. = 2^4 = 16$, and $R.S. = 4! = (4)(3)(2)(1) = 24$
Since $L.S. < R.S.$, the statement is true.

Let $n = 5$: $L.S. = 2^5 = 32$, and $R.S. = 5! = (5)(4)(3)(2)(1) = 120$
Since $L.S. < R.S.$, the statement is true.

The statement appears to be true for $n \geqslant 4$.
Try further values of n to check the conjecture that $2^n < n!$, for $n \geqslant 4$. ■

9.1 Exercises

1. Conjecture a formula for the sum of n terms of the series
$$\frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \ldots - \frac{1}{2^n}$$

2. Conjecture a formula for the sum of n terms of each of the following series.
   a) $2 + 4 + 8 + 16 + 32 + \ldots + 2^n$, for $n > 1$
   b) $1 + 2 + 4 + 8 + 16 + \ldots + 2^{n-1}$

3. Given the series $1 + 2 + 3 + \ldots + n$.
   a) List S_1, S_2, S_3, and S_4.
   b) List $\frac{2S_1}{1}$, $\frac{2S_2}{2}$, $\frac{2S_3}{3}$, and $\frac{2S_4}{4}$
   c) Guess a formula for S_n.

4. Conjecture a formula for the sum of n terms of the series
$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \ldots + \frac{1}{(2n-1)(2n+1)}$$

5. Conjecture a formula for the sum of n terms of each of the following series.
   a) $\frac{1}{1 \times 5} + \frac{1}{5 \times 9} + \frac{1}{9 \times 13} + \ldots + \frac{1}{(4n-3)(4n+1)}$
   b) $\frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \ldots + \frac{1}{(n+1)(n+2)}$
   c) $\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n}$

6. a) Evaluate the 2-term product
   $(1 + 1)\left(1 + \frac{1}{2}\right)$
   b) Evaluate the 3-term product
   $(1 + 1)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)$
   c) Evaluate the 4-term product
   $(1 + 1)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{4}\right)$
   d) Evaluate the 5-term product
   $(1 + 1)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{4}\right)\left(1 + \frac{1}{5}\right)$
   e) Conjecture a formula for value of the n-term product
   $(1 + 1)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{4}\right)\ldots\left(1 + \frac{1}{n}\right)$

7. Conjecture a formula for the value of each of the following n-term products.
   a) $\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{16}\right)\left(1 - \frac{1}{25}\right)\ldots\left(1 - \frac{1}{n^2}\right)$, $n > 1$
   b) $\left(1 + \frac{3}{1}\right)\left(1 + \frac{5}{4}\right)\left(1 + \frac{7}{9}\right)\left(1 + \frac{9}{16}\right)\ldots\left(1 + \frac{2n+1}{n^2}\right)$

8. Given $f(n) = \frac{n^3 + 3n^2 + 2n}{3}$
   a) Evaluate $f(1)$, $f(2)$, $f(3)$, and $f(4)$.
   b) Make a conjecture concerning the values of $n \in \mathbb{N}$ for which $f(n)$ is a natural number.

9. Make a conjecture concerning the values of $n \in \mathbb{N}$ for which
$$t_n = \frac{5^n - 2^n}{3}$$
is a natural number.

10. Make a conjecture concerning the values of $n \in \mathbb{N}$, for which each of the following is a natural number.
   a) $\frac{n(n+1)}{2}$
   b) $\frac{n(n+1)(n+2)}{6}$
   c) $\frac{n^3 - n}{6}$
   d) $\frac{n^5 - n}{5}$
   e) $\frac{n^2 + 2n}{8}$
   f) $\frac{n^3 + 20n}{48}$

11. Make a conjecture concerning the values of $n \in \mathbb{N}$, for which
$2^n \geq n^2$

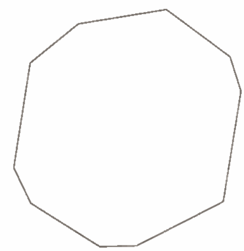
12. Make a conjecture concerning the values of $n \in \mathbb{N}$ for which each of the following inequalities is true.
    a) $3^n < n!$
    b) $3^n > 2^{n+1}$
    c) $\left(\frac{5}{6}\right)^n < \frac{5}{n}$

13. Given the series $S_n = 1 + 8 + 27 + \ldots + n^3$
    a) List $S_1, S_2, S_3, S_4, S_5, S_6,$ and S_7
    b) List $\sqrt{S_1}, \sqrt{S_2}, \sqrt{S_3}, \sqrt{S_4}, \sqrt{S_5}, \sqrt{S_6},$ and $\sqrt{S_7}$
    c) Compare your answers to part b) with $S_1, S_2, S_3, S_4, S_5, S_6,$ and S_7 of question 3.
    d) Guess a formula for S_n.

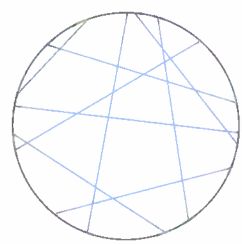
14. a) Mark 3 non-collinear points on a paper. How many different lines can you draw joining two of the points?
    b) Mark another point on the paper not collinear with the first 3 points of part a). You now have 4 non-collinear points on the paper. How many different lines can you draw joining two of the points?
    c) Mark another point on the paper not collinear with the 4 points of part b). You now have 5 non-collinear points on a paper. How many different lines can you draw joining two of the points?
    d) Conjecture a formula for the number of lines you can draw joining any two points among n non-collinear points, $n \in \mathbb{N}$. (If you need a hint read part e) of this question.)
    e) Completing the following table should help you to guess a formula for part d).

number of points (n)	number of lines	n^2	$n^2 - n$
1	0	1	0
2	1	4	2
3	3	9	6
4	*	*	*
5	*	*	*

15. a) Draw any quadrilateral and its diagonals. How many diagonals does a quadrilateral have?
    b) Draw any 5-gon, that is, a closed figure with 5 sides, and its diagonals. How many diagonals does a 5-gon have?
    c) Repeat part b) for a 6-gon and a 7-gon.
    d) Conjecture a formula for the number of diagonals that an n-gon has.



16. Draw a circle and any one chord of the circle. The interior of the circle is divided into 2 non-overlapping regions.
    a) Draw a second chord in the circle intersecting the first chord. What is the maximum number of non-overlapping regions into which the interior of the circle can be divided?



    b) Draw a third chord in the circle, intersecting the first and second chord. What is the maximum number of non-overlapping regions into which the interior of the circle can be divided?
    c) Draw a fourth chord in the circle, intersecting the three previous chords. What is the maximum number of non-overlapping regions into which the interior of the circle can be divided?
    d) Conjecture a formula for the maximum number of non-overlapping regions into which the interior of the circle can be divided by n chords.

9.2 The Principle of Mathematical Induction

In the introduction you met three stories, each of which involved starting an activity and trying to continue the activity forever. In each case the activity was repeated over and over. It was assumed that any one activity was followed immediately by the next one. You are now seeking a proof that these activities can indeed continue indefinitely.

Examine the story of the dominoes. Suppose the dominoes are numbered consecutively using the set of natural numbers $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. Suppose further that D is the set of numbers corresponding to the dominoes that will fall over after the queen has knocked over the first domino. Since the queen pushed over the first domino, you know that the number 1 is in D. Thus, $D = \{1, \ldots\}$. Now the dominoes are positioned in such a way that the kth domino falling over will push over the next, that is, the $(k + 1)$th domino. In other words, if the kth domino falls over the $(k + 1)$th will also fall over. Thus, if the number k is in D then the number $k + 1$ is also in D.

But the dominoes are infinite in number and marked with the natural numbers $1, 2, 3, 4, \ldots$. Thus, if you can show that D is actually the set $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ then you will know that all the dominoes have their numbers in D. Hence all the dominoes will fall over.

Now the set of natural numbers $\mathbb{N}$ has a very special property called the **inductive property of** $\mathbb{N}$. This property is as follows.

PROPERTY

Let T be a subset of the natural numbers $\mathbb{N}$. Then T is the *entire* set $\mathbb{N}$, if and only if *both* of the following are true.

a) 1 is a member of T.

b) If k is a member of T, then $k + 1$ is also a member of T.

But both of these are true for set D.

a) 1 is in D because the queen knocked over the first domino.

b) If the kth domino falls over it will push over the next domino, that is, the $(k + 1)$th domino. Thus, k being in D implies that $k + 1$ is also in D.

Hence by the inductive property of $\mathbb{N}$, the set D and the set $\mathbb{N}$ are the same set.

This process of using the inductive property of the set of natural numbers to prove something is true is called a **proof using mathematical induction.**

A proof using mathematical induction shows that a statement involving natural numbers is true. To help understand the method of proof, the inductive property of $\mathbb{N}$ is restated as **the principle of mathematical induction.**

PRINCIPLE

A statement involving the natural number n is true for every $n \in \mathbb{N}$ provided the following are true.

a) The statement is true for $n = 1$.

b) The truth of the statement for $n = k$ implies the statement is true for $n = k + 1$.

The principle of mathematical induction can be derived from the inductive property of $\mathbb{N}$. Suppose S is the set of natural numbers for which a statement is true. Then a) implies that $1 \in S$. But b) shows that $k \in S$ implies that $k + 1 \in S$. The inductive property of $\mathbb{N}$ allows the conclusion that $S = \mathbb{N}$.

METHOD

In practice, you should use three steps in a proof by mathematical induction.

Step 1: Show the statement is true for $n = 1$.
Step 2: Assume that the statement is true for $n = k$.
Step 3: Prove the statement is true for $n = k + 1$, *using the result of step 2*.

The principle of mathematical induction can only be used to prove a given formula is true. The principle does not help you to obtain such a formula. In Examples 1 and 2 and in Exercises 9.2 and 9.3 you will be given the opportunity to prove some of the conjectures you made in section 9.1 and in 9.1 Exercises.

Example 1

Use mathematical induction to prove the following formula for $n \in \mathbb{N}$.
$1 + 3 + 5 + \ldots + (2n - 1) = n^2$.

Solution

Step 1: Prove the statement is true for $n = 1$.
For $n = 1$, $L.S. = 1$, $R.S. = 1^2 = 1$
Since $L.S. = R.S.$, the statement is true for $n = 1$.

Step 2: Assume the formula is true for $n = k$. That is,
assume $1 + 3 + 5 + \ldots + (2k - 1) = k^2$. (*)

Step 3: Prove the formula is true for $n = k + 1$. That is, you have to prove that $1 + 3 + 5 + \ldots + (2k + 1) = (k + 1)^2$.

$$\begin{aligned} L.S. &= 1 + 3 + 5 + \ldots\ldots\ldots + (2k + 1) \\ &= 1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) && \text{the same as } L.S.\ (*) \\ &= [1 + 3 + 5 + \ldots + (2k - 1)] + (2k + 1) && \text{with } (2k + 1) \text{ added} \\ &= [k^2] + (2k + 1) && \text{by step 2} \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2 = R.S. \end{aligned}$$

Thus, by the principle of mathematical induction,
$1 + 3 + 5 + \ldots + (2n - 1) = n^2$, for all $n = \in \mathbb{N}$. ■

Example 2

Prove, by mathematical induction

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

Solution

Step 1: For $n = 1$, $L.S. = \frac{1}{1\times 2}=\frac{1}{2}$, $R.S. = \frac{1}{1+1}=\frac{1}{2}=L.S.$

Step 2: Assume $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$ ⊛

Step 3: Prove $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$

$$
\begin{aligned}
L.S. &= \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\ldots\ldots\ldots+\frac{1}{(k+1)(k+2)}\\
&= \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\ldots\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}\\
&= \left[\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\ldots+\frac{1}{k(k+1)}\right]+\frac{1}{(k+1)(k+2)}\\
&= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \quad \text{using step 2}\\
&= \frac{k(k+2)+1}{(k+1)(k+2)}\\
&= \frac{k^2+2k+1}{(k+1)(k+2)}\\
&= \frac{(k+1)(k+1)}{(k+1)(k+2)}\\
&= \frac{k+1}{k+2} = R.S.
\end{aligned}
$$

This is the *L.S.* of ⊛ with $\frac{1}{(k+1)(k+2)}$ added.

Thus, by the principle of mathematical induction, the formula is true. ■

Example 3

Use mathematical induction to prove the following formula for $n \in \mathbb{N}$.

$$\sum_{k=1}^{n}[2k-1]=n^2$$

Solution

First write the sum explicitly by letting k equal successively $1, 2, 3, 4, \ldots, n$.

$\sum_{k=1}^{n}[2k-1]=n^2$ becomes

$[2(1)-1]+[2(2)-1]+[2(3)-1]+[2(4)-1]+\ldots+[2(n)-1]=n^2$
or $1+3+5+7+\ldots+[2n-1]=n^2$.

But this now is exactly the same problem as that of Example 1, so the solution is the same. This solution will not be repeated here. ■

9.2 Exercises

1. State the three steps in a proof using mathematical induction.

2. Prove the following statement using mathematical induction, where $n \in \mathbb{N}$.
$\frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \ldots - \frac{1}{2^n} = \frac{1}{2^n}$

3. Prove the following statements using mathematical induction, where $n \in \mathbb{N}$.
   a) $2 + 4 + 8 + 16 + 32 + \ldots + 2^n = 2^{n+1} - 2$
   [You will need to use the fact that $2^{k+1} + 2^{k+1} = 2^{k+2}$ Why is this true?]
   b) $1 + 2 + 4 + 8 + 16 + \ldots + 2^{n-1} = 2^n - 1$
   c) $6 + 12 + 18 + \ldots + 6n = 3n(n + 1)$
   d) $3 + 5 + 7 + \ldots + (2n + 1) = (n + 1)^2 - 1$

4. Prove the following statements using mathematical induction, where $n \in \mathbb{N}$.
   a) $\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$
   b) $\frac{1}{1 \times 5} + \frac{1}{5 \times 9} + \ldots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$

5. All three steps are essential in a proof by mathematical induction, as the following will demonstrate.
A certain mathematician thought that he had a formula that produced prime numbers. (A prime number has exactly two divisors, 1 and the number itself.) He said that $n^2 + n + 41$ is always a prime number for $n \in \mathbb{N}$. He demonstrates the proof of his formula for $n = 1, 2, 3, \ldots 40$.
   a) Verify that the statement is true for $n = 1$, $n = 2$, $n = 3$, and $n = 4$.
   b) If you have the inclination you can show that the statement is true for all n from 1 to 40, inclusive. Nevertheless, do prove $n^2 + n + 41$ does not produce a prime number for $n = 41$.
   c) Which step(s) in a proof by mathematical induction is (are) missing in the demonstration given by the mathematician?

6. A friend tells you that the formula $7 + 9 + 11 + \ldots + (2n - 1) = n^2$ is true. He demonstrates this with the following argument.
Assume the formula is true for $n = k$, that is,
$7 + 9 + 11 + \ldots + (2k - 1) = k^2$ ①
Prove the formula is true for $n = k + 1$, that is, prove that
$7 + 9 + 11 + \ldots + (2[k + 1] - 1) = (k + 1)^2$ *
$L.S.$ of * $= [7 + 9 + 11 + \ldots + (2k - 1)] + 2k + 1$
(the $L.S.$ of ①, with $2k + 1$ added)
$= [k^2] + 2k + 1 = (k + 1)^2 = R.S.$ of *
(from ①)
   a) Which step(s), if any, in a proof by mathematical induction are missing in your friend's proof?
   b) Is your friend's formula true for all $n \in \mathbb{N}$?

7. Prove the following formulas by mathematical induction.
   a) $\sum_{s=1}^{n} s = \frac{n(n+1)}{2}$
   b) $\sum_{s=1}^{n} s^2 = \frac{n(n+1)(2n+1)}{6}$
   c) $\sum_{s=1}^{n} s^3 = \left(\frac{n(n+1)}{2}\right)^2$

8. Use mathematical induction to prove that the sum of n terms of an arithmetic series with first term a and common difference d is $S_n = \frac{n}{2}[2a + (n - 1)d]$

9. Use mathematical induction to prove that the sum of n terms of a geometric series with first term a and common ratio r is $S_n = \frac{a(r^n - 1)}{r - 1}$

9.3 Using Mathematical Induction

In section 9.1 and in 9.1 Exercises you made conjectures about formulas for the sum of series. In section 9.2 you learned how to use mathematical induction to prove your true conjectures. But in 9.1 Exercises you made conjectures about products, inequalities and geometrical conclusions. In this section you will learn how your true conjectures can be proven using mathematical induction.

The first example will deal with question 6 of 9.1 Exercises.

Example 1 Use mathematical induction to prove that

$$(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\ldots\left(1+\frac{1}{n}\right)=n+1$$

Solution

Step 1: Prove the statement is true for $n = 1$.

For $n = 1$, $L.S. = (1 + 1) = 2$, $R.S. = 1 + 1 = 2$

Since $L.S. = R.S.$, the statement is true for $n = 1$.

Step 2: Assume the statement is true for $n = k$. That is, assume

$$(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\ldots\left(1+\frac{1}{k}\right)=k+1 \quad (*)$$

Step 3: Prove the statement is true for $n = k + 1$. That is, prove

$$(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\ldots\left(1+\frac{1}{k+1}\right)=(k+1)+1$$

$$L.S. = (1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\ldots\ldots\ldots\ldots\left(1+\frac{1}{k+1}\right)$$

$$= (1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\ldots\left(1+\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right)$$

the same as the $L.S.$ of $(*)$ multiplied on the right by $\left(1+\frac{1}{k+1}\right)$

$$= \left[(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\ldots\left(1+\frac{1}{k}\right)\right]\left(1+\frac{1}{k+1}\right)$$

$$= [k+1]\left(1+\frac{1}{k+1}\right) \quad \text{by step 2}$$

$$= \left([k+1]+[k+1]\frac{1}{k+1}\right) = (k+1+1) = k+2 = R.S.$$

Thus, by the principle of mathematical induction,

$$(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\ldots\left(1+\frac{1}{n}\right)=n+1,$$

for all $n \in \mathbb{N}$. ■

In Example 2 you will prove a true conjecture about question 8 of 9.1 Exercises.

Example 2 Use mathematical induction to prove that

$$f(n) = \frac{n^3 + 3n^2 + 2n}{3} \text{ is a natural number for all } n \in \mathbb{N}.$$

Solution *Step 1*: Prove the statement is true for $n = 1$.

For $n = 1$, $f(1) = \frac{(1)^3 + 3(1)^2 + 2(1)}{3} = \frac{6}{3} = 2$, which is a natural number. Thus, the statement is true for $n = 1$.

Step 2: Assume the statement is true for $n = k$. That is,

assume $f(k) = \frac{k^3 + 3k^2 + 2k}{3}$ is a natural number.

Step 3: Prove the statement is true for $n = k + 1$. That is,

prove $f(k+1) = \frac{(k+1)^3 + 3(k+1)^2 + 2(k+1)}{3}$ is a natural number.

But
$$f(k+1) = \frac{k^3 + 3k^2 + 3k + 1 + 3(k^2 + 2k + 1) + 2k + 2}{3}$$
$$= \frac{k^3 + 3k^2 + 2k}{3} + \frac{3k^2 + 9k + 6}{3} = f(k) + k^2 + 3k + 2$$

From step 2 you know that $f(k)$ is natural number. Also, because k is a natural number, $k^2 + 3k + 2$ is also a natural number.

Thus, $f(k) + k^2 + 3k + 2 = f(k+1)$ is a natural number.

Thus, by the principle of mathematical induction,

$$f(n) = \frac{n^3 + 3n^2 + 2n}{3} \text{ is a natural number for all } n \in \mathbb{N}. \blacksquare$$

The next example shows how to prove your true conjectures about inequalities by studying the inequality of Example 3, section 9.1.

Example 3 Use mathematical induction to prove that $2^n < n!$ for $n \geqslant 4$, $n \in \mathbb{N}$.

Solution This statement is not true for $n = 1$, 2, and 3 as you can easily check. So step 1 must begin with $n = 4$.

Step 1: Prove the statement is true for $n = 4$.
For $n = 4$, $L.S. = 2^4 = 16$, $R.S. = 4! = (4)(3)(2)(1) = 32$.
Therefore $L.S. < R.S.$, so the statement is true for $n = 4$.

Step 2: Assume the statement is true for $n = k$. That is, assume $2^k < k!$

Step 3: Prove the statement is true for $n = k + 1$. That is, prove $2^{k+1} < (k+1)!$
But from step 2 you know that $2^k < k!$
Multiplying both sides by 2 gives $2^{k+1} < 2(k!)$
Since $k \geqslant 4$, $2 < k + 1$, so $2(k)! < (k+1)(k!) = (k+1)!$
Thus, $2^{k+1} < (k+1)!$, as required by step 3.

Thus, by the principle of mathematical induction, $2^n < n!$ for $n \geqslant 4$, $n \in \mathbb{N}$. $\blacksquare$

Note: If the statement to be proved is not true for the first few natural numbers, then step 1 must be done for the first number for which the statement is true.

9.3 Exercises

1. Use mathematical induction to prove that
$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\left(1+\frac{9}{16}\right)\cdots\left(1+\frac{2n+1}{n^2}\right)=(n+1)^2$$

2. Use mathematical induction to prove that for $n \geqslant 2$
$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\left(1-\frac{1}{25}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$

3. Use mathematical induction to prove that
$$\left(1-\frac{4}{1}\right)\left(1-\frac{4}{9}\right)\left(1-\frac{4}{25}\right)\cdots\left(1-\frac{4}{(2n-1)^2}\right)=\frac{2n+1}{2n-1}$$

4. Use mathematical induction to prove that each of the following is a natural number for all $n \in \mathbb{N}$.
   a) $\dfrac{n(n+1)}{2}$
   b) $\dfrac{n(n+1)(n+2)}{6}$
   c) $\dfrac{n^3-n}{6}$
   d) $\dfrac{n^5-n}{5}$

5. Use mathematical induction to prove that each of the following is a natural number, provided n is an even natural number. (hint: let $n = 2m$, $m \in \mathbb{N}$)
   a) $\dfrac{n^2+2n}{8}$
   b) $\dfrac{n^3+20n}{48}$

6. Use mathematical induction to prove that $2^n > n^2$ for $n > 4$, $n \in \mathbb{N}$.

7. Prove the following where $n \in \mathbb{N}$.
   a) $3^n < n!$ for $n \geqslant 7$
   b) $3^n > 2^{n+1}$ for $n \geqslant 2$
   c) $\left(\dfrac{5}{6}\right)^n < \dfrac{5}{n}$ for all n

8. Prove that $(1+x)^n > 1 + nx$ for $x > 0$ and $n > 1$, $x \in \mathbb{R}$, $n \in \mathbb{N}$.

9. Prove that $(1+x)^n > 1 + nx + nx^2$ for $x > 0$ and $n > 2$, $x \in \mathbb{R}$, $n \in \mathbb{N}$.

10. a) Show that
$5^{k+1} = 5 \times 5^k = 3 \times 5^k + 2 \times 5^k$.
    b) Use mathematical induction to prove that $\dfrac{5^n - 2^n}{3}$ is a natural number for all $n \in \mathbb{N}$.

11. Given a set of n points, no three of which are collinear, prove that the number of line segments that can be drawn joining these points in pairs is $\dfrac{n(n-1)}{2}$
[See question 14, 9.1 Exercises.]

12. Use mathematical induction to prove that an n-gon has $\dfrac{n(n-3)}{2}$ diagonals.
[See question 15, 9.1 Exercises.]

13. Given a circle and a set of n chords of this circle, show that the maximum number of non-overlapping regions into which the circle can be divided is $\dfrac{n^2+n+2}{2}$
[See question 16, 9.1 Exercises.]

14. Prove that $\dfrac{n}{2} \leqslant \dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{4}+\ldots+\dfrac{1}{2^n} < n$

15. Prove that $\displaystyle\sum_{k=1}^{n-1} k^3 < \frac{n^4}{4} < \sum_{k=1}^{n} k^3$

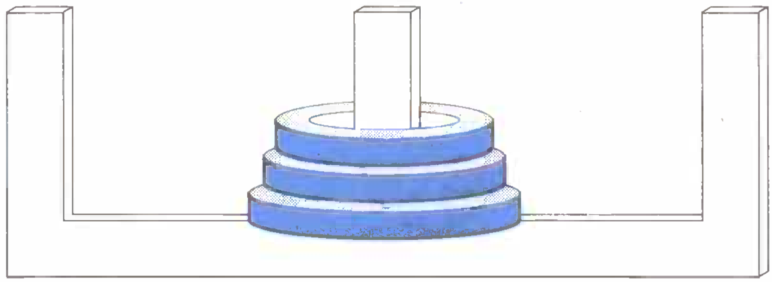
16. Use mathematical induction to prove the following.
    a) $\displaystyle\sum_{k=1}^{n} \frac{k^2}{(2k-1)(2k+1)} = \frac{n(n+1)}{2(2n+1)}$
    b) $\displaystyle\sum_{k=1}^{n} k^4 = \frac{6n^5+15n^4+10n^3-n}{30}$

17. Prove the following about matrix multiplication.
    a) $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$, $k \in \mathbb{N}$
    b) $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & kx \\ 0 & 1 \end{bmatrix}$

18. Prove $\displaystyle\frac{n!}{(n-k)!k!} = \sum_{i=k-1}^{n-1} \frac{i!}{(i-k+1)!(k-1)!}$

In Search of A Solution to the Tower of Hanoi Problem

There is an interesting and challenging puzzle called the *Tower of Hanoi*. The puzzle consists of three pegs and a set of graduated discs, as shown in the figure.



The problem posed is to transfer the discs from any one peg to another peg under the following rules.

1. Only one disc may be transferred at a time from one peg to another peg.
2. A larger disc may never be placed upon a smaller disc.

This problem can be solved using the principle of mathematical induction. Indeed, you can use this principle to calculate the minimum number of moves that would be needed for a given number n of discs.

Examine the problem for one disc, then for two discs, and finally for three discs to get some idea of the pattern involved.

One disc It is clear that one disc can be transferred in one move.

Two discs First transfer one disc leaving a peg free for the second disc. You then transfer the second disc. Finally cover the second disc with the first disc. This takes three moves.

Three discs First transfer the top two discs as above in three moves. This leaves a peg free for the third disc which is moved in one more move. Then the top two discs can be transferred onto the third disc in three moves, as above for two discs. This gives a total of seven moves.

The pattern for moving any number of discs is now clear. If you can transfer k discs you can easily transfer $k + 1$ discs. First you transfer the k discs leaving the $(k + 1)$th disc free to move to a new peg. Then the top k discs can be moved onto the $(k + 1)$th disc. Thus, the problem can be solved for any number of discs.

To determine the minimum number of moves needed to transfer n discs, observe that no disc can be moved until all of the discs above it have been transferred. Then a space is left free to which you can move that disc.

Suppose the minimum number moves for k discs is $m(k)$. To move the $(k + 1)$th disc, you will need $m(k)$ moves to the transfer the discs *above it* to another peg. Then you can transfer the $(k + 1)$th disc to the free peg. Now, to move the k discs over, to be on top of the $(k + 1)$th disc, will again take $m(k)$ moves.
Thus the total number of moves to transfer $k + 1$ discs is $m(k) + 1 + m(k)$, or $2m(k) + 1$. That is, $m(k + 1) = 2m(k) + 1$.

To use mathematical induction to determine the minimum number of moves for n discs you must now try to guess a formula for $m(n)$.

The following table gives values of $m(n)$ for n from 1 to 5.

n	1	2	3	4	5
$m(n)$	1	3	7	15	31

Adding a disc appears to 'double' the number of moves, so that this sequence of numbers should be compared with the doubling sequence 1, 2, 4, 8, 16, 32. It appears that $m(n) = 2^n - 1$.

A proof of this formula follows.
Step 1. The formula is true for $n = 1$ because $2^1 - 1 = 1 = m(1)$.
Step 2 Assume the formula is true for $n = k$.
Thus the minimum number of moves for k discs is $2^k - 1$.
Step 3 Prove the formula is true for $n = k + 1$, that is show that the minimum number of moves for $k + 1$ discs is $2^{k+1} - 1$.

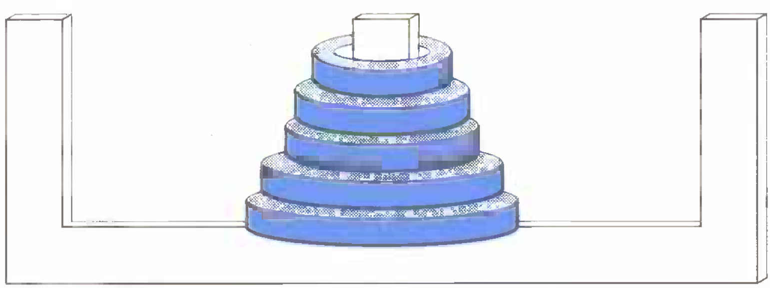
Proof: You showed above that
$m(k + 1) = 2m(k) + 1$.
Using step 2, you can say that

$$\begin{aligned} m(k + 1) &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

Hence, by the principle of mathematical induction, the minimum number of moves needed to transfer n discs is $2^n - 1$.

Activity

Try the puzzle with three discs to see if you can match the minimum number of moves. Then do the same for 4 and 5 discs.



9.4 The Binomial Theorem

There is a very important and useful formula that involves the natural numbers and the binomial $(a + x)$. This formula is called the **binomial theorem.**

The formula gives a short cut for finding values of products such as $(a + x)^2$, $(a + x)^3$, $(a + x)^4$, $(a + x)^5$, and so on.

You may already know the following products.
$(a + x)^2 = (a + x)(a + x) = a^2 + 2ax + x^2$
$(a + x)^3 = (a + x)(a + x)(a + x) = a^3 + 3a^2x + 3ax^2 + x^3$

The binomial theorem is stated as follows.

THEOREM

$$(a + x)^n = C(n,0)a^nx^0 + C(n,1)a^{n-1}x^1 + C(n,2)a^{n-2}x^2 + C(n,3)a^{n-3}x^3 + \dots + C(n,r)a^{n-r}x^r + \dots + C(n,n-1)a^1x^{n-1} + C(n,n)a^0x^n,$$

where $n \in \mathbb{N}$

Note 1 The value of $C(n,r)$ is $\dfrac{n!}{(n-r)!\,r!}$, where $n! = n(n-1)(n-2)\dots(3)(2)(1)$.

2 The expansion of the product has $n + 1$ terms.

Example 1 Expand the product $(a + x)^4$.

Solution Use the binomial theorem

$$(a + x)^n = C(n,0)a^nx^0 + C(n,1)a^{n-1}x^1 + C(n,2)a^{n-2}x^2 + C(n,3)a^{n-3}x^3 + \dots + C(n,r)a^{n-r}x^r + \dots + C(n,n-1)a^1x^{n-1} + C(n,n)a^0x^n.$$

Here $n = 4$.

Thus,
$$(a + x)^4 = C(4,0)a^4x^0 + C(4,1)a^{4-1}x^1 + C(4,2)a^{4-2}x^2 + C(4,3)a^{4-3}x^3 + C(4,4)a^{4-4}x^4$$

Now $C(4,0) = \dfrac{4!}{(4-0)!0!} = 1$ recall that $0! = 1$

$$C(4,1) = \frac{4!}{(4-1)!1!} = \frac{4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 1} = 4$$

$$C(4,2) = \frac{4!}{(4-2)!2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = 6$$

$$C(4,3) = \frac{4!}{(4-3)!3!} = 4$$

$$C(4,4) = \frac{4!}{(4-4)!4!} = 1$$

Therefore, $(a + x)^4 = a^4x^0 + 4a^3x^1 + 6a^2x^2 + 4a^1x^3 + 1a^0x^4$
$= a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$ ■

Pascal's Triangle

Observe the coefficients of the expansion of $(a + x)^n$ for $n = 1, 2, 3$, and 4.

value of n	expansion of $(a + x)^n$	Pascal's triangle
1	$1a^1 + 1x^1$	1 1
2	$1a^2 + 2ax + 1x^2$	1 2 1
3	$1a^3 + 3a^2x + 3ax^2 + 1x^3$	1 3 3 1
4	$1a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + 1x^4$	1 4 6 4 1

Note: In Pascal's triangle, the numbers on the left and right of each row are both 1. Each of the other numbers is the *sum* of the two numbers on each side of it in the line *above*.

Thus, the next line in the triangle, corresponding to $n = 5$, will be
1, 1 + 4, 4 + 6, 6 + 4, 4 + 1, 1 or 1, 5, 10, 10, 5, 1

Each line of Pascal's triangle can also be written in terms of $C(n,r)$.

$$\begin{array}{c} C(1,0)\ \ C(1,1) \\ C(2,0)\ \ C(2,1)\ \ C(2,2) \\ C(3,0)\ \ C(3,1)\ \ C(3,2)\ \ C(3,3) \\ C(4,0)\ \ C(4,1)\ \ C(4,2)\ \ C(4,3)\ \ C(4,4) \end{array}$$

For $C(n,r)$'s the note above means that, for example, $C(4,2) = C(3,1) + C(3,2)$ and $C(3,1) = C(2,0) + C(2,1)$

In general, the following is true: $C(n + 1,r) = C(n,r - 1) + C(n,r)$

You will be given the opportunity to prove this in the exercises that follow.

Example 2 Expand $(3m - 2y)^4$.

Solution Since $n = 6$ the expansion you need is

$$(a + x)^6 = C(6,0)a^6x^0 + C(6,1)a^{6-1}x^1 + C(6,2)a^{6-2}x^2 + C(6,3)a^{6-3}x^3 + C(6,4)a^{6-4}x^4 + C(6,5)a^{6-5}x^5 + C(6,6)a^{6-6}x^6$$

where $a = 3m$ and $x = -2y$. Thus,

$$\begin{aligned}(3m + (-2y))^6 &= C(6,0)(3m)^6(-2y)^0 + C(6,1)(3m)^{6-1}(-2y)^1 + C(6,2)(3m)^{6-2}(-2y)^2 \\ &\quad + C(6,3)(3m)^{6-3}(-2y)^3 + C(6,4)(3m)^{6-4}(-2y)^4 \\ &\quad + C(6,5)(3m)^{6-5}(-2y)^5 + C(6,6)(3m)^{6-6}(-2y)^6 \\ &= 1(729)m^6 + 6(243)m^5(-2)y + 15(81)m^4(4)y^2 \\ &\quad + 20(27)m^3(-8)y^3 + 15(9)m^2(16)y^4 + 6(3)m^1(-32)y^5 + 1(1)(64)y^6 \\ &= 729m^6 - 2916m^5y + 4860m^4y^2 - 4320m^3y^3 \\ &\quad + 2160m^2y^4 - 576my^5 + 64y^6 \end{aligned}$$ ■

Check that the value of the $C(n,r)$'s, make up line $n = 6$ of Pascal's triangle.

Because the binomial theorem involves the natural numbers $\mathbb{N}$, you can prove the theorem using mathematical induction. The theorem will be stated using the sigma notation Σ.

In proving the theorem you will make use of the formula
$C(n + 1,r) = C(n,r - 1) + C(n,r)$
You will also use a property of the sigma notation about changing the limits of sigma, namely,

$$\sum_{r=t}^{m} a_r = \sum_{r=t+d}^{m+d} a_{r-d}$$

Example 3 Use mathematical induction to prove the binomial theorem

$$(a+x)^n = \sum_{r=0}^{n} C(n,r)a^{n-r}x^r$$

Solution *Step 1*: Let $n = 1$. $L.S. = (a+x)^1$, $R.S. = \sum_{r=0}^{1} C(n,r)a^{1-r}x^r$, that is,

$R.S. = C(1,0)a^1x^0 + C(1,1)a^0x^1 = 1a + 1x = a + x = L.S.$

Step 2: Assume the formula is true for $n = k$.

Assume $(a+x)^k = \sum_{r=0}^{k} C(k,r)a^{k-r}x^r$ ⊛

Step 3: Prove the formula is true for $n = k + 1$, that is,

prove $(a+x)^{k+1} = \sum_{r=0}^{k+1} C(k+1,r)a^{k+1-r}x^r$

But $(a+x)^{k+1}$

$= (a+x)(a+x)^k$ This is the L.S. of ⊛ multiplied by $(a+x)$.

$= (a+x)\sum_{r=0}^{k} C(k,r)a^{k-r}x^r$ using step 2

$= a\sum_{r=0}^{k} C(k,r)a^{k-r}x^r + x\sum_{r=0}^{k} C(k,r)a^{k-r}x^r$

$= \sum_{r=0}^{k} C(k,r)a^{k-r+1}x^r + \sum_{r=0}^{k} C(k,r)a^{k-r}x^{r+1}$

Now by changing the limits of the summation,

the second summation $\sum_{r=0}^{k} C(k,r)a^{k-r}x^{r+1}$ can be written $\sum_{r=1}^{k+1} C(k,r-1)a^{k-r+1}x^r$

Thus $(a+x)^{k+1} = \sum_{r=0}^{k} C(k,r)a^{k-r+1}x^r + \sum_{r=1}^{k+1} C(k,r-1)a^{k-r+1}x^r$

In order to combine these two summations you must write each summation so that each has the same limits. You can accomplish this by removing the first term from the first summation and the last term from the second summation.

$$(a+x)^{k+1} = C(k,0)a^{k+1}x^0 + \sum_{r=1}^{k} C(k,r)a^{k-r+1}x^r + \sum_{r=1}^{k} C(k,r-1)a^{k-r+1}x^r + C(k,k)a^0x^{k+1}$$

$$= C(k,0)a^{k+1}x^0 + \sum_{r=1}^{k}[C(k,r) + C(k,r-1)]a^{k-r+1}x^r + C(k,k)a^0x^{k+1}$$

This expression may be simplified using the following facts.

1. $C(k,0) = 1 = C(k+1,0)$
2. $C(k,k) = 1 = C(k+1,k+1)$
3. $C(k,r) + C(k,r-1) = C(k+1,r)$

$$(a+x)^{k+1} = C(k+1,0)a^{k+1}x^0 + \sum_{r=1}^{k} C(k+1,r)a^{k-r+1}x^r + C(k+1,k+1)a^0x^{k+1}$$

This may be combined under one summation giving

$(a+x)^{k+1} = \sum_{r=0}^{k+1} C(k+1,r)a^{k-r+1}x^r$ which is what needed to be proven. ■

The term $C(n,r)a^{n-r}x^r$ is called the **general term** in the expansion of $(a + x)^n$. You will find questions on the general term in the exercises.

Note that since $C(n,r) = \dfrac{n!}{(n-r)!r!} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$

then $C(n,0) = 1$, $C(n,1) = n$, $C(n,2) = \dfrac{n(n-1)}{2!}$, $C(n,3) = \dfrac{n(n-1)(n-2)}{3!}\ldots$

$C(n,n-1) = n$, $C(n,n) = 1$

Hence an alternative form for the binomial theorem is

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}a^{n-r}x^r + \ldots + nax^{n-1} + x^n$$

The Binomial Theorem for n not a Natural Number

The binomial theorem has been proven for n a natural number. A similar result is true when n is *not* a natural number. In this case, however, there are three important differences.

First, the alternative form of the expansion of $(a + x)^n$, with factorials, must be used.

Second, instead of a series with a finite number of terms, you will have an *infinite* series.

Thirdly, the expansion is true only for certain values of a and x. Indeed, the expansion is true only for values of a and x such that $-1 < \dfrac{x}{a} < 1$.

The result (which will not be proven) is the following, where $n \in \mathbb{R}$ but $n \notin \mathbb{N}$.

$$(a+x)^n =$$
$$a^n x^0 + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \ldots + \ldots$$
$$+ \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}a^{n-r}x^r + \ldots \text{(an infinite number of terms)}$$

Frequently this statement is written for $a = 1$ to give the following, where $n \in \mathbb{R}$ but $n \notin \mathbb{N}$ and $-1 < x < 1$.

$$(1+x)^n =$$
$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \ldots$$
$$+ \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots \text{(an infinite number of terms)}$$

9.4 Exercises

1. a) Write the rows of Pascal's triangle for $n = 1$ to $n = 8$.
   b) Evaluate 4! and 6!
   c) Evaluate $C(7,3)$.
   d) Evaluate $C(5,0)$, $C(5,1)$, $C(5,2)$, $C(5,3)$, $C(5,4)$, $C(5,5)$.
   e) Check that your answer for part d) is the same as your answer for row $n = 5$ in part a).

2. Expand each of the following. Do not simplify the $C(n,r)$'s.
   a) $(a + x)^4$ c) $(a + x)^6$ e) $(a + x)^8$
   b) $(a + x)^5$ d) $(a + x)^7$ f) $(a + x)^9$

3. Rewrite each part of question 2 by substituting the values for the $C(n,r)$'s. You may find these either by using the formula $C(n,r) = \frac{n!}{(n-r)!r!}$ or by using the appropriate row in Pascal's triangle.

4. Expand each of the following and simplify.
   a) $(a + y)^4$ c) $(m + z)^3$ e) $(a + 1)^8$
   b) $(b - c)^4$ d) $(2 + x)^5$ f) $(3 - b)^4$

5. Expand each of the following and simplify.
   a) $(a + 2b)^4$ c) $(3 - 2m)^3$ e) $(2x + 3a)^5$
   b) $(3a + 4b)^4$ d) $(4a - 5)^3$ f) $(1 - m^2)^6$

6. Find the first four terms in the expansion of each of the following. Do not simplify.
   a) $(a + b)^{40}$ c) $(3 + x)^{23}$ e) $(2m - 3t)^{25}$
   b) $(m - k)^{39}$ d) $(4 + 2a)^{85}$ f) $(1 + b^2)^{36}$

7. Expand.
   a) $\left(x + \frac{1}{x}\right)^4$ b) $\left(x - \frac{2}{x^2}\right)^5$

8. a) Find the general term for $\left(x^2 + \frac{1}{x}\right)^6$
   b) Find the term containing x^9 in the expansion of the binomial in part a).
   c) Find the term containing x^0 in the expansion of the binomial in part a). This term is called the term independent of x.

9. Prove each of the following facts about the relationship among an element of one row in Pascal's triangle and the elements above it to the left and right.
   a) $C(4,2) = C(3,1) + C(3,2)$
   b) $C(8,5) = C(7,4) + C(7,5)$
   c) $C(n + 1,r) = C(n,r - 1) + C(n,r)$

10. Show each of the following is true by writing each sum explicitly. For example, $\sum_{r=1}^{3} a_r = a_1 + a_2 + a_3$.
    a) $\sum_{r=1}^{8} a_r = \sum_{r=5}^{12} a_{r-4}$ b) $\sum_{r=t}^{m} a_r = \sum_{r=t+d}^{m+d} a_{r-d}$

11. The first two terms in the expansion of $(3 + kx)^7$ are $2187 + 20\,412x$. Find the value of k.

12. By substituting $a = x = 1$ in the expansion of $(a + x)^n$ show that

$$C(n,0) + C(n,1) + C(n,2) + C(n,3) + \ldots + C(n,n - 1) + C(n,n) = 2^n$$

13. Use the expansion of $(a + x)^n$ to show that

$$C(n,0) - C(n,1) + C(n,2) - C(n,3) + \ldots + (-1)^n C(n,n) = 0.$$

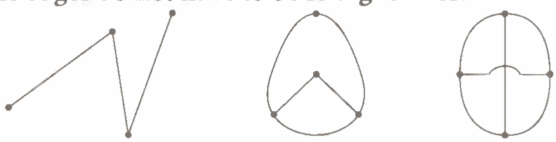
14. a) Find the first four terms in the expansion of $(1 + x)^{-2}$.
    b) Find the first four terms in the expansion of $(1 + x)^{\frac{1}{2}}$.

15. In your answers to question 14 a) and b), give x the value 0.02. Simplify these expressions to obtain approximate values for $(1.02)^{-2}$ and $(1.02)^{\frac{1}{2}}$. How do these values compare with the values of $(1.02)^{-2}$ and $(1.02)^{\frac{1}{2}}$ found using the y^x key of your calculator?

16. a) Write a computer program for the expansion of $(a + x)^n$, $n \in \mathbb{N}$.
    b) Use your program to check your answers for questions 2, 4, 5, and 6.
    c) Adjust your program so that it will handle the first few terms in the expansion of $(a + x)^n$, $n \notin \mathbb{N}$.

Graph Theory

The graphs of graph theory are different from the graphs of lines, parabolas, circles, etc, with which you are familiar. In graph theory a *graph* is defined as a set of points called *vertices*, and a set of lines called *edges*, that connect pairs of vertices. The figure shows several graphs. Notice that the edges do not have to be straight lines.



The systematic study of graph theory began in the 18th century with the famous problems of the seven bridges of Koenigsberg, which you met in the introduction to this chapter.

Graph theory is used extensively to solve problems involved in the management of complex systems such as those in business and industry. The following is a simple example.

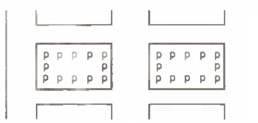


figure 1a

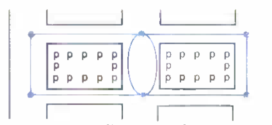


figure 1b

The map in figure 1a shows a section of a city where there are two blocks containing parking meters.

You are hired to find the most efficient route that a parking control officer should travel (on foot) to check the meters. You must consider two things.

1. The parking control officer must patrol all of the meters without retracing steps any more than is necessary.
2. The route should end at the same point it started, where the officer's car is parked.

To solve this problem you need to draw graphs. One such graph is indicated in figure 1b. Notice that each street intersection is a vertex and each sidewalk is an edge. Two possible routes covering this graph are indicated in figure 2a and 2b. It is clear that figure 2b is a better solution because it covers every edge (sidewalk) only once. A route that covers every edge only once is called an *Euler circuit*, provided that the route starts and finishes at the same vertex.

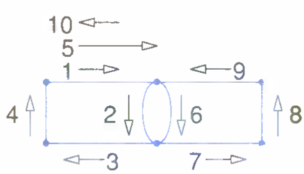


figure 2a

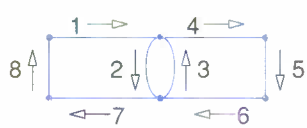


figure 2b

One of the first discoveries made in graph theory was that there are some graphs that do not have any Euler circuits. Two examples of such graphs are shown in figure 3, where it is impossible to start at a vertex and return to the same vertex unless you cover the same edge more than once.

figure 3a

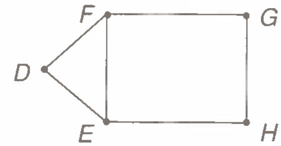


figure 3b

Euler was able to determine the conditions under which a graph had an Euler circuit. He used the concepts of valence and connectedness. The *valence* of a vertex in a graph is the number of edges meeting at that point. (Point A has valence 1, points C, D, G, and H have valence 2, while points B, E and F have valence 3.) A graph is *connected* if every pair of vertices is joined by at least one edge. The graph in figure 3a is not connected because points A and C are not joined by an edge.

Euler proved that a graph G has an Euler circuit if and only if the following two conditions are true.

1. G must be connected
2. Each vertex must have an even valence.

If you examine the graph for the parking control officer given above in figure 1b, you will see that both of these conditions are fulfilled.

It is interesting to note that the graph of the seven bridges of Koenigsberg does not have an Euler circuit. (See page 387.)

The following problems are among the many that can be solved using graph theory.

Computers, radios and TVs make use of printed circuits. These circuits are conductive paths on a sheet of nonconductive material. What conditions must hold for such a circuit to be able to be printed on a single nonconductive sheet?

A telephone company wishes to send long distance messages between cities at the least possible expense in transmission and in the construction of interconnecting telephone lines. What cities should be joined directly by telephone lines? What path should a telephone signal take to travel from city A to city B?

A salesperson must visit several cities always starting and ending at the same city. What route should be taken so that the cost of the trip will be a minimum?

What is the best way to prepare an airplane so that the airplane is on the ground for the least amount of time? Remember that passengers and baggage must be loaded and unloaded, the cabin must be cleaned, food must be brought on board and the airplane must be refueled.

Summary

- The *inductive property of* $\mathbb{N}$: let T be a subset of the natural numbers $\mathbb{N}$. Then T is the *entire* set $\mathbb{N}$, if and only if *both* of the following are true.
  a) 1 is a member of T.
  b) If k is a member of T, then $k + 1$ is also a member of T.
- *The principle of mathematical induction*: a statement involving the natural number n is true for every $n \in \mathbb{N}$ provided the following are true.
  a) The statement is true for $n = 1$.
  b) The truth of the statement for $n = k$ implies the statement is true for $n = k + 1$.
- The *three steps in a proof by mathematical induction.*
  Step 1: Show the statement is true for $n = 1$.
  Step 2: Assume that the statement is true for $n = k$.
  Step 3: Prove the statement is true for $n = k + 1$, *using the result of step 2.*

If the statement to be proved is not true for the first few natural numbers then step 1 must be done for the first number for which the statement is true.

- The principle of mathematical induction can only be used to prove a given formula is true. The principle does not help you to obtain such a formula. If a formula is not given you can try to guess a formula by examining results for $n = 1, 2, 3$, and 4. When you guess a formula you are making a *conjecture.*
- The *binomial theorem* for $n \in \mathbb{N}$:

$$(a + x)^n = C(n,0)a^n x^0 + C(n,1)a^{n-1}x^1 + C(n,2)a^{n-2}x^2 + C(n,3)a^{n-3}x^3 + \ldots + C(n,r)a^{n-r}x^r + \ldots + C(n,n-1)a^1x^{n-1} + C(n,n)a^0x^n$$

Note 1 The value of $C(n,r)$ is $\dfrac{n!}{(n-r)!r!} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$

2 The expansion of the product has $n + 1$ terms.

3 Using the sigma notation, the binomial theorem may be written

$$(a + x)^n = \sum_{r=0}^{n} C(n,r)a^{n-r}x^r$$

- *Pascal's triangle and the binomial theorem*

value of n	expansion of $(a + x)^n$	Pascal's triangle	
1	$1a^1 + 1x^1$	1 1	$C(1,0)$ $C(1,1)$
2	$1a^2 + 2ax + 1x^2$	1 2 1	$C(2,0)$ $C(2,1)$ $C(2,2)$
3	$1a^3 + 3a^2x + 3ax^2 + 1x^3$	1 3 3 1	$C(3,0)$ $C(3,1)$ $C(3,2)$ $C(3,3)$
4	$1a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + 1x^4$	1 4 6 4 1	$C(4,0)$ $C(4,1)$ $C(4,2)$ $C(4,3)$ $C(4,4)$

Note: In Pascal's triangle, the numbers on the left and right of each row are both 1. Each of the other numbers is the *sum* of the two numbers on each side of its in the line *above.*

- $C(n + 1,r) = C(n,r - 1) + C(n,r)$

Inventory

Complete each of the following statements.

1. Step 1 in the principle of mathematical induction usually shows that a statement is true for ______.

2. In step 2 in the principle of mathematical induction you assume that the statement is true for $n =$ ______, then in step 3 you ______ the statement is true for $n =$ ______.

3. Select the word in the bracket to make the statement true.
   a) If a formula is true for $n = 1$, $n = 2$, and $n = 3$, then the formula is (always, sometimes, never) true for all $n \in \mathbb{N}$.
   b) If you assume that a formula is true for $n = k$ and then are able to prove that it is true for $n = k + 1$, then the formula is (always, sometimes, never) true for all $n \in \mathbb{N}$.

4. You conjecture that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$.
   a) For $n = 1$, the *L.S.* equals ______, and the *R.S.* equals ______.
   b) For $n = k$, the *L.S.* equals ______, and the *R.S.* equals ______.
   c) For $n = k + 1$, the *L.S.* equals ______, and the *R.S.* equals ______.

5. You conjecture that
   $$(1 + 1)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{4}\right)\left(1 + \frac{1}{5}\right) \ldots \left(1 + \frac{1}{n}\right) = n + 1$$
   a) For $n = 1$, the *L.S.* equals ______, and the *R.S.* equals ______.
   b) For $n = k$, the *L.S.* equals ______, and the *R.S.* equals ______.
   c) For $n = k + 1$, the *L.S.* equals ______, and the *R.S.* equals ______.

6. You conjecture that $f(n) = \dfrac{n^3 + 3n^2 + 2n}{3}$ is a natural number for all $n \in \mathbb{N}$.
   a) For $n = 1$, the statement becomes ______.
   b) For $n = k$, the statement becomes ______.
   c) For $n = k + 1$, the statement becomes ______.

7. You conjecture that $2^n < n!$ for $n \geq 4$, $n \in \mathbb{N}$.
   a) For $n = 4$, the statement becomes ______.
   b) For $n = k$, the statement becomes ______.
   c) For $n = k + 1$, the statement becomes ______.

8. In the expansion of $(a + x)^5$, there are ______ terms. Unsimplified, these terms are ______.

9. Row $n = 8$ in Pascal's triangle is
   1 8 28 56 70 56 28 8 1
   Therefore, row $n = 9$ is ______.

Review Exercises

1. State the three steps in a proof using mathematical induction.

2. Prove the following statements using mathematical induction, where $n \in \mathbb{N}$.
   a) $4 + 11 + 18 + \ldots + (7n - 3) = \frac{n(7n + 1)}{2}$
   b) $1 + 3 + 5 + \ldots + (2n - 1) = n^2$
   c) $1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$
   d) $1(2) + 2(3) + 3(4) + \ldots + n(n + 1) = \frac{n(n + 1)(n + 2)}{3}$
   e) $1(2) + 2(4) + 3(6) + \ldots + n(2n) = \frac{n(n + 1)(2n + 1)}{3}$
   f) $1(2)3 + 2(3)4 + 3(4)5 + \ldots + n(n + 1)(n + 2) = \frac{n(n + 1)(n + 2)(n + 3)}{4}$

3. Conjecture and prove a formula for the sum of n terms of the series $1 + 7 + 19 + \ldots + (3n^2 - 3n + 1)$

4. Prove the following statements using mathematical induction, where $n \in \mathbb{N}$.
   a) $\frac{1}{1 \times 4} + \frac{1}{4 \times 7} + \frac{1}{7 \times 10} + \ldots + \frac{1}{(3n - 2)(3n + 1)} = \frac{n}{3n + 1}$
   b) $\frac{1}{1 \times 5} + \frac{1}{5 \times 9} + \frac{1}{9 \times 13} + \ldots + \frac{1}{(4n - 3)(4n + 1)} = \frac{n}{4n + 1}$
   c) $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \ldots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}$
   d) $\frac{1}{3^1} + \frac{1}{3^2} + \ldots + \frac{1}{3^n} = 1 - \frac{1}{2}\left(\frac{1}{3^n}\right)$
   e) $\sum_{s=1}^{n}(4s + 1) = \frac{n(4n + 6)}{2}$

5. Prove by mathematical induction
   a) $\sum_{s=1}^{n} s = \frac{n(n + 1)}{2}$
   b) $\sum_{s=1}^{n} s^2 = \frac{n(n + 1)(2n + 1)}{6}$
   c) $\sum_{s=1}^{n} s^3 = \left(\frac{n(n + 1)}{2}\right)^2$

6. Prove the following by mathematical induction
   $4 + 14 + 30 + 52 + \ldots + (3n^2 + n) = n(n + 1)^2$

7. Prove by mathematical induction
   $\frac{1}{3!} + \frac{5}{4!} + \frac{11}{5!} + \ldots + \frac{n^2 + n - 1}{(2n + 1)!} = \frac{1}{2} - \frac{n + 1}{(n + 2)!}$

8. Use mathematical induction to prove that each of the following is true.
   a) $\left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right)\left(1 - \frac{3}{4}\right) \ldots \left(1 - \frac{n}{n + 1}\right) = \frac{1}{(n + 1)!}$
   b) $\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{4}\right) \ldots \left(1 + \frac{1}{n + 1}\right) = \frac{n + 2}{2}$

9. Prove the following where $n \in \mathbb{N}$.
   a) $2n < 2^n$ for $n \geqslant 3$
   b) $(1.1)^n > 1 + \frac{1}{10^n}$ for $n \geqslant 2$

10. Where does mathematical induction *fail* when you try to use it to prove that $100n < n^2$ for all $n \in \mathbb{N}$?

11. Use mathematical induction to prove that $\frac{6n^5 + 15n^4 + 10n^3 - n}{30}$ is a natural number for all $n \in \mathbb{N}$.

12. Use mathematical induction to prove that $\frac{9^n - 4^n}{5}$ is a natural number for all $n \in \mathbb{N}$.

13. a) Show that if you falsely assume that $1 + 5 + 9 + \ldots + (4n - 3) = 2n^2 - n + 3$ is true for $n = k$, then the statement is also true for $n = k + 1$.
    b) Is the formula true for $n = 1$? for all $n \in \mathbb{N}$?

14. Suppose that n circles are drawn in a plane so that each circle intersects all of the others. No two circles are tangent. No three circles pass through the same point. Prove that the plane is divided into $n^2 - n + 2$ non-overlapping regions.

15. Expand each of the following and simplify.

a) $(a + x)^4$ c) $(2 + x)^3$
b) $(3a + b)^5$ d) $(2k - 5m)^6$

16. Show that the formula
$1! + 2! + 3! + \ldots + n! = 3^{n-1}$
is true for $n = 1$, $n = 2$, and $n = 3$.
Is the formula true for all $n \in \mathbb{N}$?

17. a) Turn back to the introduction to this chapter on page 384. Read again about the prince who had to open door after door. Use mathematical induction to prove that if the prince lived forever then he would be able to continue unlocking rooms.

b) Try to use mathematical induction to convince the cow with the ladder that she could climb the ladder to the moon, and beyond the moon.

18. Use mathematical induction to prove that

a) $(1)1! + (2)2! + (3)3! + (4)4! + \ldots + (n)n! = (n + 1)! - 1$

b) $(1)(2) + (2)(3) + (3)(4) + \ldots + n(n + 1) = \dfrac{n(n + 1)(n + 2)}{3}$

c) $\dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \ldots + \dfrac{1}{3^n} = \dfrac{1}{2}\left(1 - \dfrac{1}{3^n}\right)$

d) $\dfrac{1}{5} + \dfrac{1}{25} + \dfrac{1}{125} + \ldots + \dfrac{1}{5^n} = \dfrac{1}{4}\left(1 - \dfrac{1}{5^n}\right)$

19. Given n lines in a plane so that each line intersects all the other lines but no three lines are concurrent, show that the lines divide the plane into $\dfrac{n^2 + n + 2}{2}$ non-overlapping regions.

20. Use mathematical induction to prove that each of the following is a natural number for all $n \in \mathbb{N}$.

a) $\dfrac{6^n - 2^n}{4}$ b) $\dfrac{7^n - 2^n}{5}$ c) $\dfrac{8^n - 3^n}{5}$

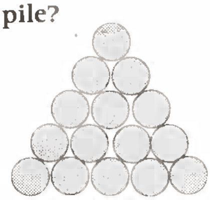
21. Three consecutive terms in the expansion of $(1 + x)^n$ have coefficients 21, 35, and 35. Find the value for n.

22. Prove, by mathematical induction or otherwise, that
$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{r}x^r + \ldots + x^n$, where n is a positive integer and for $0 < r < n$, $\binom{n}{r} = \dfrac{n!}{r!(n - r)!}$
By using this result, or otherwise, and taking $\binom{n}{n} = 1$ find the values of

a) $\sum_{r=1}^{n}\binom{n}{r}$. b) $\sum_{r=1}^{n}(-1)^r\binom{n}{r}$. c) $\sum_{r=1}^{n}r\binom{n}{r}$

(80 H)

23. i) The diagram represents a pile of cylindrical logs; if there are n logs in the lowest row, how many logs are in the pile?



ii) Show that $\dfrac{1}{r(r + 1)} = \dfrac{1}{r} - \dfrac{1}{r + 1}$ for all positive r. Hence prove that
$\sum_{r=1}^{n}\dfrac{1}{r(r + 1)} = \dfrac{n}{n + 1}$

iii) Assuming that $\sum_{r=1}^{n} r^2 = \dfrac{1}{6}n(n + 1)(2n + 1)$
show that
$\sum_{r=0}^{n}(r + 1)^2 = \dfrac{1}{6}(n + 1)(n + 2)(2n + 3)$ and
determine the value of $\sum_{r=1}^{n-1}(r + 2)^2$

(82 SMS)

VECTORS, MATRICES and COMPLEX NUMBERS

with
International Baccalaureate
questions

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and
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CHAPTER TEN

COMPLEX NUMBERS

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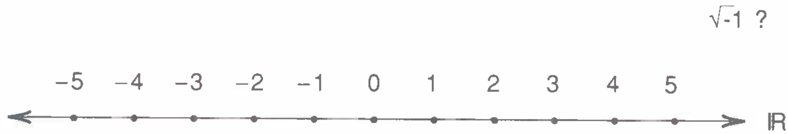
CHAPTER TEN

Complex Numbers

Is $\sqrt{-1}$ a number?

You know that $\sqrt{25} = 5$, $\sqrt{1} = 1$, $\sqrt{0} = 0$, $\sqrt{0.49} = 0.7$, etc. Also, you know that $\sqrt{3}$ cannot be written as a terminating or periodic decimal, but it is quite close to 1.732 050 808. Where could $\sqrt{-1}$ possibly fit in? You know that both positive and negative real numbers square to positive numbers. (For example, $2^2 = 4$, which is positive, and $(-1)^2 = 1$, which is positive.) Zero squares to zero. So what could possibly square to -1?? These considerations indicate that no place can be found for $\sqrt{-1}$ on the real number line.



The mathematicians who first encountered "$\sqrt{-1}$" quite naturally called it "imaginary". And so the name continues to this day, although these types of numbers are frequently used in mathematics and physics. Such numbers are now considered to be no more imaginary than irrational numbers, points, vectors, or any other mathematical object with which you are familiar.

This change came about very slowly. In general, the time-lag is such that the first person to make an important discovery does not see it fully accepted in his or her lifetime. As for many new ideas in mathematics, the introduction of numbers containing an "imaginary" component, now called **complex numbers**, had to go through at least three stages.

1. The ground had to be prepared for the discovery to take place. (In this case, simple "negative numbers" had to be accepted first.)
2. The discovery itself had to occur and be used. This meant going beyond writing "$\sqrt{-1}$", and actually attempting to 'work' with it. (This took a lot of courage, because it required going against accepted practice at the time.)
3. The new discovery had to be applied by other mathematicians—preferably well-respected mathematicians—before it could be fully accepted.

It may surprise you to learn that negative numbers did not gain a foothold in Europe until 1544, through the German mathematician Michael Stifel (1486–1567). The theory of negative numbers had in fact been completely developed more than 800 years before that, in India. However, in Europe, until the 1500s, the difference $a - b$ was deemed meaningful only for a greater than b. A first-degree equation such as $x + 3 = 7$ could be solved, but $x + 7 = 3$ was avoided because a solution was considered to be 'impossible'. And amazingly, this belief was not eradicated until the 1800s!

The theory of second-degree equations (that is, quadratic equations) was even more muddled. Writing such equations in the form $ax^2 + bx + c = 0$ did not appear until 1631, through a posthumous publication by Thomas Harriott (1560–1621). Indeed, 0 was not really considered a number that could be used like the others.

Before Harriott, quadratic equations were broken down into 'types' as follows, each with its own set of rules.
I $ax^2 = bx + c$ II $ax^2 + c = bx$ III $ax^2 + bx = c$.
Writing these equations in the form $ax^2 + bx + c = 0$ takes care of all possibilities as well as simplifying the theory of quadratic equations. This is a good example of the way that the discovery of a new entity can sometimes lead to a simplification of an entire theory.

As well as extending the concept of 'number', the acceptance of negative numbers led mathematicians to try 'solving' equations for which no real solutions could be found, such as $x^2 + 1 = x$. You will see that the solution to such an equation contains "$\sqrt{-1}$".

The first appearance of an imaginary number was in a publication by Girolamo Cardano, in his quest for a solution to a cubic equation. The time span between this first appearance, in 1545, until the full acceptance of complex numbers by the mathematical community in the mid-1800s, exceeded 300 years.

As you work through this chapter, you will have an opportunity to understand how these 'non-real' numbers grew into full acceptance. At the same time, you will extend your concept of number, and unify some of your theories in algebra.

10.1 What is a Complex Number?

The discovery of complex numbers was likely linked to the analysis of the problem of finding two numbers, knowing their sum and their product. The following will guide you through this discovery process.

Example 1 Find two numbers whose sum is 4 and whose product is 3.

Solution Let one number be x, then the other number must be $4 - x$. The product of the numbers is 3, thus

$$x(4 - x) = 3$$
$$4x - x^2 = 3$$
$$x^2 - 4x + 3 = 0$$
$$(x - 1)(x - 3) = 0$$
$$x - 1 = 0 \text{ or } x - 3 = 0$$
$$x = 1 \text{ or } \quad x = 3.$$

If $x = 1$, then $(4 - x) = 3$, and
if $x = 3$, then $(4 - x) = 1$.
Thus the required numbers are 1 and 3.

A very important part of this discovery process is the following.
Check: the sum is $1 + 3 = 4$, and
the product is $(1)(3) = 3$, as required. ■

The next example yields a more complicated solution.

Example 2 Find two numbers whose sum is 6 and whose product is 3.

Solution Let one number be x, then the other must be $6 - x$.
Thus $x(6 - x) = 3$

$$6x - x^2 = 3$$
$$x^2 - 6x + 3 = 0.$$

But this quadratic expression will not factor over the integers. By 'completing the square' (as in section 8.3, page 356),

$$x^2 - 6x + 9 - 9 + 3 = 0$$
$$(x - 3)^2 - 6 = 0$$
$$(x - 3)^2 = 6$$
$$(x - 3) = \sqrt{6} \quad \text{or } (x - 3) = -\sqrt{6}$$
$$x = 3 + \sqrt{6} \text{ or } \quad x = 3 - \sqrt{6}.$$

Thus the numbers required are $(3 + \sqrt{6})$ and $(3 - \sqrt{6})$, or approximately 5.44948...and 0.55051....Once again, although the numbers are irrational, check the result, using the exact values.

Check: the sum is $(3 + \sqrt{6}) + (3 - \sqrt{6}) = 6$, as required, and
the product is $(3 + \sqrt{6})(3 - \sqrt{6}) = 3^2 - 3\sqrt{6} + 3\sqrt{6} - \sqrt{6^2} = 9 - 6 = 3$, as required. ■

In the next example, it will appear that no numbers exist to give the required sum and product. Most mathematicians did not attempt to go beyond the seemingly insurmountable difficulty encountered, but simply classified the problem as unsolvable. Then the Italian mathematician Girolamo Cardano (1501–1576) made a breakthrough in 1572, as you shall see presently.

Example 3 Find two numbers whose sum is 6 and whose product is 10.

Solution Let one number be x, then the other is $6 - x$.

Thus
$$x(6 - x) = 10$$
$$6x - x^2 = 10$$
$$x^2 - 6x + 10 = 0.$$
Completing the square,
$$x^2 - 6x + 9 - 9 + 10 = 0$$
$$(x - 3)^2 + 1 = 0$$
$$(x - 3)^2 = -1.$$

This appears to be 'impossible'. Indeed, the square of any real number, whether positive, negative, or zero, is always greater than or equal to zero. Thus, no real number has a square of -1. Now Cardano had the insight and the courage to simply 'carry on regardless', as follows.
$$x - 3 = \sqrt{-1} \text{ or } x - 3 = -\sqrt{-1}.$$
Thus the numbers 'are' $(3 + \sqrt{-1})$ and $(3 - \sqrt{-1})$. ■

Although these numbers may appear to be meaningless, attempt the following check. Assume that $\sqrt{-1}$ is a number with the usual algebraic properties, including $(\sqrt{-1})^2 = -1$.

Check: the sum is $(3 + \sqrt{-1}) + (3 - \sqrt{-1}) = 6$, and
the product is $(3 + \sqrt{-1})(3 - \sqrt{-1})$
$$= 9 - 3\sqrt{-1} + 3\sqrt{-1} - (\sqrt{-1})^2$$
$$= 9 - (-1) = 10.$$
These 'numbers' seem to work!

As pointed out in the introduction, this breakthrough did not have immediate results. The world had to wait more than 200 years before these inventions were fully accepted. Cardano's ideas were finally formalized by Jean Argand, Leonhard Euler, Karl Friedrich Gauss and other mathematicians towards the end of the 18th century.

The Acceptance Phase

To simplify matters, the symbol i will be used to represent a number (which does not belong to $\mathbb{R}$) that has the property $i^2 = -1$.

It is now tempting to write "$i = \sqrt{-1}$", and by analogy, "$\sqrt{-4} = \sqrt{-1}\sqrt{4} = (i)(2) = 2i$", "$\sqrt{-3} = \sqrt{-1}\sqrt{3} = i\sqrt{3}$", etc. However, great care must be exercised in the use of the symbol "$\sqrt{\ }$" when dealing with roots of negative numbers. Examine the paradox illustrated by the two following 'simplifications'.

$$\sqrt{-4}\sqrt{-1} = \sqrt{(-4)(-1)} = \sqrt{4} = 2 \qquad ①$$

or $\sqrt{-4}\sqrt{-1} = (2i)(i) = 2i^2 = 2(-1) = -2 \qquad ②$

Two different results are obtained. Which of these is correct?

① uses the familiar algebraic property of real positive numbers $\sqrt{a}\sqrt{b} = \sqrt{ab}$

② makes use of the symbol i, where $i^2 = -1$.

The paradox is resolved as follows.

The algebraic property $\sqrt{a}\sqrt{b} = \sqrt{ab}$ is *true only for non-negative real numbers a, b*.

Thus, because $a = -4$ and $b = -1$, ① is *false*, but ② is *correct*.

All positive numbers have two square roots. For example, the two square roots of 9 are $\sqrt{9} = 3$ and $-\sqrt{9} = -3$.

Negative numbers also have two square roots. One must accept that i can represent either $\sqrt{-1}$ or $-\sqrt{-1}$. You will have an opportunity to verify this allegation in the exercises.

DEFINITIONS

Imaginary numbers

- i, and the scalar multiples of i, that is $3i$, $i\sqrt{2}$, $-4i$, etc, shall retain their original name of **imaginary numbers**.
- The set of imaginary numbers will be denoted by $\mathbb{I}$.

Complex numbers

- The sum of a real number and an imaginary number, that is, $z = a + bi$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$, will be called a **complex number**.
- The set of complex numbers will be denoted by $\mathbb{C}$.

Real and imaginary parts

Given the complex number $z = a + bi$, where $a \in \mathbb{R}$, $b \in \mathbb{R}$,

- a is called the **real part** of z, or $a = \text{Re}(z)$
- b is called the **imaginary part** of z, or $b = \text{Im}(z)$

If $b = 0$, z is real. If $b \neq 0$, z is non-real.

Thus it appears that the real numbers form a subset of the complex numbers.

Equality

- Two complex numbers $z = a + bi$ and $w = c + di$, where a, b, c, and d are real, are equal if and only if $a = c$ and $b = d$.

In the following example, you will save time by obtaining the solutions with the quadratic formula, instead of completing the square.

You will use the fact that the solutions of $az^2 + bz + c = 0$ are given by the formula $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example 4 Solve **a)** $2z^2 + 2z + 5 = 0$ **b)** $z^2 - 2z + 3 = 0$.

Solution **a)** Here $a = 2, b = 2, c = 5$.

Substituting in the quadratic formula gives the roots

$$z = \frac{-2 \pm \sqrt{2^2 - (4)(2)(5)}}{2(2)} = \frac{-2 \pm \sqrt{-36}}{4} = \frac{-2 \pm 6i}{4} = \frac{-1 \pm 3i}{2}$$

Thus the solutions are $z = -\frac{1}{2} + \frac{3}{2}i$ or $z = -\frac{1}{2} - \frac{3}{2}i$.

b) Here, $a = 1, b = -2, c = 3$. Using the formula, the roots are

$$z = \frac{2 \pm \sqrt{2^2 - (4)(1)(3)}}{2(1)} = \frac{2 \pm \sqrt{-8}}{2} = \frac{2 \pm i\sqrt{8}}{2} = 1 \pm \frac{i\sqrt{8}}{2}$$

Note that, although these solutions are correct, they can be simplified by writing $\sqrt{8}$ as a mixed radical, as follows.

$$\frac{2 \pm i\sqrt{8}}{2} = \frac{2 \pm i\sqrt{(4)(2)}}{2} = \frac{2 \pm 2i\sqrt{2}}{2} = \frac{2(1 \pm i\sqrt{2})}{2} = 1 \pm i\sqrt{2}.$$

Thus the solutions are $z = 1 + i\sqrt{2}$ or $z = 1 - i\sqrt{2}$. ■

The previous discussion leads to the following formulas for the *addition* and *multiplication* of complex numbers.

- Complex numbers can be added as follows:
  $(a + bi) + (c + di) = (a + c) + (b + d)i$
- Complex numbers can be multiplied as follows:
  $(a + bi)(c + di) = ac + adi + bci + bdi^2$
  but recall that $i^2 = -1$, thus
  $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

Note: that it is easier to use the processes that employ the usual rules of algebra, together with the fact that $i^2 = -1$, than to learn the above formulas.

Thus, when complex numbers are added, or multiplied, other complex numbers are produced. This can be verified in the following example.

Example 5 Given $z = 3 - 5i$ and $w = 1 + i$, calculate the following.

a) $z + w$

b) zw

Solution **a)** $z + w = 3 - 5i + 1 + i = 4 - 4i$.

b)
$$\begin{aligned} zw &= (3 - 5i)(1 + i) \\ &= (3)(1) + (3)(i) - (5i)(1) - (5i)(i) \\ &= 3 + 3i - 5i - 5i^2 \\ &= 3 - 2i - (5)(-1) \\ &= 8 - 2i. \end{aligned}$$ ■

You will be looking at operations in $\mathbb{C}$ more formally, and in more detail, in the next section.

10.1 Exercises

1. Simplify the following.
   a) $(4 + i) + (5 + 2i)$
   b) $(4 - 3i) + (-3 + 3i)$
   c) $(3 + 2i)(5 - i)$
   d) $(3 + i)(3 - i)$
   e) $(1 + 2i)^2$
   f) $(1 - 2i)^2$
   g) $i(6 + 4i)$
   h) $(1 + 2i)^3$

2. Find two numbers whose sum is 7 and whose product is $18\frac{1}{2}$.

3. Find two numbers whose sum is 4 and whose product is 5.

4. Find two numbers whose sum is -1 and whose product is 2.

5. Find the roots of the following equations.
   a) $z^2 - 8z + 25 = 0$
   b) $z^2 + 4z + 5 = 0$
   c) $3z^2 = 5z - 7$

6. The roots of an equation satisfy that equation. By substitution, verify that each of the following is a root of the given equation.
   a) $z = 2; z^2 + 3z - 10 = 0$
   b) $z = -5; z^2 + 3z - 10 = 0$
   c) $z = -i; z^2 + 1 = 0$
   d) $z = 2 - \sqrt{3}; z^2 - 4z + 1 = 0$
   e) $z = 4 + 3i; z^2 - 8z + 25 = 0$
   f) $z = 2i; z^2 + (1 - 2i)z - 2i = 0$
   g) $z = \frac{5}{3} - \frac{i}{3}; 3z^2 - 5z + iz = 0$

7. Consider the expressions $z = 4 + \sqrt{-9}$ and $w = 4 - \sqrt{-9}$. Show that $z + w = 8$ and $zw = 25$ with either of the following interpretations.
   a) let $\sqrt{-9} = 3i$
   b) let $\sqrt{-9} = -3i$

8. The equation $az^2 + bz + c = 0$ is such that $b^2 - 4ac < 0$, where a, b, and c are real. Find the sum and product of the roots of this equation in terms of a, b, and c.

9. Find the roots of the following equations.
   a) $z^2 - 4iz = 0$
   b) $z^2 - 3iz + 4 = 0$
   c) $z^2 = iz - 3$
   d) $z^2 - (1 + i)z + 2 + 2i = 0$

10. Solve the following for the real numbers x and y.
    a) $x + yi = 4 + 6i$
    b) $x + yi = 7i$
    c) $x + yi = (3 - i)(2 + 3i)$
    d) $x + yi = (5 + i)(5 - i)$
    e) $x + yi = (1 + i)^2$
    f) $(x + yi) = (4 - 3i)^2$

In the remaining questions of this exercise, use $z = a + ib$, $w = c + id$, and $u = e + if$, where a, b, c, d, e, f are all real numbers.

11. a) Calculate $z + w$ and $w + z$.
    b) Draw a conclusion concerning the commutativity of the addition of complex numbers.

12. a) Calculate $(z + w) + u$ and $z + (w + u)$.
    b) Draw a conclusion concerning the associativity of the addition of complex numbers.

13. a) Calculate zw and wz.
    b) Draw a conclusion concerning the commutativity of the multiplication of complex numbers.

14. a) Calculate $(zw)u$ and $z(wu)$.
    b) Draw a conclusion concerning the associativity of the multiplication of complex numbers.

15. a) Calculate $zw + zu$
    b) Calculate $z(w + u)$
    c) Draw a conclusion concerning the distributivity of multiplication over addition of complex numbers.

10.2 Operations in $\mathbb{C}$

Through the discovery of complex numbers in section 10.1, you learned that complex numbers could be added and multiplied. In questions 11–15 of 10.1 Exercises, you proved certain properties of these operations in $\mathbb{C}$.

In particular, multiplication in $\mathbb{C}$ is associative. That is, for any complex numbers z, w, u,

$$(zw)u = z(wu)$$

This means that a product such as zwu can be calculated without worrying about the order of the operations. Powers can be calculated in a similar fashion, as in the following example.

Example 1 Calculate a) i^8 b) $(-i)^3$

Solution a) $i^8 = i \times i \times i \times i \times i \times i \times i \times i = i^2 \times i^2 \times i^2 \times i^2 = (-1)^4 = 1.$
b) $(-i)^3 = (-i)(-i)(-i) = -(i \times i \times i) = -(i^2)i = -(-1)i = i.$ ■

You will now see that other operations can be defined in $\mathbb{C}$. The first person to use the four operations of addition, subtraction, multiplication, and division of complex numbers was Raffaello Bombelli. A contemporary of Cardano, Bombelli published his work in Bologna, Italy, in 1572.

The Subtraction of Complex Numbers

The usual rules of algebra are applied to define the subtraction of complex numbers as follows.

$$(a + bi) - (c + di) = a + bi - c - di = (a - c) + (b - d)i.$$

Example 2 Calculate $(3 + 4i) - (5 - 2i)$.

Solution As before, it is easier to go through the process than to learn the formula.
$(3 + 4i) - (5 - 2i) = 3 + 4i - 5 + 2i = -2 + 6i.$ ■

The Division of Complex Numbers

One operation that has not yet been mentioned in $\mathbb{C}$ is division.

Attempting to 'divide', say, $6 + 2i$ by i might yield $\frac{6 + 2i}{i} = \frac{6}{i} + \frac{2i}{i} = \frac{6}{i} + 2.$

This answer is not in the form $a + bi$. Is it a complex number, or something new?

Observe the following strategy.

$\frac{6}{i} = \frac{6}{i} \times \frac{i}{i} = \frac{6i}{-1} = -6i.$

Thus, division by i *does* yield a complex number!

Hence, $\frac{6 + 2i}{i} = -6i + 2$, or $2 - 6i$.

A similar trick is used to divide a more general complex number, as in the following example. Observe the solution carefully.

Example 3 Divide $(3 - 8i)$ by $(1 - 2i)$.

Solution

$$\frac{3-8i}{1-2i} = \frac{(3-8i)(1+2i)}{(1-2i)(1+2i)} = \frac{3+6i-8i-16i^2}{1-(-4)} = \frac{3-2i+16}{1+4} = \frac{19-2i}{5}$$
$$= \frac{19}{5} - \frac{2}{5}i \quad \blacksquare$$

The numbers $(1 + 2i)$ and $(1 - 2i)$ are known as **complex conjugates**, or simply **conjugates**.

DEFINITION The complex conjugate of $z = a + bi$ is $\bar{z} = a - bi$.

The quotient of two complex numbers $z = a + bi$ and $w = c + di$ is obtained as follows.

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)+(bc-ad)i}{c^2-d^2(-1)} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i.$$

Again, the above formula represents the definition of the division of two complex numbers, but it is much easier to learn the process rather than the formula. In the work above, you have used one of the important properties of conjugates. That is, given $z = c + di$, then $z\bar{z} = c^2 + d^2$ is real.

In the exercises, you will have an opportunity to prove the other properties of conjugates that are listed at the end of this section.

Example 4 Simplify the following. a) $\frac{2+i}{41i} + \frac{1-3i}{4-5i}$ b) $\frac{2}{1-i} - \frac{3i}{2+3i}$

Solution

a) $$\frac{2+i}{41i} + \frac{1-3i}{4-5i} = \frac{(2+i)(i)}{41i(i)} + \frac{(1-3i)(4+5i)}{(4-5i)(4+5i)}$$
$$= \frac{2i-1}{-41} + \frac{4+5i-12i-15(-1)}{16-(-25)}$$
$$= \frac{2i-1}{41} + \frac{19-7i}{41} = \frac{18}{41} - \frac{5}{41}i$$

b) $$\frac{2}{1-i} - \frac{3i}{2+3i} = \frac{2(1+i)}{(1-i)(1+i)} - \frac{3i(2-3i)}{(2+3i)(2-3i)}$$
$$= \frac{2+2i}{1+1} - \frac{6i+9}{4+9}$$
$$= 1 + i - \frac{(9+6i)}{13}$$
$$= \frac{13+13i-9-6i}{13} = \frac{4}{13} + \frac{7}{13}i \quad \blacksquare$$

Here is a summary of the essential properties of $\mathbb{C}$ with respect to the operations of addition and multiplication.

SUMMARY

E.	Equality	$a + bi = c + di$ if and only if $a = c$ and $b = d$
S.	Sum	$(a + bi) + (c + di) = (a + c) + (b + d)i$
P.	Product	$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

Given any numbers z, w and u of $\mathbb{C}$,

1.	Closure	$z + w$ and zw belong to $\mathbb{C}$
2.	Commutativity	$z + w = w + z$ and $zw = wz$
3.	Associativity	$(z + w) + u = z + (w + u)$ and $(zw)u = z(wu)$
4.	Distributivity	$z(w + u) = zw + zu$
5.	Neutral elements	$z + 0 = 0 + z = z$ and $(z)(1) = (1)(z) = z$
6.	Inverse elements	$z + (-z) = (-z) + z = 0$ and $z\left(\frac{1}{z}\right) = \left(\frac{1}{z}\right)z = 1$, provided that $z \neq 0$

Note 1 The neutral elements of $\mathbb{C}$ are defined as follows.
For addition: $0 = 0 + 0i$ For multiplication: $1 = 1 + 0i$

2 All of these properties apply to real numbers. You can check this by letting the imaginary part of each complex number be zero.

3 By virtue of satisfying all these properties, the set $\mathbb{C}$ is called a *field*.

Properties involving conjugates

Consider two complex numbers z, w, and their conjugates $\bar{z}$, $\bar{w}$.

1. $z + \bar{z} = 2\text{Re}(z)$
2. $z - \bar{z} = 2i\text{Im}(z)$
3. $z\bar{z} = [\text{Re}(z)]^2 + [\text{Im}(z)]^2$
4. $\overline{(z + w)} = \bar{z} + \bar{w}$
5. $\overline{(zw)} = \bar{z}\,\bar{w}$
6. $\bar{\bar{z}} = z$
7. Division: $\dfrac{z}{w} = \dfrac{z\bar{w}}{w\bar{w}} = \dfrac{z\bar{w}}{|w^2|}$

10.2 Exercises

1. Simplify the following ($n \in \mathbb{N}$).
   a) i^3
   b) i^4
   c) i^5
   d) i^6
   e) $\dfrac{1}{i}$
   f) i^{-2}
   g) i^{4n}
   h) i^{4n+1}
   i) i^{4n+2}

2. Simplify the following.
   a) $(5 - i) - (4 + 3i)$
   b) $(-1 + i) - (1 - i)$
   c) $4(3 + 2i) - 2(6 + i)$
   d) $(2 + i)^2 - (3 - 2i)^2$
   e) $(5 + 3i)(3 - i) + 3(1 + i)(1 - i) - 4(3 + 7i)i$

3. Express in the form $a + ib$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.
   a) $\dfrac{1 + 4i}{i}$
   b) $\dfrac{1 + 4i}{2 + i}$
   c) $\dfrac{7 - 3i}{-i}$
   d) $\dfrac{5 + 2i}{5 - 2i}$
   e) $\dfrac{1}{2 + 3i}$
   f) $\dfrac{i}{3 + 4i}$
   g) $\dfrac{1}{3 + 4i} + \dfrac{1}{3 - 4i}$
   h) $\dfrac{1}{6 + 5i} - \dfrac{1}{(6 + 5i)^2}$

4. Given $z = \cos\theta + i\sin\theta$ and $w = \cos\theta - i\sin\theta$, prove the following.

   (Use the formulas on page 542.)

   a) $z + w = 2\cos\theta$
   b) $z - w = 2i\sin\theta$
   c) $zw = 1$
   d) $z^2 = \cos 2\theta + i\sin 2\theta$
   e) $w^2 = \cos 2\theta - i\sin 2\theta$
   f) $\dfrac{1}{1 + w} = \dfrac{1}{2} + \dfrac{i}{2}\tan\dfrac{1}{2}\theta$

For questions 5 and 6, refer to the properties listed in section 10.2.

5. By making the imaginary part zero, verify that the following properties in $\mathbb{C}$ also hold true in $\mathbb{R}$.
   a) properties *E*, *S* and *P*.
   b) the properties of conjugates 1, 2, and 3.
   c) Is the set $\mathbb{R}$ also a field?

6. Prove the properties of conjugates 1, 2, 4, 5, and 6.

7. Prove that if $zw = 0$, then $z = 0$ or $w = 0$. (Hint: if $z = a + bi$ and $w = c + di$, you must prove $a = b = 0$ or $c = d = 0$.)

8. a) Prove that $z + \dfrac{1}{z} = \dfrac{z^2\bar{z} + \bar{z}}{|z|^2}$
   b) Simplify $3 - 2i + \dfrac{1}{3 - 2i}$

9. The equation $az^2 + bz + c = 0$ is such that $b^2 - 4ac < 0$, where a, b, and c are real. Prove that the roots of the equation are complex conjugates.

10. Solve the following for the real numbers x and y.
    a) $\dfrac{x + yi}{4 + i} = 4 - i$
    b) $x + yi = \dfrac{6 - 2i}{3 + 5i}$

11. Find the real and imaginary parts of
    a) $\dfrac{2 + i}{1 + 5i} + \dfrac{7}{1 - 5i}$
    b) $(1 + i)^4$

12. Given $z = \dfrac{5i - 4}{i} + \dfrac{3i - 4}{1 - 2i}$, find the real and imaginary parts of z, and of z^2.

13. Simplify $(1 + i)^4(4 - 3i)^2(1 - i)^4(4 + 3i)^2$.

14. Find the number b such that $\left|\dfrac{2 - 3i\sqrt{5}}{6 + bi}\right| = 2$.

15. Find z in terms of $\cos\alpha$ and $\sin\alpha$, if $z^2 - 2z\cos\alpha + 1 = 0$.

10.3 Geometric Representation of a Complex Number

The previous sections have shown you that there is some validity in working with non-real numbers. However, there is still one major difficulty.

You know how to represent an integer, a rational number, even an irrational number on a number line. Where can i be placed? Where can the multiples of i, and other non-real numbers, be represented? These questions will be answered in this section.

There is a parallel between the history of civilization and the growth of the number sets used. However, the partial list below follows a logical rather than a historical thread.

The simplest number set is the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$

Next is the set of integers, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

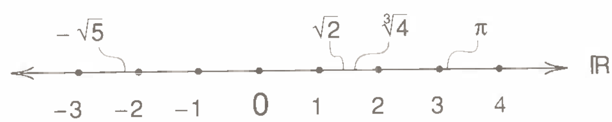
(The symbol $\mathbb{Z}$ comes from the German "zahlen", to count.)

Then the set of rational numbers,

$$\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a \in \mathbb{Z} \text{ and } b \in \mathbb{N} \right\}$$

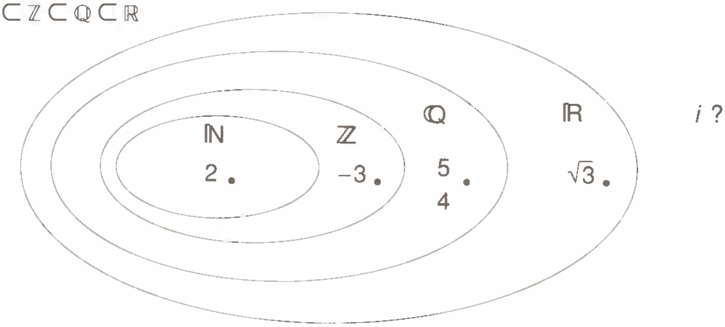
(The symbol $\mathbb{Q}$ comes from the word "quotient".)

It appears that the most complete set is the set of real numbers, $\mathbb{R}$, which is the union of $\mathbb{Q}$ and the set of all irrational numbers.



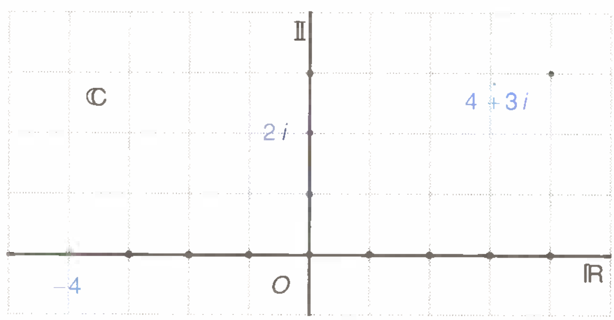
Recall that the representation of the real number line $\mathbb{R}$ is indistinguishable from $\mathbb{Q}$. However, for example, the irrational numbers $\sqrt{2}, -\sqrt{5}, \pi, \sqrt[3]{4}$ *are* elements of $\mathbb{R}$, but they do *not* belong to $\mathbb{Q}$.

Note: Each of the number sets described is a subset of its successor, as follows.
$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$



Venn diagram

As observed in the introduction to this chapter, there is certainly no appropriate spot on the real number line for i. However, $\mathbb{R} \subset \mathbb{C}$, the set of complex numbers. Indeed, note that $x \in \mathbb{R}$ can be written $1x + 0i \in \mathbb{C}$. A brilliant idea came from the Swiss mathematician Jean Argand (1768-1822), from a work published in 1806. He simply let the non-real numbers burst out of the real number line $\mathbb{R}$, by drawing another line $\mathbb{I}$ through 0, bearing the purely imaginary numbers.
Hence, any point in the *entire plane* thus created will represent a complex number. The origin, O, represents the number 0 (that is, $0 + 0i$).

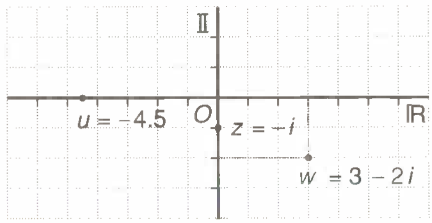


His invention bears the name **complex plane** or **Argand diagram.** Similar methods of picturing complex numbers were invented independently, at about the same time, by a Norwegian surveyor, Caspar Wessel (1745-1818), and by the famous German mathematician Karl Friedrich Gauss (1777-1855).
Note: The real number line, or real axis, is a subset of the complex plane. That is, *all numbers* can be represented by a point in this plane.

- If a number is on the $\mathbb{R}$-axis then it is real. It could be a natural number, an integer, a rational number, or an irrational number. [example: −4]
- If a number is on the $\mathbb{I}$-axis, then it is imaginary. [example: $2i$]
- If a number is *not* on the $\mathbb{R}$-axis, then it is *non-real*. [example: $4 + 3i$]

Example 1 Locate each of the following numbers in the complex plane.
$z = -i$, $w = 3 - 2i$, $u = -4.5$

Solution To plot the point representing $z = -i$, go one unit down from 0 on the $\mathbb{I}$-axis.
To plot the point representing $w = 3 - 2i$, go 3 units to the right of 0 on the $\mathbb{R}$-axis, then 2 units down, parallel to the $\mathbb{I}$-axis.
To plot the point representing $u = -4.5$, go 4.5 units to the left of 0 on the $\mathbb{R}$-axis.



■

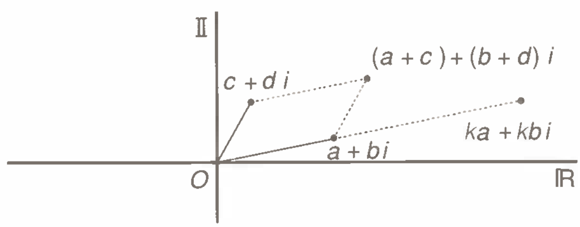
Consequences of Representation in the Complex Plane

1. *Complex numbers as two-dimensional vectors*
Note that $z = a + bi$ could be written as the ordered pair (a,b). This was first done by Sir William Hamilton in 1835.

Compare the addition of complex numbers with the addition of vectors of $\mathbb{V}_2$.

- $(a + bi) + (c + di) = (a + c) + (b + d)i$
- $\overrightarrow{(a,b)} + \overrightarrow{(c,d)} = \overrightarrow{(a + c, b + d)}$

Similarly, compare "multiplication of a complex number by a real number" with "multiplication by a scalar" in $\mathbb{V}_2$.
- $k(a + bi) = ka + kbi$
- $k\overrightarrow{(a,b)} = \overrightarrow{(ka,kb)}$



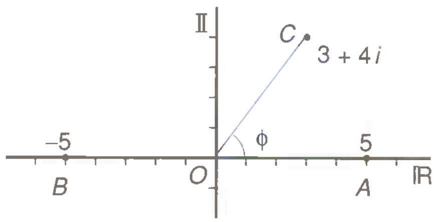
You can see that the results match exactly.
Thus, the set $\mathbb{C}$ of complex numbers can be considered a **vector space**.
All the properties of vectors of $\mathbb{V}_2$ with which you are familiar, including the geometric properties of addition and subtraction, can be applied to complex numbers. (See page 61.)

2. *The modulus of a complex number*

Consider the real number 5. It may be represented in the complex plane either by the point A, or the position vector of A, that is, $\overrightarrow{OA}$.
The absolute value of 5 is the length or magnitude of $\overrightarrow{OA}$, that is, $|\overrightarrow{OA}| = |5| = 5$.
Similarly, if B is the point representing the real number -5, the absolute value of -5 equals $|\overrightarrow{OB}| = |-5| = 5$.
In the same way, if C is the point representing the complex number $w = 3 + 4i$, then

$$|\overrightarrow{OC}| = |w| = \sqrt{3^2 + 4^2} = 5.$$

$|w|$ is called the length, magnitude, absolute value or **modulus** of the complex number w.



3. *The argument of a complex number*

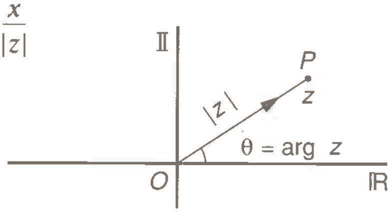
Although you know that $|w| = 5$, this fact is not sufficient to locate w precisely in the complex plane. (The numbers 5 and -5 also have a modulus of 5. Yet all three of these numbers are different, and are represented by different points). However, w *can* be fully determined by its modulus *and* the angle ϕ that it makes with the positive real axis.

In this case, $\tan \phi = \dfrac{4}{3}$ so $\phi \doteq 53°$ or $\phi \doteq 0.927$ radians.

Alternatively, ϕ can be determined by both $\sin \phi = \dfrac{4}{5}$ and $\cos \phi = \dfrac{3}{5}$, giving as before $\phi \doteq 53°$ or 0.927 rad.
ϕ is called an **argument of** w, written **arg** w.

PROPERTY In general, if $z = x + yi$, then the modulus of z, $|z| = \sqrt{x^2 + y^2}$

PROPERTY In general, if $z = x + yi$ is represented by the point P, then $\theta = \arg z$ is the angle that $\overrightarrow{OP}$ makes with the positive real axis; that is, θ is determined by $\sin \theta = \dfrac{y}{|z|}$ and $\cos \theta = \dfrac{x}{|z|}$



Note 1 Arguments of complex numbers are frequently expressed in radians. The reasons for this will be made clear in section 10.9. (Recall that π radians = 180°. There is a table of degree-radian equivalences on page 543.)

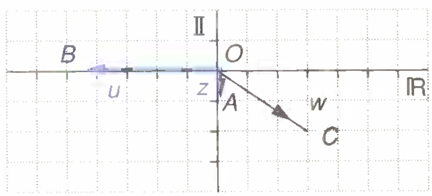
2 Any angle coterminal with θ is also an argument of z. That is, if θ is an argument of z, then so is any other angle $\theta + 2k\pi$ (or $\theta° + 360k°$), $k \in \mathbb{Z}$.

3 The principal argument of z is the angle θ chosen such that $-\pi < \theta \leqslant \pi$ (or $-180° < \theta \leqslant 180°$).
The principal argument is denoted by Arg z.

Example 2 a) Draw the following complex numbers as vectors in the complex plane.
$z = -i$, $u = -4.5$, $w = 3 - 2i$

b) Find the modulus and the principal argument for each of z, u, and w. (Give the arguments correct to the nearest degree.)

Solution a) The numbers z, u, w are represented by the points A, B, C respectively, or by the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ respectively.



b) $|z| = \sqrt{0^2 + (-1)^2} = 1$ Arg $z = -90°$

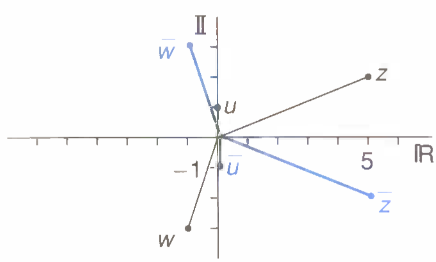
$|u| = 4 \cdot 5$ Arg $u = 180°$

$|w| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$ $\sin(\text{Arg } w) = \dfrac{-2}{\sqrt{13}}$ and $\cos(\text{Arg } w) = \dfrac{3}{\sqrt{13}}$,

so Arg $w = -34°$ ■

4. *Conjugates in the complex plane*

Consider the following complex numbers:
$z = 5 + 2i$, so $\overline{z} = 5 - 2i$; $w = -1 - 3i$, so $\overline{w} = -1 + 3i$; $u = i$, so $\overline{u} = -i$



You can see from the diagram that the conjugate of a complex number is obtained by *reflecting the complex number in the real axis*.

5. *Order in the set* $\mathbb{C}$

You are familiar with the order property of real numbers. That is, given any two distinct real numbers a and b, then either $a > b$ or $b > a$. This is interpreted on the real number line by saying that "greater than" is equivalent to "to the right of". Since $\mathbb{C}$ cannot be represented by a line, *it is impossible to "order" complex numbers*. The task of defining an order relation in $\mathbb{C}$ would be equivalent to that of defining an order relation for points in a plane.

However, since the modulus of a complex number is real, it is possible to say that the modulus of one complex number is greater than the modulus of another.

In the exercises, you will familiarize yourself more with the visual aspects of complex numbers.

SUMMARY

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

The complex plane is determined by a real axis and an imaginary axis, crossing at 0.

Complex numbers have all the properties of vectors of $\mathbb{V}_2$.

If $z = x + yi$ is represented by P in the complex plane:
The modulus of z, $|z| = \sqrt{x^2 + y^2}$
Any argument of z is the angle that $\overrightarrow{OP}$ makes with the positive real axis, that is, an angle satisfying both $\sin(\arg z) = \frac{y}{|z|}$ and $\cos(\arg z) = \frac{x}{|z|}$

The complex conjugates $z = x + yi$ and $\bar{z} = x - yi$ are reflections of each other in the real axis.

There is no order in $\mathbb{C}$.

10.3 Exercises

In these exercises, where appropriate, calculate all arguments in degrees, correct to the nearest degree.
In questions 1–6, use the numbers $z = 1 + 3i$, $w = 12 - 5i$, $p = 6i$, $q = -4 - i$, $u = -3 + 2i$.

1. a) Plot the points representing numbers z, w, p, q, u in a complex plane.
   b) Find the conjugates $\bar{z}, \bar{w}, \bar{p}, \bar{q}$, and $\bar{u}$, and plot them in the same complex plane.

2. a) Find the moduli $|z|, |w|, |p|, |q|$, and $|u|$.
   b) Find the arguments Arg z, Arg w, Arg p, Arg q, Arg u.

3. a) Find the moduli of the conjugates, namely $|\bar{z}|$ and $|\bar{w}|$.
   b) Find arguments of the conjugates, namely Arg $(\bar{z})$ and Arg$(\bar{w})$.
   c) Draw conclusions about the modulus of a conjugate and the argument of a conjugate.

4. a) Attempt to list the numbers z, w, p, q, u in order, from smallest to largest.
   b) Attempt to list the moduli of these numbers in order, from smallest to largest.

5. a) Calculate the number $z + w$.
   b) Draw z, w, and $z + w$ as vectors in a complex plane.
   c) Use the diagram in b) to explain how $z + w$ could be considered an addition of vectors.

6. a) Calculate $z + \bar{z}$, $z - \bar{z}$ and $z\bar{z}$.
   b) Plot $z, \bar{z}, z + \bar{z}, z - \bar{z}$, and $z\bar{z}$ in a complex plane.
   c) Verify that
   $z + \bar{z} = 2\text{Re}(z)$,
   $z - \bar{z} = 2i\text{Im}(z)$, and
   $z\bar{z} = |z|^2$.

7. Describe the modulus, the argument, and the conjugate of the following.
   a) a real number
   b) an imaginary number

8. If θ is any angle, calculate the modulus of $z = \cos\theta + i\sin\theta$ and $w = 3\cos\theta - 3i\sin\theta$.

9. a) Plot the points A and B representing the numbers $z = -2 + 3i$ and $w = 8 - i$ respectively in a complex plane.
   b) Calculate $u = \frac{1}{2}z + \frac{1}{2}w$ and plot the point M representing u on the same diagram.
   c) Calculate $v = \left(\frac{3}{4}\right)z + \frac{1}{4}w$ and plot the point N representing v on the same diagram.
   d) Describe M and N geometrically with reference to A and B.

10. Consider the numbers
   $z = 1 + i\sqrt{3}$, $z_1 = iz$, $z_2 = iz_1$, and $z_3 = iz_2$.
   a) Calculate the numbers z_1, z_2, and z_3.
   b) Draw all four numbers as vectors in a complex plane.
   c) Calculate the modulus and an argument of all four numbers.
   d) Draw conclusions on the effect of i as a multiplier in the complex plane.

11. Given $z = 1 + i\sqrt{3}$,
   a) calculate z^2 and z^3,
   b) plot z, z^2, and z^3 in a complex plane
   c) discuss the statement: "$\sqrt[3]{-8} = 1 + i\sqrt{3}$".

12. a) If $z = 3 + 3i$, find $|z|$ and Arg z.
   b) Verify that z could be expressed as $z = 3\sqrt{2}(\cos 45° + i\sin 45°)$
   (For an exact solution, use the table on page 543.)

13. If $|z| = r$ and arg $z = \theta$, show that the number z can be represented in the form $z = r(\cos\theta + i\sin\theta)$. (This is known as the **polar form** or **modulus-argument form** of a complex number.)

10.4 Equations in $\mathbb{C}$

Recall the following vocabulary.

- $az^2 + bz + c = 0$, where $a \neq 0$, is a quadratic equation, or a polynomial equation of degree 2.
- $az^3 + bz^2 + cz + d = 0$, where $a \neq 0$, is a cubic equation, or a polynomial equation of degree 3.
- $a_n z^n + a_{n-1} z^{n-1} + \ldots + a_2 z^2 + a_1 z + a_0 = 0$, where $a_n \neq 0$ (*) is a polynomial equation of degree n.

Consider the quadratic equation $az^2 + bz + c = 0$. Recall that the solutions are given by $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If the variable $z \in \mathbb{R}$, three cases need to be considered.

1. If $b^2 - 4ac > 0$, then z can take two different real values.
2. If $b^2 - 4ac = 0$, then z has a single real value.
3. If $b^2 - 4ac < 0$, then there are no values for z.

Now if you let z take any values in $\mathbb{C}$, roots of $az^2 + bz + c = 0$ will always exist. The three previous cases can be replaced by the following single statement.
All quadratic equations *have two roots*
(which may or may not be real, and may or may not be equal).

This result can be extended to the following general case, which is one version of the **fundamental theorem of algebra**.

THEOREM A polynomial equation of degree n always has n complex roots.

Note 1 The coefficients $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are not necessarily real.
2 Recall that "complex roots" includes real roots.
3 Some of the roots may be equal.
4 It was not possible to make such a clean statement before the advent of complex numbers. In this way, complex numbers have *simplified* our view of algebra.
5 Given that the roots of the polynomial equation (*) are $z_1, z_2, \ldots, z_n$, then (*) is expressible in the factored form $a_n(z - z_1)(z - z_2)\ldots(z - z_n) = 0$, or $(z - z_1)(z - z_2)\ldots(z - z_n) = 0$, since $a_n \neq 0$.

The Factor Theorem

Consider the polynomial $p(z) = (z - z_1)(z - z_2)\ldots(z - z_n)$. You can see that $p(z_k) = 0$ where $k \in \{1, \ldots, n\}$. The **factor theorem** is stated as follows.

THEOREM If $p(z_k) = 0$, then $(z - z_k)$ is a factor of $p(z)$.

This theorem is exactly the same as for $\mathbb{R}$. The factor theorem can be used as an aid in factoring polynomials.

Example 1 Form a quadratic equation whose roots are

a) $3 + 2i$ and $3 - 2i$

b) $3 + 2i$ and $1 - i$

Solution **a)** In factored form, a quadratic equation is

$$(z - [3 + 2i])(z - [3 - 2i]) = 0$$
$$z^2 - [3 + 2i + 3 - 2i]z + [3 + 2i][3 - 2i] = 0$$
$$z^2 - 6z + 13 = 0.$$

Notice that the coefficients of this equation are real.

b) $(z - [3 + 2i])(z - [1 - i]) = 0$

$$z^2 - [3 + 2i + 1 - i]z + [3 + 2i][1 - i]$$
$$z^2 - [4 + i]z + 5 - i = 0 \quad ■$$

Notice that the coefficients of this equation are not all real.

Example 2 By solving the equation $z^3 = 1$, find the three cube roots of 1.

Solution The equation is equivalent to $z^3 - 1 = 0$, a cubic. By the fundamental theorem of algebra, you know that there are three (not necessarily distinct) roots.

To solve the equation, express it in factored form.
[Recall that $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$]

$$z^3 - 1 = 0$$
$$\Rightarrow (z - 1)(z^2 + z + 1) = 0$$

Thus, $z - 1 = 0$ or $z^2 + z + 1 = 0$

$$z = 1 \qquad z = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)}$$
$$z = \frac{-1 \pm i\sqrt{3}}{2}$$

Hence, the cube roots of 1 are $1, -\frac{1}{2} + \frac{i\sqrt{3}}{2}, -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ ■

One of the most useful aspects of working with complex numbers is that each equation in $\mathbb{C}$ incorporates two equations in $\mathbb{R}$, because of the definition of the equality of two complex numbers. This will be illustrated in the following examples.

Example 3 Solve the equation $(2 + i)z - 4i = 0$ by writing $z = x + iy$, where $x, y \in \mathbb{R}$, and solving a system of equations in x and y.

Solution

$$(2 + i)z - 4i = 0$$
$$(2 + i)(x + iy) - 4i = 0$$
$$2x + 2iy + ix - y - 4i = 0$$
$$(2x - y) + i(2y + x - 4) = 0$$

That is, $2x - y = 0$ ①

and $x + 2y - 4 = 0$ ②

Thus, the original equation in $\mathbb{C}$ has produced *two* equations in $\mathbb{R}$.

$2 \times$ ① + ② gives $5x - 4 = 0$, so $x = \frac{4}{5}$. Substituting this into ① gives $y = \frac{8}{5}$.

Thus, $z = x + iy = \frac{4}{5} + \frac{8}{5}i$. ■

Note: This equation could also be solved by writing

$z = \frac{4i}{2 + i}$ and simplifying, to obtain $z = \frac{4}{5} + \frac{8}{5}i$.

Example 4 Solve the equation $z^2 = 16 - 30i$.

Solution $z \in \mathbb{C}$. Hence, let $z = x + iy$, where x and y are real.

Thus

$$(x + iy)^2 = 16 - 30i$$
$$x^2 + 2xyi - y^2 = 16 - 30i$$
$$(x^2 - y^2) + 2xyi = 16 - 30i$$

That is, $x^2 - y^2 = 16$ ①

and $2xy = -30$ ②

Once again, the original equation in $\mathbb{C}$ has produced *two* equations in $\mathbb{R}$.

From ②, $y = -\frac{30}{2x} = -\frac{15}{x}$ ③

Substituting into ①, $x^2 - \left(-\frac{15}{x}\right)^2 = 16$

$$x^4 - 225 = 16x^2 \quad \text{multiplying both sides by } x^2$$
$$x^4 - 16x^2 - 225 = 0$$
$$(x^2 - 25)(x^2 + 9) = 0$$
$$x^2 = 25 \quad \text{or } x^2 = -9 \text{ (which is impossible, since } x \text{ is real)}$$
$$x = 5 \text{ or } x = -5$$

and so $y = -3$ or $y = 3$, from ③

Thus, $z = 5 - 3i$ or $z = -5 + 3i$. ■

Note: These numbers can be considered the 'square roots' of the number $(16 - 30i)$, since the original equation was $z^2 = 16 - 30i$. However, the notation "$\sqrt{(16 - 30i)} = 5 - 3i$" will be avoided, since there is more than one square root. The term "principal square root" can only be used in relation to a positive real number.

10.4 Exercises

1. State the roots of the following equations.
   a) $(z - 2)(z + 3) = 0$
   b) $(z - 1 - i)(z - 1 + i) = 0$
   c) $(4z - 1)(z + i)(z - i) = 0$
   d) $z(z + 2i)(2z - 3 - 4i) = 0$

2. Which of the equations of question 1 are polynomial equations with real coefficients?

3. Find quadratic equations in the form $az^2 + bz + c = 0$ with the following roots.
   a) $4i$ and $2 + i$
   b) $p + qi$ and $p - qi$

4. The quadratic equation $az^2 + bz + c = 0$ is such that the coefficients a, b, c, are real, and $b^2 - 4ac < 0$.
   a) Prove that the roots of this quadratic equation must be conjugates.
   b) Use this fact to show that the non-real roots of any polynomial equation with real coefficients must be conjugates, in pairs.

5. Find cubic equations in the form $az^3 + bz^2 + cz + d = 0$ with the following roots.
   a) $4i$, $2 + i$, and $1 - 3i$
   b) 0, $p + qi$, and $p - qi$.

6. Prove that a cubic equation with real coefficients always has at least one real root.

7. a) Verify that $w = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ is a cube root of 1.
   b) Locate w in the complex plane.

8. Solve the equation $(3 - 5i)z + 1 + 2i = 0$ by writing $z = x + iy$, $x, y \in \mathbb{R}$, and solving a system of equations in x and y.

9. Repeat question 8 for the equation $(a + bi)z + c + di = 0$, $a, b, c, d, \in \mathbb{R}$. Does this equation *always* have a unique root?

10. By solving $z^2 = i$, find the two square roots of i. Locate these roots in the complex plane.

11. Given that the square roots of i are $\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$, use the quadratic formula to solve the equation $z^2 - 3z - iz + 2 = 0$. Express the roots in the form $a + bi$, $a, b \in \mathbb{R}$.

12. a) By solving $z^3 + 1 = 0$, find the three cube roots of -1.
    b) Locate these roots in the complex plane.

13. Discuss the validity of the following statements.
    a) $z^2 + w^2 = 0 \Rightarrow z = 0$ and $w = 0$.
    b) $z^3 - w^3 = 0 \Rightarrow z = w$.

14. a) Use the factor theorem to show that $(3z - 2)$ and $(2z + 1)$ are factors of the polynomial $p(z) = 6z^4 - 25z^3 + 32z^2 + 3z - 10$.
    b) Hence solve $p(z) = 0$.

15. It is given that $2 + i$ and $-2 + i$ are two of the roots of the equation $z^4 - 6z^2 + 25 = 0$.
    a) Use this information to find all the roots of the equation.
    b) Show that the representations of these roots in a complex plane are the vertices of a rectangle.

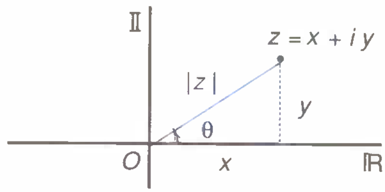
16. Show that the equation $z^2 - rz - iz + ir = 0$, $r \in \mathbb{R}$, has exactly one real root.

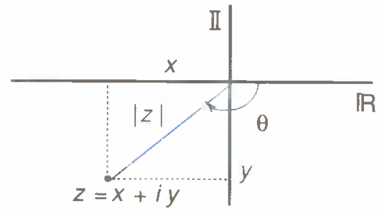
17. $z = \frac{r}{1 + i}$, $w = \frac{s}{1 + 2i}$, where r and s are real, and it is given that $z + w = 1$.
    a) Calculate the values of r and s.
    b) Calculate $|z - w|$.

18. Given $z = a + bi$ and $w = c + di$, where a, b, c, and d are real, prove that
$$|z + w| = |z - w| \Rightarrow \frac{iz}{w} \text{ is real.}$$

10.5 Complex Numbers and Trigonometry

Until now, you have used $z = x + yi$ to represent the complex number z. This is called the **Cartesian form** of z.
z can also be expressed by using its modulus $|z| = r$ and its argument θ.





Recall that the number z can be represented by the point with coordinates (x,y) in the complex plane.
The definition of angles in standard position tells you that $x = r\cos\theta$ and $y = r\sin\theta$, no matter what the position of z in the complex plane.
(See page 541.)
Thus $z = x + yi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$.
This is known as the **polar form**, or **modulus-argument form**, of a complex number.

Cartesian form: $z = x + yi$

polar form: $z = r(\cos\theta + i\sin\theta)$,

where $r = |z| = \sqrt{x^2 + y^2}$, $\cos\theta = \dfrac{x}{r}$, and $\sin\theta = \dfrac{y}{r}$

Note: The polar form of representation is not unique. For example,

$$2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left(\cos\frac{13\pi}{6} + i\sin\frac{13\pi}{6}\right), \text{ or}$$
$$2(\cos 30^\circ + i\sin 30^\circ) = 2(\cos 390^\circ + i\sin 390^\circ).$$

If the complex number z is represented in polar form by its modulus r and its argument θ, then any other argument of z, that is, any angle $\theta + 2k\pi$ or $\theta^\circ + 360k^\circ$ (with $k \in \mathbb{Z}$) could be substituted for θ.

In the exercises, you will have an opportunity to prove the equality principle for complex numbers expressed in polar form. That is, you will prove that

$r(\cos\theta + i\sin\theta) = p(\cos\phi + i\sin\phi)$
implies
$r = p$ and $\theta = \phi + 2k\pi$ (or $\theta^\circ = \phi + 360k^\circ$), $k \in \mathbb{Z}$.

Recall that the principal argument θ is such that $-\pi < \theta \leqslant \pi$, or $-180^\circ < \theta^\circ \leqslant 180^\circ$.

Example 1 State the principal arguments of the following complex numbers.

a) $z = 3\left(\cos \frac{13\pi}{3} + i \sin \frac{13\pi}{3}\right)$

b) $w = 5(\cos[-200°] + i \sin[-200°])$

Solution a) Since $\frac{13\pi}{3}$ is an argument of z, then $\frac{13\pi}{3} + 2k\pi$, $k \in \mathbb{Z}$, are its other arguments.

Since the principal argument θ is such that $-\pi < \theta \leqslant \pi$, you must select $k = -2$.

Thus $\text{Arg } z = \frac{13\pi}{3} - 2(2\pi) = \frac{13\pi - 12\pi}{3} = \frac{\pi}{3}$

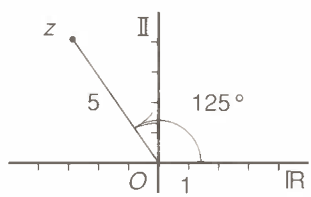
b) The arguments of w are $-200° + 360k°$, $k \in \mathbb{Z}$.
For the principal argument, you must select $k = 1$.

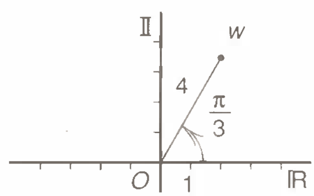
Therefore $\text{Arg } w = -200° + (1)360° = 160°$ ■

For the examples that follow, you may wish to refer to the tables of values of the trigonometric ratios of special angles, and the table of radian and degree equivalences, on page 543.

Example 2 Find the Cartesian form of the following numbers.

a) $z = 5(\cos 125° + i \sin 125°)$ b) $w = 4\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$



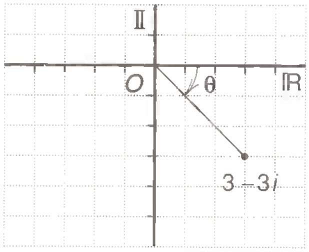


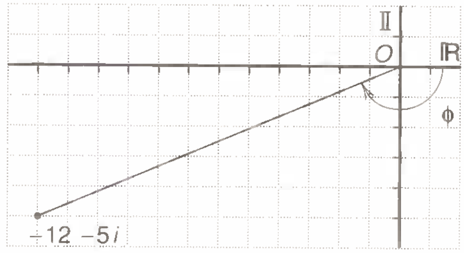
Solution a) $z = 5(-0.5735\ldots) + 5i(0.8191\ldots) \doteq -2.9 + 4.1i$, correct to 1 decimal place.

b) $w = 4\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = 2 + 2i\sqrt{3}$, using exact values. Alternatively,
$w = 4(0.5 + [0.8660\ldots]i) \doteq 2 + 3.5i$,
correct to 1 decimal place. ■

Example 3 Find the polar form of the following numbers.

a) $z = 3 - 3i$ b) $w = -12 - 5i$





Solution a) $|z| = \sqrt{3^2 + 3^2} = 3\sqrt{2}$; $\sin\theta = -\frac{3}{3\sqrt{2}} = -\frac{1}{\sqrt{2}}$ and $\cos\theta = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$

Thus, $\theta = -\frac{\pi}{4}$ or $-45°$.

Hence $z = 3\sqrt{2}(\cos[-45°] + i\sin[-45°])$
or $z \doteq 4.24(\cos[-45°] + i\sin[-45°])$.

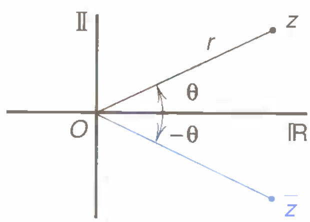
Note: Any angle coterminal with $-45°$ would also be correct. For example, $-45° + 360° = 315°$ could have been used.

b) $|w| = \sqrt{12^2 + 5^2} = 13$; $\sin\phi = -\frac{5}{\sqrt{13}}$ and $\cos\phi = -\frac{12}{13}$

Thus, $\phi \doteq -157°$, correct to the nearest degree.
Hence $z \doteq 13(\cos[-157°] + i\sin[-157°])$ ■

Conjugates

The reflection in the real axis of
a complex number of modulus r, argument θ, is
a complex number of modulus r, argument $-\theta$.



Thus if $z = r(\cos\theta + i\sin\theta)$, then $\bar{z} = r(\cos[-\theta] + i\sin[-\theta])$
or $\bar{z} = r(\cos\theta - i\sin\theta)$,
since for any angle θ, $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$ (see page 541).

This form is also used routinely for complex numbers with *negative* arguments.
That is, the complex number z of modulus r and argument $-\theta$, where $\theta > 0$, can be written

$$z = r(\cos[-\theta] + i\sin[-\theta]) \quad \text{OR} \quad z = r(\cos\theta - i\sin\theta)$$

Multiplication and Division in Polar Form

The most useful aspect of the polar form is the stunning result obtained when complex numbers are multiplied or divided. You will observe this presently.

Let $z = p(\cos\theta + i\sin\theta)$ and $w = q(\cos\phi + i\sin\phi)$.

Then $zw = pq(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$

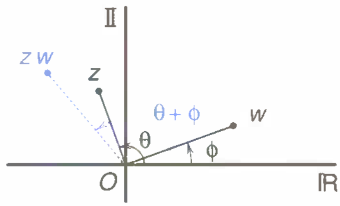
$= pq(\cos\theta\cos\phi + i\cos\theta\sin\phi + i\sin\theta\cos\phi - \sin\theta\sin\phi)$

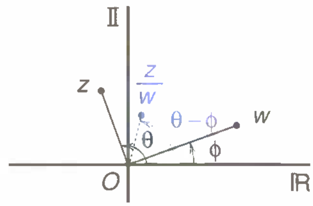
$= pq([\cos\theta\cos\phi - \sin\theta\sin\phi] + i[\sin\theta\cos\phi + \sin\phi\cos\theta])$

so $zw = pq(\cos[\theta + \phi] + i\sin[\theta + \phi])$

from the formulas for the cosine and sine of compound angles on page 542.

The product obtained is a complex number in polar form, whose *modulus is pq, and whose argument is* $\theta + \phi$.





RULE 1

Hence, when two complex numbers are *multiplied, the modulus of their product is the product of their moduli*, and *the argument of their product is the sum of their arguments.*

Similarly, you will have an opportunity to prove in the exercises that

$$\frac{z}{w} = \frac{p}{q}(\cos[\theta - \phi] + i\sin[\theta - \phi])$$

The quotient obtained is a complex number in polar form, whose *modulus is* $\frac{p}{q}$*, and whose argument is* $\theta - \phi$.

RULE 2

Thus, when two complex numbers are *divided, the modulus of their quotient is the quotient of their moduli*, and *the argument of their quotient is the difference of their arguments.*

Example 4 Given $z = 12(\cos 160^\circ + i\sin 160^\circ)$ and $w = 3(\cos 35^\circ + i\sin 35^\circ)$, find zw and $\frac{z}{w}$ in Cartesian form, correct to 1 decimal place.

Solution

$zw = (12)(3)(\cos[160^\circ + 35^\circ] + i\sin[160^\circ + 35^\circ])$

$= 36(\cos 195^\circ + i\sin 195^\circ)$

$= 36([-0.9659\ldots] + i[-0.2588\ldots]) \doteq -34.8 - 9.3i$

$\frac{z}{w} = \frac{12}{3}(\cos[160^\circ - 35^\circ] + i\sin[160^\circ - 35^\circ])$

$= 4(\cos 125^\circ + i\sin 125^\circ)$

$= 4([-0.5735\ldots] + i[0.8191\ldots]) \doteq -2.3 + 3.3i$ ■

Example 5 Find the exact values of zw and $\dfrac{z}{w}$

if $z = 4\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right)$ and $w = 2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

Solution

$$\begin{aligned} zw &= (4)(2)\left(\cos\left[\frac{5\pi}{6} + \frac{\pi}{3}\right] + i\sin\left[\frac{5\pi}{6} + \frac{\pi}{3}\right]\right) \\ &= 8\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) \\ &= 8\left(-\frac{\sqrt{3}}{2} - \frac{1i}{2}\right) \\ &= -4\sqrt{3} - 4i \end{aligned}$$

$$\begin{aligned} \frac{z}{w} &= \frac{4}{2}\left(\cos\left[\frac{5\pi}{6} - \frac{\pi}{3}\right] + i\sin\left[\frac{5\pi}{6} - \frac{\pi}{3}\right]\right) \\ &= 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \\ &= 2(0 + 1i) \\ &= 2i \quad \blacksquare \end{aligned}$$

The next example shows how rules 1 and 2 can be used advantageously in different situations.

Example 6 Given the complex number $z = 3 - 3i$ from Example 3a), calculate the exact values of

a) the modulus and argument of z^2

b) the modulus and argument of $\dfrac{1}{z}$

Solution $z = 3 - 3i = 3\sqrt{2}(\cos[-45^\circ] + i\sin[-45^\circ])$ from Example 3a).

That is, $|z| = 3\sqrt{2}$ and $\arg z = -45^\circ$

a) Using rule 1,
the modulus of z^2 is $(3\sqrt{2})(3\sqrt{2}) = (3\sqrt{2})^2 = 18$
the argument of z^2 is $[-45^\circ] + [-45^\circ] = -90^\circ$.

b) The complex number 1 has modulus 1, argument 0.
Thus, using rule 2,

the modulus of $\dfrac{1}{z}$ is $\dfrac{1}{3\sqrt{2}}$

the argument of $\dfrac{1}{z}$ is $0 - (-45^\circ) = 45^\circ$. $\blacksquare$

10.5 Exercises

1. Plot each of the following numbers in the complex plane and find their Cartesian forms. Use 3 significant digit accuracy.
$a = 4(\cos 50° + i \sin 50°)$
$b = 4(\cos 50° - i \sin 50°)$
$c = 2(\cos 145° + i \sin 145°)$

2. Plot each of the following numbers in the complex plane and find their Cartesian forms. Use exact values.
$d = 8\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$
$e = \left(\cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4}\right)$
$f = -\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$

3. State the modulus and an argument of z in the following cases.
a) $z = 3i$ b) $z = 4$ c) $z = -17$ d) $z = -i$

4. State the modulus and an argument of $\bar{z}$ using the numbers of question 3.

5. State the principal argument of the following numbers.
a) $\cos 115° + i \sin 115°$
b) $\cos 425° + i \sin 425°$
c) $6\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$
d) $2\left(\cos \frac{11\pi}{6} - i \sin \frac{11\pi}{6}\right)$

6. Plot each of the following numbers in the complex plane and find their polar forms. Use degrees.
$s = 4 + 3i$ $v = -2$
$t = -1 + 2i$ $z = -15 - 8i$
$u = 5i$ $w = 4 - 9i$

7. Find the exact polar form of z in the following cases. Use radians.
a) $z = -1 + i$ c) $z = 2\sqrt{3} - 6i$
b) $z = \sqrt{3} + i$ d) $z = -3 - \sqrt{3}i$

8. Given any complex number z, state the possible values of the argument of $z + \bar{z}$.

9. Find r and a value of θ in the following cases.
a) $r(\cos 30° + i \sin 30°) = 5(\cos \theta + i \sin \theta)$
b) $6 \cos \theta + 6i \sin \theta = r(\cos 328° + i \sin 328°)$
c) $\cos \frac{\pi}{8} - i \sin \frac{\pi}{8} = r \cos \theta + r \sin \theta$

10. Given $z = p(\cos \theta + i \sin \theta)$ and $w = q(\cos \phi + i \sin \phi)$, prove that
$$\frac{z}{w} = \frac{p}{q}(\cos[\theta - \phi] + i \sin[\theta - \phi])$$
$\left[\text{Hint: Recall that } \frac{z}{w} = \frac{z\bar{w}}{w\bar{w}}\right]$

11. Given $z = 10(\cos 71° + i \sin 71°)$ and $w = 5(\cos 34° + i \sin 34°)$, express the following in polar form.
a) zw b) $\frac{z}{w}$ c) $\frac{w}{z}$

12. Given $z = 4 - 5i$ and $w = -2 + 3i$,
a) express z and w in polar form (use degrees).
b) Hence express zw, $\frac{z}{w}$, and $\frac{w}{z}$ in polar form.

13. Calculate the exact modulus and an exact argument of each of the numbers $z = -1 + \sqrt{3}i$ and $w = 2\sqrt{3} + 2i$.

14. Use the results of question 13 to express the following in polar form.
a) z^2 b) w^2 c) zw d) $\frac{w}{z}$

15. a) State the modulus and argument of i.
b) Describe geometrically what happens to the vector representation of a complex number that is multiplied by i.

16. a) If $z = \cos \theta + i \sin \theta$, state an argument of z^2.
b) Hence show that $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and that $\sin 2\theta = 2 \sin \theta \cos \theta$.

10.6 De Moivre's Theorem

The investigations of the last section lead to the most important theorem concerning complex numbers. This theorem was published by Abraham De Moivre (1667-1754) in 1730, well before the advent of the complex plane.

By the multiplication principle, recall that
$[r(\cos\theta + i\sin\theta)]^2 = r^2(\cos 2\theta + i\sin 2\theta)$.
Similarly,
$[r(\cos\theta + i\sin\theta)][r^2(\cos 2\theta + i\sin 2\theta)] = r^3(\cos 3\theta + i\sin 3\theta)$.

De Moivre's theorem extends this principle as follows.

THEOREM

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

This can be proved by induction for $n \in \mathbb{N}$ as follows.

Step 1: Show the statement is true for $n = 1$.
For $n = 1$, $L.S. = [r(\cos\theta + i\sin\theta)]^1$, $R.S. = r^1(\cos 1\theta + i\sin 1\theta)$.
Since $L.S. = R.S.$, the statement is true for $n = 1$.

Step 2: Assume the statement is true for some $n = k \in \mathbb{N}$. That is, assume $[r(\cos\theta + i\sin\theta)]^k = r^k(\cos k\theta + i\sin k\theta)$ is true.

Step 3: Prove the statement is true for $n = k + 1$. That is, prove $[r(\cos\theta + i\sin\theta)]^{k+1} = r^{k+1}(\cos[k+1]\theta + i\sin[k+1]\theta)$.

$$\begin{aligned} L.S. &= [r(\cos\theta + i\sin\theta)]^k[r(\cos\theta + i\sin\theta)] \\ &= [r^k(\cos k\theta + i\sin k\theta)][r(\cos\theta + i\sin\theta)] && \text{from step 2} \\ &= (r^k)(r)(\cos[k\theta + \theta] + i\sin[k\theta + \theta]) && \text{by multiplication property} \\ &= r^{k+1}(\cos[k+1]\theta + i\sin[k+1]\theta) = R.S. \end{aligned}$$

Thus, by the principle of mathematical induction, $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$ is true for all $n \in \mathbb{N}$.

Example 1

Calculate in Cartesian form

a) $(\cos 50° + i\sin 50°)^8$ b) $(1 + i)^{24}$

Solution

a) Note that the modulus of $(\cos 50° + i\sin 50°)$ is 1.

$$\begin{aligned} (\cos 50° + i\sin 50°)^8 &= [1(\cos 50° + i\sin 50°)]^8 \\ &= 1^8(\cos[8\times 50°] + i\sin[8\times 50°]) && \text{de Moivre} \\ &= 1(\cos 400° + i\sin 400°) \\ &\doteq 0.77 + 0.64i \end{aligned}$$

b) $|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\arg(1 + i) = 45°$

$$\begin{aligned} \text{Thus } (1 + i)^{24} &= [\sqrt{2}(\cos 45° + i\sin 45°)]^{24} \\ &= (\sqrt{2})^{24}(\cos 1080° + i\sin 1080°) && \text{de Moivre} \\ &= 2^{12}(1 + 0i) \\ &= 4096 \end{aligned}$$ ■

Negative Exponents

You will now use the division principle and De Moivre's theorem to find the polar form of z^{-n}, where $z = r(\cos\theta + i\sin\theta)$ and $n \in \mathbb{N}$.

$$z^{-n} = \frac{1}{z^n} = \frac{1(\cos 0 + i\sin 0)}{[r(\cos\theta + i\sin\theta)]^n} \quad \text{since 1 has modulus 1 and argument 0}$$

$$= \frac{1(\cos 0 + i\sin 0)}{r^n(\cos n\theta + i\sin n\theta)} \quad \text{by De Moivre's theorem}$$

$$= \frac{1}{r^n}(\cos[0 - n\theta] + i\sin[0 - n\theta] \quad \text{by division property}$$

so $\quad z^{-n} = r^{-n}(\cos[-n\theta] + i\sin[-n\theta])$

This last statement is the expression of De Moivre's theorem for a negative integer.

Thus, De Moivre's theorem is true for any $n \in \mathbb{Z}$.

Example 2 Calculate in Cartesian form $(-\sqrt{3} + i)^{-9}$.

Solution $|-\sqrt{3} + i| = \sqrt{3 + 1} = 2 \quad$ and $\arg(-\sqrt{3} + i) = 150°$

Thus $(-\sqrt{3} + i)^{-9} = [2(\cos 150° + i\sin 150°)]^{-9}$

$$= 2^{-9}[\cos(-1350°) + i\sin(-1350°)]$$

$$= \frac{1}{512}(0 - 1i)$$

$$= -\frac{i}{512} \quad ■$$

De Moivre's theorem can also be used in conjunction with the binomial theorem to establish certain trigonometrical identities. This is one of the applications of complex numbers to other areas of mathematics.

Example 3 Find expressions for $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

Solution $\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3 \quad$ by De Moivre's theorem

$$= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta$$

by the binomial theorem

$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

Thus, by equating real and imaginary parts,

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

and $\quad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta. \quad ■$

De Moivre's Theorem for Rational Exponents

De Moivre's theorem is true not only for all positive and negative integer exponents, but also for *all rational exponents* (with a reservation), as the following indicates.

Assume that ϕ is such that
$(\cos\theta + i\sin\theta)^{\frac{p}{q}} = \cos\phi + i\sin\phi,$ ①
where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

Then raising each side of ① to the exponent q gives

$$(\cos\theta + i\sin\theta)^p = (\cos\phi + i\sin\phi)^q$$

so $$\cos p\theta + i\sin p\theta = \cos q\phi + i\sin q\phi.$$

Equating real and imaginary parts shows that this is satisfied by
$p\theta = q\phi + 2k\pi$, that is, $\phi = \dfrac{p\theta - 2k\pi}{q}$, $k \in \mathbb{Z}$

If $k = 0$, $\phi = \dfrac{p}{q}\theta$
The statement ① now gives

$$(\cos\theta + i\sin\theta)^{\frac{p}{q}} = \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$$

Thus, De Moivre's theorem appears to be true for a rational exponent. *The reservation* is that, if n is not an integer, then there is *more than one value possible for z^n*, namely

$$z^n = r^n\left(\cos\frac{p\theta - 2k\pi}{q} + i\sin\frac{p\theta - 2k\pi}{q}\right),\ k \in \mathbb{Z}.$$

This will be clarified by Example 1 in section 10.7.

The Norwegian mathematician Niels Henrik Abel (1802-1829) showed that De Moivre's theorem can be extended to include all real, and even all complex exponents.

NOTATION

In some texts, the short form "cis θ" is used as an abbreviation for "$\cos\theta + i\sin\theta$".

SUMMARY

De Moivre's theorem:
$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\,\theta)$, $n \in \mathbb{Q}$.

If n is not an integer, then $(\cos\theta + i\sin\theta)^n$ is *not unique*.

10.6 Exercises

1. Express the following in Cartesian form (use 3 significant digit accuracy).
   a) $(\cos 130° + i \sin 130°)^{10}$
   b) $[3(\cos 20° - i \sin 20)]^6$
   c) $[4(\cos 257° + i \sin 257°)]^5$

2. Express the following in Cartesian form (use exact values).
   a) $(1 - i)^{32}$
   b) $(1 + i\sqrt{3})^{12}$
   c) $\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)^9$
   d) $\left(\cos \frac{5\pi}{6} - i \sin \frac{5\pi}{6}\right)^8$

3. Express the following in Cartesian form (use 3 significant digit accuracy).
   a) $(\cos 130° + i \sin 130°)^{-10}$
   b) $[3(\cos 20° - i \sin 20°)]^{-6}$
   c) $[4(\cos 257° + i \sin 257°)]^{-5}$

4. Express the following in Cartesian form (use exact values).
   a) $(1 - i)^{-32}$
   b) $(1 + i\sqrt{3})^{-12}$
   c) $\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)^{-9}$
   d) $\left(\cos \frac{5\pi}{6} - i \sin \frac{5\pi}{6}\right)^{-8}$

5. Calculate the following in Cartesian form.
   a) $(1 + i)^8(\sqrt{3} - i)^6$
   b) $\dfrac{i^{50}(-1 + i)^{20}}{(1 + i)^{10}}$

6. Simplify the following expressions.
   a) $\dfrac{\left(\cos \frac{\pi}{7} + i \sin \frac{\pi}{7}\right)^5}{\left(\cos \frac{\pi}{7} - i \sin \frac{\pi}{7}\right)^2}$
   b) $\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)^{100}\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)^{100}$

7. Find expressions for $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos \theta$ and $\sin \theta$.

8. Given $z = \cos \theta + i \sin \theta$,
   a) use De Moivre's theorem to prove the following.
   $\frac{1}{z} = \cos \theta - i \sin \theta$,
   $z^3 = \cos 3\theta + i \sin 3\theta$,
   $\frac{1}{z^3} = \cos 3\theta - i \sin 3\theta$
   b) show that $\left(z + \frac{1}{z}\right)^3 = 8 \cos^3 \theta$
   c) by expanding $\left(z + \frac{1}{z}\right)^3$, prove that
   $2 \cos 3\theta + 6 \cos \theta = 8 \cos^3\theta$
   d) hence find $\cos 3\theta$ in terms of powers of $\cos \theta$.

9. Given $z = \cos \theta + i \sin \theta$,
   a) show that $\left(z - \frac{1}{z}\right)^3 = -8i \sin^3\theta$
   b) hence find $\sin 3\theta$ in terms of powers of $\sin \theta$.

10. Given $z = \cos \theta + i \sin \theta$,
    a) expand and simplify $\left(z + \frac{1}{z}\right)^4$
    b) hence prove that
    $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4 \cos 2\theta + 3)$
    c) hence find $\cos 4\theta$ in terms of powers of $\cos \theta$.

11. If $z = -1$, verify that $z^{\frac{1}{3}}$ may take more than one value in $\mathbb{C}$ as follows.
    $w = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ is one of the values of $z^{\frac{1}{3}}$,
    $u = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ is one of the values of $z^{\frac{1}{3}}$,
    $v = -1$ is one of the values of $z^{\frac{1}{3}}$.

12. If $z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$, verify that $z^{\frac{1}{5}}$ may take more than one value in $\mathbb{C}$ as follows.
    $p = \cos 9° + i \sin 9°$
    is one of the values of $z^{\frac{1}{5}}$,
    $q = \cos 153° + i \sin 153°$
    is another value of $z^{\frac{1}{5}}$.

10.7 Quest for Roots in $\mathbb{C}$

You know that $1^3 = 1$.
Hence, 1 is a solution of the equation $z^3 = 1$ or $z^3 - 1 = 0$.

According to your experience of mathematics thus far, it may seem that 1 is the only root.

However, recall from the fundamental theorem of algebra (section 10.4) that this equation has *three* roots in $\mathbb{C}$. The roots cannot all be 1, since that would imply that the equation $z^3 - 1 = 0$ could be rewritten in factored form as $(z - 1)^3 = 0$.
(You know that $z^3 - 1 \neq (z - 1)^3$.)

The search for all the roots of this equation will be investigated in the following example.

Example 1 Use De Moivre's theorem to determine all the cube roots of 1 in $\mathbb{C}$.

Solution The cube roots of 1 are the roots of the equation $z^3 - 1 = 0$, or $z^3 = 1$.
You can solve this equation by writing each side in polar form.
Now $|1| = 1$, and $\arg 1 = 0$, thus $1 = 1(\cos 0 + i \sin 0)$.
Let $z = r(\cos\theta + i\sin\theta)$.
Thus you must solve $z^3 = 1$

$$\Rightarrow \quad [r(\cos\theta + i\sin\theta)]^3 = 1(\cos 0 + i\sin 0)$$

$$\Rightarrow \quad r^3(\cos 3\theta + i\sin 3\theta) = 1(\cos 0 + i\sin 0)$$

Recall from section 10.5 that if two complex numbers are equal, then their moduli are equal and their arguments differ by a multiple of 2π.
Thus $r^3 = 1$ and $3\theta = 0 + 2k\pi$, $k \in \mathbb{Z}$,

$$\Rightarrow \quad r = 1 \text{ (since } r \text{ is real)} \quad \text{and} \quad \theta = \frac{2k\pi}{3}, \text{ where } k \text{ is any integer.}$$

That is, $z = 1\left(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right) = \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}$,
where k is any integer.

You will now see that this expression for z represents *different* complex numbers, depending on the value chosen for k.
Let these numbers be represented by w_k, then, by substituting successively the values 0, 1, 2, 3,. . ., you obtain

$w_0 = \cos 0 + i\sin 0 = 1 + 0i = 1$

$w_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, a value different from w_0

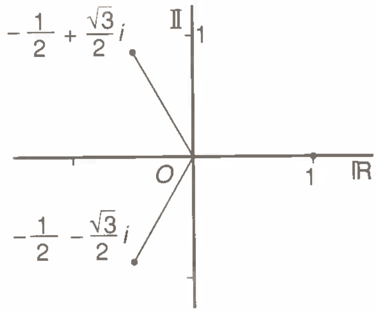
$w_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, a value different from w_0 or w_1

$w_3 = \cos 2\pi + i\sin 2\pi = 1 + 0i = 1 = w_0$

Continuing the process yields the values $w_1, w_2, w_0, w_1, \ldots$ etc.

Thus the three cube roots of 1 are $1, -\frac{1}{2}+\frac{\sqrt{3}}{2}i, -\frac{1}{2}-\frac{\sqrt{3}}{2}i$ ■

In the exercises, you will have an opportunity to verify that each of these numbers, when cubed, yields 1.



- Observe from the figure that the roots have a rotational symmetry about the origin, of angle $\frac{2\pi}{3}$. That is, each root, if rotated counterclockwise through $\frac{2\pi}{3}$ about the origin, has for image another root.
- Also observe that $w_1^2 = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = w_2$.
- Compare also with the solutions formed earlier, using the quadratic formula (Example 2, section 10.4).

Method to Find the nth Roots of Unity

The above method can be applied to solving the equation $z^n - 1 = 0$, or $z^n = 1$, where n is any natural number. According to the fundamental theorem of algebra, this equation will have n roots in ℂ. These roots are called the **nth roots of 1** or **nth roots of unity**.

Let $z = r(\cos\theta + i\sin\theta)$, then

$$z^n = 1$$

$$\Rightarrow [r(\cos\theta + i\sin\theta)]^n = 1(\cos 0 + i\sin 0)$$
$$\Rightarrow r^n(\cos n\theta + i\sin n\theta) = 1(\cos 0 + i\sin 0)$$

Thus $r^n = 1$ and $n\theta = 0 + 2k\pi$, $k \in \mathbb{Z}$,

$\Rightarrow \quad r = 1$ (since r is real) and $\theta = \frac{2k\pi}{n}$, where k is any integer.

That is, $z = w_k = 1\left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right) = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$,

where k is any integer.

By substituting successively the values 0, 1, 2, . . . , $n - 1$ of k in
$w_k = \left(\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}\right)$, you find
$w_0 = \cos 0 + i \sin 0 = 1 + 0i = 1$
$w_1 = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$, a value different from w_0
$w_2 = \cos \frac{4\pi}{n} + i \sin \frac{4\pi}{n}$, a value different from w_0 or w_1
. . . .
$w_{n-1} = \cos \frac{2(n-1)\pi}{n} + i \sin \frac{2(n-1)\pi}{n}$, a value different from all previous
$w_n = \cos \frac{2n\pi}{n} + i \sin \frac{2n\pi}{n} = \cos 2\pi + i \sin 2\pi = w_0$.
Successive values of k will again yield solutions equal to
$w_1, w_2, \ldots,$ in turn.

Thus the nth roots of unity are given by the n numbers $w_0, w_1, w_2, \ldots, w_{n-1}$.

Once again, observe that $w_1^2 = w_2$.
Furthermore, $w_1^3 = w_3$, $w_1^4 = w_4$, etc.

Rational Powers of z

You are familiar with a result in $\mathbb{R}$ such as "$32^{\frac{1}{5}} = 2$". In $\mathbb{C}$, however, the expression $32^{\frac{1}{5}}$ may take *five different values*. The number 2, which is real and positive, is called the **principal root**. In order to distinguish the principal root from the others when working in $\mathbb{C}$, you can use the notation $\sqrt[5]{32}$ for the principal root. That is,
$32^{\frac{1}{5}}$ may take five different values, including 2, but
$\sqrt[5]{32} = 2$ (a positive real number).

Note: If z is not a positive real number, then there is no principal root of $z^{\frac{1}{n}}$, where $n \in \mathbb{N}$. The ambiguity can occur only if z is a positive real number.

The method of searching for roots can now be extended to any rational power of z, that is, $z^{\frac{p}{q}}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$. This is illustrated in the following example.

Example 2 Find all the values of $z = [16(1 + i\sqrt{3})]^{\frac{2}{5}}$, and sketch them in the complex plane.

Solution The numbers required are the solutions of the equation

$$z^5 = [16(1 + i\sqrt{3})]^2$$

Let $z = r(\cos\theta + i\sin\theta)$, and let $u = 16(1 + i\sqrt{3})$.

Now $|u| = 16\sqrt{1^2 + \sqrt{3}^2} = 16\sqrt{4} = 32$, and $\arg u = \dfrac{\pi}{3}$ or $60°$.

Thus $\quad z^5 = [32(\cos 60° + i\sin 60°)]^2$

$\Rightarrow \quad r^5(\cos 5\theta + i\sin 5\theta) = 32^2\,(\cos[2 \times 60°] + i\sin[2 \times 60°])$

Thus $\quad r^5 = 32^2$ and $5\theta = 120° + 360k°$, $k \in \mathbb{Z}$

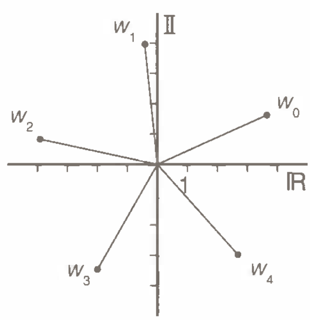
$\Rightarrow \quad r = \sqrt[5]{32^2} = 2^2 = 4$ (since r is real; it is the principal fifth root of 32^2)

and $\quad \theta = \dfrac{1}{5}(120° + 360k°) = 24° + 72k°$, where k is any integer.

That is, $z = w_k = 4[\cos(24° + 72k°) + i\sin(24° + 72k°)]$

Now substitute the values 0, 1, 2, 3, and 4 of k in w_k.

$w_0 = 4(\cos 24° + i\sin 24°) = 4(0.913\ldots + 0.406\ldots i) \doteq 3.65 + 1.63i$
$w_1 = 4(\cos 96° + i\sin 96°) = 4(-0.104\ldots + 0.994\ldots i) \doteq -0.42 + 3.98i$
$w_2 = 4(\cos 168° + i\sin 168°) = 4(-0.978\ldots + 0.207\ldots i) \doteq -3.91 + 0.83i$
$w_3 = 4(\cos 240° + i\sin 240°) = 4(-0.5 + 0.866\ldots i) \doteq -2 - 3.46i$
$w_4 = 4(\cos 312° + i\sin 312°) \doteq 4(+0.669\ldots - 0.743\ldots i) \doteq 2.68 - 2.97i$



■

Notice again the symmetry of the roots. However, in this case, $w_1^2 \neq w_2$. You will investigate this further in the exercises.

10.7 Exercises

In the following, leave numerical answers correct to 3 significant digits, where you cannot find exact values.

1. By finding w_0^3, w_1^3 and w_2^3, verify that each of the following numbers is a cube root of 1.

$$w_0 = 1, \quad w_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad w_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

2. Find the following roots of unity in Cartesian form and represent them in a complex plane.
   a) the fourth roots of unity
   b) the fifth roots of unity
   c) the tenth roots of unity

3. Find the following roots in Cartesian form and represent them in a complex plane.
   a) the square roots of i
   b) the cube roots of i
   c) the square roots of $-i$
   d) the cube roots of $27(\cos 72° + i \sin 72°)$
   e) the fourth roots of $81(\cos 72° + i \sin 72°)$
   f) the sixth roots of $64(\cos 102° - i \sin 102°)$

4. Two of the roots of the equation $z^5 - 32 = 0$ are $2\left(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}\right)$ and $2\left(\cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}\right)$. State the other roots.

5. a) Solve the equation $z^5 - 1 = 0$.
   b) Use these solutions to express $z^5 - 1$ in factored form.
   c) Express $z^5 - 1$ in the factored form $(z - 1)p(z)q(z)$, where $p(z)$ and $q(z)$ are quadratic expressions with real coefficients.

6. Find the real factors of
   a) $z^5 + 1$ b) $z^7 - 1$ c) $z^6 - 1$

7. a) Show that $\sqrt{3} - i$ is a fourth root of $-8(1 + i\sqrt{3})$.
   b) Hence solve the equation $z^4 + 8 + 8i\sqrt{3} = 0$.

8. a) If w is a non-real seventh root of unity, show that the other roots are w^2, w^3, w^4, w^5, w^6, and 1.
   b) Prove that $1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$
   c) Do similar properties hold for all other nth roots of unity?

9. In section 10.7, Example 2, it was shown that the five values of $[16(1 + i\sqrt{3})]^{\frac{2}{5}}$ could be represented by $w_k = 4[\cos(24° + 72k°) + i \sin(24° + 72k°)]$, $k \in \{0,1,2,3,4\}$.
   a) Show that $\frac{w_{k+1}}{w_k}$ is a constant.
   b) Use your answer to a) to explain the symmetry of the representatives in the complex plane.

10. a) Express $z^7 + 1 = 0$ in factored form, using factors with real coefficients.
    b) Hence show that

$$\cos \frac{\pi}{7} + \cos \frac{3\pi}{7} + \cos \frac{5\pi}{7} = \frac{1}{2}$$

11. Solve the equation $z^6 - 2z^3 + 4 = 0$.

10.8 Graphing and Complex Numbers

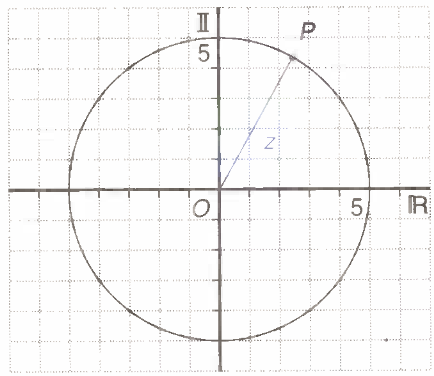
In this section, you will be describing the set of points representing a complex number z that is subject to certain conditions. Such a set of points is called the **locus** of z in the complex plane. This topic will give you the opportunity to work with complex numbers in a variety of ways.

Example 1 a) Describe the locus of the points $z = x + iy$ in the complex plane, given that $|z| = 5$.

b) Find an equation in x and y that represents this locus.

Solution a) The modulus of a complex number is its length, that is, its distance from the origin.

If $|z| = 5$, then z must lie on a circle of centre O and radius 5.



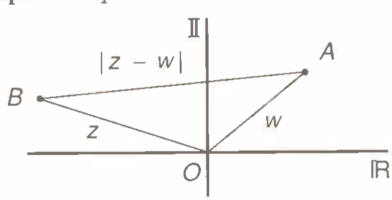
b) $|z| = 5 \Rightarrow |x + iy| = 5$

$\Rightarrow \sqrt{x^2 + y^2} = 5$

$\Rightarrow x^2 + y^2 = 25$ ■

Distance Between Two Points in the Complex Plane

Recall that complex numbers can be represented by vectors. Let the complex numbers w and z be represented in the complex plane by the points A and B respectively.

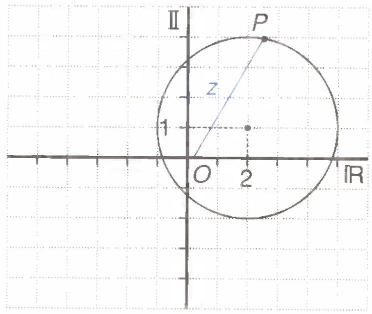


Then the distance $AB = |\overrightarrow{AB}| = |\overrightarrow{OB} - \overrightarrow{OA}| = |z - w|$.

Thus, the modulus of $(z - w)$ represents the distance between the points representing z and w in the complex plane.

Example 2

a) Describe the locus represented by the equation $|z - 2 - i| = 3$.
b) If $z = x + iy$, find an equation for this locus in terms of x and y.



Solution

a) $|z - 2 - i| = |z - (2 + i)|$.
Thus, this expression gives the distance between the points representing z and $(2 + i)$. The locus is therefore a circle with centre $(2 + i)$ and radius 3.

b) Since $z = x + iy$,

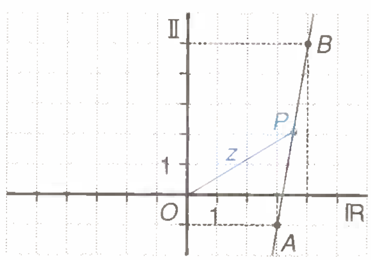
$$\begin{aligned} |z - 2 - i| &= 3 \text{ becomes} \\ |x + iy - 2 - i| &= 3 \\ |(x - 2) + i(y - 1)| &= 3 \\ \sqrt{(x-2)^2 + (y-1)^2} &= 3 \\ (x-2)^2 + (y-1)^2 &= 9 \end{aligned}$$

■

The next example illustrates the link between complex numbers and vectors. Recall that the vector equation of a line is $\vec{r} = \vec{r}_0 + k\vec{m}$,
where $\vec{r}$ is the position vector of any point on the line,
$\vec{r}_0$ is the position vector of a given point on the line,
and $\vec{m}$ is a direction vector of the line.

Example 3

a) Determine a complex number equation for the line passing through the points A and B representing $3 - i$ and $4 + 5i$ respectively.
b) Deduce parametric equations for the line AB.



Solution

a) Let z represent any point P on the line AB.

Since $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are collinear, $\overrightarrow{AP} = k\overrightarrow{AB}, k \in \mathbb{R}$

or $$z - (3 - i) = k[(4 + 5i) - (3 - i)]$$

$\Rightarrow$ $$z = (3 - i) + k(1 + 6i) \qquad ①$$

This is the required equation.

b) Let $z = x + iy$, and rewrite ① as follows.

$$x + iy = 3 + k + i(-1 + 6k)$$

Equating real and imaginary parts gives

$x = 3 + k$

$y = -1 + 6k$

These are parametric equations for the line AB. ■

The next example shows that straight lines, or parts thereof, can be described in a totally different way with complex numbers.

Example 4

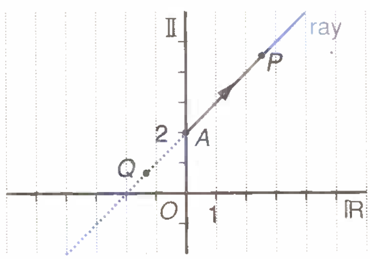
a) Describe the locus of z if $\arg(z - 2i) = 45°$

b) Find an equation in terms of x and y for this locus, given that $z = x + iy$.

Solution

a) Let A be the point representing $2i$, and P be the point representing z. Then the complex number $z - 2i$ is represented by the vector $\overrightarrow{AP}$.

$\arg(z - 2i) = 45°$ means that the vector $\overrightarrow{AP}$ must make an angle of 45° with the positive x-axis.



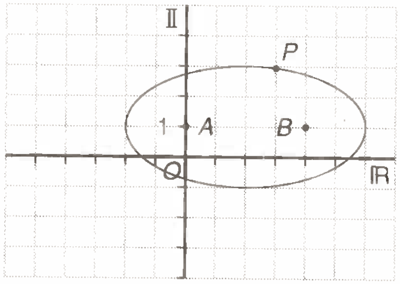
Thus, P is on the part-line, or ray, shown in the diagram.

b) This ray has slope 1, and it passes through (0,2). Recall that the equation of a line of slope m passing through the point (x_0, y_0) is $y - y_0 = m(x - x_0)$.

Thus the equation of the ray is $y - 2 = 1(x - 0)$, or $y = x + 2$, with the condition that $x > 0$. ■

Note: The point Q, which is on the line, does *not* satisfy $x > 0$. Indeed, the angle between $\overrightarrow{AQ}$ and the positive x-axis is 135°, *not* 45°. Thus points such as Q do not satisfy the original complex equation.

Example 5 An ellipse has a major axis of length 8 and foci at the points A and B, representing i and $4 + i$ respectively. Find a complex equation for this ellipse.



Solution A property of an ellipse is that the sum of the distances from the foci to any point on the ellipse is equal to the length of the major axis.
Let P be any point on the ellipse, represented by the number z.

Thus $|\overrightarrow{AP}| + |\overrightarrow{BP}| = 8,$

or $|z - i| + |z - (4 + i)| = 8$ is the required equation. ■

Example 6 Find an equation in x and y for the locus described by $z^2 - (1 + i)^2 = \bar{z}^2 - (1 - i)^2$, where $z = x + iy$.

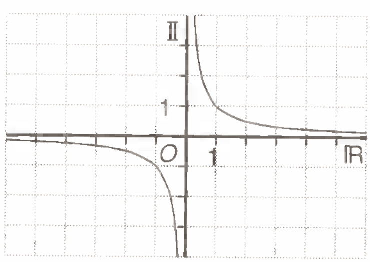
Solution Substituting $z = x + iy$ and $\bar{z} = x - iy$ gives

$$(x + iy)^2 - (1 + i)^2 = (x - iy)^2 - (1 - i)^2 \text{ or}$$
$$(x + iy)^2 - (x - iy)^2 = (1 + i)^2 - (1 - i)^2$$
$$(x + iy + x - iy)(x + iy - x + iy) = (1 + i + 1 - i)(1 + i - 1 + i)$$

factoring as difference of squares

$$(2x)(2iy) = (2)(2i)$$
$$xy = 1$$

This represents a rectangular hyperbola centred at the origin, with the real axis and the imaginary axis as asymptotes.



■

10.8 Exercises

In the following, let $z = x + iy$ wherever appropriate.

1. Sketch the locus of the point P representing the complex number z in the following cases.
   a) $|z| = 4$
   b) $|z - 1| = 4$
   c) $|z - i| = 4$
   d) $|z - 5 - 2i| = 1$
   e) $\arg z = \frac{\pi}{2}$
   f) $\arg(z - 1) = \frac{5\pi}{6}$
   g) $|z + 1| + |z - 1| = 4$
   h) $|z| + |z - 4 - i| = 6$
2. Find a complex number equation for the following.
   a) a circle centre O, radius 6
   b) a circle centre $-1 + 3i$, radius 5
   c) a circle centre u, radius a, with $u \in \mathbb{C}$, $a \in \mathbb{R}$
3. Find equations in x and y of the loci described by the following.
   a) $|z + 4 + 3i| = 2$
   b) $|z - i| = 3|z + i|$
4. A point moves so that its distance from the origin is twice its distance from $3 - i$. Show that the locus is a circle, and find its centre and its radius.
5. Determine a complex number equation for the straight line through the points A and B represented respectively by $-2 + 5i$ and $-2 - i$.
6. Describe the locus represented by $\text{Re}\left(z - \frac{1}{z}\right) = 0$.
7. Describe the locus represented by $\arg(z - 4 - 2i) = 120°$.
8. Describe the locus represented by $\text{Im}(z^2) = 0$.
9. Describe the locus represented by $\text{Im}\left(z - 1 + \frac{2}{z}\right) = 0$.
10. Find a complex number equation for the perpendicular bisector of the line segment AB where A and B are represented respectively by the following complex numbers.
    a) $2, -6$
    b) $2 + i, 3 - 2i$
11. Given that $|z - w| = |z + w|$, show that $|\arg z - \arg w| = 90°$.
12. Find an equation in x and y for the following.
    a) $\left|\frac{z - 3}{z - 6}\right| = 1$
    b) $\left|\frac{z + 4i}{z - 2}\right| = 2$
    c) $\arg\left(\frac{z}{z + 2}\right) = \frac{\pi}{4}$
    d) $\arg\left(\frac{z - 1 - i}{z + 2 + i}\right) = \frac{\pi}{2}$
13. Describe the locus of z if $\text{Im}(z^2) = 2$.
14. Describe the locus represented by the following.
    a) $|z| < 5$
    b) $|z - 5 + 3i| \leqslant 3$
    c) $\text{Re}(z^2) > 2$
    d) $2 \leqslant |z - 2i| \leqslant 3$
    e) $|z - 1 - i| + |z + 2 - 4i| < 10$
15. a) Describe the locus represented by $|z - 1| = \text{Re}(z) + 1$.
    b) Find an equation in x and y for this locus.
16. Describe the locus represented by each of the following.
    a) $|z - 2 - 3i| = 4$
    b) $\text{Re}(z) = 2$ and $-\frac{\pi}{3} \leqslant \arg z \leqslant \frac{\pi}{4}$
17. For each locus in question 16, find the greatest value of $|z|$.

In Search of Graphical Representation of Non-real Solutions of Equations

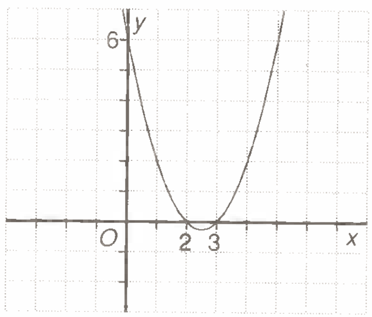
Solutions to quadratic equations in ℝ can be seen graphically as shown in the examples A and B that follow.

A. $x^2 - 5x + 6 = 0$

$\Rightarrow \quad (x - 3)(x - 2) = 0$

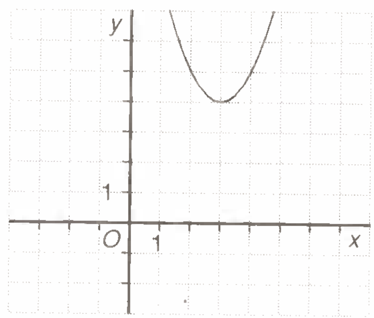
$\Rightarrow \quad x = 3$ or $x = 2$.

Graphically, these solutions can be viewed as the points where the parabola $y = x^2 - 5x + 6$ ① intersects with the line $y = 0$, that is, the x-axis.



B. $x^2 - 6x + 13 = 0$ has no real solutions. The quadratic formula yields $x = 3 + 2i$ or $x = 3 - 2i$.

The parabola $y = x^2 - 6x + 13$ ② does *not* intersect the x-axis. Is there any geometric significance in this context for $3 + 2i$ and $3 - 2i$?



Pursuing the question asked in B, proceed as follows. Allow the x-values in the parabola ② to extend into ℂ, that is, let x take the form $a + bi$, with $a, b \in$ ℝ.

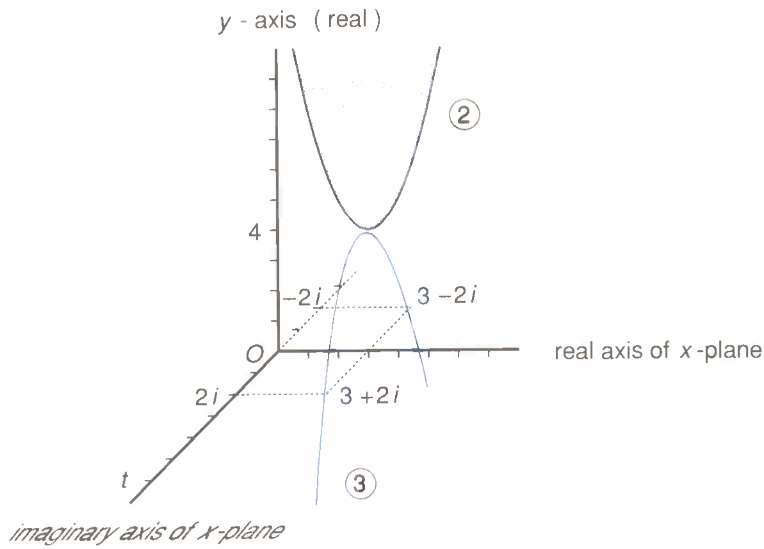
You now have a complex plane, the x-plane, taking the place of the old x-axis. (Note that the old x-axis is contained in this complex plane.)

Unfortunately, y will also now take on non-real values, and a four-dimensional situation is set up.

However, it is still possible to view a part of this, as follows.

You already know that the solutions to $y = 0$ are $x = 3 \pm 2i$. Hence, the real part of each solution is 3. Allow x to take the form $3 + ti$, with $t \in \mathbb{R}$.

Then $y = (3 + ti)^2 - 6(3 + ti) + 13$
$\quad = 9 + 6ti - t^2 - 18 - 6ti + 13$
or $\quad y = 4 - t^2$ ③



Now t is the variable along the imaginary axis of the x-plane.

The equation ③ thus represents a parabola whose plane is perpendicular to the plane of the original parabola ②.

Also, this parabola punctures the x-plane at the points $3 + 2i$ and $3 - 2i$.

Thus, you can see that the non-real intersections of a parabola with the x-axis are "somewhere in front of, or behind, the paper"!

A Canadian mathematician, Richard Dewsbury, is presently researching the geometrical aspect of extensions to $\mathbb{C}$ of equations in $\mathbb{R}$.

10.9 Exponential Form of a Complex Number

A geometric series with first term a, and common ratio r, has an 'infinite sum' $S = \frac{a}{1-r}$, provided $|r| < 1$.

Consider the infinite series $S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$.

This is a geometric series with $a = 2$, $r = \frac{1}{2}$

Thus

$$S = \frac{2}{1 - \frac{1}{2}} = 4$$

No finite sum of this series has a value 4. However, the sum of a finite part of this series will get as close to 4 as you want, provided that you add a sufficient number of terms.

In chapter 9, you saw that the binomial expansion becomes an infinite series if the exponent is not a natural number, that is,

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

If $|x| < 1$, the series approaches the value of $(1 + x)^n$ as closely as you like, by taking a sufficient number of terms. The series is said to *converge*.

If $|x| \geq 1$, the series does *not* approximate $(1 + x)^n$. In fact, the series may change value considerably for each extra term added. In that case, the series is said to *diverge*.

The theory of infinite series developed most significantly after the invention of calculus. Around 1700, the mathematicians Brook Taylor (1685-1731) and Colin Maclaurin (1698-1746) developed formulas to find series expansions, or polynomial approximations, to many functions in mathematics.

Three of these series, valid for all $x \in \mathbb{R}$, follow.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

For the trigonometrical functions, x is in 'natural measure', that is *radians*, not degrees.

The Swiss mathematician Leonhard Euler (1707-1783) broke from tradition by attempting to use these expansions for $x \in \mathbb{C}$, in 1748.

Using $z = x + iy$, he wrote

$e^z = e^{x+iy} = e^x e^{iy} = re^{iy}$, where $r = e^x \in \mathbb{R}$. ①

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \frac{(iy)^6}{6!} + \ldots$$

$$= 1 + iy - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} + \frac{iy^5}{5!} - \frac{y^6}{6!} - \ldots$$

$$= 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \ldots + i\left[y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots\right]$$

But these are the series for $\cos y$ and for $\sin y$,

FORMULA

so $e^{iy} = \cos y + i \sin y$ ②

or, using ①,

FORMULA

$e^z = re^{iy} = r(\cos y + i \sin y)$, ③

where $z = x + iy$ and $r = e^x$, $x, y \in \mathbb{R}$.

Thus e^{x+iy} is a complex number whose modulus is e^x and whose argument in radians is y, that is,

PROPERTIES

$|e^{x+iy}| = e^x$ and $\arg(e^{x+iy}) = y$

The identities ② and ③ are known as **Euler's formulas**. They show that any complex number can be written in exponential form. The formula ② is the special case where the modulus is 1.

One extraordinary consequence of these formulas is the following identity, obtained by substituting $y = \pi$ in ②.

$e^{i\pi} = \cos \pi + i \sin \pi = -1 + 0$, or

IDENTITY

$e^{i\pi} = -1$

This wonderful relation links π, e, and i, three of the most important numbers that evolved in the history of mathematics. This is another example of the simplification, or rather 'unification', that may result after extended research into new areas.

Recall that

- π is the length of half the circumference of a unit circle (that is, a circle of radius 1). [$\pi \doteq 3.141\,592\,654$]
- e is the base of natural logarithms (the area under the curve $y = \frac{1}{x}$ is a natural logarithm) [$e \doteq 2.718\,281\,828$]
- i is a square root of -1.

Note: Just as the polar form of a complex number is not unique, so the exponential form is also not unique.
That is, since any argument θ can always be replaced by another argument $\theta + 2k\pi$, $k \in \mathbb{Z}$, then $e^{i\theta} = e^{i[\theta+2k\pi]}$

Example 1

Write the complex number $z = 5 + 2i$ in exponential form. (Leave numbers in your answer correct to 2 decimal places.)

Solution

$z = re^{iy}$, where $r = |z|$ and $y = \arg z$.
Now $|z| = \sqrt{5^2 + 2^2} = \sqrt{29} \doteq 5.39$,

and $\tan(\arg z) = \dfrac{2}{5} = 0.4$. Since z is in the first quadrant, $\arg z = 0.38$.

Hence $z \doteq 5.39e^{0.38i}$ ■

Example 2

Write the complex number $w = -\sqrt{3} + i$ in exponential form. (Use exact values.)

Solution

$|w| = \sqrt{\sqrt{3}^2 + 1^2} = 2.$

$\tan(\arg w) = -\dfrac{1}{\sqrt{3}}$ and w is in the second quadrant, so $\arg w = \dfrac{5\pi}{6}$

Hence $w = 2e^{\frac{5i\pi}{6}}$ ■

De Moivre's Theorem in Exponential Form

For clarity, consider De Moivre's theorem for a complex number of modulus 1.

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

One immediate result of Euler's formulas is the expression of De Moivre's theorem as follows.

$$(e^{i\theta})^n = e^{in\theta}$$

Thus De Moivre's theorem can be seen as the extension of a normal exponent rule of $\mathbb{R}$ to $\mathbb{C}$!

However, recall that if n is not an integer, then $z = (e^{i\theta})^n$ is *not unique*. In this case, z is called a **multiple-valued function**. One of these values is $e^{in\theta}$, the complex number of modulus 1, with argument $n\theta$.

Example 3

Given $z = e^{\frac{i\pi}{8}}$, find the following in Cartesian form.

a) z^2 b) z^3 c) z^8 d) $z^{\frac{1}{2}}$

Solution

a) $z^2 = (e^{\frac{i\pi}{8}})^2 = e^{\frac{2i\pi}{8}} = e^{\frac{i\pi}{4}} = \cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}} \doteq 0.71 + 0.71i$

b) $z^3 = (e^{\frac{i\pi}{8}})^3 = e^{\frac{3i\pi}{8}} = \cos\dfrac{3\pi}{8} + i\sin\dfrac{3\pi}{8} \doteq 0.38 + 0.92i$

c) $z^8 = (e^{\frac{i\pi}{8}})^8 = e^{\frac{8i\pi}{8}} = e^{i\pi} = -1$

d) Since $\frac{1}{2} \notin \mathbb{Z}$, $z^{\frac{1}{2}}$ is not unique.

You must proceed as you did when using De Moivre's theorem to find roots.

If $u = z^{\frac{1}{2}}$, then u is a solution of the equation $u^2 = z$.

Let $u = re^{i\theta}$, then $r^2 e^{2i\theta} = e^{\frac{i\pi}{8}}$

Thus $\quad r^2 = 1$ and $2\theta = \frac{\pi}{8} + 2k\pi,\ k \in \mathbb{Z}$

$\Rightarrow \quad r = 1$ (since r is real)

and $\quad \theta = \frac{\pi}{16} + k\pi$, where k is any integer.

That is, $u = w_k = 1e^{i(\frac{\pi}{16}+k\pi)}$

Now substitute successively the values 0 and 1 for k in w_k.

$$w_0 = e^{\frac{i\pi}{16}} = \cos\frac{\pi}{16} + i\sin\frac{\pi}{16} \doteq 0.98 + 0.20i$$

$$w_1 = e^{\frac{17i\pi}{16}} = \cos\frac{17\pi}{16} + i\sin\frac{17\pi}{16} \doteq -0.98 - 0.20i$$

These are the two values of $z^{\frac{1}{2}}$. ■

In Search of Other Results using the Exponential Form

The link that Euler made between complex numbers and the exponential series gave birth to the theory of complex variables, an extensive branch of mathematics that you will have an opportunity to touch upon from the following.

1. Definition of Sine and Cosine using Exponential Forms

$$e^{i\theta} = \cos\theta + i\sin\theta \quad ①$$

and $e^{-i\theta} = \cos\theta - i\sin\theta \quad ②$

Adding ① and ② yields

FORMULA

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \quad \text{or} \quad \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

Subtracting ① – ② yields

FORMULA

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta \quad \text{or} \quad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

Thus $\cos\theta$ and $\sin\theta$, which are real, can be defined in terms of non-real exponentials.

2. *The Link with Hyperbolic Functions*

The functions cos θ and sin θ are called **circular functions.** One link is that the circle of equation $x^2 + y^2 = 1$ can be represented parametrically by the system of equations $\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}$

The two functions defined as follows, are pronounced "cosh" and "shine".

DEFINITIONS $\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$ and $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$ are called **hyperbolic functions.**

This name is used because the hyperbola of equation $x^2 - y^2 = 1$ can be represented parametrically by the system of equations

$$\begin{cases} x = \cosh\theta \\ y = \sinh\theta \end{cases}$$

The definitions of cosh θ and sinh θ are deemed to hold also when θ is non-real.

In the exercises you will have an opportunity to verify the following identities.

$\cosh iz = \cos z$ and $\sinh iz = i\sin z$

3. *The Meaning of z^w*

Given $z = x + iy$ and $w = a + ib$, where $x, y, a, b \in \mathbb{R}$, you will have an opportunity to demonstrate in 10.9 Exercises that one value of the complex power of a complex number z^w is

$$z^w = e^{ax - b\arg z}\, e^{i(bx - a\arg z)}$$

4. *Complex Numbers and Calculus*

If the formulas for differentiation are applied to complex numbers, it can be shown that $\frac{d}{d\theta}(e^{i\theta}) = \frac{d}{d\theta}(\cos\theta + i\sin\theta)$.

You will have an opportunity to do this in 10.9 Exercises.

It is hoped that, from this very sketchy introduction to the theory of complex variables, you will be spurred on to study further this area of mathematics in depth in the years to come.

10.9 Exercises

1. If $z = re^{i\theta}$, find the following in exponential form.

   a) z^2 b) z^3 c) $\frac{1}{z}$ d) $\bar{z}$

2. Write the following in exponential form (leave numbers correct to 2 decimal places).

   a) $z = 2 + i$ b) $w = -1 - 3i$

3. Write the following in exponential form, using exact values.

   a) $u = 5 - 5i\sqrt{3}$ b) $v = -3 + 3i$

4. Given $z = 2e^{\frac{2i\pi}{5}}$, simplify the following

   a) z^2 c) z^{-1} e) $z^{\frac{1}{2}}$
   b) z^5 d) z^{-2} f) $z^{2.5}$

5. If $z = x + iy$ and $w = a + ib$, prove that $e^z e^w = e^{z+w}$.

6. A student claims to have calculated the value of i as follows.
   $e^{2i\pi} = 1$ and $e^0 = 1$,
   thus $2i\pi = 0$ or $i = 0$.
   What is wrong with this demonstration?

7. Express $e^{i\theta} \times e^{i\phi}$, where $\theta, \phi \in \mathbb{R}$, in terms of sines and cosines of θ and ϕ in two different ways, and use your result to prove that
   $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$.

8. Use the definitions $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ to prove the following identities.

   a) $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

   b) $\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

   c) $\cos^2\theta + \sin^2\theta = 1$

9. Using the definitions of $\cos\theta$ and $\sin\theta$ given in question 8, verify that

   a) $\sin(-\theta) = -\sin\theta$ b) $\cos(-\theta) = \cos\theta$

10. Use the definitions of $\cos\theta$ and $\sin\theta$ given in question 8 to solve the following equations.

    a) $\sin\theta = 0$ b) $\cos\theta = 0$

11. If z is any complex number, show that $z^{\frac{1}{2}}$ always has two values, w_0 and w_1, such that $w_0 + w_1 = 0$.

12. Given $z = 6e^{\frac{i\pi}{3}}$, prove the following.

    a) $iz = -3\sqrt{3} + 3i$
    b) $|e^{iz}| = e^{-3\sqrt{3}}$

13. Using the definitions of $\cosh z$ and $\sinh z$ given on page 460, prove the following identities.

    a) $\cosh iz = \cos z$ b) $\sinh iz = i\sin z$

14. Given $z = x + iy$ and $w = a + ib$, where $x, y, a, b \in \mathbb{R}$, show that one value of z^w is $e^{ax-b\arg z}e^{i(bx-a\arg z)}$.

15. If $z = e^w$, then $w = \ln z$, called the *natural logarithm* of z. If $z = re^{iy}$, then $w = \ln z = \ln r + i(y + 2k\pi)$, $k \in \mathbb{Z}$.
    Use these definitions to show that the values of $\ln(1 - i)$ are given by
    $\ln(1 - i) = \frac{1}{2}\ln 2 + \frac{i\pi(7 + 8k)}{4}$, $k \in \mathbb{Z}$

16. Consider a complex-valued function that can be written in the two forms $f(\theta) = e^{i\theta}$ ① or $f(\theta) = \cos\theta + i\sin\theta$ ②.
    Using the normal rules of differentiation, find $f'(\theta)$ using each of the forms ① and ② and show that these derivatives are equal.

17. Use the ideas on pages 454–455, and your knowledge of the exponential form of a complex number, to graph $z = e^{ix}$, $x \in \mathbb{R}$. (Use a complex z-plane and a real x-axis.)

MAKING

The Visual Display of Data

René Descartes' (1596–1650) invention of coordinate geometry was the first link established between algebra and geometry. The link is visual, since it gives us a 'picture' of algebraic relations. Today, 'graphing' is the general term used to indicate that any information is displayed visually, rather than by words alone.

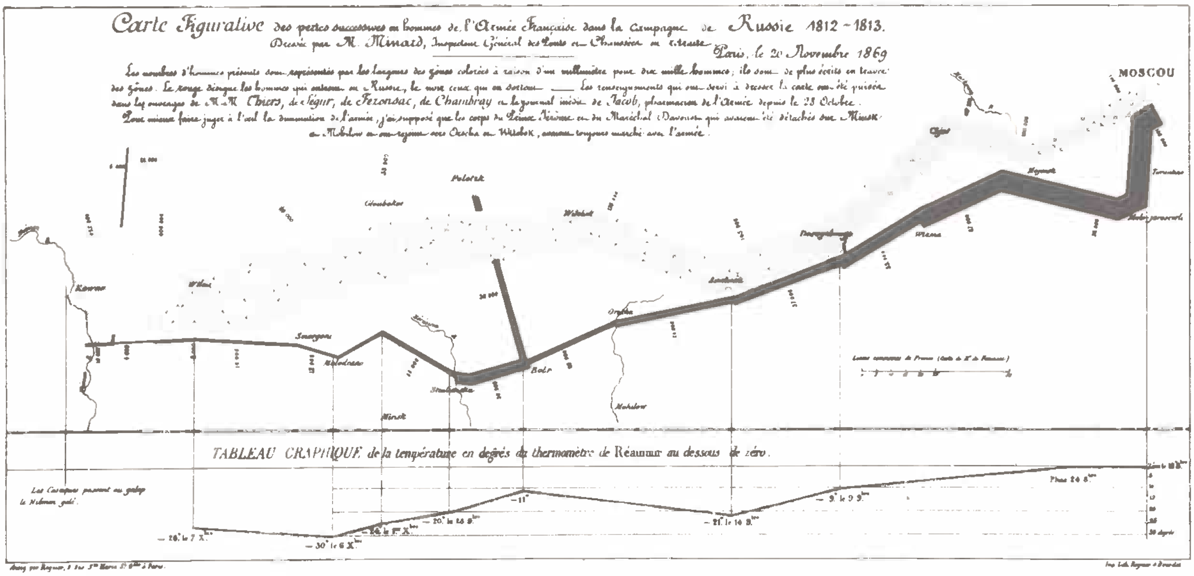
Graphing has made extraordinary advances since the time of Descartes. The recent advent of computers is leading to another great increase in the availability of visual displays of information.

Unfortunately, visual displays are not always good representations of what they try to portray. When used for advertising purposes, only some aspects of the data may be emphasized, while information that is not helpful to the advertiser is either not displayed, or cleverly disguised.

Possibly one of the most eloquent graphics ever drawn is the 'figurative map' drawn in 1869 by the French engineer Charles Joseph Minard (1781–1870) to describe Napoleon's Russian campaign of 1812.

Minard started with a map of the region extending from the Niemen river (the Russian-Polish border at the time) to Moscow. He then indicated not only the route taken by Napoleon's Grande Armée, but also superimposed the size of the army as it progressed towards Moscow. (The size of the army is indicated by the width of the shaded band.) The Russian armies sacked, burned and deserted most cities before Napoleon could reach them, thus cutting off supplies needed by the French. This had a devastating effect on the Grande Armée. Of the 422 000 men who started the campaign at the Niemen river in June 1812, only 100 000 made it to Moscow in September.

The retreat, which started on October 19, also had to contend with an unusually cold winter. Minard shows the retreating army with the darker band, and adds to his graphic a time scale (from October 24 to December 7), and a temperature scale indicating degrees below freezing. Note the immense losses suffered at the Berezina river, swelled by a sudden thaw. The Russians had destroyed the bridge. The Grande Armée made it back to Poland with about 10 000 men.



Observe that six variables are represented on this single diagram: the geographical location of the army (two dimensions), its size, its direction, a time-scale, and a temperature scale for the retreat from Moscow. Few graphics contain so much clearly displayed information.

Notes 1 The "lieue commune" is about 4444 metres.

2 The Réaumur temperature scale is such that water freezes at 0°R, boils at 80°R. Thus, to convert from Réaumur degrees to Celsius degrees, multiply by $\frac{100}{80}$ or 1.25.

3 The abbreviations 8^{bre}, 9^{bre} and X^{bre} refer to October, November and December respectively.

Summary

First Definitions and Properties

- $i^2 = -1$. i is called an *imaginary number*.
- $z = a + bi$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$, is called a *complex number*.
- a is the real part of z, or $a = \text{Re}(z)$.
  b is the imaginary part of z, or $b = \text{Im}(z)$.
  If $b = 0$, z is real. If $b \neq 0$, z is non-real.
- The set of all complex numbers is denoted by $\mathbb{C}$.
- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$
- The *complex plane* is determined by a real axis and an imaginary axis, crossing at 0.
- Complex numbers have all the properties of vectors of $\mathbb{V}_2$.
- There is no order relation in $\mathbb{C}$.

Modulus and Argument—Conjugates

- If $z = x + yi$ is represented by the point P, or the vector $\overrightarrow{OP}$, in the complex plane:
  the *modulus* of z, $|z| = |\overrightarrow{OP}| = \sqrt{x^2 + y^2}$
  the *argument* of z is the angle that $\overrightarrow{OP}$ makes with the positive real axis,
  that is, $\sin(\arg z) = \dfrac{y}{|z|}$ and $\cos(\arg z) = \dfrac{x}{|z|}$
- The *complex conjugates* $z = x + yi$ and $\bar{z} = x - yi$ are reflections of each other in the real axis.
- $|z - w|$ represents the distance between the points representing z and w in the complex plane.

Properties of $\mathbb{C}$

E. Equality	$a + bi = c + di$ if and only if $a = c$ and $b = d$
S. Sum	$(a + bi) + (c + di) = (a + c) + (b + d)i$
P. Product	$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

Given any numbers z, w and u of $\mathbb{C}$,

1. Closure	$z + w$ and zw belong to $\mathbb{C}$
2. Commutativity	$z + w = w + z$ and $zw = wz$
3. Associativity	$(z + w) + u = z + (w + u)$ and $(zw)u + z(wu)$
4. Distributivity	$z(w + u) = zw + zu$
5. Neutral elements	$z + 0 = 0 + z = z$ and $(z)(1) = (1)(z) = z$
6. Inverse elements	$z + (-z) = (-z) + z = 0$ and $z\left(\dfrac{1}{z}\right) = \left(\dfrac{1}{z}\right)z = 1$, provided that $z \neq 0$

Properties involving Conjugates

Consider two complex numbers z, w, and their conjugates $\bar{z}$, $\bar{w}$.

1. $z + \bar{z} = 2\text{Re}(z)$
2. $z - \bar{z} = 2i\text{Im}(z)$
3. $z\bar{z} = |z|^2$
4. $(\overline{z + w}) = \bar{z} + \bar{w}$
5. $(\overline{zw}) = \bar{z}\bar{w}$
6. $(\bar{\bar{z}}) = z$
7. Division: $\dfrac{z}{w} = \dfrac{z\bar{w}}{w\bar{w}} = \dfrac{z\bar{w}}{|w|^2}$

Polar Form

- $z = a + bi = r(\cos\theta + i\sin\theta)$

  Cartesian form polar form

- $r(\cos\theta + i\sin\theta) = p(\cos\phi + i\sin\phi)$ implies $r = p$ and $\theta = \phi + 2k\pi$ (or $\theta° = \phi° + 360k°$), $k \in \mathbb{Z}$.
- $z = r(\cos\theta + i\sin\theta) \Rightarrow \bar{z} = r[\cos(-\theta) + i\sin(-\theta)]$ or $\bar{z} = r(\cos\theta - i\sin\theta)$

Multiplication and Division in Polar Form

- $[p(\cos\theta + i\sin\theta)][q(\cos\phi + i\sin\phi)] = pq(\cos[\theta + \phi] + i\sin[\theta + \phi])$
  The modulus of the product is the product of the moduli.
  The argument of the product is the sum of the arguments.
- $\dfrac{p(\cos\theta + i\sin\theta)}{q(\cos\phi + i\sin\phi)} = \dfrac{p}{q}(\cos[\theta - \phi] + i\sin[\theta - \phi])$
  The modulus of the quotient is the quotient of the moduli.
  The argument of the quotient is the difference of the arguments.

De Moivre's Theorem

- $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$, $n \in \mathbb{Q}$.
  If n is not an integer, then $(\cos\theta + i\sin\theta)^n$ is *not unique*.
- Given $z, w \in \mathbb{C}$, and $n \in \mathbb{N}$, the n values of z satisfying $z^n = w$ are called the nth roots of w.

The Fundamental Theorem of Algebra

- A polynomial equation of degree n always has n complex roots.

The Factor Theorem

- If $p(z_k) = 0$, then $(z - z_k)$ is a factor of $p(z)$.

Exponential Form

- $e^z = re^{iy} = r(\cos y + i\sin y)$,
  where $z = x + iy$ and $r = e^x$, $x, y \in \mathbb{R}$. (y in radians)
- $|e^{x+iy}| = e^x$, and $\arg(e^{x+iy}) = y$

Inventory

Complete each of the following statements.

1. $i^2 =$ ____.
2. The numbers i, $4i$, $i\sqrt{2}$ are called ____.
3. Using the real numbers a, b and the number i, a complex number can be written ____.
4. The sets of numbers $\mathbb{R}$ and $\mathbb{C}$ are related such that $\mathbb{R}$ ____ $\mathbb{C}$.
5. The two axes of the complex plane are called ____.
6. A complex number can be real, imaginary, or ____.
7. If the complex numbers $a + bi$ and $c + di$ are equal then ____ and ____.
8. Given $z = a + bi$, $\mathrm{Re}(z) =$ ____, $\mathrm{Im}(z) =$ ____, the complex conjugate $\bar{z} =$ ____, the modulus $|z| =$ ____ the argument arg z is such that tan(____) = ____.
9. Complex conjugates are ____ of each other in the ____ of the complex plane.
10. The conjugate of the conjugate of z is equal to ____.
11. ____ numbers are added like vectors of $\mathbb{V}_2$.
12. A complex number z whose modulus is r and whose argument is θ can be represented in polar form as $z =$ ____.
13. If two complex numbers are equal, then their moduli are ____ and their arguments differ by ____.
14. When two complex numbers are multiplied, the modulus of the product is the ____ of the moduli.
15. When two complex numbers are divided, the argument of the quotient is the ____ of the arguments.
16. De Moivre's theorem: $[r(\cos\theta + i\sin\theta)]^n = r^n($____$)$.
17. The fifth roots of unity are the numbers z satisfying the equation ____.
18. A polynomial equation of degree n has ____ complex roots, some of which may be equal.
19. Each equation in $\mathbb{C}$ incorporates ____ equations in $\mathbb{R}$.
20. The distance between the points representing z and w in the complex plane is ____.
21. If $z = x + iy$, then the complex number e^z has modulus ____ and argument ____.
22. $e^{i\pi} =$ ____.

Review Exercises

1. Simplify the following.
   a) $(7 + 2i) + (3 - 2i)$
   b) $(7 + 2i)(3 - 2i)$
   c) $(11 - i)^2$
   d) $(1 + i)^3$
   e) i^7
   f) i^{-4}
   g) $\dfrac{1}{-i}$
   h) $(2 + i) - (4 - 5i)$
   i) $4(-1 + i) - 3(1 + i)$
   j) $(1 + 6i)^2 - (1 - 6i)^2$
   k) $i(i - 1) - (2 + i)(4 + 3i)$

2. Express in the form $a + ib$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.
   a) $\dfrac{1 - 3i}{i}$
   b) $\dfrac{1 - 3i}{1 + 3i}$
   c) $\dfrac{8 + 5i}{-i}$
   d) $\dfrac{5 + i}{4 - 2i}$
   e) $\dfrac{1}{3 - 4i}$
   f) $\dfrac{i}{2 + i}$
   g) $\dfrac{1}{6 + i} + \dfrac{1}{6 - 4i}$
   h) $\dfrac{1}{(9 - 2i)^2} - \dfrac{1}{9 - 2i}$

3. Simplify the following expressions.
   a) $(a + bi)^2 - (a - bi)^2$
   b) $\dfrac{1}{a - bi} - \dfrac{1}{a + bi}$
   c) $a + bi + \dfrac{1}{a + bi}$

4. Find two numbers whose sum is 10 and whose product is 29.

5. Find the roots of the following equations.
   a) $z^2 - 12z + 37 = 0$
   b) $z^2 + 4z + 20 = 0$
   c) $z^2 = 3z - 5$

6. Find the roots of the equation $z^2 - (4 + i)z + 4i = 0$ by factoring in $\mathbb{C}$.

7. Show that a quadratic equation whose roots are $z = \alpha$ and $z = \beta$ can be written $z^2 - (\alpha + \beta)z + \alpha\beta = 0$.

8. Simplify $(2 + i)^3(5 - 12i)^2(2 - i)^3(5 + 12i)^2$.

9. Find the number k such that
$$\left|\frac{2 - ki}{1 + i}\right| = 5.$$

10. Let $z = 2 + i$,
    $w = 3 - 4i$,
    $p = -5i$,
    $q = -6 - i$,
    $u = 4$.
    a) Plot the points representing numbers z, w, p, q, u in a complex plane.
    b) Find the conjugates $\overline{z}, \overline{w}, \overline{p}, \overline{q}$, and $\overline{u}$, and plot them in the same complex plane.
    c) Find the moduli $|z|, |w|, |p|, |q|$, and $|u|$.
    d) Find the arguments arg z, arg w, arg p, arg q, arg u.

11. a) State the complex conjugate $\overline{z}$ of the number $z = a + bi$.
    b) Prove that the sum of a complex number and its conjugate is always real.
    c) Prove that the product of a complex number and its conjugate is always real.

12. a) What is the argument of the number -1?
    b) Describe the geometric effect of -1 as a multiplier in the complex plane. Does your description apply to real numbers?

13. Given two complex numbers z and w, use a vector analogy to illustrate the following inequalities geometrically.
    a) $|z + w| \leqslant |z| + |w|$ (the triangle inequality)
    b) $|z - w| \leqslant |z| + |w|$
    c) $|z - w| \geqslant ||z| - |w||$

14. Given numbers z and w, use a vector analogy to find an interpretation in the complex plane of
    a) $\dfrac{1}{2}z + \dfrac{1}{2}w$
    b) $\dfrac{m}{m + n}z + \dfrac{n}{m + n}w$

15. a) If $z = a + bi$ and $w = c + di$, prove that $|z - w| = \sqrt{(a - c)^2 + (b - d)^2)}$

b) Use a vector analogy to illustrate geometrically that the distance between the points representing z and w in the complex plane is $|z - w|$.

16. Find quadratic equations in the form $az^2 + bz + c = 0$ with the following roots.

a) $-i$ and $5 - i$ b) $a + bi$ and $c - di$

17. a) Prove that a polynomial equation of degree n with real coefficients always has at least one real root if n is odd.

b) How many real roots are there if n is even?

18. By solving $z^2 + 4i = 0$, find the two square roots of $-4i$. Locate these roots in the complex plane.

19. Given $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$,

a) calculate z^2

b) plot z and z^2 in a complex plane

c) discuss the statement: "$\sqrt{i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$".

20. The difference of two squares can be factored, but the sum of two squares cannot be factored. Discuss.

21. It is given that $1 + 3i$ is a root of the equation $2z^3 - 9z^2 + 30z - 50 = 0$.

a) Use this information to find all the roots of the equation.

b) Show that the representations of these roots in a complex plane are the vertices of an isosceles triangle.

22. a) Verify that $w = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ is a cube root of 1.

b) Calculate w^2 and show that w^2 is also a cube root of 1.

23. Consider the numbers $z = r(\cos\theta + i\sin\theta)$ and $w = p(\cos\phi + i\sin\phi)$, where θ and ϕ are measured in radians. Prove that if $z = w$, then $r = p$ and $\theta = \phi + 2k\pi$, where $k \in \mathbb{Z}$.

24. Consider the equation in z
$z^2 - uz + v = 0$,
where u and v are known to be non-real. Determine whether or not it is possible for this equation to have a real root.

25. a) If $z = \cos 45° + i\sin 45°$, calculate z^2.

b) Calculate $(-z)^2$.

c) Use your results to a) and b) to state the two square roots of i in Cartesian form.

d) Calculate z^4.

26. a) Verify the identity
$9\cos^2\theta - \sin^2\theta - 8 = \cos^2\theta - 9\sin^2\theta$.

b) Use this identity to solve the equation
$z^2 - (3\cos\theta + i\sin\theta)z + 2 = 0$.

27. Given $z = r(\cos\theta + i\sin\theta)$, verify that $z\bar{z} = r^2$.

28. Find the modulus and an argument of $z = x + iy$ in the following cases.

a) $x = 0, y > 0$

b) $x < 0, y = 0$

29. Given $z = 3(\cos 67° + i\sin 67°)$ and $w = 2(\cos 123° - i\sin 123°)$, express the following in polar form.

a) $\bar{z}$

b) $\bar{w}$

c) zw

d) $\bar{z}\,\bar{w}$

e) $\frac{z}{w}$

f) $\frac{w}{z}$

30. Given $z = 10 + i$ and $w = 4 - 7i$, express the following in polar form.

a) z

b) w

c) zw

d) $\frac{z}{w}$

e) $\frac{w}{z}$

31. a) Calculate the exact modulus and an exact argument of each of the numbers $z = -1 + i\sqrt{3}$ and $w = -1 - i$.

b) Hence state the values of z^3 and w^4.

32. Use the results of question 31 to express the following in polar form.

a) zw **b)** $\frac{z}{w}$ **c)** $\frac{w}{z}$

33. a) If $z = \cos\theta + i\sin\theta$, state an argument of z^3.

b) Hence find expressions for $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

34. Calculate in Cartesian form

a) $(\cos 30° + i\sin 30°)^{12}$

b) $(\cos 20° - i\sin 20°)]^{-6}$

c) $(1 + i)^{10}$

d) $(-1 - i\sqrt{3})^{-2}$

e) $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5$

f) $\left(\cos\frac{7\pi}{12} - i\sin\frac{7\pi}{12}\right)^6$

35. Simplify $\frac{\cos 3\theta + i\sin 3\theta}{(\cos\theta + i\sin\theta)^2}$

36. Find the modulus and argument of

$$\frac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}$$

37. a) Compare the expressions for $(\cos\theta + i\sin\theta)^5$ given by De Moivre's theorem and the binomial expansion to prove that
$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

b) By considering the equation $\cos 5\theta = 0$, prove that $\cos\left(\frac{\pi}{10}\right)\cos\left(\frac{3\pi}{10}\right) = \frac{1}{4}\sqrt{5}$.

38. a) Verify that each of the following numbers is a sixth root of unity.

$$\alpha = \frac{1}{2} + \frac{\sqrt{3}}{2}i,\ \beta = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

b) State the six roots of the equation $z^6 - 1 = 0$.

39. Find the fifth roots of -1 in Cartesian form and represent them in a complex plane.

40. a) Find a complex number equation for a circle of centre $3 + 4i$ and radius 5.

b) Show that this circle passes through O.

41. Find equations in x and y of the loci described by the following, where $z = x + iy$.

a) $|z - 1 + 3i| = 1$

b) $|2z + i| = 5|z - i|$

42. Describe the locus of a point that moves in the complex plane in such a way that its distance from $-1 + 2i$ is half its distance from the origin.

43. Write the following in exponential form, using exact values.

a) $u = -2 - 2i\sqrt{3}$

b) $v = 5 - 5i$

44. Use the definitions $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ to prove that

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta.$$

45. i) Solve the equation $2z^3 - 3z^2 + 2z + 2 = 0$ given that $z = 1 + i$ is a solution

ii) The complex numbers w and z are related by the equation

$$w = \frac{z - 6i}{z + 8},$$

and the points W and Z in the Argand diagram correspond to w and z respectively.

a) Given that the real part of w is zero, show that Z lies on a circle, and find the centre and radius of this circle.

b) Given that the imaginary part of w is zero, show that Z lies on a straight line, and give the equation of this line.

(88 H)

46. i) By first putting $z^2 = w$, or otherwise, find the values of z for which
$z^4 + 2z^2 + 25 = 0, z \in \mathbb{C}$,
giving your answers in the form
$z = x + iy, x, y \in \mathbb{R}$.

ii) It is given that
$z = 2r(\cos\theta + i\sin\theta)$,
$w = z + \dfrac{r^2}{z}, r \in \mathbb{R}^+; w, z \in \mathbb{C}; -\pi < \theta \leqslant \pi$.

a) If $w = u + iv, u, v \in \mathbb{R}$,
show that $\left(\dfrac{2u}{5r}\right)^2 + \left(\dfrac{2v}{3r}\right)^2 = 1$.

b) Find the four values of θ where
$|w| = 2r$,
giving your answers correct to two decimal places.

(88 S)

47. Let

$w = \cos\dfrac{2\pi}{5} + i\sin\dfrac{2\pi}{5}$.

a) Show that $1, w, w^2, w^3$ and w^4 are the 5 roots of the equation
$z^5 = 1, z \in \mathbb{C}$.

b) By factorizing $(z^5 - 1)$, or otherwise, prove that
$1 + w + w^2 + w^3 + w^4 = 0$.

c) Show, by multiplying out and using parts a) and b), that
$(1 - w)(1 - w^2)(1 - w^3)(1 - w^4) = 5$.

d) i) Use the given expression for w to prove that

$(1 - w)(1 - w^4) = 4\sin^2\dfrac{\pi}{5}$.

ii) Work out a similar expression for $(1 - w^2)(1 - w^3)$.

iii) Deduce, from parts c), d)(i) and d)(ii), that

$\sin\dfrac{\pi}{5}\sin\dfrac{2\pi}{5} = \dfrac{1}{4}\sqrt{5}$.

(88 S)

48. a) Find the complex roots of the equation
$z^2 - z + 1 = 0$
in the form $p + iq, p - iq$, where $p, q \in \mathbb{R}$.

b) Express the two roots obtained in part a) in the form $r(\cos\theta + i\sin\theta)$ and $r(\cos\theta - i\sin\theta)$, where r and θ are to be determined, $r \in \mathbb{R}^+, 0 \leqslant \theta \leqslant \pi$.

c) Show that
$(z + 1)(z^2 - z + 1) = (z^3 + 1)$.

d) i) Use the results already obtained to write down the modulus and argument of each of the three roots of the equation
$z^3 + 1 = 0$.

ii) Hence plot these roots on a carefully labelled Argand diagram.

iii) Prove that the three plotted points lie at the vertices of an equilateral triangle.

e) By expressing
$z^3 - 3z^2 + 3z = (z - 1)^3 + 1$,
or otherwise, prove that the roots of the equation
$z(z^2 - 3z + 3) = 0$,
when plotted on an Argand diagram, also lie at the vertices of an equilateral triangle.

(86 S)

49. i) a) Find, in the form $a + bi$, all the solutions of the equation
$z^3 + 6z = 20$.

b) The points in the Argand diagram, representing the three solutions found in part a) are the vertices of a triangle. Find the angles of this triangle.

c) Show that two of the solutions found in part a) have modulus $\sqrt{10}$ and find their arguments.

ii) Given that $0° \leqslant \theta \leqslant 360°$ solve the equation
$\sin 3\theta + \sin\theta = \cos\theta$

iii) Given that $0° \leqslant \theta \leqslant 360°$ solve, correct to the nearest degree, the equation
$3\cos\theta + 4\sin\theta + 2 = 0$.

(87 H)

50. The complex number z is given by $z = x + iy$, $x, y \in \mathbb{R}$.

a) Find z^2 in terms of x and y.

b) Given that $z^2 = 9 + 40i$,

i. find the possible values of x and y, and

ii. hence solve, for z, the equation $z^2 = 9 + 40i$.

c) On a clearly labelled Argand diagram plot the points P and Q which represent the solutions obtained in part b), placing P in the first quadrant. Plot also the point R representing z^2.

d) Find OP and OQ, leaving your answers in surd form.

e) Determine, to the nearest degree, the value of the angle that (OP) makes with the positive direction of the real axis.

f) Determine, to the nearest degree, the value of the angle POR.

(85 S)

51. i) Solve the simultaneous equations

$$\left.\begin{aligned} z + 2w &= 7 \\ iz + w &= 1 \end{aligned}\right\}$$

and show the solutions on an Argand diagram.

ii) Given that $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, where $r_2 \neq 0$, prove that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}.$$

If $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$ find the modulus and argument of

$$\frac{z_1}{z_2} \text{ and } \frac{1}{z_2^6}$$

(83 H)

52. i) a) Solve the equation $z^3 = 4\sqrt{3} - 4i$, giving your answers in modulus-argument form.

b) The equation $z^3 - z^2 + 3z + 5 = 0$ has $z = -1$ as one of its roots. Find the other two roots, giving your answers in the form $z = a + bi$.

ii) The complex number z satisfies each of the inequalities

a. $-\frac{1}{2}\pi \leqslant \arg z \leqslant 0$,

b. $|z - 1| \leqslant 2$,

c. $|z - 3| \leqslant |z - 1|$.

Show, on a clearly labelled Argand diagram, the region containing the set of points satisfying the three inequalities simultaneously.

(84 H)

53. i) Show that the set of complex numbers which satisfy the equation $|z + 1| = 2|z - 1|$ lie on a circle in the Argand diagram. Find the centre and radius of this circle.

ii) Use the fact that $\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$ to prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. Hence, without using a calculator, prove that

$$\cos 18° = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}$$

and find a similar expression for $\cos 54°$.

(86 H)

VECTORS, MATRICES and COMPLEX NUMBERS

with
International Baccalaureate
questions

Jean-Paul GINESTIER
and
John EGSGARD

PROBLEM SUPPLEMENT
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PROBLEM SUPPLEMENT

1. In quadrilateral $ABCD$, E and F are the midpoints of the diagonals AC and BD respectively. P is the midpoint of EF. If O is any point, prove that
$4\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}$.

2. $ABCD$ is a rectangle. The midpoints of sides AB, BC, CD, and DA are M, N, P, and Q respectively. Prove that $MNPQ$ is a rhombus.

3. Given any triangle ABC, and any point O, a point G is positioned so that
$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \vec{0}$.
Use vector subtraction, with origin O, to prove that
$\overrightarrow{OG} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$
(The point G is known as the **centroid** or **centre of mass** of the triangle ABC.)

4. Determine whether or not the three vectors in each of the following are linearly dependent. In each case state the geometric significance of the result.
   a) $\overrightarrow{(2,0,6)}$, $\overrightarrow{(1,1,-3)}$, and $\overrightarrow{(2,1,-1)}$
   b) $\overrightarrow{(3,2,1)}$, $\overrightarrow{(4,11,6)}$, and $\overrightarrow{(14,1,0)}$
   c) $\overrightarrow{(4,9,1)}$, $\overrightarrow{(-3,1,1)}$, and $\overrightarrow{(6,29,3)}$

5. Given the vectors $\vec{a} = \overrightarrow{(5,-6)}$ and $\vec{b} = \overrightarrow{(4,2)}$.
   a) Prove the two vectors form a basis for $\mathbb{V}_2$.
   b) Express the vector $\overrightarrow{(3,-7)}$ as a linear combination of $\vec{a}$ and $\vec{b}$.

6. Express the vector $\vec{a} = \overrightarrow{(7,8,16)}$ as a linear combination of the vectors $\vec{b} = \overrightarrow{(1,-2,3)}$, $\vec{c} = \overrightarrow{(0,2,5)}$, and $\vec{d} = \overrightarrow{(2,2,1)}$.

7. Find the value for m if the vectors $\vec{a} = \overrightarrow{(2,7,-4)}$, $\vec{b} = \overrightarrow{(4,m,3)}$, and $\vec{c} = \overrightarrow{(0,1,-2)}$ are coplanar.

8. a) A point P divides the line segment AB internally in the ratio $3:5$. Express $\overrightarrow{OP}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.
   b) If point Q divides segment PB of part a) externally in the ratio $7:4$, then express $\overrightarrow{OQ}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

9. a) Four points M, K, T, and R are given such that $\overrightarrow{TR} = -\frac{3}{2}\overrightarrow{TK} + \frac{5}{2}\overrightarrow{TM}$. Draw conclusions about the points M, K, T, and R.
   b) If T has coordinates $(0,0,0)$, while K and M have respectively coordinates $(1,-3,4)$ and $(5,0,2)$, find the coordinates of point R.

10. Points A, B, C, and D are points in 3-space with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$. $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly independent and $\vec{d} = -7\vec{a} + 3\vec{b} + 5\vec{c}$. Prove that points A, B, C, and D are coplanar.

11. $MTRS$ is a parallelogram. Point A divides TR internally in the ratio $8:5$. Segments MR and SA intersect at point B. Use vector methods to find the ratio into which point B divides segment MR.

12. Given the vectors $\vec{a} = \overrightarrow{(-1,-4,7)}$, $\vec{u} = \overrightarrow{(2,3,-1)}$, and $\vec{v} = \overrightarrow{(4,1,11)}$.
    a) Prove the vectors are linearly dependent.
    b) Find a vector coplanar with $\vec{u}$ and $\vec{v}$, that is perpendicular to vector $\vec{a}$.

13. Given the vectors $\vec{u} = \overrightarrow{(1,1,1)}$, $\vec{v} = \overrightarrow{(1,-2,1)}$, and $\vec{w} = \overrightarrow{(2,0,-1)}$.

a) Prove that $\vec{u}$, $\vec{v}$, and $\vec{w}$ form a basis for $\mathbb{V}_3$.

b) Find the components of $\vec{a} = \overrightarrow{(5,3,2)}$ with respect to the basis in part a).

c) If $m\vec{u} + p\vec{v} + r\vec{w} = \overrightarrow{(x,y,z)}$, then show that $m + p + r = 1$ if and only if $2x + z = 3$.

14. Points P, D, and R are collinear and O is any point such that $\overrightarrow{OD} = 3m\overrightarrow{OP} + 4k\overrightarrow{OR}$, and $2m - 3k = 16$. Find the values of k and m.

15. Prove that the components of a vector $\vec{d}$ with respect to the $\mathbb{V}_3$ basis $\{\vec{a},\vec{b},\vec{c}\}$ are unique.

16. Express $\vec{e} = \overrightarrow{(11,6,1)}$ as a linear combination of $\vec{a} = \overrightarrow{(3,1,0)}$, $\vec{b} = \overrightarrow{(-2,0,4)}$, $\vec{c} = \overrightarrow{(0,1,2)}$, and $\vec{d} = \overrightarrow{(1,1,1)}$.

17. Prove that $\vec{a}$, $\vec{b}$, and $\vec{c} = k\vec{b}$, $k \in \mathbb{R}$ are linearly dependent in $\mathbb{V}_3$.

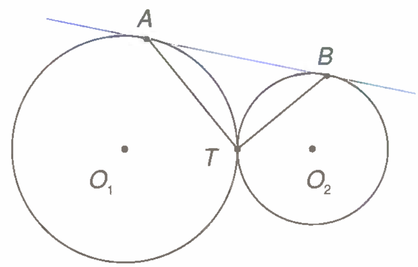
18. G is the point of intersection of the medians of a triangle ABC. The point G divides the medians AD, BE, and CF internally in the ratio 2 : 1. Prove that $\overrightarrow{AG} + \overrightarrow{BG} + \overrightarrow{CG} = \vec{0}$

19. O, P, D, R, and S are five points in 3-space such that $\overrightarrow{OP} = 2\overrightarrow{OD} + 2\overrightarrow{OR} - 3\overrightarrow{OS}$. Prove that the four points P, D, R, and S are coplanar.

20. O, A, B, and C are four points in 3-space such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$ where $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly independent. If D lies in the plane determined by points A, B, and C, then express $\overrightarrow{OD}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$.
(You will need to use two scalars.)

21. Triangle ABC is right-angled at B, and M is the midpoint of AC. If $\overrightarrow{BA} = \vec{a}$ and $\overrightarrow{BC} = \vec{c}$, express $\overrightarrow{BM}$ and $\overrightarrow{AM}$ in terms of $\vec{a}$ and $\vec{c}$, and hence show that $|\overrightarrow{AM}| = |\overrightarrow{BM}|$.

22. Two circles, of centres O_1 and O_2, touch externally at T. A common tangent to the two circles touches them at A and B respectively. Use the dot product to prove that ATB is a right angle. (You may assume that O_1TO_2 is a straight line and that the angles O_1AB and O_2BA are right angles.)



23. $OABC$ is a tetrahedron with OA perpendicular to BC, and OC perpendicular to AB. If $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, and $\overrightarrow{OC} = \vec{c}$, express $\overrightarrow{AB}$ and $\overrightarrow{BC}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$, and hence prove that $\overrightarrow{OB}$ is perpendicular to $\overrightarrow{AC}$.

24. A right square pyramid $ABCDT$ whose base has a side $2s$ and whose height is h is positioned in a three-space coordinate system as follows. $A(s,s,0)$, $B(-s,s,0)$, $C(-s,-s,0)$, $D(s,-s,0)$ and $T(0,0,h)$.

a) If θ is the angle between a slant edge and the base, show that
$$\cos\theta = \frac{h}{\sqrt{2s^2 + h^2}}$$

b) If ϕ is the angle between two triangular faces, show that
$$\cos\phi = -\frac{s^2}{s^2 + h^2}$$

25. If $\vec{w} = (\vec{v} \cdot \vec{e})\vec{e}$ and $\vec{z} + \vec{w} = \vec{v}$, prove that $\vec{z} \cdot \vec{e} = 0$. Draw a figure that shows how the vectors $\vec{v}$, $\vec{e}$, $\vec{w}$, and $\vec{z}$ are related.

26. Resolve the vector $\vec{v} = \overrightarrow{(-2,3)}$ onto the vectors $\vec{a} = \overrightarrow{(1,1)}$ and $\vec{b} = \overrightarrow{(-1,1)}$.

27. In 3.2 Exercises you proved that for any vectors $\vec{u}$ and $\vec{v}$, $|\vec{u} \cdot \vec{v}| \leqslant |\vec{u}||\vec{v}|$. Given that $\vec{u} = \overrightarrow{(u_1,u_2,u_3)}$, and $\vec{v} = \overrightarrow{(v_1,v_2,v_3)}$, write out this inequality using components. (This is known as the **Cauchy-Schwarz inequality**.)

28. The vector $\vec{n} = \overrightarrow{(2,4,-3)}$ is normal (that is, perpendicular) to the plane Π. $A(1,5,5)$ is a given point of Π, and $P(x,y,z)$ is any point of Π. Let the position vectors of A and P be $\vec{a}$ and $\vec{r}$ respectively.
   a) Explain why $\overrightarrow{AP} \cdot \vec{n}$ must be zero.
   b) Substitute the given values into the equation $\overrightarrow{AP} \cdot \vec{n} = 0$ to show that an equation representing the plane Π is $2x + 4y - 3z = 7$.

29. A motor boat is travelling upstream at full power. As the boat passes under a low bridge, the woman at the helm loses her hat. However, she does not notice this until 6 minutes later. At this time, she turns her boat around and goes downstream at full power. She retrieves her hat 1 km downstream from the bridge where it got knocked off. Calculate the speed of the current.

30. The diagonals of a parallelogram can be represented by the vectors $\overrightarrow{(6,6,0)}$ and $\overrightarrow{(1,-1,2)}$.
   a) Prove that the parallelogram is a rhombus.
   b) Calculate the length of its sides and the values of its angles.

31. Given any four points A, B, C, D in 3-space, prove that $\overrightarrow{AB} \cdot \overrightarrow{CD} + \overrightarrow{AC} \cdot \overrightarrow{BD} + \overrightarrow{AD} \cdot \overrightarrow{BC} = 0$. Deduce from this that the three altitudes of a triangle are concurrent. (That is, the three altitudes intersect at the same point.)

32. OAB is an isosceles triangle with $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, and $|\vec{a}| = |\vec{b}|$. M is the midpoint of OA, and N is the midpoint of OB. Express $\overrightarrow{AN}$ and $\overrightarrow{BM}$ in terms of $\vec{a}$ and $\vec{b}$, and use the dot product to show that $|\overrightarrow{AN}| = |\overrightarrow{BM}|$.

33. Two forces $\vec{E}$ and $\vec{F}$ act upon a particle in such a way that the resultant force $\vec{R}$ has a magnitude equal to that of $\vec{E}$, and makes an angle of 90° with $\vec{E}$.
   Given that $|\vec{E}| = |\vec{R}| = 50$ N, calculate the magnitude and direction of the second force relative to $\vec{E}$.

34. A river flows due east with a speed of 2.5 km/h. A woman rows a boat across the river, her velocity relative to the water being 3 km/h due north.
   a) What is her velocity relative to the Earth?
   b) If the river is 250 m wide, how far east of her starting point will she reach the opposite bank?
   c) How long will she take to cross the river?

35. The woman in the previous question decides, on another day, to row in such a way that the boat reaches a point on the opposite bank directly north of her starting point.
   a) In what direction should she head the boat?
   b) What is her velocity relative to the Earth?
   c) How long will she take to cross the river?

36. An airplane pilot wishes to fly along bearing 158°. A wind of 100 km/h is blowing from the west. If the plane's airspeed is 350 km/h, find the following.

a) the plane's groundspeed

b) the heading that the pilot should take

37. Find vector equations, parametric equations, and symmetric equations of the following lines.

a) the line through the point $A(2,5,3)$ having direction numbers 2, 3, 6

b) the line through the points $(-2,-1,4)$ and $(3,2,2)$.

38. a) Determine whether the lines L_1: $\vec{r} = \overrightarrow{(1,3,0)} + t\overrightarrow{(-2,1,4)}$ and L_2: $\vec{r} = \overrightarrow{(4+3k, 2-k, -1+k)}$ are skew, or if they intersect.

b) If the lines in a) intersect, then find their point of intersection. If the lines are skew, then find the shortest distance between the lines.

39. Find a vector equation of the line that is perpendicular to the vector $\vec{u} = \overrightarrow{(1,2,3)}$ and also perpendicular to the line $\vec{r} = \overrightarrow{(0,-2,5)} + k\overrightarrow{(4,-2,1)}$ at the point corresponding to the parameter value $k = 2$.

40. Find a vector equation of the line in 2-space that is perpendicular to the line $3x + 5y + 7 = 0$, and passes through the point of intersection of the lines $\vec{r} = \overrightarrow{(4,5)} + t\overrightarrow{(1,-3)}$ and $\vec{r} = \overrightarrow{(4+2k, 3-4k)}$.

41. The line L passes through the point $A(1,3,2)$ and has direction vector $\overrightarrow{(2,1,-2)}$. Find the coordinates of two points on L that are 3 units from point A.

42. a) Find a value of c such that the lines
$$\begin{cases} x = 1 + 7t \\ y = 2 + 3t \\ z = 1 + ct \end{cases} \quad \text{and} \quad \begin{cases} x = 3 + 21k \\ y = 2 + 9k \\ z = 5 + 8k \end{cases}$$
are parallel.

b) Show that there is no value of c for which the lines in part a) intersect.

43. Determine which of the following sets of points are collinear.

a) $A(0,5,-3)$ $B(3,-1,12)$ $C(-2,9,-13)$
b) $P(5,1,9)$ $Q(-1,5,0)$ $R(8,-1,13)$
c) $L(4,2,3)$ $M(2,-4,5)$ $N(6,8,1)$

44. a) A vector through the origin makes equal angles of 60° with both the positive x-axis and positive y-axis. Find the angle the vector makes with the z-axis.

b) A vector through the origin makes equal angles of θ with the positive x-axis and with the positive y-axis. What is the smallest possible value for θ? What is the largest possible value for θ?

45. a) For what values of t, if any, will the line $\vec{r} = \overrightarrow{(3,0,5)} + k\overrightarrow{(2,t,1)}$ intersect with the line $\vec{r} = \overrightarrow{(1,2,-1)} + s\overrightarrow{(-1,-4,2)}$?

b) If a point of intersection exists in a), find its coordinates.

46. A line passes through the point $A(2,1)$ and makes an angle of 45° with the vector $\vec{u} = \overrightarrow{(3,4)}$.

a) Explain why there are two lines satisfying these conditions.

b) Find vector equations of the two lines of part a).

47. Find a vector equation of the plane that contains the point $A(-1,2,1)$ and is perpendicular to the vector $\vec{u} = \overrightarrow{(-3,0,2)}$.

48. Find a vector equation of the plane that contains the line L: $\vec{r} = \overrightarrow{(1,2,3)} + t\overrightarrow{(1,-2,2)}$ and is perpendicular to the plane $x + 2y - 3z = 8$.

49. Find the point of intersection of the plane $2x + y - z = 2$ and the line of intersection of the two planes $x - y + z = 3$ and $x + 2y - 2z = -3$.

50. Find a Cartesian equation of the plane, containing the points $T(0,1,-2)$ and $S(1,-1,3)$, that is parallel to the vector $\vec{u} = \overrightarrow{(-5,1,3)}$.

51. Find a Cartesian equation of the plane, containing the points $W(1,0,-4)$ and $N(0,2,1)$, that is perpendicular to the plane $3x + 2y - 2z = 5$.

52. Find the intersection of the line $\vec{r} = \overrightarrow{(-3+t,-6+2t,1-t)}$ and the plane $\vec{r} = \overrightarrow{(2+k-3m,1+2k-3m,1-3k+2m)}$. Describe the intersection geometrically.

53. Show that the point of intersection of the line $\frac{x+2}{2} = 2z - 3,\ y = 3$ with the plane $x + 2y - 3z = 11$ lies on the plane $2x - y + z = 98$.

54. Given the three planes

$$\begin{aligned} x + 2y + z &= 0 \\ (k+2)x + (k+2)y + (k+1)z &= 20 \\ (k+2)x + (2k+3)y &= -20 \end{aligned}$$

a) For what values of k will the planes intersect in a point?

b) For what values of k will the planes intersect in a line?

c) For what values of k will the planes form a triangular prism?

55. Find a Cartesian equation of the plane through the points $K(3,2,1)$, $R(0,-1,1)$, and $S(-4,-3,2)$.

56. Given the point $M(1,-2,4)$.

a) Find the distance between the point M and the plane $3x + 4y - z = 5$.

b) Find the value of k so that the distance between the point M and the line $\vec{r} = \overrightarrow{(1,0,-1)} + t\overrightarrow{(3,0,k)}$ will be equal to the distance found in part a).

57. Consider the set of real numbers as the vector space $\mathbb{V}_1$ of dimension 1.

a) Show that the function defined by $f: \vec{x} \to 5\vec{x}$ is a linear transformation of $\mathbb{V}_1$.

b) Show that the function defined by $g: \vec{x} \to \overrightarrow{3x-2}$ is *not* a linear transformation of $\mathbb{V}_1$.

(g is known as an **affine transformation** of $\mathbb{V}_1$.)

58. Determine the possible dimensions of the matrices A and B if *both* the products AB and BA are defined.

59. The transformation matrix $S = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ is known as a **vertical shear of factor** k. The properties of a shear will appear as answers to the following questions.

a) Find the images of $(0,1)$ and $(0,b)$, where $b \in \mathbb{R}$, and describe how any point of the y-axis is transformed.

b) Find the images of $(1,0)$ and $(1,b)$, and describe how any point on the line $x = 1$ is transformed.

c) Find the image of the point $(a,0)$, where $a \in \mathbb{R}$, and describe how any point on the x-axis is transformed.

d) Find $\det(S)$, and describe the area and orientation of a figure transformed by a shear.

60. The line of equation $y = (\tan \theta)\, x$ has slope $\tan \theta$. Thus, the angle between this line and the positive x-axis is $\theta°$. Consider the transformation of matrix R that reflects the plane in the line whose equation is $y = (\tan 30°)\, x$.
   a) Find the image of $\vec{i}$ under R.
   b) Find the image of $\vec{j}$ under R.
   c) Hence write the matrix R.
   d) Write the matrix of the reflection in the line of equation $y = (\tan \theta)\, x$.

61. Given that M_ϕ represents a reflection in the line $y = (\tan \phi°)\, x$, compare M_{60} and M_{240}. Explain.

62. Given the conic C: $16x^2 + y^2 = 16$.
   a) Name the type of conic.
   b) Find, in general form, an equation for C', the image of C, under the translation $(x,y) \rightarrow (x - 3, y + 1)$.
   c) Sketch a graph of the conic C and its image C'.

63. Given the conic C: $16x^2 = y$.
   a) Name the type of conic.
   b) Find, in general form, an equation for C', the image of C, under the translation $(x,y) \rightarrow (x + 1, y + 5)$.
   c) Sketch a graph of the conic C and its image C'.

64. Given the conic C: $9x^2 - y^2 = 36$.
   a) Name the type of conic.
   b) Find, in general form, an equation for C', the image of C, under the translation $(x,y) \rightarrow (x + 2, y - 4)$.
   c) Sketch a graph of the conic C and its image C'.

65. Given the conic C: $8x^2 + y^2 + 16x + 8y + 9 = 0$.
   a) Name the type of conic.
   b) Determine the translation that changes the equation into standard form.
   c) State an equation for C', the image of C.
   d) Graph the image conic C' and then the given conic C.

66. Given the conic C: $x^2 + 4x + 8y + 12 = 0$.
   a) Name the type of conic.
   b) Determine the translation that changes the equation into standard form.
   c) State an equation for C', the image of C.
   d) Graph the image conic C' and then the given conic C.

67. Given the conic C: $4x^2 - y^2 + 32x - 4y + 64 = 0$
   a) Name the type of conic.
   b) Determine the translation that changes the equation into standard form.
   c) State an equation for C', the image of C.
   d) Graph the image conic C' and then the given conic C.

68. Given the conic C: $x^2 + 4y^2 = 16$.
   a) Find an equation for C', the image of C under a rotation of $30°$ about $(0,0)$.
   b) Sketch a graph of C and C'.

69. Given the conic C: $4x^2 - 25y^2 = 100$.
   a) Find an equation for C', the image of C under a rotation of $70°$ about $(0,0)$.
   b) Sketch a graph of C and C'.

70. Given the conic C: $15x^2 + 34xy + 15y^2 = 128$.
   a) Determine the type of conic.
   b) Find an angle of rotation about $(0,0)$ that will eliminate the xy term.
   c) Find an equation of the image curve C'.
   d) Sketch the graph of the image curve C', then the graph of C, the original curve.

71. Given the conic C: $109x^2 + 236xy + 151y^2 = 1000$
   a) Determine the type of conic.
   b) Find an angle of rotation about $(0,0)$ that will eliminate the xy term.
   c) Find an equation of the image curve C'.
   d) Sketch the graph of the image curve C', then the graph of C, the original curve.

72. Prove using mathematical induction.

a) $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

b) $\sum_{i=1}^{n}(4i^2 - 3i + 2) = \frac{n(8n^2 + 3n + 7)}{6}$

73. Prove by mathematical induction that

a) $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$, for all $n \in \mathbb{N}$.

b) Evaluate $(8)(9) + (9)(10) + (10)(11) + (11)(12) + \ldots + (98)(99)$.

74. Use mathematical induction to prove

$\left(1 + \frac{3}{1}\right)\left(1 + \frac{5}{4}\right)\left(1 + \frac{7}{9}\right)\ldots\left(1 + \frac{2n+1}{n^2}\right) = (n+1)^2$

75. Prove by mathematical induction that $\frac{(3n+1)7^n - 1}{9}$ is a natural number.

76. Prove that a convex polygon of n sides has $\frac{n}{2}(n-3)$ diagonals.

77. Prove that $\frac{1}{n!} < \left(\frac{1}{2}\right)^{n-1}$ for n any natural number greater than 2.

78. Prove by induction that $x^{2n} - y^{2n}$ is divisible by $x + y$, where $n \in \mathbb{N}$.

79. Expand each of the following and simplify.

a) $(a + x)^5$

b) $(4a - 3)^4$

80. **a)** Show that the solution set of $z^6 + z^3 + 1 = 0$ is a subset of the solution set of $z^9 - 1 = 0$, and hence find the solutions of $z^6 + z^3 + 1 = 0$ in polar form.

b) Deduce the values of θ between 0 and 2π that satisfy the system of equations

$$\begin{cases} \cos 6\theta + \cos 3\theta + 1 = 0 \\ \sin 6\theta + \sin 3\theta = 0 \end{cases}$$

81. The equation $z^2 + (a + bi)z + (c + di) = 0$, where a, b, c, and d are non-zero real numbers, has exactly one real root. Show that $abd = b^2c + d^2$.

82. w is one of the non-real cube roots of unity.

a) Find the two possible values of the expression $1 + w + w^2$.

b) Simplify each of the expressions $(1 + w + 3w^2)^2$ and $(1 + 3w + w^2)^2$.

c) Show that the product of the two expressions in b) is 16, and that the sum is -4.

83. The matrices A and B and the non-zero vector $\vec{t}$ are given by

$$A = \begin{bmatrix} 2 & 1 \\ -7 & 1 \end{bmatrix},\ B = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \text{ and } \vec{t} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}.$$

a) i) Find the inverse of the matrix A.

ii) Use the inverse of the matrix A to solve the simultaneous equations

$$\begin{cases} 2x + y = 7 \\ -7x + y = -11, \text{ where } x, y \in \mathbb{R}. \end{cases}$$

b) Given $\vec{v} = B\vec{t}$,

i) express the components of $\vec{v}$ in terms of t_1 and t_2, and

ii) calculate the possible values of the scalar λ given that $\vec{v} = \lambda\vec{t}$ also.

c) Given $\vec{v} = B\vec{t}$ and $\vec{w} = A\vec{t}$,

i) prove that $\vec{v}$ and $\vec{w}$ are not perpendicular, and

ii) calculate the values of t_1 and t_2 when $\vec{v} - \vec{w} = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$.

d) Given $\vec{t} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, calculate the components of the vector $\vec{z}$, where $\vec{z} = AB\vec{t}$.

(85 SMS)

84. The position vectors of points A, B and C are respectively $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{b} = 3\vec{i} + \vec{j} + \vec{k}$, and $\vec{c} = 2\vec{i} - \vec{j} - 2\vec{k}$ with respect to an origin O. ($\vec{i}, \vec{j}, \vec{k}$ are mutually perpendicular unit vectors.)

i) Show that O, A, B and C are coplanar by proving the linear dependence of $\vec{a}$, $\vec{b}$ and $\vec{c}$.

ii) Prove that $OABC$ is a parallelogram.

iii) Find angles ABC and ABO. Show that $OABC$ is not a rhombus.

(80 S)

85. The position vectors of four distinct points A, B, C and D are $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ respectively.

a) The mid points of [AB], [CD], [BC], [AD], [AC] and [BD] are E, F, G, H, L and M respectively. Find, in terms of $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$, the position vectors of the mid points of [EF], [GH] and [LM]. What does your result indicate about the lines (EF), (GH) and (LM)?

b) (AB) is perpendicular to (CD). Show that

$$\vec{a}\cdot\vec{d} + \vec{b}\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{d}$$

Given also that (AC) is perpendicular to (BD), prove that (AD) is perpendicular to (BC).

(87 H)

86. Two lines L_1 and L_2 have equations

$$\frac{x+3}{3} = \frac{y+4}{2} = \frac{z-6}{-2} \text{ and}$$

$$\frac{x-4}{-3} = \frac{y+7}{4} = \frac{z+3}{-1} \text{ respectively.}$$

a) Find the coordinates of a point P_1 on the line L_1 and a point P_2 on the line L_2 such that the line (P_1P_2) is perpendicular to both the lines L_1 and L_2.

b) Show that the length of [P_1P_2] is 7.

c) Find the equation of the plane which contains the line L_1 and has no point of intersection with the line L_2. Give your answer in the form $ax + by + cz = d$.

(84 H)

87. i) The set of planes Π_k is given with equations
$x + 2y - 2z = 3k, k \in \mathbb{R}$.

a) Find a vector $\vec{n}$ of unit length perpendicular to the planes Π_k.

b) i) Express the equations of the planes Π_k in the form $\vec{r} \cdot \vec{n} = p$, where $\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and where the value of p is to be found in terms of k.

ii) What is the geometrical significance of

a) the magnitude of k, and

b) the sign of k?

c) Show that the perpendicular distance between the planes Π_{-2} and Π_5 is 7 units (i.e. the planes with $k = -2$ and $k = 5$ respectively).

d) Find the coordinates of the point P of intersection of the line with equation
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, y \in \mathbb{R},$$
and the plane Π_1.

e) Find the length of $[OP]$, where O is the origin, leaving your answer as an irrational number.

ii) The set of planes Π_t is given with equations
$x + ty - tz = 3, t \in \mathbb{R}$.

a) Prove that all planes of the set contain a certain line, and

b) find its equation in parametric form.

(86 S)

88. In a rectangular Cartesian coordinate system with origin O and unit basis vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$, two straight lines l and l' have respective parametric equations:

$x = 6 - t,\ y = 0,\ z = t$

and $x = 0,\quad y = t',\ z = 2t'$

a) $\vec{u} = \lambda\vec{i} + \mu\vec{k}$ is the unit vector parallel to l for which $\lambda > 0$.
Find λ and μ.
Find also a unit vector $\vec{u'}$ which is parallel to l'.

b) H is a point on l with parameter t and K is a point on l' with parameter t'. If $\vec{HK}$ is perpendicular to $\vec{u}$ show that $t' = t - 3$.

c) If $\vec{HK}$ is also perpendicular to $\vec{u'}$, find the coordinates of H and K.

(81 S)

89. In a rectangular Cartesian coordinate system the points O, A, B and C have coordinates (0,0), (4,2), (−4,2) and (2,−2) respectively.

a) Prove that $|\vec{OB}| = |\vec{AC}|$.

b) i) Write in column vector form each of the vectors $\vec{OA}$, $\vec{OB}$ and $\vec{OC}$.

ii) Hence, given that the vector $\vec{OA} + k\vec{OB} + l\vec{OC}$ is the zero-vector, where $k, l \in \mathbb{R}$, find the values of k and l.

c) Given that $\vec{v} = \vec{OB} + n\vec{OC}$, where $n \in \mathbb{R}$.

i) If $\vec{v}$ is perpendicular to $\vec{BC}$, show that $n = 1.6$.

ii) If $\vec{v}$ is parallel to $\vec{BC}$, find the value of n.

d) It is given that matrix $M = \begin{bmatrix} -p & p \\ 0 & p \end{bmatrix}$, where $p \in \mathbb{R}$, and that $M(\vec{OA})$ means the product of M and $\vec{OA}$.

i) Given that $M(\vec{OA}) = \vec{OC}$, find the value of p.

ii) Prove that for all n and p:
$M(\vec{OA}) \neq \vec{OB} + n\vec{OC}$, where $n \in \mathbb{R}$.

(86 SMS)

90. Using a rectangular Cartesian coordinate system with origin O, a transformation $T: P \rightarrow P'$ is defined such that the coordinates of the point $P(x,y)$ are transformed to the coordinates of the point $P'(x',y')$ by means of the equation
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$
a) Find the equation of the straight line l' which is the image of the straight line l whose equation is $2x + 3y = 6$.
b) Determine the coordinates of Q, the point of intersection of the lines l and l'.
c) Prove that the straight line (OQ) consists of points that are their own images.
d) Find the coordinates of the point R' that is the image of a variable point R on the line perpendicular to (OQ) through O.
e) Give a geometrical description of the effect of the transformation T.
(85 S)

91. In a shear transformation parallel to the line with equation $y = x$, the point $A(1,1)$ is its own image and the point $B(-1,1)$ has as its image the point $B'(0,2)$. Find the (2×2) matrix that represents this shear.
(84 S)

92. For each of the following assertions concerning 2×2 matrices A and B state whether it is true or false. Prove any assertion that you consider true, and give a counterexample for any assertion you consider false.
a) For all A and B, $(A - B)(A + 2B) = A^2 + AB - 2B^2$.
b) For all A, $(A - I)(A + 2I) = A^2 + A - 2I$, where I is the unit 2×2 matrix.
c) For all A and B, $(AB)^T = A^TB^T$ where A^T is the transpose of the matrix A.
d) If $A^2 = A$ then A is a singular matrix.
(82 H)

93. The matrix M and non-zero vector $\vec{v}$ are given by
$$M = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}.$$
a) Find the components of the vector $\vec{w}$ where $\vec{w} = M\vec{v}$.
b) It is given that $\vec{w} = \lambda\vec{v}$ where λ is a scalar, $\lambda \in \mathbb{R}$.
i) Obtain two simultaneous equations in x and y.
ii) Show that the condition for a non-zero vector $\vec{v}$ to exist is $\lambda^2 - 9 = 0$.
iii) Solve the equation $\lambda^2 - 9 = 0$. For each value of λ find the corresponding set of vectors $\vec{v}$ and give a unit vector in that set.
c) Given the matrix $N = \begin{bmatrix} 1 & 4 \\ -2 & -1 \end{bmatrix}$ find the components of the vector $\vec{u}$ where $\vec{u} = N\vec{v}$. Does there exist a value of the scalar μ, $\mu \in \mathbb{R}$, such that $\vec{u} = \mu\vec{v}$?
(83 SMS)

94. It is given that
$$M = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$$
a) Find vectors $\vec{u}$ and $\vec{v}$ such that $M\vec{u} = \vec{u}$ and $M\vec{v} = -\vec{v}$.
b) A transformation $T: P \rightarrow P'$ is defined such that the coordinates of the point $P(x,y)$ are transformed to the coordinates of the point $P'(x',y')$ by means of the equation
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = M\begin{bmatrix} x \\ y \end{bmatrix}.$$
By using the result of part a), or otherwise, find the equations of the lines through the origin that are invariant under the transformation T.
c) Describe geometrically the effect of the transformation T and by applying the geometry of this result explain why $T^{-1} = T$
(84 S)

95. M is the (2×2) matrix given by

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, 0 \leqslant \theta < 360°$$

a) Show that $M^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$.

b) Using the result of part a), or otherwise, find a (2×2) matrix X with real elements such that $X^2 + I = 0$, where I is the (2×2) unit matrix and 0 is the (2×2) zero matrix.

c) i) In the case when $\theta = 60°$, find the matrices M^2, M^4 and M^6, expressing your answers *exactly* in numerical form.

ii) What do each of the three matrices obtained in part c)i) represent when considered a linear transformations of a plane?

(87 S)

96. A point P is reflected in the line with equation $y = x$, to give an image point P'. P' is given a positive (anticlockwise) quarter turn about the origin to give an image point P''. Find the matrix M corresponding to the single reflection that would map P to P'' but keep the origin fixed.

(82 S)

97. Given that $A = \begin{bmatrix} 5 & 2 \\ 9 & 8 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ form the quadratic equation in λ given by $\det(A - \lambda I) = O$. Find the roots λ_1, λ_2 of this equation, where $\lambda_1 < \lambda_2$. Find a vector $\vec{e}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ such that $A\vec{e}_1 = \lambda_1\vec{e}_1$, and a vector $\vec{e}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ such that $A\vec{e}_2 = \lambda_2\vec{e}_2$.

Using these results

a) find the equations of two distinct lines which are mapped onto themselves by the transformation represented by the matrix A,

b) if $P = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$ find $P^{-1}AP$ and deduce that $P^{-1}A^nP = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$.

(83 H)

98. Given $z = x + iy$, $z' = x' + iy'$ and $z' = (2 + i)z$ find the (2×2) matrix M such that $\begin{bmatrix} x' \\ y' \end{bmatrix} = M\begin{bmatrix} x \\ y \end{bmatrix}$

(84 S)

99. The linear transformation L represents an anti-clockwise rotation of θ about the origin, where $\theta \in \left[0, \frac{\pi}{2}\right]$, and the linear transformation M represents a reflection in the line with equation $y = x \tan\alpha$, where $\alpha \in \left[0, \frac{\pi}{2}\right]$.

i) Show that

a) $L\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

b) $M\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos 2\alpha \\ \sin 2\alpha \end{bmatrix}$

and find $L\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $M\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

ii) Hence, or otherwise, find the 2×2 matrices A and B which represent the linear transformations L and M respectively.

iii) Evaluate and simplify the matrices A^2 and B^2, giving a geometrical interpretation of your results.

iv) Prove, by induction, that

$$A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

(85 H)

100. a) Show that the value of the determinant

$$\begin{vmatrix} 1 & a & 1 \\ 1 & 1 & a \\ a+1 & a & 1 \end{vmatrix} \text{ is } a(a^2 - 1).$$

b) Find such solutions as exist of the simultaneous linear equations

$$x + ay + z = 2a,$$
$$x + y + az = 0,$$
$$(a + 1)x + ay + z = a,$$

in the cases (i) $a = 0$ and (ii) $a = -1$. Give a geometrical interpretation of your answer in each case.

(83 S)

101. a) Find, in terms of d, the value of the determinant

$$\begin{vmatrix} d & 5 & -1 \\ 1 & 3 & d \\ 1 & 4 & 7 \end{vmatrix}.$$

b) Calculate the values of d for which the value of the determinant is zero.

c) Find the solutions, if they exist, of the simultaneous linear equations.

$dx + 5y - z = 0,$
$x + 3y + dz = d,$
$x + 4y + 7z = 6,$

in each of the following cases:

i) $d = 2;$
ii) $d = 0;$
iii) $d = \frac{9}{2}$

(86 S)

102. A is the (2×2) matrix given by

$$A = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$$

a) Write down the value of the determinant of A.

b) Write down the (2×2) matrix inverse to A.

c) Using a matrix method, solve the simultaneous equations $3x + 4y = 12$, $4x + 5y = 20$.

d) Find the area of the closed region in the first quadrant bounded by the axes Ox, Oy and the lines l_1 and l_2 whose equations are $3x + 4y = 12$ and $4x + 5y = 20$ respectively.

e) i) Find a unit vector in the direction of each of the lines l_1 and l_2, both considered in the sense of x increasing, and

ii) hence, or otherwise, show that the value of the acute angle between these lines is 1.8°, correct to the nearest tenth of a degree

(86 S)

103. In a two dimensional rectangular Cartesian co-ordinate system the points A and B are given such that

$$\overrightarrow{OA} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ and } \overrightarrow{OB} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

a) The matrix L, where $L = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, maps the points A and B onto the points C and D respectively.

Find $\overrightarrow{OC}$ and $\overrightarrow{OD}$, giving your answers in column vector form.

b) The 2×2 matrix M represents the transformation which reflects in the line $y = x$ all points of the plane.

i) Find the matrix M.

ii) Given that the transformation M maps the points C and D onto the points E and F respectively, find $\overrightarrow{OE}$ and $\overrightarrow{OF}$, giving your answers in column vector form.

iii) Given the matrix $N = M \times L$ show that $N = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

iv) a) Find the value of det N and

b) hence, or otherwise, compare the areas of the triangles OAB and OEF.

c) The 2×2 matrix R represents the transformation which rotates through 180° about the origin O all points of the plane.

i) Find the matrix R.

ii) The matrix S is given such that $S = R \times N$.

a) Find the matrix S.

b) Find the matrix S^{-1}.

c) The transformation represented by the matrix S^{-1} maps the points A and B onto the points G and H respectively.

i) Show that $GH^2 = 2$ and

ii) Calculate the angle between $\overrightarrow{OG}$ and $\overrightarrow{OH}$, giving your answer correct to the nearest degree.

(87 SMS)

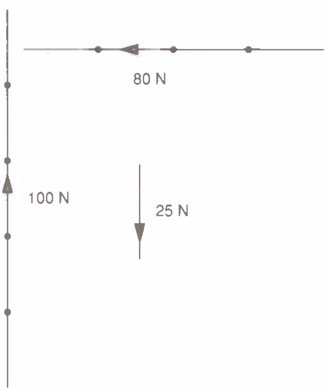
ANSWER KEY

Generally, answer is not provided where this is implied in the question.
Answers which will vary are also not provided.
Some answers have been left in unsimplified form, as a hint.
In general, numerical answers which are approximations are given with 3 significant digit accuracy, and angles to nearest degree, unless otherwise specified in the question.

Chapter One
An Introduction to Vectors

1.1 Exercises, pages 9–10

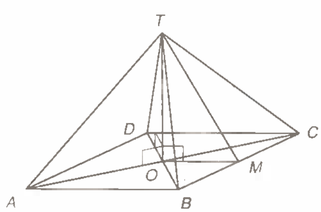
1. a) c) i) k) l)
2. a) b) ; none
3. a) $|\vec{u}| = 1$
   b) $|\vec{w}| = 3$
   c) $|\overrightarrow{AB}| = 2.5$
4. a)
   b)
   c)



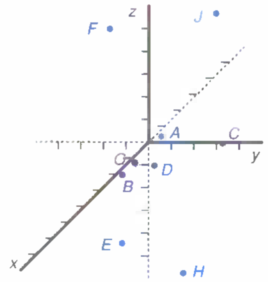
5. $AB = PQ$
6. a) Yes; equal vectors have same magnitude.
   b) No; direction could be different.
7. a) 245 km, 221°
   b) 122.5 km, 221°
8. a) no
   b) yes
   c) no
   d) no
   e) yes
   f) no
9. a) F
   b) F
   c) T
10. a) same length, same direction
    b) $\overrightarrow{PM} = \overrightarrow{MQ}$
11. $\overrightarrow{PB} = \vec{u}$, $\overrightarrow{QC} = \vec{v}$
12. a) $\overrightarrow{SR}$ d) $\overrightarrow{SP}$
    b) none e) $\overrightarrow{SI}$
    c) $\overrightarrow{IR}$ f) none
13. a) 5 b) 4 c) 3
14. $\vec{a} = \overrightarrow{BD} = \overrightarrow{EG} = \overrightarrow{CF}$
    $\vec{b} = \overrightarrow{AD} = \overrightarrow{FG} = \overrightarrow{CE}$
    $\vec{c} = \overrightarrow{BE} = \overrightarrow{DG} = \overrightarrow{AF}$
15. a) $\overrightarrow{AA'} = \overrightarrow{BB'} = \overrightarrow{CC'}$
    b) $\overrightarrow{PP'} = \overrightarrow{QQ'} = \overrightarrow{RR'} = \overrightarrow{SS'}$
    Your sketches will be of three vectors equal to
    a) $\overrightarrow{AA'}$ b) $\overrightarrow{PP'}$
16. The plane would be entirely 'shaded' in.

1.2 Exercises, page 16

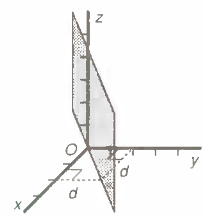
1. a) $TW \| UV \| QR \| PS$
   b) $WS \| VR \| UQ \| TP$
   c) $PR \| AB \| TV$
2. a) $TP \perp TW$ and $WS \perp TW$
   b) $TW \perp WS$ and $RS \perp WS$
   c) $TP \perp AB$ and $UQ \perp AB$
   (There are others.)
3. a) parallel
   b) skew
   c) intersecting
   d) skew
   e) intersecting
   f) skew
4. No; use a different scale, or a different direction for $\overrightarrow{QR}$, and redraw.
5.



6. $TM = 5$ cm; $TB = \sqrt{34}$ cm
7. a) TB, AC (There are others.)
   b) $ABCD$, TBC, TBA
8.



9. +++, ++−, +−+, −++, +−−, −+−, −−+, −−−
10. a) $y = z = 0$
    b) $x = z = 0$
    c) $x = y = 0$
11. a) $z = 0$
    b) $x = 0$
    c) $y = 0$
12. a plane through the origin, containing the z-axis, equidistant from the x- and y-axes



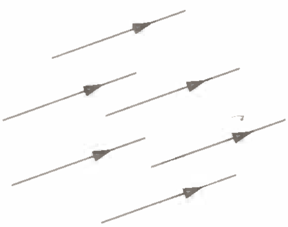
In Search of, page 20

1. a) $\alpha = 42°$
   b) $\beta = 53°$
   c) $\gamma = 26°$
2. a) $\measuredangle A = 59°$
   b) $\theta = 43°$
   c) $\phi = 53°$
3. a) 0.6 m
   b) 13°
4. a) 81.1 m
   b) 10°
5. 10.4 m; 27°; 23°

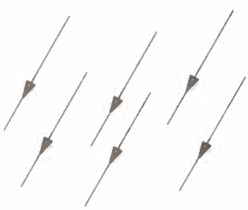
1.3 Exercises, page 25

1. $\vec{a} = \overrightarrow{(1,3)}$
   $\vec{b} = \overrightarrow{(1,0)}$
   $\vec{c} = \overrightarrow{(1,-2)}$
   $\vec{d} = \overrightarrow{(-2,-3)}$
   $\vec{s} = \overrightarrow{(0,-4)}$
   $\vec{t} = \overrightarrow{(-2,0)}$
   $\vec{u} = \overrightarrow{(-2,-2)}$
   $\vec{v} = \overrightarrow{(3,1)}$
   $\vec{w} = \overrightarrow{(-3,-1)}$

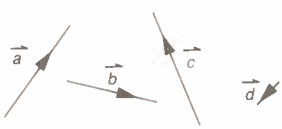
2.



3.

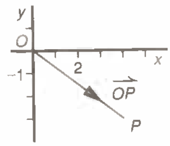


4.



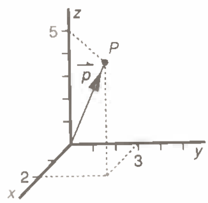
5. $|\vec{a}| = 5, |\vec{c}| = \sqrt{29},$
   $|\vec{b}| = \sqrt{17}, |\vec{d}| = \sqrt{2}$

6. a)

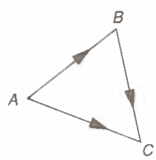


   b) $Q(-1,7)$
   c) $R(0,2,-2)$

7.



8. a)



   b) $\overrightarrow{AC} = \overrightarrow{(5,-2)}$
   c) $\overrightarrow{(x,y)} = \overrightarrow{(4,3)} + \overrightarrow{(1,-5)}$

9. $P'(5,3)$

11. $x = \sqrt{21}$

12. a) $\sqrt{13}$
    b) $\sqrt{12}$ or $2\sqrt{3}$

14. a) 700 m
    b) 500 m

15. $x = -6; w = 3$

16. $k = 7, m = 2, n = -1$

17. $h = \frac{1}{3}; k = -\frac{10}{3}$

18. a) $\overrightarrow{(3,2)}$ d) $\overrightarrow{(-4,0)}$
    b) $\overrightarrow{(-3,-2)}$ e) $\overrightarrow{(-6,-1)}$
    c) $\overrightarrow{(1,5)}$ f) $\overrightarrow{(c-a,d-b)}$

1.4 Exercises, pages 30–31

1. a) $\overrightarrow{PR}$
   b) $\overrightarrow{PR}$

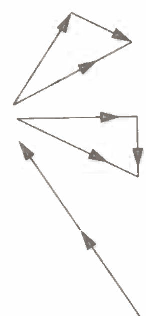
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

3. a) $\vec{a}$
   b) $\vec{b}$
   c) $\vec{a} + \vec{b}$
   d) $\vec{b} + \vec{a}$
   e) $\vec{c} + \vec{b}$
   f) $\vec{b} + \vec{c}$

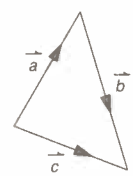
4. a) $\overrightarrow{AC}$
   b) $\overrightarrow{AC}$
   c) $\overrightarrow{BG}$
   d) $\overrightarrow{AG}$
   e) $\overrightarrow{DE}$
   f) $\overrightarrow{EF}$
   (There are others.)

5. a) $\vec{u} + \vec{w}$
   b) $\vec{u} + \vec{w}$
   c) $\vec{v} + \vec{w}$
   d) $\vec{u} + \vec{v} + \vec{w}$

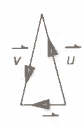
6. a) b) c)



7.



8. a) $\vec{w} = \overrightarrow{(-2,0)}$
   b)



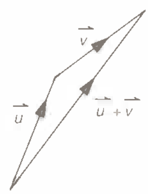
1.4 Exercises, pages 30–31, continued

9. a) $\overrightarrow{(1,8)}$
   b) $\overrightarrow{(5,-1)}$
   c) $\overrightarrow{(2,1)}$
   d) $\overrightarrow{(2,1)}$

10. a) $\overrightarrow{(3,-2,5)}$
    b) $\overrightarrow{(-4,-4,-1)}$
    c) $\overrightarrow{(-1,-8,8)}$
    d) $\overrightarrow{(-1,-8,8)}$

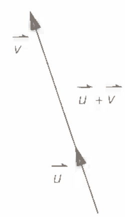
11. Vector addition is commutative.

12. a) $\overrightarrow{(6,8)}$
    b)



c) $|\vec{u}| = \sqrt{29}; |\vec{v}| = 5;$
$|\vec{u} + \vec{v}| = 10$
d) no (triangle inequality)

13. a) $\overrightarrow{(-3,9)}$
    b)



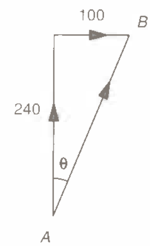
c) $|\vec{u}| = \sqrt{10}; |\vec{v}| = 2\sqrt{10};$
$|\vec{u} + \vec{v}| = 3\sqrt{10}$
d) $|\vec{u} + \vec{v}| = |\vec{u}| + |\vec{v}|$
(vectors in a line)

14. a) $\overrightarrow{(7,6)}$
    b) $\overrightarrow{(7,6)}$
    $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w});$
    Vector addition is associative.

15. $p = 4, q = -9$

16. $\vec{a} = \overrightarrow{(1,0,0)}$

17.



$|\overrightarrow{AB}| = 260$; bearing 023°

18. 8.26; bearing 226°

19. a) $|\vec{p} + \vec{q}| = 6.61$
    b) $\theta = 25°$

20. a) $4\sqrt{10}$ or 12.6
    b) 13

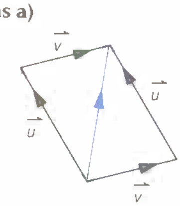
21. $12.6 < 12 + 4$;
    A, B, C collinear

1.5 Exercises, page 37

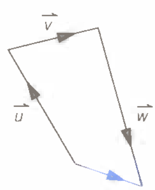
1. a) $\overrightarrow{AC}$
   b) $\overrightarrow{AC}$

2. a) $\overrightarrow{AA}$
   b) $\overrightarrow{AC}$
   c) $\overrightarrow{AD}$
   d) $\overrightarrow{AE}$
   e) $\overrightarrow{AC}$

3. a) $\vec{u} + \vec{v} = \vec{v} + \vec{u} = \overrightarrow{(1,6)}$
   c) as a)
   b)



4. b)



a) c) $(\vec{u} + \vec{v}) + \vec{w} =$
$\vec{u} + (\vec{v} + \vec{w}) = \overrightarrow{(3,-1)}$

5.

	commutative	associative
+	yes	yes
×	yes	yes
−	no; $8 - 3 \neq 3 - 8$	no; $(8 - 3) - 1 \neq 8 - (3 - 1)$
÷	no; $2 \div 8 \neq 8 \div 2$	no; $2 \div (8 \div 4) \neq (2 \div 8) \div 4$

6. a) no b) $(2^3)^4 \neq 2^{3^4}$

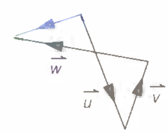
7. a) $\overrightarrow{AC}$
   b) $\overrightarrow{XZ}$
   c) $\overrightarrow{AB}$
   d) $\overrightarrow{QQ}$

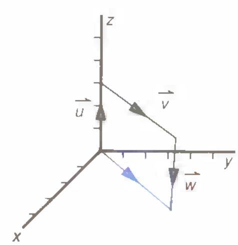
8. a) A diagram gives sufficient explanation.
   b) $\overrightarrow{PP}$
   c) 0
   d) no

9. a) $\sqrt{38}$
   b) $\sqrt{38}$
   c) $\sqrt{38}$
   d) 5

10. a) $\overrightarrow{(-3,-1)}$
    b) $\overrightarrow{(5,5,0)}$

11. a)



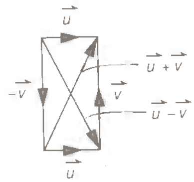


b) neither

12. a) $\sqrt{89}$
    b) $\sqrt{105}$
    c) 23°

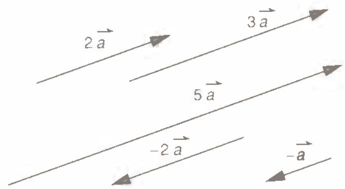
1.6 Exercises, page 42

1. a) F b) T c) T d) F
2. $\overrightarrow{RS} = \overrightarrow{QS} - \overrightarrow{QR}$
3. a) $\overrightarrow{RP}$
   b) $\overrightarrow{XZ}$
   c) $\overrightarrow{AB}$
   d) $\vec{0}$
   e) $\overrightarrow{AC}$
4. a) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$
   b) $\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ}$
   c) $\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR}$
   d) $\overrightarrow{RP} = \overrightarrow{OP} - \overrightarrow{OR}$
5. $\vec{d} = \vec{a} + \vec{c} - \vec{b}$
7. a) $\overrightarrow{(3,-2)}$
   b) $\overrightarrow{(-4,9)}$
   c) $\overrightarrow{(1,-7)}$
   d) $\overrightarrow{(-1,7)}$
8. a) $\overrightarrow{(-3,4,-9)}$
   b) $\overrightarrow{(7,2,6)}$
   c) $\overrightarrow{(-4,-6,3)}$
   d) $\overrightarrow{(4,6,-3)}$
9. $(\vec{r} - \vec{p}) = -(\vec{p} - \vec{r})$
10. a) $\overrightarrow{(-2,12)}$
    b) $\overrightarrow{(6,-6)}$
    c) $\overrightarrow{(-7,-1,-8)}$
    d) $\overrightarrow{(1,7,-6)}$
11. $m = 6, n = -2, k = 9$
12. a) $\overrightarrow{(5,1)}$
    b) $\overrightarrow{(-8,-3)}$
    c) $\overrightarrow{(3,2)}$
13. $\vec{0}$ (by vector addition)
14. a) $\overrightarrow{(1,2,16)}$
    b) $\overrightarrow{(-5,-6,-3)}$
    c) $\overrightarrow{(4,4,-13)}$
15. (3,1)
16. (−4,−4,−4)
17. (−3,−2)
18. a) $|\vec{u} + \vec{v}| = \sqrt{|\vec{u}|^2 + |\vec{v}|^2}$
    b)

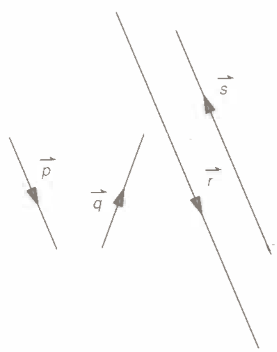


1.7 Exercises, pages 46-47

1.

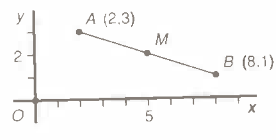


2. a) b)
3. a) $2\vec{u} + \vec{v}$
   b) $2\vec{v}$
   c) $2\vec{v}$
   d) $2\vec{u} - 2\vec{v}$
   e) $-2\vec{u} - 2\vec{v}$
4.



$\vec{p} \| \vec{r} \| \vec{s}$
$\vec{p}$ and $\vec{r}$ in same direction

5. a) $\overrightarrow{BC} = 2\overrightarrow{AB}$
   b) $\overrightarrow{AB} = \frac{1}{2}\overrightarrow{BC}$
   c) same slope
6. a) $\overrightarrow{BA} = \overrightarrow{(-2,3)}$
   b) $\overrightarrow{AC} = \overrightarrow{(6,-9)}$
   c) $\overrightarrow{CA} = \overrightarrow{(-6,9)}$
7. a) $\overrightarrow{(9,0)}$
   b) $\overrightarrow{(-3,-12,21)}$
   c) $\overrightarrow{\left(\frac{17}{2}, \frac{3}{2}, 3\right)}$
8. a) $\overrightarrow{(10,5,-15)}$
   b) $\vec{0}$
   c) coplanar
9. a) $5\sqrt{14}$
   b) 0
10. Answers will vary.
11. a) $\vec{v}$
    b) $2\vec{u}$
    c) $-\vec{v}$
    d) $\vec{0}$
12. a) $\overrightarrow{AB}$
    b) $\vec{0}$
    c) $\overrightarrow{BA}$
15. 1
16. a) $\overrightarrow{(5,2)}$
    b) c)



17. a) $\overrightarrow{MA} = -\overrightarrow{MB}$
    b) M midpoint of AB
    c) $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}$
18. a) 16
    b) 16
    c) 5
    d) 8.73

1.8 Exercises, page 51

1. $\overrightarrow{AB} = \vec{u}$; $\overrightarrow{CB} = 2\vec{v} - \vec{u}$;
   parallelogram
2. rhombus
3. $\overrightarrow{AB} = \frac{1}{2}\vec{c}$; $\overrightarrow{CB} = \vec{a} - \frac{1}{2}\vec{c}$
   trapezoid

1.8 Exercises, page 51, continued

4. a) $\overrightarrow{OS} = 2\vec{p}$
   $\overrightarrow{OT} = 4\vec{q}$
   $\overrightarrow{QP} = \vec{p} - \vec{q}$
   $\overrightarrow{TS} = 2\vec{p} - 4\vec{q}$
   $\overrightarrow{QR} = 3(\vec{p} - \vec{q})$
   $\overrightarrow{OR} = 3\vec{p} - 2\vec{q}$
   $\overrightarrow{TR} = 3\vec{p} - 6\vec{q}$
   b) $\overrightarrow{TR} = \frac{3}{2}\overrightarrow{TS}$

12. c) parallelogram

1.9 Exercises, page 56

1. a) $\overrightarrow{(1,1)}$
   b) $\overrightarrow{(-5,1)}$
   c) $\overrightarrow{(0,-2)}$
2. a) $\vec{u} = 2\vec{i} - 7\vec{j}$
   b) $\vec{v} = 6\vec{i} + \vec{j}$
   c) $\vec{w} = -3\vec{i}$
3. a) $\overrightarrow{(1,2,3)}$
   b) $\overrightarrow{(4,0,-1)}$
   c) $\overrightarrow{(0,-1,-1)}$
4. a) $\vec{u} = 2\vec{i} - 4\vec{j} + 6\vec{k}$
   b) $\vec{v} = -\vec{j} - \vec{k}$
   c) $\vec{w} = 10\vec{j}$
5. a) $18\vec{i} - 6\vec{j}$
   b) $2\vec{i} - 2\vec{j} + 2\vec{k}$
   c) $15\vec{i} - 5\vec{k}$
   d) $2\vec{i} + 4\vec{j}$
6. a) $-\vec{i} + 8\vec{j}$
   b) $-2\vec{i} - 11\vec{k}$
   c) $-8\vec{k}$
7. $\vec{b}, \vec{c}, \vec{d}, \vec{f}$
8. $\vec{e}_u = \frac{1}{5}\overrightarrow{(3,-4)}$
   $\vec{e}_v = \overrightarrow{(1,0,0)}$
   $\vec{e}_w = \frac{1}{\sqrt{41}}\overrightarrow{(2,6,-1)}$
   $\vec{e}_z = \frac{1}{\sqrt{3}}\overrightarrow{(1,1,1)}$
   $\vec{e}_p = \frac{1}{4}\overrightarrow{(\sqrt{7},3)}$
   $\vec{e}_q = \frac{1}{\sqrt{41}}\overrightarrow{(5,-4)}$
10. a) $\overrightarrow{PQ} = \overrightarrow{(7,-5)}$
    $\vec{e}_{PQ} = \frac{1}{\sqrt{74}}\overrightarrow{(7,-5)}$
    b) $\overrightarrow{PQ} = \overrightarrow{(-1,2,-5)}$
    $\vec{e}_{PQ} = \frac{1}{\sqrt{30}}\overrightarrow{(-1,2,-5)}$
11. $\vec{e}_u = \frac{1}{13}\overrightarrow{(-5,12)}$
    $\vec{e}_v = \frac{2}{13}\overrightarrow{\left(\frac{3}{2},2,-6\right)}$
    $\vec{e}_w = \frac{1}{\sqrt{53}}\overrightarrow{(2,0,7)}$
12. a) $\sqrt{2}$ b) $\sqrt{5}$
13. a) $\frac{1}{\sqrt{33}}\overrightarrow{(4,4,1)}$
    b) $\frac{1}{15}\overrightarrow{(12,0,-9)}$
    c) $\frac{1}{2}\overrightarrow{(0,0,-2)}$
15. $\overrightarrow{AB} = -\vec{i} + 4\vec{j}$

In Search of, page 59

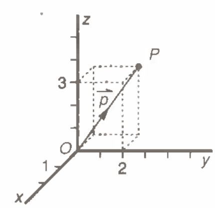
2. a) triangular shapes
   b) Canadian families
   c) the school's classes
   d) directions (pointing either way)
   e) lengths
   f) integers, e.g. $(4,1) \in 3$ but $(1,4) \notin -3$
   g) rational numbers, e.g. $(4,6) \in \frac{2}{3}$

Inventory, page 62

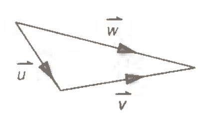
1. direction
2. vector
3. scalar
4. directed; ordered pair; ordered triple
5. length (or magnitude); direction
6. −3; 2
7. points; non-skew
8. verticals; parallels
9. intersecting; parallel
10. $\overrightarrow{(2,-3,4)}$
11. $|\vec{v}|$; 3
12. $\overrightarrow{(0,2)}$
13. $\overrightarrow{FH}$
14. $\overrightarrow{LK}$
15. $\overrightarrow{(10,5,20)}$
16. parallel; $|k|\vec{u}|$
17. $\overrightarrow{PQ} = \overrightarrow{(-2,-2)}$
18. unit
19. $4\vec{i} - 3\vec{j}$
20. unit; $\frac{1}{5}\overrightarrow{(4,-3)}$

Review Exercises, page 63–65

1. a) equal lengths $AP = PM$; same direction
   b) $\overrightarrow{AP} = \overrightarrow{PM} = \overrightarrow{MQ} = \overrightarrow{QB}$
   c) $\overrightarrow{BM} = \overrightarrow{MA} = \overrightarrow{QP}$
2. a) $\overrightarrow{PB} = \vec{u}$
   b) $\overrightarrow{RC} = \vec{u}$
   c) $\overrightarrow{DR} = \vec{u}$
   d) $\overrightarrow{QC} = \vec{v}$
   e) $\overrightarrow{AS} = \vec{v}$
3. c) ∢*DON*, ∢*DOM*, ∢*DOP*
4. a) 60°
   b) 55°
5. a) *DB* and *AC* (There are others.)
   b) *DAB*, *DBC*, *DCA*
6.



7. $P'(6,0,-1)$
8. $\pm\frac{\sqrt{11}}{6}$
9. $\pm\sqrt{122}$
10. a) $\vec{w} = \overrightarrow{(8,-2)}$
    b)

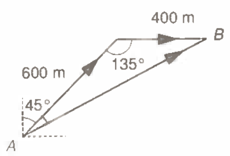


    c) $|\vec{w}| = 2\sqrt{17}$

11. a) $\vec{u}$
b) $\vec{v}$
c) $\vec{v} + \vec{w} + \vec{z}$
d) $\vec{v} + \vec{u}$
e) $\vec{u} + \vec{v} + \vec{w} + \vec{z}$

12. $k = -4, n = 0, m = 6$

13.



$|\overrightarrow{AB}| = 927$ m, bearing 063°

14. a) $\overrightarrow{RP}$
b) $\vec{0}$

15. $\overrightarrow{LM} = \overrightarrow{(-13,5,7)}$

16. a) $\overrightarrow{AC} - \overrightarrow{AB}$
b) $\overrightarrow{AD} - \overrightarrow{AC}$
c) $-\overrightarrow{AB}$

17. $\vec{s} - \vec{p} - \vec{r}$

18. a) $\overrightarrow{(3,-6,-7)}$
b) $\overrightarrow{(3,-2,9)}$
c) $\overrightarrow{(-6,8,-2)}$
d) $\overrightarrow{(4,-12,-18)}$
e) $\overrightarrow{(6,0,-10)}$
f) $\overrightarrow{\left(\frac{5}{2},-7,\frac{9}{2}\right)}$

19. a) $\overrightarrow{QR} = 2\overrightarrow{PQ}$
b) collinear

20. $P(3,0,-1)$
$Q(7,-1,1)$
$R(15,-3,5)$

21. a) 12.5
b) 10
c) $\sqrt{29}$

22. a) $\overrightarrow{DC}$
b) $\overrightarrow{DB}$
c) $\vec{0}$
d) $\overrightarrow{DC}$
e) $\overrightarrow{AD}$

25. a) $-15\vec{i} + 24\vec{j}$
b) $32\vec{i} + 16\vec{j} - 47\vec{k}$

26. a) $\frac{1}{\sqrt{29}}\overrightarrow{(2,5)}$
b) $\frac{1}{3\sqrt{2}}\overrightarrow{(1,-4,1)}$
c) $\frac{1}{K}\overrightarrow{(d-a, e-b, f-c)}$, where
$K = \sqrt{(d-a)^2 + (e-b)^2 + (f-c)^2}$

27. a) $\sqrt{2}$
b) $\sqrt{3}$
c) $\sqrt{10}$
d) $\sqrt{14}$

29. E

30. a) $\overrightarrow{OP} = \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix}$ $\overrightarrow{OQ} = \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix}$
b) $OP^2 = 45, OQ^2 = 81$
$OQ^2 \neq OP^2 + PQ^2$
c) $RS = \frac{3\sqrt{14}}{2}$
d) $6\sqrt{2}$

Chapter Two
Linear Dependence

2.1 Exercises, pages 71-72

1. a) $\vec{a} \| \vec{b}$
   b) $\vec{a} = s\vec{b}$, $m\vec{a} + k\vec{b} = \vec{0}$
   c) $s \in \mathbb{R}$, m, k not both zero
2. $\vec{x} = s\vec{y}$ or $m\vec{x} + k\vec{y} = \vec{0}$
3. The line segments can be parallel and distinct.
4. $\vec{z}$ and $\vec{d}$ are linearly dependent.
5. $\vec{m} \| \vec{k}$
6. a) $\vec{a} \| \overrightarrow{PR}$
   b) $\vec{a} \nparallel \overrightarrow{PT}$
7. $\vec{u}, \vec{w}, \vec{r}\ \vec{t}$
8. $\vec{a} \| \vec{c} \| \vec{u} \| \vec{w} \| \vec{r} \| \vec{t}$
   $\vec{b} \| \vec{v}$
9. a) $\overrightarrow{(4,6)}$, $\overrightarrow{(6,9)}$, $\overrightarrow{(1,1.5)}$
   b) same as a)
   c) $\overrightarrow{(1,3)}$
10. a) $\overrightarrow{(8,2,6)}$, $\overrightarrow{(12,3,9)}$, $\overrightarrow{(2,0.5,1.5)}$
    b) same as a)
    c) $\overrightarrow{(0,1,3)}$
11. $\vec{c}, \vec{e}$
12. a) $\vec{a} \nparallel \vec{b}$
    b) yes
13. a) $r = v = 0$
    b) no
14. a) c) d) e)
15. a) $\overrightarrow{DC}$
    b) $\overrightarrow{AE}$, $\overrightarrow{EC}$
    c) $\overrightarrow{EC}$, $\overrightarrow{AC}$
    d) $\overrightarrow{BE}$, $\overrightarrow{ED}$
16. None are parallel.
17. a) $\overrightarrow{PQ} = \overrightarrow{(-1,3)}$
    $\overrightarrow{PR} = \overrightarrow{(-2,6)}$
    b) $2\overrightarrow{PQ} = \overrightarrow{PR}$
    c) collinear
18. a) $\overrightarrow{AB} = \overrightarrow{(4,2,6)}$
    $\overrightarrow{AC} = \overrightarrow{(-2,-1,-3)}$
    b) $\overrightarrow{AB} = -2\overrightarrow{AC}$
    c) collinear
19. a) $m = k = 0$; no
    b) if $\vec{a} = \vec{b}$, then $m = -k$
    if $\vec{a} = -\vec{b}$, then $m = k$;
    yes
20. a) $s = t = 0$
    b) $r = 0, m = 3$
    c) $x = 1, y = -2$
    d) $z = 3, k = -\frac{7}{3}$
21. $m = 3, k = -5$
22. $d = -2, c = -\frac{4}{5}$
23. $\vec{f} = 2\vec{a}$
26. Use $\overrightarrow{AC} = 3\overrightarrow{AB}$

2.2 Exercises, page 80

1. a) coplanar
   b) $\vec{a} = s\vec{b} + r\vec{c}$;
   $m\vec{a} + k\vec{b} + t\vec{c} = \vec{0}$
   c) m, k, t not all zero
2. $k\vec{x} + m\vec{y} + t\vec{z} = \vec{0}$
   $\vec{x} = a\vec{y} + b\vec{z}$
3. A vector can be represented by many parallel line segments.
4. a) coplanar
   b) not all zero
8. a) $\vec{c} = 3\vec{a} - 2\vec{b}$
   b) $\vec{f} = -\vec{d} - 3\vec{e}$
   c) $\vec{h} = -2\vec{g} + 5\vec{n}$
   d) $\vec{p} = 0\vec{q} - 2\vec{r}$
9. 17
10. 0
11. a) $\vec{a} = 3\vec{u}$
    b) impossible
    c) $\vec{a} = 3\vec{u} + 0\vec{v}$
    d) $\vec{a} = 3\vec{u} + 0\vec{v} + 0\vec{w}$
12. a) impossible
    b) impossible
    c) $\vec{a} = 2\vec{u} - \vec{v}$
    d) $\vec{a} = 2\vec{u} - \vec{v} + 0\vec{w}$
13. a) $\vec{a} = 0\vec{u} + 3\vec{v}$
    b) $\vec{a} = 0\vec{u} + 3\vec{v} + 0\vec{w}$
    c) $\vec{a} = 0\vec{u} + 3\vec{v} + 0\vec{w} + 0\vec{t}$
14. a) impossible
    b) $\vec{a} = \frac{7}{3}\vec{u} + \vec{v} + \frac{1}{3}\vec{w}$
    c) $\vec{a} = \frac{7}{3}\vec{u} + \vec{v} + \frac{1}{3}\vec{w} + 0\vec{t}$
16. $m = 2, k = 0, p = 1$

2.3 Exercises, page 87

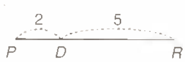
1. $w = x = z = 0$
2. $\vec{a}, \vec{b}, \vec{d}$
4. $\vec{u} = \frac{1}{2}\vec{v} + \frac{1}{2}\vec{w}$
6. a) dependent, coplanar
   b) independent, not coplanar
   c) independent, not coplanar
   d) dependent, coplanar
   e) dependent, coplanar
7. b) $\vec{c} = 2\vec{a} + \vec{b}$,
   $\vec{d} = 5\vec{a} - 4\vec{b}$,
   $\vec{e} = 3\vec{a} + 0\vec{b}$
8. a) b)
9. b) $\vec{d} = 2\vec{a} + \vec{b} - 3\vec{c}$
   $\vec{e} = 0\vec{a} - 3\vec{b} + 4\vec{c}$
   $\vec{f} = \vec{a} - \vec{b} + 2\vec{c}$
12. a) $w = 3$
    b) $w \neq 3$
13. a) $m = 1$ or $m = -1$
    b) $m \neq 1$ and $m \neq -1$
14. a) At least one is a linear combination of the other three.

Making Connections, page 89

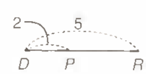
3. 5
4. 366 (377 in leap year)
5. 1096 (1099 in leap year)

2.4 Exercises, page 94

1. a)



   b)



2. a) b) $\overrightarrow{OD} = \frac{3}{7}\overrightarrow{OP} + \frac{4}{7}\overrightarrow{OR}$

3. a) $\frac{5}{8}\overrightarrow{OP} + \frac{3}{8}\overrightarrow{OR}$
 b) $\frac{1}{5}\overrightarrow{OP} + \frac{4}{5}\overrightarrow{OR}$
 c) $\frac{2}{9}\overrightarrow{OP} + \frac{7}{9}\overrightarrow{OR}$
 d) $\frac{11}{17}\overrightarrow{OP} + \frac{6}{17}\overrightarrow{OR}$
 e) $\frac{1}{2}\overrightarrow{OP} + \frac{1}{2}\overrightarrow{OR}$

4. a) b) $\overrightarrow{OD} = -3\overrightarrow{OP} + 4\overrightarrow{OR}$

5. a) $\frac{5}{2}\overrightarrow{OP} - \frac{3}{2}\overrightarrow{OR}$
 b) $-\frac{1}{3}\overrightarrow{OP} + \frac{4}{3}\overrightarrow{OR}$
 c) $-\frac{2}{5}\overrightarrow{OP} + \frac{7}{5}\overrightarrow{OR}$
 d) $\frac{11}{5}\overrightarrow{OP} - \frac{6}{5}\overrightarrow{OR}$
 e) $2\overrightarrow{OP} - \overrightarrow{OR}$

6. $\frac{12}{5}\overrightarrow{OP} - \frac{7}{5}\overrightarrow{OQ}$

7. a) between PR, closer to R
 b) outside PR, closer to R
 c) $D = R$

8. $\left(\frac{37}{10}, \frac{71}{10}\right)$

9. a) $\left(\frac{27}{7}, \frac{50}{7}\right)$
 b) $\left(\frac{14}{13}, \frac{83}{13}\right)$
 c) $\left(\frac{3}{2}, \frac{13}{2}\right)$
 d) $\left(-\frac{65}{8}, \frac{31}{8}\right)$
 e) $\left(\frac{89}{8}, \frac{73}{8}\right)$
 f) (18,11)

10. $\left(\frac{23}{6}, \frac{11}{3}, \frac{11}{6}\right)$

11. a) $\left(\frac{31}{7}, \frac{34}{7}, \frac{17}{7}\right)$
 b) $\left(\frac{33}{18}, \frac{17}{4}, \frac{17}{8}\right)$
 c) (4,4,2)
 d) $\left(\frac{12}{5}, \frac{4}{5}, \frac{2}{5}\right)$
 e) (9,14,7)
 f) (7,10,5)

12. a) $\frac{6}{11}$
 b) 5 : 6

13. $\overrightarrow{OT} = \frac{1}{5}\overrightarrow{OA} + \frac{4}{5}\overrightarrow{OR}$

14. a) $\overrightarrow{(3,-1)}, \overrightarrow{(5,6)}, \overrightarrow{(1,1)}$
 b) (3,2)
 c) (3,2)
 d) (3,2)
 e) concurrent; divide each other in ratio 2 : 1

In Search of, page 96

a) (1,2,3)
b) (3,2,−2)
c) $\left(\frac{27-7t}{13}, \frac{34+9t}{13}, t\right), t \in \mathbb{R}$
d) no solution

2.5 Exercises, page 101

1. a) Any two of $\overrightarrow{PQ}, \overrightarrow{PR}, \overrightarrow{QR}$ are linearly dependent.
 b) Any two of $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{AC}$ are linearly dependent.

2. $\overrightarrow{AB} = -2\overrightarrow{AC}$

4. $\overrightarrow{PR} = -2\overrightarrow{PQ}$

6. a) collinear
 b) not collinear
 c) not collinear
 d) collinear

7. Scalars sum to 1.
 a) 2 : 1
 b) 3 : 4
 c) 4 : −1
 d) −7 : 8

8. $\frac{1}{3}$

9. a) $m = 5$
 b) $n = \frac{2}{9}$
 c) $s = -\frac{2}{3}$

10. a) $\overrightarrow{PD}, \overrightarrow{PR}, \overrightarrow{PS}$ are linearly dependent.
 b) $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are linearly dependent.

11. $\overrightarrow{PS} = 2\overrightarrow{PQ} + 3\overrightarrow{PR}$

13. a) coplanar; $\overrightarrow{PS} = 3\overrightarrow{PQ} - 2\overrightarrow{PR}$
 b) coplanar; $\overrightarrow{WZ} = \overrightarrow{WX} + 2\overrightarrow{WY}$
 c) not coplanar
 d) coplanar; $\overrightarrow{KN} = -2\overrightarrow{KM} + 0\overrightarrow{KL}$

14. $k\overrightarrow{AB} + p\overrightarrow{AC} = \vec{0}$

15. a) $\overrightarrow{AB} = \vec{b} - \vec{a}, \overrightarrow{AC} = \vec{c} - \vec{a}$, $\overrightarrow{AZ} = -6\vec{a} + 2\vec{b} + 3\vec{c}$
 b) $\overrightarrow{AZ} = 2\overrightarrow{AB} + 3\overrightarrow{AC}$

2.6 Exercises, page 105

1. 2 : 1
2. 7 : 2
3. 2 : 1
4. 1 : 1
5. 1 : 2
6. 1 : k
7. 5 : 2; 6 : 1
8. 5 : 7
9. 2 : 3
10. 1 : k + 1
11. 1 : 2

Inventory, page 108

1. collinear; $k\vec{a}$; $t\vec{b}$
2. coplanar; combination; $s\vec{a} + t\vec{b}$; zero; $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$
3. independent
4. independent
5. a) 0; 0
 b) 0; 0; 0
6. a) $\overrightarrow{(6,10)}$
 b) $\overrightarrow{(3,0)}$
 c) $\overrightarrow{(4,6,2)}$
 d) $\overrightarrow{(2,3,0)}$
 Answers may vary.
7. 0; $2m + 4k = 0$
8. linearly dependent
9. linearly dependent
10. a) $\frac{7}{12}, \frac{5}{12}$
 b) $\frac{7}{2}, -\frac{5}{2}$
11. a) collinear
 b) coplanar
12. linearly independent
13. two; independent
14. three; independent

Review Exercises, pages 109–111

1. a) parallel
   b) $\vec{a} = k\vec{b}$; $m\vec{a} + t\vec{b} = \vec{0}$
   c) $k \in \mathbb{R}$;
   not both m, t can be zero, $m, t \in \mathbb{R}$
2. a) coplanar
   b) $\vec{a} = k\vec{b} + m\vec{c}$;
   $s\vec{a} + t\vec{b} + p\vec{c} = \vec{0}$
   c) $k, m, s, t, p \in \mathbb{R}$;
   not all of s, t, p can be zero.
3. $a = b = 0$
4. $s = t = r = 0$
5. $2\vec{a}, -8\vec{b}, -7\vec{d}$
6. a) $\vec{x} \nparallel \vec{y}, \vec{x} \nparallel \vec{z}, \vec{x} \nparallel \vec{w}$,
   b) $\vec{q}, \vec{y}, \vec{z}$
7. $\vec{a}, \vec{b}, \vec{d}$
8. $\vec{z}, \vec{h}, \vec{w}$
9. a) $\overrightarrow{(-3,-5)}, \overrightarrow{(6,10)}\ \overrightarrow{(1.5,2.5)}$
   b) $\overrightarrow{(3,0)}$
   Answers may vary.
10. a) $\overrightarrow{(-7,-1,2)}, \overrightarrow{(14,2,-4)}$
   $\overrightarrow{(21,3,-6)}$
   b) $\overrightarrow{(7,1,0)}, \overrightarrow{(7,0,-2)}$
   Answers may vary.
12. b) $\overrightarrow{(-3,2)} = -2\overrightarrow{(1.5,-1)}$
   c) $\overrightarrow{(9,3,6)} = 3\overrightarrow{(3,1,2)}$
13. $3\vec{a} + 4\vec{b}$
14. $2\vec{a} - \vec{b} + 3\vec{c}$
15. not coplanar
16. yes
18. a) $\vec{a}, \vec{b}$ are linearly independent
   b) $\vec{a}, \vec{b}, \vec{c}$ are linearly independent
19. a) linearly independent, not coplanar
   b) linearly dependent, coplanar
   c) linearly independent, not coplanar
20. b) $4\vec{a} + 2\vec{b}$
21. b) $3\vec{a} + 4\vec{b} + \frac{1}{2}\vec{c}$
22. $k = 3$ $(\vec{c} = -2\vec{a} + 3\vec{b})$
23. a) $\frac{5}{16}\overrightarrow{OP} + \frac{11}{16}\overrightarrow{OR}$
   b) $\frac{7}{12}\overrightarrow{OP} + \frac{5}{12}\overrightarrow{OR}$
   c) $\frac{1}{8}\overrightarrow{OP} + \frac{7}{8}\overrightarrow{OR}$
   d) $\frac{11}{17}\overrightarrow{OP} + \frac{6}{17}\overrightarrow{OR}$
24. a) $\frac{5}{4}\overrightarrow{OA} - \frac{1}{4}\overrightarrow{OB}$
   b) $-\frac{3}{4}\overrightarrow{OA} + \frac{7}{4}\overrightarrow{OB}$
   c) $\frac{9}{5}\overrightarrow{OA} - \frac{4}{5}\overrightarrow{OB}$
25. $2\overrightarrow{OP} - \overrightarrow{OQ}$
26. a) $(-2.5, 2, 5.75)$
   b) $\left(10, \frac{31}{3}, 12\right)$
27. $\overrightarrow{OT} = -\frac{13}{11}\overrightarrow{OA} + \frac{24}{11}\overrightarrow{OR}$
28. a) $\overrightarrow{PQ} = k\overrightarrow{PR}$
   b) $\overrightarrow{AB} = m\overrightarrow{AC} + t\overrightarrow{AD}$
29. a) collinear, $\overrightarrow{AC} = 3\overrightarrow{AB}$
   b) not collinear
30. a) coplanar, $\overrightarrow{AB} = 3\overrightarrow{AC} - 2\overrightarrow{AD}$
   b) not coplanar
31. 2 : 5
32. $k = 3, m = -1$
34. a) $\overrightarrow{AB} = \vec{b} - \vec{a}$,
   $\overrightarrow{AC} = \vec{c} - \vec{a}$,
   $\overrightarrow{AZ} = 3\vec{a} + 4\vec{b} - 7\vec{c}$
   b) $\overrightarrow{AZ} = 4\overrightarrow{AB} - 7\overrightarrow{AC}$
35. 11 : 4
36. 5 : 6
37. 3 : 2
38. 3 : 2; 2 : 1
39. 7 : 6
41. a) $\overrightarrow{AB} = \vec{b} - \vec{a}$
   $\overrightarrow{AC} = \vec{c} - \vec{a}$
   $\overrightarrow{AZ} = -5\vec{a} + 2\vec{b} + 3\vec{c}$
   b) $\overrightarrow{AZ} = 2\overrightarrow{AB} + 3\overrightarrow{AC}$
   c) coplanar
42. $\frac{11}{3}\vec{i} + 2\vec{j} - \frac{13}{3}\vec{k}$
43. $\frac{2}{5}$
44. A
45. D

Chapter Three The Multiplication of Vectors

3.1 Exercises, page 120

1. a) 56°
   b) 0°
   c) 145°
2. a) 3.830; 3.213
   b) −2.394; 6.578
   c) 0, 13
3. a) 1.50; $1.50\vec{e}_v$
   b) 1.12; $1.12\vec{e}_u$
4. −0.342
5. a) $2\vec{i}, -3\vec{j}$; 2, −3
   b) $\vec{i}, \vec{0}$; 1, 0
   c) $-15\vec{i}, 3\vec{j}$; −15, 3
6. a) $\vec{i}, -4\vec{j}, \vec{k}$; 1,−4,1
   b) $2\vec{i}, \vec{0}, 3\vec{k}$; 2,0,3
   c) $-2\vec{i}, -2\vec{j}, \vec{0}$; −2,−2,0
7. $\vec{u} = 4\vec{i} + 5\vec{j} + 0\vec{k}$
   $\vec{v} = -2\vec{i} - 3\vec{j} + \vec{k}$
8. a) a b) $b\vec{j}$ c) $-c\vec{k}$
9. a) $\frac{1}{\sqrt{2}}$ b) $\frac{1}{\sqrt{2}}$
11. a) 2
    b) $\frac{2}{\sqrt{29}}$
    c) $4\vec{j}$
    d) $\frac{4}{25}\overrightarrow{(-3,4)}$
12. $\vec{u} \perp \vec{v}$, or $|\vec{u}| = 0$
13. a) $|\vec{u}| = |\vec{v}|$ or $\vec{u} \perp \vec{v}$
    b) $\vec{u} = \vec{v}$ or $\vec{u} \perp \vec{v}$
14. 1 and 1
15. $\sqrt{\frac{41}{5}}$
16. a) 1
    b) $\overrightarrow{\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)}$
17. a) 13
    b) $-12\vec{j} + 2\vec{k}$

3.2 Exercises, page 124

1. a) 12
   b) 18.7
   c) 0
   d) 32
   e) −3.14
2. a) 6
   b) −0.5
   c) 3
3. a) 0
   b) 1
   c) −1
4. a) vector
   b) scalar
   c) vector
   d) vector
   e) scalar
   f) scalar
   g) vector
   h) vector
   i) vector
10. No: $\vec{u}$ or $\vec{v}$ could be $\vec{0}$.
11. a) 50
    b) −50
12. a) 0°
    b) 100°
    c) 90°
13. b) e) f) are meaningful,
    a) c) d) are not.
14. a) 4; 4
    b) no

3.3 Exercises, page 128

1. a) 14
   b) −12
   c) 0
   d) 6
   e) 0
   perpendicular
2. 1
3. a) 0
   b) 1
4. $t = 2$
5. a) 40°
   b) 90°
   c) 105°
   d) 82°
6. a) 3
   b) $-\frac{2}{5}$
7. a) $k = -5$ or $k = 3$
   b) $\theta = 88°$ or $\theta = 85°$
8. a) 13
   b) 26
   c) $x^2 + y^2 + z^2$
9. a) $\vec{p} = \overrightarrow{(5,4)}$
   (or any multiple)
   b) $\vec{e}_p = \frac{1}{\sqrt{41}}\overrightarrow{(5,4)}$
10. a) −10
    b) 9
    c) 12
    d) −2
    e) 38
13. a) 7
    b) $\frac{1}{\sqrt{7}}(\vec{p} - 3\vec{q})$
15. $k = 2.566$ or 0.234

3.4 Exercises, page 131

1. a) $\alpha = 71°, \beta = 61°, \gamma = 144°$
   b) 2, 3, −5
2. a) 81°
   b) 40°
   c) 40°
   d) 135°
3. a) 1
   b) 6
   c) 6
   d) −4
4. a) $\frac{7}{\sqrt{5}}; \frac{7}{\sqrt{34}}$
   b) $-\frac{3}{\sqrt{45}}; -\sqrt{3}$
5. $\overrightarrow{BA} \cdot \overrightarrow{BC} < 0$
6. a) $\angle P = 123°$,
   $\angle Q = 29°$,
   $\angle R = 28°$
   b) $\angle P = 154°$,
   $\angle Q = 16°$,
   $\angle R = 10°$
7. $\vec{u} = 4\vec{i} + 5\vec{j} + 0\vec{k}$
   $\vec{v} = -2\vec{i} - 3\vec{j} + \vec{k}$
8. a) $\overrightarrow{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$ b) $\overrightarrow{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$
9. a) Either the vectors have the same magnitude, or they are perpendicular.
   b) Either the vectors are equal, or they are perpendicular.
10. a) −7.21
    b) $-2\overrightarrow{(-3,-2)} = \overrightarrow{(6,4)}$

3.4 Exercises, page 131, continued

11. $\overrightarrow{(-2,3)} = \frac{1}{2}\overrightarrow{(1,1)} + \frac{5}{2}\overrightarrow{(-1,1)}$

12. a) yes
 b) no
 c) yes

13. a) $\overrightarrow{OP} = -\vec{r}$
 b) $\overrightarrow{RC} = \vec{c} - \vec{r}$; $\overrightarrow{PC} = \vec{c} + \vec{r}$
 c) 0, since $|\vec{r}| = |\vec{c}|$
 d) $\overrightarrow{RC} \perp \overrightarrow{PC} \Rightarrow \measuredangle C$ a right angle

14. $\cos \measuredangle ABD = \frac{1}{\sqrt{10}} = \cos \measuredangle ACD$
 $\measuredangle B = \measuredangle C \Rightarrow A, B, C, D$ concyclic

3.5 Exercises, page 135

1. a) $5\vec{e}$; $\vec{u}, \vec{v}, \vec{e}$ *RH* system, with $\vec{e} \perp \vec{u}$ and $\vec{e} \perp \vec{v}$
 b) $42\vec{e}$; as in part a)
 c) $16.1\vec{e}$; as in part a)
 d) $17.3\vec{e}$; as in part a)
 e) $33\vec{i}$
 f) $-33\vec{i}$

2. a) $\vec{0}$
 b) $\vec{i}$
 c) $-\vec{k}$

3. 26.8

4. 74.8

5. b) c) e) are meaningful, a) d) f) are not.

11. b) no
 c) no

15. a) No, but $\vec{a}, \vec{b}, \vec{c}$ must be coplanar, with $|\vec{b}| \sin\theta = |\vec{c}| \sin\phi$.
 b) Yes, since $\theta = \phi$ and $\vec{a}, \vec{b}, \vec{e}$, in same RH system as $\vec{a}, \vec{c}, \vec{e}_2$.

16. Both sides equal $\vec{i} - \vec{j}$.

3.6 Exercises, page 145

1. a) $\vec{k}$
 b) $3\overrightarrow{(4,-1,-1)}$
 c) $\overrightarrow{(-46,-11,-14)}$
 d) $\vec{0}$

2. a) $\pm\vec{k}$
 b) $\pm\frac{1}{\sqrt{18}}\overrightarrow{(4,-1,-1)}$
 c) $\pm\frac{1}{\sqrt{2433}}\overrightarrow{(46,11,14)}$
 d) any vector $\frac{1}{\sqrt{a^2+b^2+c^2}}\overrightarrow{(a,b,c)}$, such that $a - 3b + 2c = 0$

3. a) $\vec{p} \times \vec{q}$
 b) $\vec{0}$
 c) 0
 d) $\vec{p} \times \vec{r} \cdot \vec{q}$

4. 64.8

5. a) 9.76
 b) 61.0

6. 2

7. $\frac{32}{3}$

8. a) $\alpha = 24°$
 b) $\beta = 17°$
 c) $\alpha = 28°$
 d) $\theta = 81°$

9. No; $\sin\theta = \frac{7}{10}$ gives two angles in [0°, 180°]

10. a) $(\vec{u} \times \vec{v}) \times \vec{w} = \overrightarrow{(-28,-14,-14)}$
 $\vec{u} \times (\vec{v} \times \vec{w}) = \overrightarrow{(-30,-20,-4)}$
 b) not associative

11. $\vec{i} \times (\vec{u} \times \vec{k}) = \overrightarrow{(0,0,-1)} \neq \vec{0}$

13. $\vec{a} \times \vec{c} = -5(\vec{a} \times \vec{b})$, and if normals collinear then $\vec{a}, \vec{b}, \vec{c}$ coplanar

15. $4\overrightarrow{(3,3,1)}$

Inventory, page 147

1. tail
2. $2\vec{i}$
3. -3
4. resolving
5. scalar
6. vector
7. cross
8. $\vec{v}$ on $\vec{u}$
9. parallelogram; $\vec{u}$; $\vec{v}$
10. cross
11. 1
12. $\vec{0}$
13. 0
14. negative
15. triple scalar product
16. $\vec{a} \cdot \vec{b} \times \vec{c}$
17. perpendicular

Review Exercises, pages 148–151

1. a) $6\vec{i}, -5\vec{j}, -3\vec{k}$; 6, −5, −3
 b) $-\sqrt{2}\vec{i}, \vec{j}, \sqrt{3}\,\vec{k}$; $-\sqrt{2}$, 1, $\sqrt{3}$
 c) $-3\vec{i}, 3\vec{j}, -6\vec{k}$; −3, 3, −6

2. a) 1.147; 1.638
 b) −3,782; 1.302
 c) −5; 0

3. a) $\vec{u}, \vec{v}$ collinear, same direction
 b) $\vec{u}, \vec{v}$ collinear, opposite direction

4. a) 5.73
 b) −5
 c) −92.1

5. a) 5.73
 b) −1
 c) −9.21

6. a) 2
 b) 2
 c) 2
 d) 4
 e) 0
 f) 8

7. a) 12
 b) −39
 c) 0
 d) 0

8. $k = 1$

12. $\vec{e}_v = \frac{1}{\sqrt{5+2\sqrt{2}}}(2\vec{a} - \vec{b})$

13. a) 97°
 b) $-\frac{1}{\sqrt{13}}$
 c) $-\frac{1}{13}\overrightarrow{(-2,-3)}$
 d) $-\frac{1}{\sqrt{5}}$
 e) $-\frac{1}{5}\overrightarrow{(2,-1)}$

14. 2.24 or −3.57

15. c) 3

16. **a)** $\measuredangle P = 4°$
$\measuredangle Q = 29°$
$\measuredangle R = 147°$
b) $\measuredangle P = 90°$,
$\measuredangle Q = 35°$,
$\measuredangle R = 55°$

17. **a)** $\overrightarrow{AN} = \frac{1}{2}\vec{b} - \vec{a}$;
$\overrightarrow{BN} = \frac{1}{2}\vec{a} - \vec{b}$

18. $2|\vec{a}|^2 + 2|\vec{c}|^2$

19. **a)** $2.54\,\vec{e}$; $\vec{u}, \vec{v}, \vec{e}$ *RH* system with $\vec{e}\perp\vec{u}$ and $\vec{e}\perp\vec{v}$
b) $3.76\,\vec{e}$; as in part a)
c) $29.6\,\vec{e}$; as in part a)
d) $\overrightarrow{(4,12,-3)}$

20. $\vec{e}_1 = \frac{1}{\sqrt{2241}}\overrightarrow{(-14,37,-26)}$
$\vec{e}_2 = -\frac{1}{\sqrt{2241}}\overrightarrow{(-14,37,-26)}$

21. 54.2

22. no

23. **a)** yes
b) no
c) no
d) yes
e) yes

25. **a)** $\vec{p} = \overrightarrow{(5,-4,12)}$
(or any multiple)
b) $\vec{e}_p = \frac{1}{\sqrt{185}}\overrightarrow{(5,-4,12)}$

26. **a)** $\theta = 0°$
b) $\theta = 180°$

27. θ not defined

29. **c)** $\vec{v}$ must be perpendicular to $\vec{u}$ and to $\vec{w}$

30. **b)** $\vec{0} + \vec{p}\times\vec{q} + \vec{p}\times\vec{r} = \vec{0}$
c) $\vec{q}\times\vec{p} + \vec{0} + \vec{q}\times\vec{r} = \vec{0}$
$\vec{r}\times\vec{p} + \vec{r}\times\vec{q} + \vec{0} = \vec{0}$

33. **a)** $\gamma = 109.5°$
b) $OA = \sqrt{3}$, $OB = \sqrt{3}$, $AB = \sqrt{8}$
$\gamma = 109.5°$, $\alpha = \beta \doteq 35°$
c) each side $\sqrt{8}$,
each angle 60°
d) $\overrightarrow{OD} = \vec{i}$
$\overrightarrow{OE} = \vec{j}$
$\overrightarrow{OF} = \vec{k}$
Volume $= \frac{1}{6}$ units^3

34. **a)** sketch not provided
b) $\begin{bmatrix}4\\8\end{bmatrix}\cdot\begin{bmatrix}4\\-2\end{bmatrix} = 0$
c) $D(-4,-8)$
d) $\alpha = -\frac{1}{2}$
e) $E(0,10)$; $F(-4,2)$
f) Area $OAEF = 40$
Area $OBDC = 40$

35. **ii)** $AG : GL = 2 : 1$

36. **i)** $p = 4, q = 3$
$D(-3,-3)$
ii) $E\left(-\frac{17}{8}, \frac{25}{8}\right)$ $F\left(-\frac{21}{4}, \frac{25}{4}\right)$

37. **a)** 8
b) 5

Chapter Four
Applications of Vectors

4.1 Exercises, page 160

1.

	a)	b)	c)	d)	e)
$\lvert\vec{R}\rvert$	13	17	17.7	40.3	424
resultant bearing	067°	107°	316°	358°	179°
equilibrant bearing	247°	287°	136°	178°	359°

2. 73.7 N, 52° between P and R (or 23° between Q and R)
3. 295 N; 24° to horizontal (or 66° to vertical)
4. 90° between 10 and 24
   157° between 24 and 26
   113° between 26 and 10
5. 56° between 15 and 11
   147° between 11 and 23
   157° between 23 and 15
6. 57.2 N, bearing 061°
7. 70.7 N
8. 49.7 N
9. 63.0 N for 40° string
   75.1 N for 50° string
10. 78.7 N for 29° string
    91.2 N for 41° string
11. 30.0 N for 80 cm cord
    35.5 N for 70 cm cord
12. 5 N
13. bearing 125°

4.2 Exercises, pages 168–170

1. a) 10; bearing 037°
   b) 91.2; bearing 351°
3. a) 11.0
   b) $\alpha = 63°, \beta = 35°, \gamma = 69°$
4. a) $17.3\vec{e}_1 + 10\vec{e}_2$
   b) $98.5\vec{e}_1 + 17.4\vec{e}_2$
   c) $17.4\vec{e}_1 + 98.5\vec{e}_2$
   d) $-32.1\vec{e}_1 + 38.3\vec{e}_2$
5. 73.7 N, 52° between P and R (or 23° between Q and R)
6. 57.2 N, bearing 061°
7. 20.3, bearing 021°
8. 52.1 N, bearing 070°
9. $-8.94\vec{i} - 49.1\vec{j}$
10. a) $26.2\vec{i} - 236\vec{j} + 183\vec{k}$
    b) $\alpha = 85°, \beta = 142°, \gamma = 52°$
11. 252 000 N;
    $\alpha = 83°, \beta = 37°, \gamma = 53°$
12. a) $\overrightarrow{(18,0,-6)}$
    b) $\overrightarrow{(-3,-4,2)}$
13. a) $180° - \theta$
    b) $|\vec{r}||\vec{F}| \sin\theta\, \vec{e}$,
    where $\vec{r}, \vec{F}, \vec{e}$ form a RH system, and $|\vec{e}| = 1$
    c) maximum at $\theta = 90°$; minumum at $\theta = 0°$ or 180°
14. $7.07\vec{e}_1 + 7.07\vec{e}_2$
15. $4.30\vec{e}_1 + 6.45\vec{e}_2$;
    $\theta = 34°$
16. 843 N
17. bearing 125°
18. 52 N
19. a) $|\vec{N}| = 176$ N
    $|\vec{F}| = 85.9$ N
    b) 0.488
    c) 77.2 N
20. a) $|\vec{N}| = mg\cos\theta$
    $|\vec{F}| = mg\sin\theta$
    b) $\mu = \tan\theta$
    c) $mg\sin\theta\cos\theta$
    (or $\frac{1}{2}mg\sin 2\theta$)
21. a) 56.6 N (horizontal); 113 N
    b) 117 N (horizontal); 152 N

4.3 Exercises, page 174

1. a) 140
   b) 15
   c) 294
2. a) $743\vec{e}_1 + 669\vec{e}_2$
   b) 22 300 J
3. 1 360 000 J
4. 507 J
5. 0; situation 'impossible', since force acts perpendicularly on intended direction of motion.
6. 421
7. a) 384 J
   b) 697 J
8. a) 2.5 m
   b) 6.26 m/s
9. a) 1.47 J
   b) 5.42 m/s

4.4 Exercises, page181

1. a) 47 km/h, bearing 058°
   b) 106 km/h, bearing 261°
   c) 888 km/h, bearing 093°
   d) 13 km/h, bearing 030°
2. a) 105 km/h, westward
   b) 85 km/h, westward
   c) 101 km/h, bearing 263°
3. 206 km/h, bearing 104°
4. 4 h 10 min
5. a) bearing 239°
   b) 28 min
6. a) at an angle of 70° to AB, upstream
   b) 5 min
7. a) at an angle of θ° to AB, upstream
   b) $\dfrac{w}{v \sin 2\theta}$ min
8. a) 50 km/h, bearing 287°
   b) 50 km/h, bearing 107°
9. 23 km/h

4.5 Exercises, pages 186–187

1. a) $\overrightarrow{(1,6)}$; 6.08
   b) $\overrightarrow{(-4,-5,7)}$; 9.49
   c) $\overrightarrow{(-12,4,4)}$; 13.3
2. a) 340 km/h, bearing 115°
   b) 340 km/h, bearing 295°
3. $\overrightarrow{(43,45,-2)}$; 62.3 km/h
4. a) $\overrightarrow{(-21,47,0)}$; 51.5
   b) $\overrightarrow{(-22,48,-2)}$; 52.8
   c) $\overrightarrow{(-20,46,-1)}$; 50.2
5. 206 km/h, bearing 104°
6. a) bearing 239°
   b) 28 min
7. a) $\vec{0}$
   b) $-20\vec{i}$
   c) $-9.33\vec{j}$
   d) $-20\vec{i} - 9.33\vec{j}$; 22.1 m/s
8. a) 50 km/h, bearing 287°
   b) 50 km/h, bearing 107°
9. 15.017 km/h, bearing 070°

10. a) 547 km/h, bearing 106°
    b) 547 km/h, bearing 106°
    (Wind makes no difference, here)
11. 613 km/h, bearing 072°
12. 53.3 km/h, wind from bearing 214°

In Search of, page 191

2. $\dfrac{v^2 \sin^2 \alpha}{2g}$
3. $\dfrac{2v^2 \cos\alpha \sin\alpha}{g} \left(\text{or } \dfrac{v^2 \sin 2\alpha}{g}\right)$
4. a) describes a circle
   b) $\overrightarrow{v(t)} = -a \sin t\vec{i} + a \cos t\vec{j}$ perpendicular to $\overrightarrow{r(t)}$, with same magnitude a
   c) $-\overrightarrow{r(t)}$ centripetal
5. a) describes an ellipse
   b) $\overrightarrow{v(t)} = -a \sin t\vec{i} + b \cos t\vec{j}$
   c) $-\overrightarrow{r(t)}$ centripetal

Inventory, pages 192–193

1. vector
2. particle
3. newtons
4. 9.8 N
5. added
6. resultant
7. equilibrant
8. equilibrium
9. space (or position)
10. $\vec{v} = \cos\alpha\vec{i} + \sin\alpha\vec{j}$
11. $\vec{v} = \cos\alpha\vec{i} + \cos\beta\vec{j} + \cos\gamma\vec{k}$
12. components
13. parallel
14. direction; $\vec{i}, \vec{j}, \vec{k}$
15. magnitude; displacement
16. scalar
17. joules
18. dot product
19. zero
20. relative
21. $\vec{v}_{BC}$; $\vec{v}_{AB}$; $\vec{v}_{BC}$

Review Exercises, pages 194–195

1. 112 N, at an angle of 23° to $\vec{p}$
2. a) 739 N, at an angle of 51° to horizontal
   b) 51° *below* horizontal; 739 N
3. 158° between 85 and 40
   39° between 40 and 50
   163° between 50 and 85
4. No; no possible triangle
5. a) $|\vec{P}| = 83.6$ N; $|\vec{Q}| = 33.4$ N;
   b) $|\vec{P}| = 24.6$ N; $|\vec{Q}| = 49.2$ N;
6. 161°
7. $|\vec{T}_1| = 47.3$ N
   $|\vec{T}_2| = 65.7$ N
8. a) 81.3 N
   b) $\alpha = 36°, \beta = 116°, \gamma = 112°$
9. $|\vec{J}| \doteq 2253$, bearing 340°
10. $\vec{F} \doteq 22.8\vec{i} - 76\vec{j} + 60.8\vec{k}$
11. $\vec{F} = \dfrac{100}{\sqrt{3}}\vec{e}_1 + \dfrac{100}{\sqrt{3}}\vec{e}_2 + \dfrac{100}{\sqrt{3}}\vec{e}_3$
12. 526 N, up slope
13. $\vec{N} \doteq 416$ N
    $\vec{F} \doteq 260$ N
14. a) 328 N for 45° string
    268 N for 60° string
    b) tension 80.6 N
    thrust 63.2 N
16. a) 1270 J
    b) 495 J
17. a) $\vec{F} \doteq 111\vec{e}_1 + 792\vec{e}_2$
    b) 1110 J
18. 1830 J
19. 98.5 km/h, bearing 294°
20. a) 088°
    b) 1 hr 56 min
21. a) 55.7 km/h, bearing 351°
    b) 55.7 km/h, bearing 171°
22. $\vec{v}_{FS} = 24\vec{i} - \dfrac{95}{2}\vec{j} - \vec{k}$
    $|\vec{v}_{FS}| = 53.2$ km/h

Chapter Five Equations of Lines in 2- and 3-Space

5.1 Exercises, page 202

1. a) $(2,-1)$; $\overrightarrow{(4,2)}$
   b) $(8,-3)$; $\overrightarrow{(5,-4)}$
   c) $(3,-1,4)$; $\overrightarrow{(5,-2,1)}$
   d) $(-4,7,5)$; $\overrightarrow{(1,0,-8)}$
   (Answers will vary.)
2. a) $\vec{r} = \overrightarrow{(3,7)} + k\overrightarrow{(1,5)}$
   b) $\vec{r} = \overrightarrow{(-2,0)} + k\overrightarrow{(-9,-2)}$
   c) $\vec{r} = \overrightarrow{(6,9)} + k\overrightarrow{(-2,4)}$
   d) $\vec{r} = \overrightarrow{(3,2,7)} + k\overrightarrow{(1,5,3)}$
   e) $\vec{r} = \overrightarrow{(0,-2,0)} + k\overrightarrow{(-9,-2,5)}$
   f) $\vec{r} = \overrightarrow{(2,4,-3)} + k\overrightarrow{(0,0,6)}$
3. a) $\vec{r} = \overrightarrow{(4,-6)} + k\overrightarrow{(-2,13)}$
   b) $\vec{r} = \overrightarrow{(4,-6,2)} + k\overrightarrow{(-5,8,5)}$
4. a) $\vec{r} = \overrightarrow{(-5,2)} + k\overrightarrow{(2,3)}$
   b) $\vec{r} = \overrightarrow{(3,-5,2)} + k\overrightarrow{(2,3,6)}$
5. a) $\vec{r} = \overrightarrow{(3,-1)} + k\overrightarrow{(-3,2)}$
   b) $\vec{r} = \overrightarrow{(3,-1,-5)} + k\overrightarrow{(0,-3,2)}$
6. $\vec{r} = \overrightarrow{(3,-1)} + k\overrightarrow{(2,3)}$
7. $\vec{r} = \overrightarrow{(3,0,2)} + k\overrightarrow{(3,14,1)}$
8. $\vec{r} = \overrightarrow{(3,-1,2)} + k\overrightarrow{(11,19,1)}$
9. a) 12
   b) $\frac{13}{3}$
10. b) between P_1 and P_2
    c) $k > 1 \Rightarrow$ outside segment P_1P_2, closer to P_2
    $k < 0 \Rightarrow$ outside segment P_1P_2, closer to P_1
11. a) $\vec{r} = \overrightarrow{(3,8)} + k\overrightarrow{(1,0)}$
    b) $\vec{r} = \overrightarrow{(3,8,1)} + k\overrightarrow{(0,1,0)}$
    c) $\vec{r} = \overrightarrow{(3,8,1)} + k\overrightarrow{(0,0,1)}$

5.2 Exercises, pages 206–207

1. a) $(5,2)$ $\overrightarrow{(2,4)}$
   b) $(-3,1)$ $\overrightarrow{(8,-5)}$
   c) $(-2,0)$ $\overrightarrow{(1,3)}$
   d) $(-4,1)$ $\overrightarrow{(6,0)}$
   e) $(5,2,2)$ $\overrightarrow{(2,4,-5)}$
   f) $(-3,1,-2)$ $\overrightarrow{(8,-5,5)}$
   g) $(-2,0,4)$ $\overrightarrow{(1,3,2)}$
   h) $(-4,1,0)$ $\overrightarrow{(6,0,-7)}$
2. a) 2, 4
   b) 8, −5
   c) 1, 3
   d) 6, 0
   e) 2, 4, −5
   f) 8, −5, 5
   g) 1, 3, 2
   h) 6, 0, −7
3. a) $(-6,3)$ $(-2,1)$ $(2,-1)$
   b) $\begin{cases} x = -2 + 4k \\ y = 1 - 2k \end{cases}$ $\begin{cases} x = 2 + 4s \\ y = -1 - 2s \end{cases}$ $\begin{cases} x = -6 + 8t \\ y = 3 - 4t \end{cases}$
4. a) $(-6,3,5)$ $(-2,1,4)$ $(-10,5,6)$
   b) $\begin{cases} x = -6 + 8k \\ y = 3 - 4k \\ z = 5 - 2k \end{cases}$ $\begin{cases} x = -2 + 4s \\ y = 1 - 2s \\ z = 4 - s \end{cases}$ $\begin{cases} x = -10 + 4t \\ y = 5 - 2t \\ z = 6 - t \end{cases}$
5. a) $\vec{r} = \overrightarrow{(-3,4)} + k\overrightarrow{(5,1)}$
   $\begin{cases} x = -3 + 5k \\ y = 4 + k \end{cases}$
   b) $\vec{r} = \overrightarrow{(-3,4)} + t\overrightarrow{(10,-2)}$
   $\begin{cases} x = -3 + 10t \\ y = 4 - 2t \end{cases}$
   c) $\vec{r} = \overrightarrow{(-3,4)} + s\overrightarrow{(6,-2)}$
   $\begin{cases} x = -3 + 6s \\ y = 4 - 2s \end{cases}$
   d) $\vec{r} = \overrightarrow{(7,2)} + a\overrightarrow{(4,-5)}$
   $\begin{cases} x = 7 + 4a \\ y = 2 - 5a \end{cases}$
   e) $\vec{r} = \overrightarrow{(8,-3)} + m\overrightarrow{(1,0)}$
   $\begin{cases} x = 8 + m \\ y = -3 \end{cases}$
   f) $\vec{r} = \overrightarrow{(8,-3)} + k\overrightarrow{(0,1)}$
   $\begin{cases} x = 8 \\ y = -3 + k \end{cases}$
6. a) $\vec{r} = \overrightarrow{(5,-3,4)} + k\overrightarrow{(-6,2,1)}$
   $\begin{cases} x = 5 - 6k \\ y = -3 + 2k \\ z = 4 + k \end{cases}$
   b) $\vec{r} = \overrightarrow{(5,-3,4)} + t\overrightarrow{(2,5,-5)}$
   $\begin{cases} x = 5 + 2t \\ y = -3 + 5t \\ z = 4 - 5t \end{cases}$
   c) $\vec{r} = \overrightarrow{(5,-3,4)} + s\overrightarrow{(6,7,-2)}$
   $\begin{cases} x = 5 + 6s \\ y = -3 + 7s \\ z = 4 - 2s \end{cases}$
   d) $\vec{r} = \overrightarrow{(7,2,-1)} + m\overrightarrow{(4,-5,1)}$
   $\begin{cases} x = 7 + 4m \\ y = 2 - 5m \\ z = -1 + m \end{cases}$
   e) $\vec{r} = \overrightarrow{(8,-3,4)} + k\overrightarrow{(1,0,0)}$
   $\begin{cases} x = 8 + k \\ y = -3 \\ z = 4 \end{cases}$
   f) $\vec{r} = \overrightarrow{(8,-3,4)} + s\overrightarrow{(0,1,0)}$
   $\begin{cases} x = 8 \\ y = -3 + s \\ z = 4 \end{cases}$
   g) $\vec{r} = \overrightarrow{(8,-3,4)} + t\overrightarrow{(0,0,1)}$
   $\begin{cases} x = 8 \\ y = -3 \\ z = 4 + t \end{cases}$
7. yes: *A*, *D*
   no: *B*, *C*, *E*
8. yes: *A*, *C*, *D*
   no: *B*, *E*
9. a) no
   b) yes
   c) no
13. a) $\begin{cases} x = k \\ y = 2.5 - 1.5k \end{cases}$
    b) $\begin{cases} x = 3 + 2t \\ y = -2 - 3t \end{cases}$
14. a) $\begin{cases} x = -3 - 4k \\ y = 2 + 7k \end{cases}$
    b) $k = \frac{x+3}{-4}$
    $k = \frac{y-2}{7}$
    c) $\frac{x+3}{-4} = \frac{y-2}{7}$
15. a) $\begin{cases} x = -3 + 2k \\ y = 2 - 4k \\ z = 1 + 7k \end{cases}$
    b) $k = \frac{x+3}{2}$
    $k = \frac{y-2}{-4}$
    $k = \frac{z-1}{7}$
    c) $\frac{x+3}{2} = \frac{y-2}{-4} = \frac{z-1}{7}$
16. c) $\left(\frac{41}{9}, \frac{30}{9}\right)$

5.3 Exercises, pages 213–214

1. a) (3,2); 5, 6
   b) (−1,4); 2, −7
   c) (2,0); −8, 3
   d) (3,3); 2.5, −4
   e) (2,4,1); 3, 8, 5
   f) (−3,2,1); −3, 3.5, −9
   g) (2,4,7); 3, 5, 0
   h) (1,2,−2); −3, 0, 4
2. a) $\vec{r} = \overrightarrow{(3,2)} + k\overrightarrow{(5,6)}$
   b) $\vec{r} = \overrightarrow{(-1,4)} + k\overrightarrow{(2,-7)}$
   c) $\vec{r} = \overrightarrow{(2,0)} + k\overrightarrow{(-8,3)}$
   d) $\vec{r} = \overrightarrow{(3,3)} + k\overrightarrow{(2.5,-4)}$
   e) $\vec{r} = \overrightarrow{(2,4,1)} + k\overrightarrow{(3,8,5)}$
   f) $\vec{r} = \overrightarrow{(-3,2,1)} + k\overrightarrow{(-3,3.5,-9)}$
   g) $\vec{r} = \overrightarrow{(2,4,7)} + k\overrightarrow{(3,5,0)}$
   h) $\vec{r} = \overrightarrow{(1,2,-2)} + k\overrightarrow{(-3,0,4)}$
3. a) $\frac{x-3}{-2} = \frac{y-4}{5}$
   b) $\frac{x-6}{4} = \frac{y+3}{-1}$
   c) $\frac{x-2}{4} = \frac{y-1}{2} = \frac{z}{3}$
   d) $\frac{x+3}{-1} = \frac{y-4}{5}, z = 2$
   e) none exists
   f) $\frac{x}{-4} = \frac{y+1}{3}$
4. a) $\frac{x-5}{-6} = \frac{y+3}{2} = \frac{z-4}{1}$
   b) $\frac{x-4}{1} = \frac{z}{-3}, y = -1$
   c) $\frac{x-5}{2} = \frac{y+3}{5} = \frac{z-4}{-5}$
   d) $\frac{x-7}{4} = \frac{y-2}{-5} = \frac{z+1}{1}$
   e) f) g) none exist
5. $\vec{r} = \overrightarrow{(1,-3,5)} + k\overrightarrow{(4,2,-3)}$
6. 2
7. −0.5
9. 3
10. a) 3, 4
    b) 5, −2
    c) 1, −5
    d) 2, 0
    e) 0, 1
11. a) $-4x + y + 13 = 0$
    b) $6x + 4y + 20 = 0$
    c) $2x - 3y + 29 = 0$
    d) $-2x - 4 = 0$
12. parallel: a) d) e)
    perpendicular: b) c) f)
13. b) $\vec{r} = \overrightarrow{(3,5)} + k\overrightarrow{(5,-2)}$
14. a) $\vec{r} = \overrightarrow{(1,-3)} + t\overrightarrow{(7,4)}$
    b) $\vec{r} = \overrightarrow{(1,-3)} + k\overrightarrow{(4,-7)}$
15. 7
16. −7.5
17. $6x - 5y - 8 = 0$
18. $\frac{x-1}{7} = \frac{y-1}{-4}$
19. b) no

5.4 Exercises, page 217

1. a) −3, −2, 4; 6, 4, −8; 1.5, 1, −2 (Answers may vary.)
   b) 0.5571, 0.3714, −0.7428
   c) 56°, 68°, 138° (or 124°, 112°, 42°)
2. a) −0.1690, 0.5071, 0.8452
      100°, 60°, 32°
   b) 0, −0.5300, 0.8480;
      90°, 122°, 32°
   c) 0.9370, −0.3123, −0.1562;
      20°, 108°, 99°
   d) −0.4472, 0, 0.8944;
      117°, 90°, 27°
3. a) $\frac{5}{\sqrt{30}}, -\frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}$
   b) $\frac{1}{\sqrt{65}}, 0, -\frac{8}{\sqrt{65}}$
   c) $\frac{2}{\sqrt{45}}, \frac{4}{\sqrt{45}}, -\frac{5}{\sqrt{45}}$
   d) $\frac{8}{\sqrt{114}}, -\frac{5}{\sqrt{114}}, \frac{5}{\sqrt{114}}$
   e) $\frac{3}{\sqrt{98}}, \frac{8}{\sqrt{98}}, \frac{5}{\sqrt{98}}$
   f) $\frac{-3}{\sqrt{102.25}}, \frac{3.5}{\sqrt{102.25}}, \frac{-9}{\sqrt{102.25}}$
   g) $-\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}$
4. ± 0.8660
5. 69° (or 111°)
6. $\vec{r} = k\overrightarrow{(\pm 0.6855, 0.7071, 0.1736)}$
7. 60°
8. 63°
9. $-\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}};$
   $\frac{4}{\sqrt{525}}, \frac{22}{\sqrt{525}}, \frac{5}{\sqrt{525}}$
11. $-\frac{12}{\sqrt{229}}, -\frac{6}{\sqrt{229}}, \frac{7}{\sqrt{229}}$
13. $\overrightarrow{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}$

5.5 Exercises, page 221

1. a) Lines do not intersect: no solutions.
   b) Lines are identical: an infinity of solutions.
   c) Lines intersect in a point: one solution.
2. a) intersect at (3,−2)
   b) parallel and distinct
   c) parallel and identical
   d) intersect at (1.2,1.4)
3. a) consistent and independent
   b) inconsistent
   c) consistent and dependent
   d) consistent and independent
4. a) (−1,5)
   b) (4,−2)
   c) parallel and distinct
   d) (3,−1)
5. (−2,−1)
6. (5,15)
7. (−1,2)
8. a) $\vec{r} = \overrightarrow{(1,4)} + k\overrightarrow{(2,2)}$
   b) $\vec{r} = \overrightarrow{(-5,6)} + t\overrightarrow{(7,-5)}$
   c) $\vec{r} = \overrightarrow{(3,-2)} + s\overrightarrow{(5,-7)}$
   d) $\left(-\frac{1}{3}, \frac{8}{3}\right)$
9. a) $\vec{r} = k\overrightarrow{(8,4)}$;
      $\vec{r} = \overrightarrow{(5,0)} + t\overrightarrow{(-2,4)}$
   b) (4,2)
10. a) $\vec{r} = \overrightarrow{(3,10)} + t\overrightarrow{(4,3)}$
    b) $\left(\frac{127}{25}, \frac{289}{25}\right)$
    c) $\frac{13}{5}$
11. $\vec{r} = k\overrightarrow{\left(\frac{7}{8}, \frac{9}{4}\right)}$

5.6 Exercises, pages 227–228

1. a) (−1,5,4)
   b) (5,−2,4)
   c) skew
   d) parallel
   e) (3,−1,−4)
   f) skew

5.6 Exercises, pages 227–228, continued

2. consistent and independent: a) b) e)
   inconsistent: c) d) f)
3. (2,−1,5)
4. (5,2,15)
5. (3,−1,2)
6. $\frac{4}{\sqrt{1466}}$; $\frac{126}{\sqrt{405}}$
7. b) $\frac{39}{\sqrt{390}}$
8. b) $\sqrt{44}$
   c) $P_1(2,3,-2)$; $P_2(4,-3,-4)$
9. a) (4,−1,3)
   b) $\cos^{-1}\frac{25}{\sqrt{1470}} = 49°$
10. The lines intersect at (−2,4,−1).
11. a) L_1 at (6,11,1)
    b) $\frac{68}{\sqrt{326}}$
    c) (2−1,3), (4,1,−3)
12. a) $\vec{r} = k\left(\frac{8}{7}, 1, \frac{-4}{7}\right)$
    b) $\left(\frac{8}{7}, 1, \frac{-4}{7}\right)$
13. $\frac{1}{\sqrt{2}}\, d$
15. a) yes b) no c) no
16. a) $\overrightarrow{(5,-3,-1)}$
    d) For L_1, let $a = 0$ and $b = -\frac{1}{3}$.
    For L_2, let $p = 1$ and $q = -\frac{1}{3}$.
    (Answers may vary.)

5.7 Exercises, page 233

See answers to 2.6 Exercises

Inventory, pages 237–238

1. position vector of any point on the line;
   position vector of a given point on the line;
   a direction vector of the line;
   a parameter
2. (1,4); $\overrightarrow{(3,5)}$; k
3. (5,−1,0); $\overrightarrow{(-2,3,-6)}$; t
4. (−4,2); $\overrightarrow{(3,-5)}$; s; $-\frac{5}{3}$
5. (0,−1,4); (−6,1,0); t
6. (3,−2); $\overrightarrow{(4,-3)}$
7. (−1,0,3); $\overrightarrow{(4,4,1)}$
8. $-\frac{4}{3}$; $\overrightarrow{(4,3)}$
9. 1;
   the angle between the line and the x-axis direction;
   the angle between the line and the y-axis direction;
   the angle between the line and the z-axis direction
10. 1
11. parallel
12. parallel (or) skew
13. an infinite number of;
    an infinity of
14. one; one
15. no; no
16. the shortest distance between two skew lines L_1 and L_2;
    a point on L_1, a point on L_2;
    $\vec{m}_1 \times \vec{m}_2$
17. $\overrightarrow{(2,4,7)}$

Review Exercises, pages 239–243

1. a) $\vec{r} = \overrightarrow{(2,-5)} + k\overrightarrow{(3,2)}$
   $\begin{cases} x = 2 + 3k \\ y = -5 + 2k \end{cases}$
   b) $\vec{r} = \overrightarrow{(-5,-4)} + t\overrightarrow{(6,-2)}$
   $\begin{cases} x = -5 + 6t \\ y = -4 - 2t \end{cases}$
   c) $\vec{r} = \overrightarrow{(-7,0)} + s\overrightarrow{(2,-5)}$
   $\begin{cases} x = -7 + 2s \\ y = -5s \end{cases}$
   d) $\vec{r} = \overrightarrow{(2,1)} + k\overrightarrow{(-3,-2)}$
   $\begin{cases} x = 2 - 3k \\ y = 1 - 2k \end{cases}$
   e) $\vec{r} = \overrightarrow{(-4,1)} + k\overrightarrow{(1,0)}$
   $\begin{cases} x = -4 + k \\ y = 1 \end{cases}$
   f) $\vec{r} = \overrightarrow{(-4,1)} + t\overrightarrow{(0,1)}$
   $\begin{cases} x = -4 \\ y = 1 + t \end{cases}$
2. yes: A, B
   no: C, D
3. a) $\frac{x-1}{-2} = \frac{y-3}{-4}$
   b) $\frac{x-3}{-2} = \frac{y-1}{1}$
   c) $\frac{x+2}{1} = \frac{y+4}{-1}$
4. a) $4x + y + 9 = 0$
   b) $8x - 2y - 32 = 0$
   c) $4y + 20 = 0$
5. a) perpendicular
   b) parallel
   c) perpendicular
6. $\vec{r} = \overrightarrow{(-3,6,1)} + k\vec{c}$, where $\vec{c}$ is any linear combination of $\vec{a}$ and $\vec{b}$
7. b) $\vec{r} = k(\vec{b} + \vec{d})$
   c) $\vec{r} = \vec{b} + t(\vec{d} - \vec{b})$
8. c) $\vec{r} = \overrightarrow{(3,4)} + k\overrightarrow{(7,2)}$
9. a) $\frac{x+4}{-3} = \frac{y-3}{-1}$
   b) $\frac{x+4}{1} = \frac{y-3}{-3}$
10. $-\frac{6}{5}$
11. −2
12. a) $\vec{r} = \overrightarrow{(2,3,0)} + k\overrightarrow{(4,2,-2)}$
    $x = 2 + 4k$
    $y = 3 + 2k$
    $z = -2k$
    $\frac{x-2}{4} = \frac{y-3}{2} = \frac{z}{-2}$
    b) $\vec{r} = \overrightarrow{(-2,3,5)} + t\overrightarrow{(5,-4,1)}$
    $x = -2 + 5t$
    $y = 3 - 4t$
    $z = 5 + t$
    $\frac{x+2}{5} = \frac{y-3}{-4} = \frac{z-5}{1}$
    c) $\vec{r} = \overrightarrow{(1,0,10)} + s\overrightarrow{(1,2,-4)}$
    $x = 1 + s$
    $y = 2s$
    $z = 10 - 4s$
    $\frac{x-1}{1} = \frac{y}{2} = \frac{z-10}{-4}$
    d) $\vec{r} = \overrightarrow{(5,6,10)} + m\overrightarrow{(-2,1,3)}$
    $x = 5 - 2m$
    $y = 6 + m$
    $z = 10 + 3m$
    $\frac{x-5}{-2} = \frac{y-6}{1} = \frac{z-10}{3}$
    e) $\vec{r} = \overrightarrow{(-4,3,6)} + k\overrightarrow{(1,0,0)}$
    $x = -4 + k$
    $y = 3$
    $z = 6$
    none exists
    f) $\vec{r} = \overrightarrow{(-4,3,6)} + t\overrightarrow{(0,0,1)}$
    $x = -4$
    $y = 3$
    $z = 6 + t$
    none exists

13. a) yes
b) no
c) yes

14. a) $\vec{r} = \overrightarrow{(3,-2,1)} + k\overrightarrow{(1,2,6)}$

15. $\vec{r} = \overrightarrow{(-1,-2,4)} + s\overrightarrow{(-3,1,4)}$

16. $\vec{r} = \overrightarrow{(4,1,5)} + k\overrightarrow{(-25,14,2)}$

17. $\vec{r} = \overrightarrow{(3,-1,3)} + k\overrightarrow{(4,5,-1)}$

18. $\frac{3}{17}$

19. −2

20. 3

21. a), d)

22. a) $\vec{r} = \overrightarrow{(1,-2,3)} + k\overrightarrow{(1,3,2)}$
b) $\vec{r} = \overrightarrow{(1,-2,3)} + k\overrightarrow{(1,3,-2)}$

24. $\frac{k - 2m}{\sqrt{14k^2 + 2km + 5m^2}}$
$\frac{2k}{\sqrt{14k^2 + 2km + 5m^2}}$
$\frac{3k + m}{\sqrt{14k^2 + 2km + 5m^2}}$
$k, m \in \mathbb{R}$ with k and m not both zero

25. a) $\frac{2}{\sqrt{29}}, -\frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}$
b) $-\frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}$
c) $\frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

26. 57°

27. $\vec{r} = k\overrightarrow{(0.9063,0.0839,0.4135)}$

28. a) intersect at (2,1)
b) parallel and distinct
c) parallel and identical

29. a) consistent and independent
b) inconsistent
c) consistent and dependent

30. a) intersect at (6,−1)
b) intersect at (17,5)
c) parallel and distinct

31. a) (−2,14)
b) 27°

32. $\vec{r} = \vec{d} + k(\vec{d} - \vec{b})$

33. a) $S(0,4,0)$, $T(0,0,3)$, $U(2,0,3)$
b) $A(0,0,1.5)$, $B(2,4,1.5)$

34. a) intersect at (2,9,1)
b) skew
c) the same line
d) skew

35. a) consistent and independent
b) inconsistent
c) consistent and dependent
d) inconsistent

36. (4,6,2)

37. (3,−4,3)

38. b) $\frac{35}{\sqrt{83}}$

39. a) (1,7,−5)
b) 38°

40. intersect at (−7,−2,0); direction vectors are linearly dependent

41. intersect at (8,3,5); direction vectors are linearly independent

42. a) 7
b) $\vec{r} = \overrightarrow{(1,1,-3)} + k\overrightarrow{(2,3,6)}$

43. a) $x = \frac{y-2}{\frac{1}{3}} = \frac{z+2}{\frac{1}{2}}$
b) $x = \frac{y}{2} = \frac{-z+5}{4}$

44. 11 : 4

45. 5 : 6

46. 3 : 2

47. 3 : 2; 2 : 1

48. 7 : 6

49. $Q_1(3,5,7)$
$Q_2(-1,-3,-1)$

50. a) $P_1\ (3,0,2)$
$P_2\ (1,-3,-4)$

51. (i) (1,1,−1);
$2x - 2y + z = 1$
(ii) $\frac{1}{3}$

52. $\vec{r} = \overrightarrow{(-1,2,1)}$

53. E

54. a) $2y - x = 18$
b) $3y - 4x = -3$
c) $H(12,15)$
e) The three altitudes are concurrent at H.
f) $S(1,2)$

Chapter Six Equations of Planes

6.1 Exercises, pages 249–250

1. plane: a) b) d)
line: c)

2. **a)** $(3,4,5)$, $\overrightarrow{(0,-4,6)}\overrightarrow{(2,1,5)}$
b) $(0,1,9)$, $\overrightarrow{(2,3,-4)}\overrightarrow{(-1,-2,9)}$
c) $(4,-2,0)$, $\overrightarrow{(3,4,-1)}\overrightarrow{(2,0,5)}$

3. **a)** $\vec{r} = \overrightarrow{(3,2,7)} + k\overrightarrow{(4,6,2)} + s\overrightarrow{(0,-2,5)}$
$\begin{cases} x = 3 + 4k \\ y = 2 + 6k - 2s \\ z = 7 + 2k + 5s \end{cases}$
b) $\vec{r} = \overrightarrow{(2,-3,1)} + k\overrightarrow{(5,-3,-1)} + s\overrightarrow{(2,3,1)}$
$\begin{cases} x = 2 + 5k + 2s \\ y = -3 - 3k + 3s \\ z = 1 - k + s \end{cases}$
c) $\vec{r} = \overrightarrow{(3,2,-7)} + k\overrightarrow{(1,5,-8)} + s\overrightarrow{(5,-3,-1)}$
$\begin{cases} x = 3 + k + 5s \\ y = 2 + 5k - 3s \\ z = -7 - 8k - s \end{cases}$

4. **a)** $(5,-9,17)$

5. $\vec{r} = \overrightarrow{(4,0,5)} + k\overrightarrow{(-1,2,4)} + s\overrightarrow{(3,2,-6)}$

6. $\vec{r} = \overrightarrow{(3,2,1)} + k\overrightarrow{(2,-3,0)} + s\overrightarrow{(0,5,-2)}$

7. $\vec{r} = \overrightarrow{(5,6,7)} + t\overrightarrow{(1,0,5)} + k\overrightarrow{(2,0,-1)}$

8. $\vec{r} = \overrightarrow{(2,-3,1)} + k\overrightarrow{(3,4,-8)} + s\overrightarrow{(5,-2,4)}$

10. **a)** $\vec{r} = \overrightarrow{(2,5,1)} + k\overrightarrow{(3,-6,2)} + s\overrightarrow{(4,-2,1)}$
b) $\vec{r} = \overrightarrow{(1,2,-3)} + k\overrightarrow{(3,-6,2)} + s\overrightarrow{(4,-2,1)}$

11. $\vec{r} = \overrightarrow{(3,0,0)} + k\overrightarrow{(3,2,0)} + s\overrightarrow{(3,0,-7)}$

12. $\vec{r} = \overrightarrow{(2,3,1)} + k\overrightarrow{(3,-7,1)} + s\overrightarrow{(0,0,1)}$

13. **a)** $\overrightarrow{CD} \parallel \vec{a}$
b) $\vec{r} = \overrightarrow{(3,5,8)} + k\overrightarrow{(-2,2,5)} + s\vec{a}$, where $\vec{a} \nparallel \overrightarrow{(-2,2,5)}$

14. $\vec{r} = \overrightarrow{(2,3,1)} + k\overrightarrow{(6,3,0)} + s\overrightarrow{(0,4,6)}$

15. $\vec{r} = \overrightarrow{(1,3,2)} + k\overrightarrow{(2,1,-2)} + s\overrightarrow{(2,-5,10)}$

16. none

17. All sets are coplanar.

6.2 Exercises, pages 256–257

1. **a)** $\overrightarrow{(3,-5,4)}$
b) line in plane
c) $\overrightarrow{(1,-5,4)}$
d) $\overrightarrow{(1,-2,-3)}$
e) $\overrightarrow{(2,0,-6)}$
f) $\overrightarrow{(0,2,5)}$
g) $\overrightarrow{(-6,10,-8)}$
h) $\overrightarrow{(2,-4,-6)}$

2. parallel and distinct: a) g)
parallel and identical: d) h)

4. $3x + 6y + z - 21 = 0$

5. $4x - 2y + 7z = -46$

6. **a)** $A + 4B - 6C + D = 0$
b) $3A + 5C + D = 0$
c) $\overrightarrow{(A,B,C)} \cdot \overrightarrow{(3,-4,2)} = 0$
d) $36x + 29y + 4z - 128 = 0$

7. $8x - y + 4z = 0$

8. $x - 56y - 20z + 103 = 0$

9. $2x - y + z = 3$

10. $2y - z = 0$

11. $z = 0$, $x = 0$, $y = 0$

12. $8x - 33y - 7z + 47 = 0$

13. $x + 2z = 4$

14. $2x + 8y + 13z = 5$

15. $\vec{r} = \overrightarrow{(1,0,-1)} + k\overrightarrow{(0,3,1)} + s\overrightarrow{(4,8,-1)}$

16. $25x + 3y + 6z = 15$

17. $7x - 2y - 13z + 49 = 0$

18. $\frac{4}{7}$

20. $-\frac{14}{5}$

21. $x - 4y + 3z = 16$

22. 26°

23. $x - y - 2z - 7 = 0$

24. **a)** 62°
b) 66°

25. $x - z = -3$

26. $14x + 16y + 3z + 47 = 0$

27. $14x - 19y - 4z + 58 = 0$

28. $k = \frac{9}{7}$, $m = -\frac{29}{7}$

29. $19x - 7y - 22z = 26$

30. 6

31. $5x - 2y + 11z = 0$

32. **b)** $6x - 7y + 2z + 11 = 0$

33. $2x - y = 0$

34. $4y + z - 7 = 0$

35. $2x - y + z = 6$

36. **a)** $A = B$
b) $B = 0$
c) $-7A + 18B + 11C + D$
d) $A = B = 0$
e) $4A - B + 3C = 0$
f) $\frac{A}{4} = \frac{B}{-1} = \frac{C}{3}$

37. **a)** $\cos\theta = \dfrac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{(A_1^2 + B_1^2 + C_1^2)(A_2^2 + B_2^2 + C_2^2)}}$

6.3 Exercises, page 262

1. **a)** point $(1,10,4)$
b) see d)
c) point $(2,2,-7.5)$
d) infinity of solutions: line lies in plane
e) point $(3,5,-1)$
f) g) no solution: line parallel to, but not in, plane

3. **b)** $(1,3,5)$
c) $\sqrt{14}$

4. $-2x + 3y - z = 3$

5. $-\frac{1}{6}$

6. $\vec{r} = \overrightarrow{(4,0,0)} + k\overrightarrow{(0,2,0)}$

7. **a)** $x = 2 + 3k$
$y = 3 - k$
$z = 2k$
b) $(5,2,2)$
c) $\sqrt{14}$

8. $(-6,4,1)$

10. $(2,-3,0)$

6.4 Exercises, page 266

1. **a)** intersect
b) parallel, distinct
c) parallel, distinct
d) parallel, identical
e) intersect
f) intersect

2. a) $\begin{cases} x = \dfrac{20-19t}{13} \\ y = \dfrac{14+14t}{13} \\ z = t \end{cases}$

e) $\begin{cases} x = \dfrac{5-t}{5} \\ y = \dfrac{1-t}{3} \\ z = t \end{cases}$

f) $\begin{cases} x = \dfrac{2t-1}{7} \\ y = \dfrac{-50t+25}{14} \\ z = t \end{cases}$

3. a) $\dfrac{3x-20}{-19} = \dfrac{13y-14}{14} = \dfrac{z}{1}$

e) $\dfrac{5x-5}{-1} = \dfrac{3y-1}{-1} = \dfrac{z}{1}$

f) $\dfrac{7x+1}{14} = \dfrac{14y-25}{-350} = \dfrac{z}{7}$

4. $13x + 4y - 10z + 21 = 0$

5. $-6x - 5y - 4z + 13 = 0$

6. a) $\begin{cases} x = 2t+1 \\ y = t-3 \\ z = 5t \end{cases}$

b) $(0,-3.5,-2.5)$, $(7,0,15)$, $(1,-3,0)$

8. $\vec{r} = (-3,0,1) + t\overrightarrow{(2,4,-3)}$

9. a) $k = 4$

b) $k = -2.5$

13. $k = 7$

14. a) $\begin{cases} x = 2t+1 \\ y = -t-3 \\ z = 3t+2 \end{cases}$

c) $k = 3, m = 11$

$23x - 26y - 14z + 17 = 0$

6.5 Exercises, page 274

1. a) ① ③ parallel, and distinct

② not parallel

b) ① ② identical,

③ parallel and distinct

c) identical

d) parallel and distinct

2. a) point $(2,1,-1)$

b) intersect in a line

c) point $(3,-2,-1)$

d) intersect as a triangular prism

e) point $\left(-2,1,\frac{1}{2}\right)$

f) intersect in a line

3. a) consistent, independent

b) consistent, dependent

c) consistent, independent

d) inconsistent

e) consistent, independent

f) consistent, independent

4. $k \neq -\dfrac{69}{19}$

5. 8

6. 3

7. $Ax + By + Cz + D = 0$,

where $B = -2A$, $C = 3A$, $D \neq -5A$

8. a) line

b) triangular prism

c) point

9. a) $m \neq \pm 1$

b) $m = -1$

c) $m = 1$

10. a) linearly independent

b) coplanar

c) coplanar

6.6 Exercises, pages 278–279

1. a) $\dfrac{7}{\sqrt{35}}$

b) $\dfrac{22}{\sqrt{29}}$

c) $\dfrac{5}{\sqrt{10}}$

2. a) $\dfrac{71}{\sqrt{105}}$

b) $\dfrac{10}{\sqrt{200}}$

c) $\dfrac{88}{\sqrt{630}}$

3. a) $\dfrac{\sqrt{195}}{\sqrt{14}}$

b) $\dfrac{\sqrt{650}}{5}$

c) $\dfrac{\sqrt{932}}{\sqrt{38}}$

d) $\dfrac{\sqrt{329}}{\sqrt{13}}$

4. b) $\dfrac{7}{\sqrt{29}}$

5. a) $\dfrac{10}{\sqrt{21}}$

b) $\dfrac{7}{\sqrt{38}}$

c) $\dfrac{9}{\sqrt{44}}$

6. b) $\dfrac{\sqrt{200}}{\sqrt{18}}$

7. a) $\dfrac{\sqrt{131}}{\sqrt{70}}$

b) $\dfrac{\sqrt{4418}}{\sqrt{66}}$

8. -2 or $-\dfrac{2}{3}$

9. 3 or -15

10. $\dfrac{10}{\sqrt{2}}$

11. $\dfrac{3}{\sqrt{5}}$

12. $(3 \pm 4\sqrt{6},0,0)$

13. a) $\dfrac{13}{\sqrt{18}}$

b) $\left(\dfrac{44}{9},\dfrac{41}{18},-\dfrac{103}{18}\right)$

14. a) $\vec{r} = \overrightarrow{(-4,1,0)} + k\overrightarrow{(4,0,5)}$

b) $\sqrt{\dfrac{210}{41}}$

c) $\dfrac{1}{2}\sqrt{210}$

15. a) $5x - 13y - 4z + 33 = 0$

b) $\dfrac{33}{\sqrt{210}}$

c) $\dfrac{11}{2}$

16. 55.69 m

Inventory, page 282

1. the position vector of any point on the plane;
   the position vector of a given point on the plane;
   vectors parallel to the plane;
   parallel to;
   parameters
2. $(1,-2,6)$, $\overrightarrow{(2,5,-7)}$, $\overrightarrow{(3,-4,1)}$
3. $(3,0,-3)$, $\overrightarrow{(2,5,9)}$, $\overrightarrow{(-4,2,2)}$
4. two; line
5. $\overrightarrow{(3,4,5)}$
6. parallel
7. in a point
8. in a line
9. a triangular prism
10. $3A + 2B + 4C = 0$
11. a point
12. the line itself
13. the distance from a point P to a plane Π;
    a point on Π;
    a normal to Π
14. the distance from the point P to a line L;
    a point on L;
    a direction vector of L

Review Exercises, pages 283–287

1. a) $\vec{r} = \overrightarrow{(1,3,6)} + k\overrightarrow{(0,2,2)} + s\overrightarrow{(1,6,5)}$
   $$\begin{cases} x = 1 + s \\ y = 3 + 2k + 6s \\ z = 6 + 2k + 5s \end{cases}$$
   b) $\vec{r} = \overrightarrow{(5,-1,4)} + k\overrightarrow{(2,2,4)} + s\overrightarrow{(4,-1,2)}$
   $$\begin{cases} x = 5 + 2k + 4s \\ y = -1 + 2k - s \\ z = 4 + 4k + 2s \end{cases}$$
   c) $\vec{r} = \overrightarrow{(1,2,3)} + k\overrightarrow{(0,3,3)} + s\overrightarrow{(1,-5,-7)}$
   $$\begin{cases} x = 1 + s \\ y = 2 + 3k - 5s \\ z = 3 + 3k - 7s \end{cases}$$
2. $\vec{r} = \overrightarrow{(0,1,4)} + k\overrightarrow{(-3,4,2)} + s\overrightarrow{(2,-5,1)}$
3. $\vec{r} = \overrightarrow{(-1,0,1)} + k\overrightarrow{(2,0,-4)} + s\overrightarrow{(0,2,-1)}$
4. $\vec{r} = \overrightarrow{(0,6,1)} + k\overrightarrow{(1,2,5)} + s\overrightarrow{(3,2,-1)}$
5. $\vec{r} = \overrightarrow{(0,3,-1)} + k\overrightarrow{(4,8,-3)} + s\overrightarrow{(2,-1,5)}$
6. parallel and distinct: a) f)
   parallel and identical: b) d)
7. $x + 3y + 5z = 1$
8. $\left(\frac{1}{3}, 1, \frac{8}{3}\right)$
9. $2x - 26y - 19z + 5 = 0$
10. $x + 17y + 13z = 5$
11. $6x + y - z = 5$
12. $y - 2z = 0$
13. $x - y + 2z + 9 = 0$
14. $\vec{r} = \overrightarrow{(0,1,0)} + k\overrightarrow{(2,3,0)} + s\overrightarrow{(0,1,2)}$
15. $21°$
16. $25°$
17. $4x + 3y = 0$
18. $z = 3$
19. a) point $(-5,3,2)$
    b) point $(2,3,0)$
    c) line lies in plane
20. $-\frac{1}{2}$
22. a) Each line has the same normal $\overrightarrow{(A,B)}$. Different values of C give different lines.
    b) Each plane has the same normal $\overrightarrow{(A,B,C)}$. Different values of D give different planes.
24. $2x - 2y - z = 3$
25. $x + z = 0$;
    $5x - 4y - 3z = 0$
26. $\vec{r} = \overrightarrow{(4,0,0)} + k\overrightarrow{(0,2,0)}$
27. $(4,8,3)$
28. a) parallel and distinct
    b) parallel and identical
    c) $$\begin{cases} x = 1 - \frac{13}{11}t \\ y = \frac{14}{11}t \\ z = t \end{cases}$$
    d) $$\begin{cases} x = \frac{4}{5} - \frac{3}{5}t \\ y = t \\ z = 1 - 3t \end{cases}$$
30. $2x + 3y - z - 6 = 0$
31. $x - y - z = -6$
    $2x + y - z = 1$
32. a) -30
    b) $\frac{15}{7}$
33. a) parallel and distinct
    b) parallel and identical
34. a) point $(4,0,-2)$
    b) line $$\begin{cases} x = \frac{3t + 13}{5} \\ y = \frac{6t - 9}{5} \\ z = t \end{cases}$$
    c) a triangular prism
    d) point $(3,0,-3)$
35. a) consistent and independent
    b) consistent and dependent
    c) inconsistent
    d) consistent and independent
36. a) $k \neq -3, k \neq -1$
    b) $k = -3$
    c) $k = -1$
37. $a = 2$
    $b = 8$
    $c = 7.5$
    $d \in \mathbb{R}, d \neq 4.5$ and $d \neq 2.25$
39. a) $\frac{3}{\sqrt{14}}$
    b) $\frac{10}{\sqrt{83}}$
    c) $\frac{44}{\sqrt{336}}$
    d) 0 (point in plane)
40. a) $\frac{\sqrt{1443}}{\sqrt{38}}$
    b) $\frac{\sqrt{509}}{\sqrt{37}}$
    c) $\frac{\sqrt{944}}{\sqrt{38}}$
    d) $\frac{\sqrt{699}}{\sqrt{66}}$
    e) $\frac{\sqrt{329}}{\sqrt{13}}$
42. $27a - 7b = 0$
43. $$\begin{cases} x = \frac{-19 + 5y}{t} \\ y = t \\ z = 0 \end{cases}$$
44. $\frac{\sqrt{1350}}{\sqrt{59}}$

45. -2 or $-\frac{2}{3}$

46. a) $a = 2$
b) $a \neq 2$
c) no a exists

47. $\frac{7}{3}$

48. $\vec{r} = \overrightarrow{(1,0,1)} + k\overrightarrow{(7,1,-5)}$

49. $x = \dfrac{11p - 7q + 6w}{33}$

$y = \dfrac{13p - 2q - 3w}{33}$

$z = \dfrac{3p - 3q + w}{11}$

50. a) any plane that passes through $(2,1,-2)$, for example $x - y + z = -1$
b) $x + y + z + k(2x - 3y + z) = 1 + 3k$, for all k, for example $3x - 2y + 2z = 4$
c) $x + y + z + k(2x - 3y + z) = t$, where $t \neq 1 + 3k$, for example $3x - 2y + 2z = 1$

51. $(14,6,0)$

52. $a - 2b + 10 = 0$

55. a) $x + y + z = 1$
b) $x - y - z = -1$
c) $71°$
d) $\frac{2}{\sqrt{3}}$
e) $\frac{1}{3}$

56. a) $\begin{cases} x = 2 - 7t \\ y = 1 + 5t \\ z = 3 + t \end{cases}$
b) $c = 1$
c) $9x + 16y - 17z + 17 = 0$

57. b) $3\vec{i} + 3\vec{j} + 4\vec{k}$
c) $x = p + 1, y = 3, z = 3p - 2$ (Answers will vary.)

58. a) $P(-1,2,1)$
b) $\overrightarrow{(7,-3,-5)}$
$7x - 3y - 5z + 18 = 0$
c) 1.98 units

59. a) $42°$
b) $\dfrac{x}{2} = \dfrac{y - \frac{1}{2}}{1} \pm \dfrac{z - \frac{7}{2}}{-5}$
c) $\overrightarrow{OB} = \vec{i} + 4\vec{j} + 2\vec{k}$

60. a) $\overrightarrow{(6,-1,3)}$
b) $x = 1 + 6t, y = -t, z = 2 - 3t$
$H(-5,1,5)$
c) $q = \frac{5}{3}$

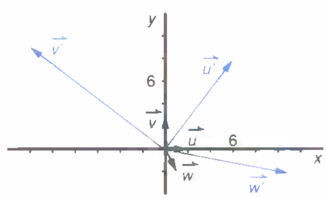
Chapter Seven Matrices and Linear Transformations

7.1 Exercises, page 294

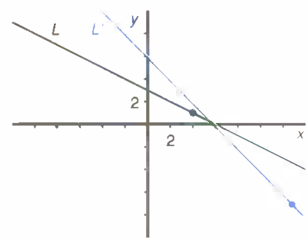
1. A 2×4
   B 3×2
   C 4×1
2. a) $a_{11} = 3, a_{14} = 1, a_{23} = 5, a_{33} = 8$
   b) $a_{13} = 0, a_{32} = 0$
3. a) $x = 4, y = 2, w = 0, z = 3$
   b) $x = 3, y = -2, w = 8, z = 8$
4. a) $x = 1, y = 9, w = 5, z = 1$
   b) $a = 16, b = -3, c = 0, d = -12$
   c) $a = 17, b = -\frac{2}{3}, c = \frac{8}{3}, d = -29$
   d) $x = 3, y = 2, z = 0$
5. a) $\begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$
   b) $\begin{bmatrix} -15 & 0 \\ 9 & 0 \end{bmatrix}$
   c) $\begin{bmatrix} -17 & -3 \\ 10 & 1 \end{bmatrix}$
   d) $\begin{bmatrix} 4 & 12 \\ -2 & 5 \end{bmatrix}$
   e) $\begin{bmatrix} -\frac{33}{2} & -18 \\ \frac{19}{2} & -12 \end{bmatrix}$
   f) $\begin{bmatrix} 18 & 24 \\ -10 & 10 \end{bmatrix}$
   g) $\begin{bmatrix} 18 & 24 \\ -10 & 10 \end{bmatrix}$
   h) $\begin{bmatrix} -30 & 0 \\ 18 & 0 \end{bmatrix}$
   i) $\begin{bmatrix} -30 & 0 \\ 18 & 0 \end{bmatrix}$
6. a) $\begin{bmatrix} -\frac{5}{2} & 0 \\ \frac{3}{2} & 0 \end{bmatrix}$
   b) $\begin{bmatrix} 6 & 9 \\ -3 & -3 \end{bmatrix}$
   c) $\begin{bmatrix} 7 & 3 \\ -4 & -1 \end{bmatrix}$
   d) $\begin{bmatrix} -1 & -\frac{9}{2} \\ \frac{1}{2} & -3 \end{bmatrix}$
7. $x = 2, y = 5$ contradicts $x - y = 4$
8. Answers will vary.

7.2 Exercises, page 301

1. F, G, T: yes
   H, R, S: no
2. $F = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$
   $G = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
   $T = \begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix}$
4. a) $\begin{bmatrix} 3 \\ 14 \end{bmatrix}$
   b) $\begin{bmatrix} 2 \\ -6 \end{bmatrix}$
5. a) $x = -\frac{3}{2}, y = -\frac{5}{2}$
   b) $x = 2, y = 1$
   c) $x = y = 0$
   d) $x = \pm 3, y = \pm 1$, z can have any value
6. $x^2 + y^2 = 0 \Rightarrow x = y = 0$ contradicts $3x + 5y = -1$
7. $M\vec{u} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$
   $M\vec{v} = \begin{bmatrix} -12 \\ 9 \end{bmatrix}$
   $M\vec{w} = \begin{bmatrix} 11 \\ -2 \end{bmatrix}$

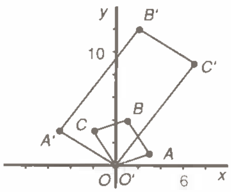


8. a) $\begin{bmatrix} 18 \\ 24 \end{bmatrix}$
   b) $\begin{bmatrix} 17 \\ 11 \end{bmatrix}$
   c) $\begin{bmatrix} 10 \\ 66 \end{bmatrix}$
   d) $\begin{bmatrix} 174 \\ -102 \end{bmatrix}$
   so $M(N\vec{v}) \neq N(M\vec{v})$
9. a) $M\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
   b) $M\vec{i} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$
   c) $M\vec{j} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$
   so $M\vec{0} = \vec{0}$
   $M\vec{i}$ = first column of M
   $M\vec{j}$ = second column of M
10. a) $A\vec{r}_0 = \begin{bmatrix} 13 \\ -7 \end{bmatrix}$
    $A\vec{m} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$
    b)



11. For example,
    a) $A(0,1)$ $B(1,3)$
    b) $A'(5,0)$ $B'(12,2)$
    c) $y = \frac{2}{7}x - \frac{10}{7}$
    d) L'

12. a) $C(-2,3)$
b) $A'(-5,3)$
$B'(2,12)$
$C'(7,9)$
$O'(0,0)$
c) a parallelogram

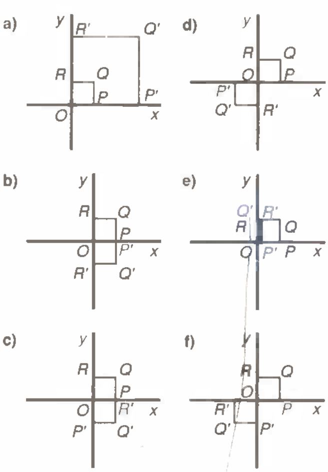


7.3 Exercises, pages 306–307

2. I leaves plane unchanged
$0_{2\times 2}$ maps plane to O

3. a) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$
b) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
d) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
e) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
f) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

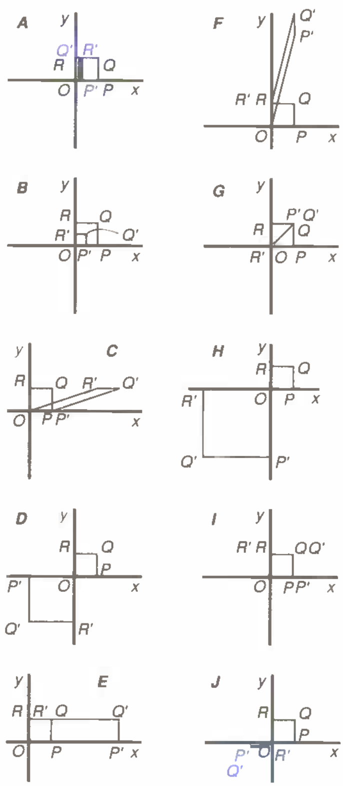
4.



5. same transformation

6. A: projection onto y-axis
B: dilation $\times \frac{1}{2}$
C: shear$\|x$-axis of factor 3
D: dilatation $\times -2$
E: stretch$\|x$-axis of factor 4
F: shear$\|y$-axis of factor 4
G: maps to line $y = x$
H: dilatation $\times -3$ and reflection in $y = x$
I: identity
J: projection onto x-axis and reflection in y-axis

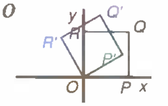
7.



8. O

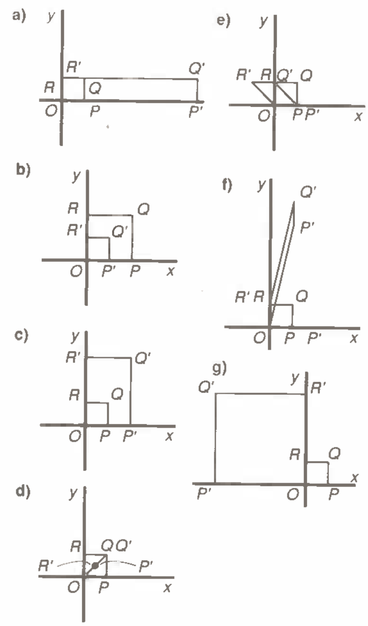
9. a) $(1,0) \to (1,0)$
$(a,0) \to (a,0)$
b) remains unchanged
c) $(0,1) \to (2,1)$
$(a,1) \to (a+2,1)$
d) moves 2 to the right
e) $(0,b) \to (2b,b)$
f) if $b > 0$: moves to the right, by a factor of $2b$
if $b < 0$: moves to the left, by a factor of $2b$

10. counterclockwise rotation by $30°$, about O



11. a) $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$
b) $\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$
c) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$
d) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$
e) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$
f) $\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$
g) $\begin{bmatrix} -4 & 0 \\ 0 & 4 \end{bmatrix}$

12.



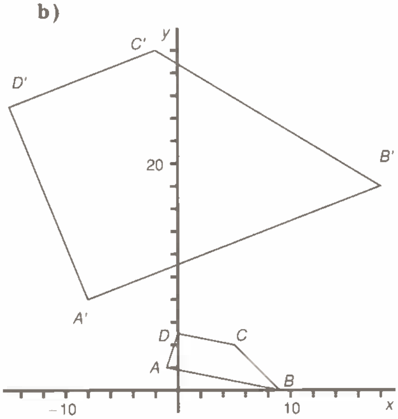
7.3 Exercises, pages 306–307, continued

13. $\begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix}$ or $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

14. a) slope AB = slope $DC = -\frac{1}{5}$

$AB \| DC$

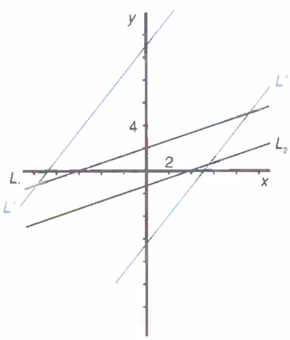
b)



c) slope $A'B'$ = slope $D'C' = \frac{5}{13}$

$A'B' \| D'C'$

15. a) direction vectors collinear



b) $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$

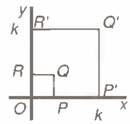
$\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

$\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \end{bmatrix}$

c) new direction vectors

$\begin{bmatrix} 8 \\ 10 \end{bmatrix}$ and $\begin{bmatrix} -8 \\ -10 \end{bmatrix}$

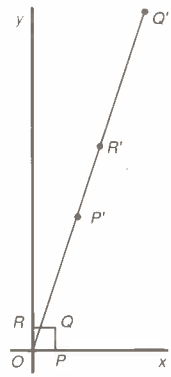
16. a) dilatated by k



b) $s\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix} = k\begin{bmatrix} x \\ y \end{bmatrix}$

c) entire plane dilatated by k

17. a)



b) $s\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + 3y \\ 6x + 9y \end{bmatrix} = \begin{bmatrix} a \\ 3a \end{bmatrix}$

c) entire plane mapped to line $y = 3x$

7.4 Exercises, pages 313–314

1. a) $\begin{bmatrix} 0.77 & -0.64 \\ 0.64 & 0.77 \end{bmatrix}$

b) $\begin{bmatrix} 0.17 & -0.98 \\ 0.98 & 0.17 \end{bmatrix}$

c) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

d) $\begin{bmatrix} -0.34 & -0.94 \\ 0.94 & -0.34 \end{bmatrix}$

e) $\begin{bmatrix} -0.94 & 0.34 \\ -0.34 & -0.94 \end{bmatrix}$

f) $\begin{bmatrix} 0.95 & 0.31 \\ -0.31 & 0.95 \end{bmatrix}$

2. a) $\begin{bmatrix} 0.77 & 0.64 \\ 0.64 & -0.77 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} -0.94 & -0.34 \\ -0.34 & 0.94 \end{bmatrix}$

3. b) det(1a)) = 1
det(2a)) = −1

4. b) det(1e)) = 1
det(2c)) = −1

5. a) $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

b) $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

c) $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

6. a) $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

b) $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

c) $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

7. $R_{300} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = R_{-60}$

8. $M_{30} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = M_{210}$

9. $\det(I) = 1$
same area and orientation
$\det(0_{2\times 2}) = 0$
zero area, orientation not defined

10.

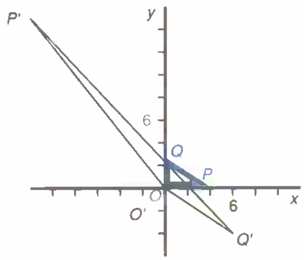
	det.	a.s.f.	orientation
A	0	0	undefined
B	0.25	0.25	same
C	1	1	same
D	4	4	same
E	4	4	same
F	1	1	same
G	0	0	undefined
H	−9	9	reversed
I	1	1	same
J	0	0	undefined

11. a) $k = 4$
b) $k = 1$
c) $k = \frac{3}{2}$

12. $\det(S) = 1$
same area and orientation

13. $\det(M) = 0$
area zero

14.



a) 3
b) $O'(0,0)$
$P'(-12,15)$
$Q'(6,-4)$
c) 21
d) OPQ and $O'P'Q'$ have reverse orientation

15. a) $\overrightarrow{OP} \cdot \overrightarrow{OQ} = 0$
b) $\overrightarrow{O'P'} \cdot \overrightarrow{O'Q'} = -132$
$\measuredangle P'O'Q' = 162°$
c) no

17. a) $\cos\theta = \frac{4}{5}$, $\sin\theta = \frac{3}{5}$

b) $\begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$

c) $\begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{bmatrix}$

18. K: rotation through 37°
L: rotation through $-60°$ (or 300°)
M: reflection in line
$$y = \left(\tan \frac{1}{2}\theta\right)x,$$
where $\theta = -37°$
that is, line $y = -\frac{1}{3}x$
N: rotation through 67° + 180°, that is, 247°

7.5 Exercises, page 318

1. A, G

2. $B^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$C^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

$D^{-1} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$

$E^{-1} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}$

$F^{-1} = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$

$H^{-1} = \begin{bmatrix} 0 & -\frac{1}{3} \\ -\frac{1}{3} & 0 \end{bmatrix}$

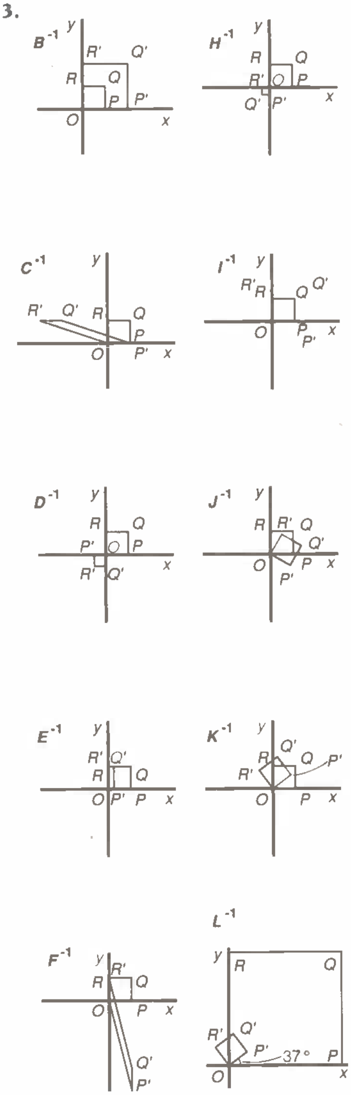
$I^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$J^{-1} = \begin{bmatrix} \cos 30° & \sin 30° \\ -\sin 30° & \cos 30° \end{bmatrix}$

$K^{-1} = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$

$L^{-1} = \frac{1}{25}\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$

3.



7.5 Exercises, page 318, continued

4.

	a.s.f	orientation
B^{-1}	4	same
C^{-1}	1	same
D^{-1}	$\frac{1}{4}$	same
E^{-1}	$\frac{1}{4}$	same
F^{-1}	1	same
H^{-1}	$\frac{1}{9}$	reversed
I^{-1}	1	same
J^{-1}	1	same
K^{-1}	1	same
L^{-1}	$\frac{1}{25}$	same

5. a) J and K

b) $R_\theta^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ the transpose of R

6. a) $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

b) $R^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$

7. $k = \frac{5}{2}$

8. $I^{-1} = I$

9. $S^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

horizontal shear of factor -2

10. a) $M^{-1} = \begin{bmatrix} 3 & -10 \\ -2 & 7 \end{bmatrix}$

b) $M\vec{v} = \begin{bmatrix} 13 \\ 4 \end{bmatrix}$

c) $M^{-1}\vec{v}' = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

11. a) $M^{-1} = \frac{1}{17}\begin{bmatrix} 2 & 1 \\ -5 & 6 \end{bmatrix}$

b) $M\vec{v} = \begin{bmatrix} -8 \\ -1 \end{bmatrix}$

c) $M^{-1}\vec{v}' = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

13. 1

15. a) $\begin{bmatrix} 3x - y \\ 12x - 4y \end{bmatrix} = \begin{bmatrix} a \\ 4a \end{bmatrix}$

b) $y = 4x$

c) no, $\det(S) = 0$

7.6 Exercises, pages 324–325

1. $LM = \begin{bmatrix} 14 & 13 \\ 32 & 25 \end{bmatrix}$

$ML = \begin{bmatrix} 1 & 4 \\ 26 & 38 \end{bmatrix}$

$LN = \begin{bmatrix} 2 & 5 \\ 10 & 15 \end{bmatrix}$

$NL = \begin{bmatrix} 21 & 32 \\ -2 & -4 \end{bmatrix}$

$MN = \begin{bmatrix} 26 & 20 \\ 16 & 25 \end{bmatrix}$

$NM = \begin{bmatrix} 49 & 29 \\ -8 & 2 \end{bmatrix}$

2. a) A, G

b) I

c) B

d) C

e) J, K

f) C, I, J, K

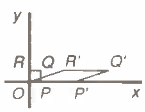
3. a) $IB = B$, $IC = C$, $IK = K$

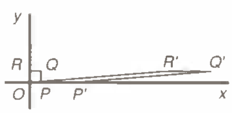
4. a) $CE = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix}$

$EC = \begin{bmatrix} 4 & 12 \\ 0 & 1 \end{bmatrix}$

b) CE: stretch, then shear

EC: shear, then stretch





c) 4, for both

5. a) $BJ = \begin{bmatrix} \frac{1}{2}\cos 30° & -\frac{1}{2}\sin 30° \\ \frac{1}{2}\sin 30° & \frac{1}{2}\cos 30° \end{bmatrix} = JB$

b) rotation then dilatation = dilatation then rotation

6. a) $JK = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = KJ$

b) rotation 30° then rotation 60° = rotation 60° then rotation 30° = rotation 90°

7. a) $AK = \begin{bmatrix} 0 & 0 \\ \sin 60° & \cos 60° \end{bmatrix}$

$KA = \begin{bmatrix} 0 & -\sin 60° \\ 0 & \cos 60° \end{bmatrix}$

b) 0, for both; both singular

8. a) $I^2 = I$

$A^2 = A$

b) identity then identity = identity

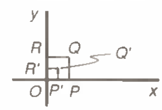
projection to y-axis then projection to y-axis = projection to y-axis

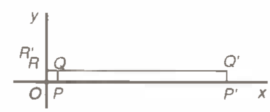
9. a) $B^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$

$E^2 = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}$

b) B^2: dilatation $\times \frac{1}{4}$

E^2: stretch $\times$ 16





10. a) $J^2 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

$J^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$J^6 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

b) rotation 30° then rotation 30° = rotation 60°

rotation 30°, then 30°, then 30° = rotation 90°

rotation 30°, six times, = rotation 180°

12. a) $M^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

b) $MM^{-1} = M^{-1}M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}$

c) M and M^{-1} commute. Their product is the 2×2 identity matrix.

14. $AB = BA = I$

15. $AB = BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_{3\times3}$

16. $KL = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}$

$LM = \begin{bmatrix} pw + qy & px + qz \\ rw + sy & rx + sz \end{bmatrix}$

$(KL)M =$

$\begin{bmatrix} apw + brw + aqy + bsy & apx + brx + aqz + bsz \\ cpw + drw + cqy + dsy & cpx + drx + cqz + dsz \end{bmatrix}$

$= K(LM)$

17. $\det(K) = ad - bc$

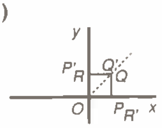
$\det(L) = ps - qr$

$\det(KL) = adps - adqr -$
$bcps + bcqr$
$= (ad - bc)(ps - qr)$

18. a) $K^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

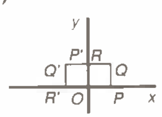
c) yes

19. a)



b) c) $M^2 = I, M^3 = M, M^4 = I$

20. a)



b) $M^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(rotation 180°)

$M^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(rotation 270°)

$M^4 = I$

(rotation 360°)

21. a) back to original

b) $T^2 = I$

22. a) rotation through 2θ

b) $R^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} =$

$\begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$

23. a) P: rotation through $\alpha = -53°$

$\left(\tan\alpha = -\dfrac{4}{3}\right)$

Q: rotation through $\alpha = 53°$

$\left(\tan\alpha = \dfrac{4}{3}\right)$

b) $PQ = I$

c) $P^{-1} = Q, Q^{-1} = P$

24. a) $AB = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} BA = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$

b) $A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

$B^{-1} = \dfrac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

c) $A^{-1}B^{-1} = \begin{bmatrix} 4 & -1 \\ -\dfrac{3}{2} & \dfrac{1}{2} \end{bmatrix}$

d) $(A^{-1}B^{-1})(AB) = \begin{bmatrix} 24 & 10 \\ -\dfrac{17}{2} & -\dfrac{7}{2} \end{bmatrix}$

$(A^{-1}B^{-1})(BA) = I$

e) $(A^{-1}B^{-1})^{-1} = BA$

7.7 Exercises, pages 331–332

1. a) $L^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

$M^{-1} = \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}$

b) $LM = \begin{bmatrix} 7 & -5 \\ 3 & -2 \end{bmatrix}$

$L^{-1}M^{-1} = \begin{bmatrix} -1 & -7 \\ 1 & 6 \end{bmatrix}$

$M^{-1}L^{-1} = \begin{bmatrix} -2 & 5 \\ -3 & 7 \end{bmatrix}$

c) $(LM)^{-1}\begin{bmatrix} -2 & 5 \\ -3 & 7 \end{bmatrix}$

d) $(LM)(M^{-1}L^{-1}) = I$

$(LM)(L^{-1}M^{-1}) \neq I$

2. $B^{-1}A^{-1} = A^{-1}B^{-1} = \begin{bmatrix} \dfrac{1}{2} & 0 \\ 0 & \dfrac{1}{2} \end{bmatrix}$

reflection in x-axis then
dilatation × 2 =
dilatation × 2 then
reflection in x-axis

7.7 Exercises, pages 331–332, continued

3. a) $x = 4, y = \frac{7}{2}$
   b) no solution
   c) $x = k, y = \frac{2k - 4}{7}$
   d) none
   e) $x = k, y = \frac{7 + k}{4}$
   f) $x = \frac{3}{17}, y = \frac{32}{17}$
4. a) (only possible) $x = 4, y = \frac{7}{2}$
5. a) none
   b) $\begin{bmatrix} 2k - 4 \\ k \end{bmatrix}$
   c) $\begin{bmatrix} 3 + 2k \\ k \end{bmatrix}$
   d) none
6. $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ -2x + 2 \end{bmatrix} =$
   $\begin{bmatrix} 2x - 2x + 2 \\ 4x - 4x + 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$
7. a) $L + M = \begin{bmatrix} p + w & q + x \\ r + y & s + z \end{bmatrix}$
   $KL = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}$
   $KM = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$
   $K(L + M) =$
   $\begin{bmatrix} ap + br + aw + by & aq + bs + ax + bz \\ cp + dr + cw + dy & cq + ds + cx + dz \end{bmatrix}$
   $LK = \begin{bmatrix} pa + qc & pb + qd \\ ra + sc & rb + sd \end{bmatrix}$
   $MK = \begin{bmatrix} wa + xc & wb + xd \\ ya + zc & yb + zd \end{bmatrix}$
   $(L + M)K =$
   $\begin{bmatrix} pa + qc + wa + xc & pb + qd + wb + xd \\ ra + sc + ya + zc & rb + sd + yb + zd \end{bmatrix}$
8. a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
   b) no; a) is a counterexample
9. a) $PQ = RQ \Rightarrow (PQ)Q^{-1} = (RQ)Q^{-1}$
   $\Rightarrow P(QQ^{-1}) = R(QQ^{-1})$
   $\Rightarrow P = R$
   b) If $\det(Q) = 0$, P does not have to equal R.
10. no, in general
11. $\begin{bmatrix} \pm 2 & 0 \\ 0 & \pm 2 \end{bmatrix}$ or
    $\begin{bmatrix} x & y \\ \frac{4 - x^2}{y} & -x \end{bmatrix}$, $x, y \in \mathbb{R}, y \neq 0$
12. a) $A(I - A) = I \Rightarrow I - A = A^{-1}$
    b) $A^3 = A^2A = (A - I)(A)$
    $= A^2 - A$
    $= -I$
    c) $p = 2, q = -1$
13. a) $PQ = \begin{bmatrix} 2a + 4b & 5b + a \\ 4a + 20b & 19b + 5a \end{bmatrix}$
    $= QP$
    b) $Q = \begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix}$
14. a) $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
    $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
    b) $RM = MR \Rightarrow \sin\theta = -\sin\theta$
    $\Rightarrow 2\sin\theta = 0$
    $\Rightarrow \theta \in \{0°, 180°\}$
18. b) $x = 99, y = -52, z = -36$
19. c) $y = \frac{3}{2}x$, and $y = -x$

In Search of, page 335

1. a) 5, −1
   b) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$
   c) $y = x$ and $y = -\frac{1}{2}x$
2. a) 3
   b) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
   c) $y = -x$
   a) $\pm i$
   b) none
   c) none
4. b) $a = b$ and $h = 0$
5. a) $s\vec{v} \cdot \vec{u} = s\vec{u} \cdot \vec{v}$
   b) $p\vec{u} \cdot \vec{v} = q\vec{v} \cdot \vec{u}$
   c) Since $p \neq q$, $\vec{u} \cdot \vec{v} = 0$

Inventory, page 338

1. 4
2. $p \times q$
3. n; n
4. 6; 3; 2
5. $T\vec{u} + T\vec{v}$; $k(T\vec{v})$
6. the origin

7. parallelism

8. $\vec{i}$ by M;
image of $\vec{j}$ by M

9. $\begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}$

10. B; A

11. commutative

12. $q = r$; $p \times s$

13. $\det(M) \neq 0$;
$\dfrac{1}{\det(M)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

14. $\det(M) = 0$

15. reversed

16. singular transformation

17. $\det(M)$

18. $\det(M) > 0$; $\det(M) < 0$

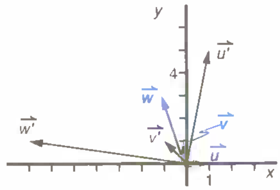
Review Exercises, pages 339–341

1. a) $\begin{bmatrix} -1 & -3 \\ -2 & 0 \end{bmatrix}$
   b) $\begin{bmatrix} 18 & 0 \\ -6 & -9 \end{bmatrix}$
   c) $\begin{bmatrix} -1 & 15 \\ 12 & 3 \end{bmatrix}$
   d) $\begin{bmatrix} 12 & 2 \\ -4 & 4 \end{bmatrix}$
   e) $\begin{bmatrix} -19 & 7 \\ 18 & -28 \end{bmatrix}$
   f) $\begin{bmatrix} \frac{11}{2} & -\frac{1}{2} \\ -3 & 2 \end{bmatrix}$
   g) $\begin{bmatrix} 24 & -6 \\ -22 & 39 \end{bmatrix}$
   h) $\begin{bmatrix} -8 & -24 \\ -16 & 0 \end{bmatrix}$
   i) $\begin{bmatrix} -8 & -24 \\ -16 & 0 \end{bmatrix}$

2. a) $\begin{bmatrix} 2 & 0 \\ -\frac{2}{3} & -1 \end{bmatrix}$
   b) $\begin{bmatrix} 2 & 6 \\ 4 & 0 \end{bmatrix}$
   c) $\begin{bmatrix} -4 & 4 \\ 6 & -7 \end{bmatrix}$
   d) $\begin{bmatrix} \frac{13}{4} & \frac{3}{4} \\ -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$

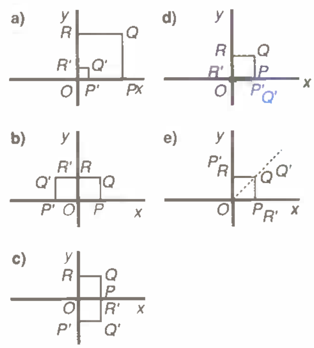
3. a) $x = 1, y = -\dfrac{11}{4}$
   b) $x = y = 0$

4. $M\vec{u} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $M\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$,
   $M\vec{w} = \begin{bmatrix} -7 \\ 1 \end{bmatrix}$

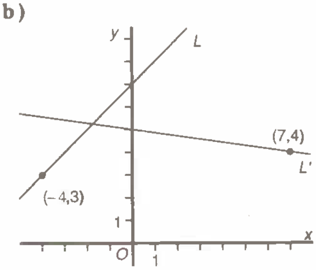


5. a) $\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$
   b) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
   c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
   d) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
   e) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

6.

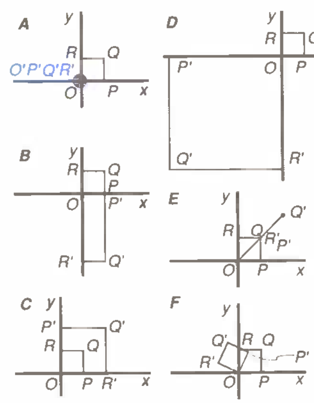


7. a) $\vec{r} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} + k\begin{bmatrix} 7 \\ -1 \end{bmatrix}$
   b)



8. A: null transformation
   B: stretch $\parallel$ y-axis of factor 3, and reflection in x-axis
   C: reflection in $y = x$ and dilatation $\times$ 2
   D: reflection in origin and dilatation $\times$ 5
   E: mapping to $y = x$
   F: counterclockwise rotation of 65° about origin

9.



10. a)

	a.s.f.	orientation
A	0	undefined
B	3	reversed
C	4	reversed
D	25	same
E	0	undefined
F	1	same

   b) A and E
   c) $B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}$
   $C^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$
   $D^{-1} = \begin{bmatrix} -\frac{1}{5} & 0 \\ 0 & -\frac{1}{5} \end{bmatrix}$
   $F^{-1} = \begin{bmatrix} \cos 65° & \sin 65° \\ -\sin 65° & \cos 65° \end{bmatrix}$

Review Exercises, pages 339–341, continued

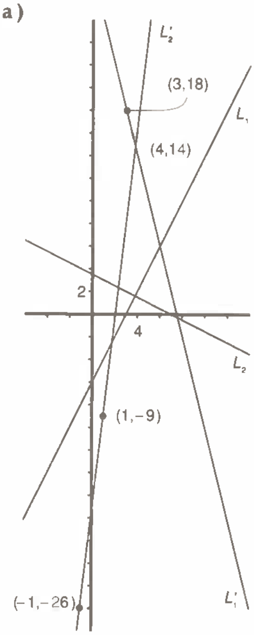
11. a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
b) $\begin{bmatrix} 0 & -10 \\ -10 & 0 \end{bmatrix}$
c) $\begin{bmatrix} -5 & -5 \\ -5 & -5 \end{bmatrix}$
d) $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

12. a) $\begin{bmatrix} 0.64 & -0.77 \\ 0.77 & 0.64 \end{bmatrix}$
b) $\begin{bmatrix} -0.57 & -0.82 \\ 0.82 & -0.57 \end{bmatrix}$
c) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

13. a) $\begin{bmatrix} -0.77 & 0.64 \\ 0.64 & 0.77 \end{bmatrix}$
b) $\begin{bmatrix} -0.10 & -0.99 \\ -0.99 & 0.10 \end{bmatrix}$
c) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

14. $R_{225} = R_{-135}$, same rotation

16. a)



b) L_1': $\vec{r} = \begin{bmatrix} 3 \\ 18 \end{bmatrix} + k\begin{bmatrix} 1 \\ -4 \end{bmatrix}$
L_2': $\vec{s} = \begin{bmatrix} -1 \\ -26 \end{bmatrix} + p\begin{bmatrix} 2 \\ 17 \end{bmatrix}$
c) no

17. M: rotation through α,
$\tan \alpha = \frac{y}{x}$
N: reflection in $y = \left(\tan\frac{1}{2}\alpha\right)x$,
$\tan \alpha = \frac{y}{x}$

18. a) $M_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$
b) $M_\theta^{-1} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$
c) $M_\theta^{-1} = M_\theta$

19. $\begin{bmatrix} 2y \\ y \end{bmatrix}$ (or $\begin{bmatrix} 2k \\ k \end{bmatrix}$)

20. a) $A^{-1} = A$, $B^{-1} = B$, $C^{-1} = C$
b) Reflections are self-inverses.

21. a) Plane is brought back to original status.
b) I

22. a) $AB = \begin{bmatrix} 1 & -3 \\ -10 & -6 \end{bmatrix}$
$BA = \begin{bmatrix} 8 & -4 \\ 17 & -13 \end{bmatrix}$
b) $\det(A) = -3$
$\det(B) = 12$
$\det(AB) = -36 = \det(BA)$

23. no, for example
$\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, but
$\begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 14 & 0 \end{bmatrix}$

24. a) yes
b) no (sometimes many, sometimes none)

25. $\begin{bmatrix} 0 & bt \\ -ct & 0 \end{bmatrix}$

27. a) $N = \frac{1}{4-4p}\begin{bmatrix} 2 & -p \\ -4 & 2 \end{bmatrix}$
$p = 1$
c) $p = -2, q = -1$
d) $p = -1$ or $p = 3$

28. a) sketch not provided
b) $\lambda = \frac{33}{14}$ $\mu = \frac{16}{7}$
d) $D\left(8, \frac{1}{2}\right)$
$|\overrightarrow{BA}| = \sqrt{17}, |\overrightarrow{BC}| = \frac{3\sqrt{17}}{2}$
e) $A'(3,-4)$ $B'(-2,5)$
$C'\left(\frac{5}{2}, 2\right)$ $D'\left(\frac{15}{2}, -7\right)$
$\alpha = -2$

29. C

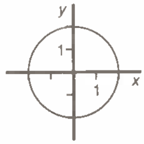
30. $\begin{bmatrix} 7 \\ 0 \end{bmatrix}$

31. a) $O'(0,0)$ $A'(1,2)$ $B'(4,3)$ $C'(3,1)$
d) shear with invariant line $y = 2x$; of factor 1
e) image of A under T^{-1}: (1,2)
image of C under T^{-1}: (1,−3)

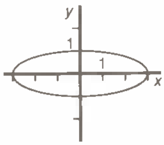
Chapter Eight
Transformations of Conics

8.1 Exercises, page 348

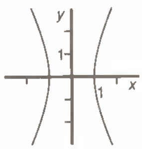
1. a)



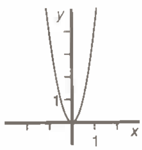
b)



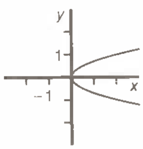
c)



d)



e)



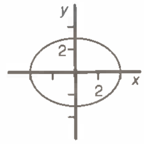
2. Circles as follows

	centre	radius
a)	(0,0)	4
b)	(0,0)	3
c)	(0,0)	2
d)	(0,0)	$\sqrt{5}$
e)	(0,0)	2
f)	(0,0)	$\sqrt{2}$

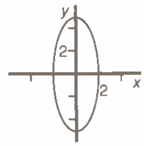
$g = f = 0$ for each

	a	b	c
a)	1	1	−16
b)	1	1	−9
c)	4	4	−16
d)	3	3	−15
e)	−2	−2	8
f)	−5	−5	10

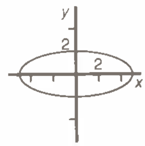
3. a)



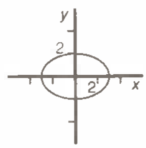
b)



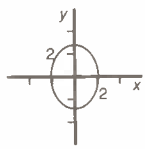
c)



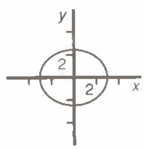
d)



e)



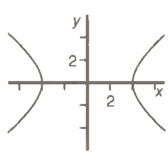
f)



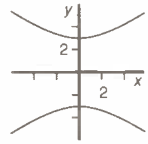
$g = f = 0$ for each

	a	b	c
a)	9	16	−144
b)	25	4	−100
c)	4	25	−100
d)	−4	−9	36
e)	−32	−18	144
f)	−2	−3	18

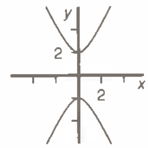
4. a)



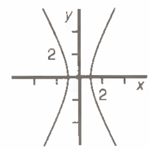
b)



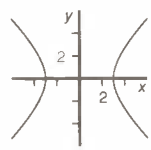
c)



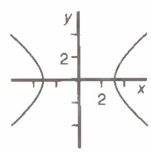
d)



e)



f)

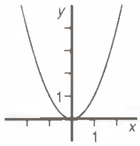


$g = f = 0$ for each

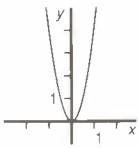
	a	b	c
a)	9	−16	−144
b)	9	−16	144
c)	4	1	4
d)	−8	2	8
e)	−1	1	9
f)	3	−5	−30

8.1 Exercises, page 348, continued

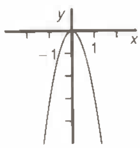
5\. a)



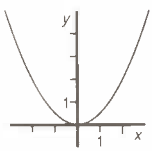
b)



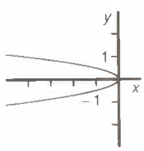
c)



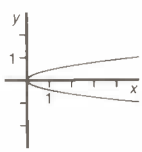
d)



e)



f)



	a	b	c	g	f
a)	1	0	0	0	$-\frac{1}{2}$
b)	4	0	0	0	$-\frac{1}{2}$
c)	3	0	0	0	$\frac{1}{2}$
d)	−1	0	0	0	1
e)	0	4	0	$\frac{1}{2}$	0
f)	0	5	0	0	$-\frac{1}{2}$

6\. a) circle
b) hyperbola
c) parabola
d) hyperbola
e) circle
f) ellipse
g) hyperbola
h) parabola

circles
a) (0,0), 5
e) (0,0), 4

ellipses
f) (0,0), (±3,0), x-axis
g) (0,0), (0,±4), y-axis

hyperbolas
b) (0,0), (±3,0), x-axis
d) (0,0), (0,±2), y-axis

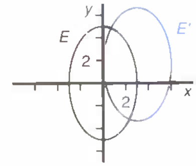
parabolas
c) (0,0), (positive) y-axis
h) (0,0), (negative) x-axis

7\. a) ellipse
b) hyperbola
c) parabola
d) circle
e) parabola
f) hyperbola
g) ellipse
h) circle
i) hyperbola
j) ellipse

8.2 Exercises, pages 354–355

1\. a) (6,11)
b) (−3,11)
c) (5,−2)
d) (0,0)

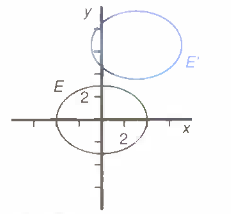
2\. a) $25x^2 + 9y^2 - 150x - 18y + 9 = 0$
b)



c)

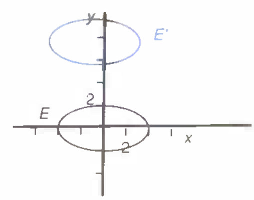
	a	b	c	g	f
E	25	9	−225	0	0, $ab > 0$
E'	25	9	9	−75	−9, $ab > 0$

3\. a) $9x^2 + 16y^2 - 54x - 192y + 513 = 0$



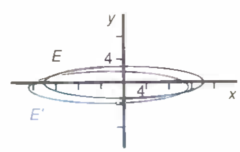
	a	b	c	g	f
E	9	16	−144	0	0, $ab > 0$
E'	9	16	513	−27	−96, $ab > 0$

b) $x^2 + 4y^2 + 2x - 56y + 181 = 0$



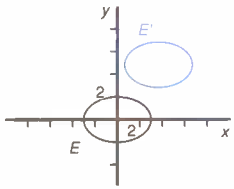
	a	b	c	g	f
E	1	4	−16	0	0, $ab > 0$
E'	1	4	181	1	−28, $ab > 0$

c) $8x^2 + 200y^2 + 48x + 800y - 728 = 0$



	a	b	c	g	f
E	8	200	−1600	0	0, $ab > 0$
E'	8	200	−728	24	400, $ab > 0$

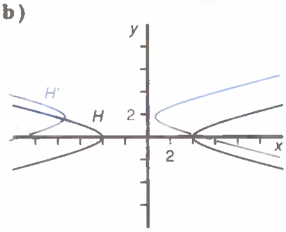
d) $4x^2 + 9y^2 - 32x - 90y + 253 = 0$



	a	b	c	g	f
E	4	9	−36	0	0, $ab > 0$
E'	4	9	253	−16	−45, $ab > 0$

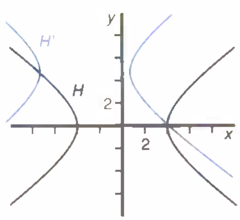
4. a) $x^2 - 16y^2 + 6x + 64y - 71 = 0$

b)



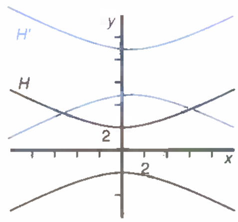
	a	b	c	g	f
H	1	−16	−16	0	0, $ab < 0$
H'	1	−16	−71	3	32, $ab < 0$

5. a) $9x^2 - 16y^2 + 54x + 160y + 463 = 0$



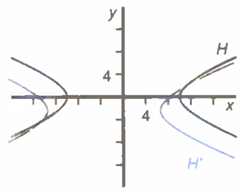
	a	b	c	g	f
H	9	−16	−144	0	0, $ab < 0$
H'	9	−16	463	27	80, $ab < 0$

b) $x^2 - 4y^2 - 2x + 56y - 179 = 0$



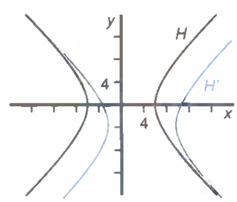
	a	b	c	g	f
H	1	−4	−16	0	0, $ab < 0$
H'	1	−4	−179	−1	28, $ab < 0$

c) $8x^2 - 50y^2 + 48x - 200y - 928 = 0$



	a	b	c	g	f
H	8	−50	−800	0	0, $ab < 0$
H'	8	−50	−928	24	−100, $ab < 0$

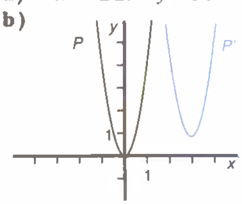
d) $x^2 - y^2 - 8x - 6y - 29 = 0$



	a	b	c	g	f
H	1	−1	−36	0	0, $ab < 0$
H'	1	−1	−29	−4	−3, $ab < 0$

6. a) $4x^2 - 24x - y + 37 = 0$

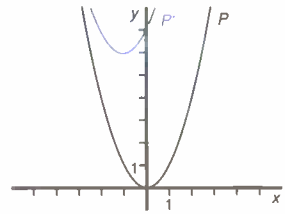
b)



c)

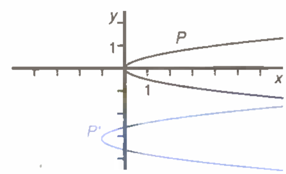
	a	b	c	g	f
P	4	0	0	0	$-\frac{1}{2}$, $ab = 0$
P'	4	0	37	−12	$-\frac{1}{2}$, $ab = 0$

7. a) $x^2 + 2x - y + 7 = 0$



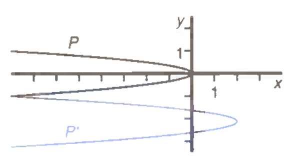
	a	b	c	g	f
P	1	0	0	0	$-\frac{1}{2}$, $ab = 0$
P'	1	0	7	1	$-\frac{1}{2}$, $ab = 0$

b) $4y^2 - x + 24y + 35 = 0$



	a	b	c	g	f
P	0	4	0	$-\frac{1}{2}$	0, $ab = 0$
P'	0	4	35	$-\frac{1}{2}$	12, $ab = 0$

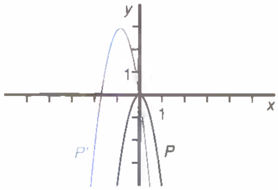
c) $8y^2 + x + 32y + 30 = 0$



	a	b	c	g	f
P	0	8	0	$\frac{1}{2}$	0, $ab = 0$
P'	0	8	30	$\frac{1}{2}$	16, $ab = 0$

8.2 Exercises, pages 354–355, continued

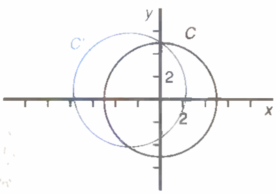
d) $4x^2 + 8x + y + 1 = 0$



	a	b	c	g	f
P	4	0	0	0	$\frac{1}{2}$, $ab = 0$
P'	4	0	1	4	$\frac{1}{2}$, $ab = 0$

8. a) $x^2 + y^2 + 6x - 2y - 15 = 0$

b)



c)

	a	b	c	g	f
C	1	1	−25	0	0, $ab > 0$
C'	1	1	−15	3	−1, $ab > 0$

9. a) $x^2 + y^2 - 6x + 2y - 134 = 0$

C: centre (0,0) radius 12

C': centre (3,−1) radius 12

	a	b	c	g	f
C	1	1	−144	0	0, $ab > 0$
C'	1	1	−134	−3	1, $ab > 0$

b) $x^2 + y^2 + 10x + 4y + 4 = 0$

C: centre (0,0) radius 5

C': centre (−5,−2) radius 5

	a	b	c	g	f
C	1	1	−25	0	0, $ab > 0$
C'	1	1	4	5	2, $ab > 0$

c) $8x^2 + 8y^2 - 48x + 32y + 88 = 0$

C: centre (0,0) radius $\sqrt{2}$

C': centre (3,−2) radius $\sqrt{2}$

	a	b	c	g	f
C	8	8	−16	0	0, $ab > 0$
C'	8	8	88	−24	16, $ab > 0$

d) $-4x^2 - 4y^2 + 32x + 40y - 148 = 0$

C: centre (0,0) radius 2

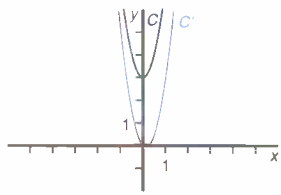
C': centre (4,5) radius 2

	a	b	c	g	f
C	−4	−4	16	0	0, $ab > 0$
C'	−4	−4	−148	16	20, $ab > 0$

11. a) $4x^2 - y = 0$

b) parabola

c)



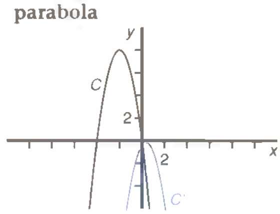
12. a) $x^2 + y^2 - 25 = 0$

C': circle centre (0,0) radius 5

C: circle centre (−2,−3) radius 5

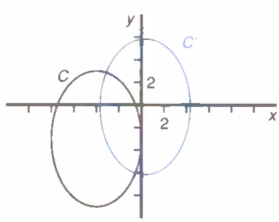
b) $2x^2 + y = 0$

parabola



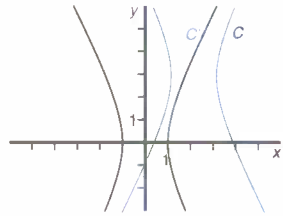
c) $9x^2 + 4y^2 - 144 = 0$

ellipse



d) $4x^2 - y^2 - 4 = 0$

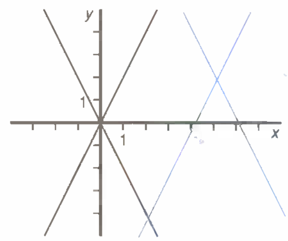
hyperbola



13. a) $2x - y = 0$, $2x + y = 0$

b) $4x^2 - y^2 - 40x + 4y + 96 = 0$

c)



d) two lines, rather than a circle, ellipse, parabola, or hyperbola

14. a) $ax^2 + by^2 - 2ahx + 2bky + ah^2 + bk^2 + t = 0$

15. $(x,y) \rightarrow (x + 2, y + 3)$

17. b) (h,k)

c) $2a$; $2b$ (if $a > b$)

$2b$; $2a$ (if $b > a$)

18. a) $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

b) $y - k = a(x - h)^2$

8.3 Exercises, page 360

1. ellipse: a) e)

circle: d)

hyperbola: b) f)

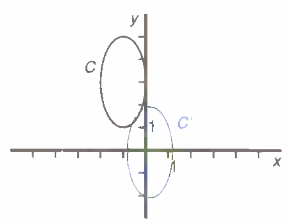
parabola: c)

2. a) ellipse

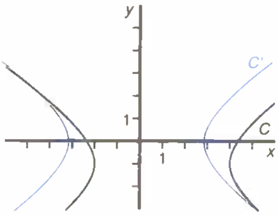
b) $(x,y) \rightarrow (x + 1, y - 3)$

c) $4x^2 + y^2 = 4$

d)

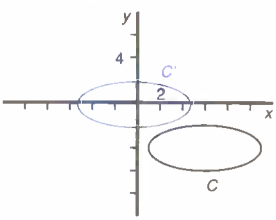


3. a) hyperbola
$(x,y) \rightarrow (x-1, y+1)$
$4x^2 - 9y^2 = 36$

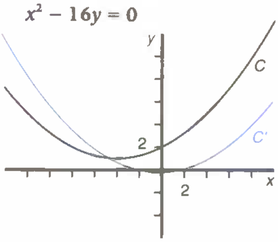


b) circle
$(x,y) \rightarrow (x-3, y+5)$
$x^2 + y^2 = 36$
C: circle centre $(3,-5)$
radius 6
C': circle centre $(0,0)$
radius 6

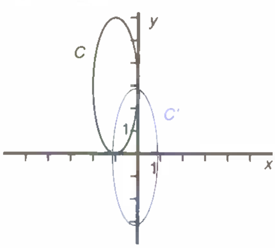
c) ellipse
$(x,y) \rightarrow (x-6, y+4)$
$4x^2 + 25y^2 = 100$



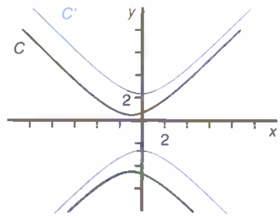
d) parabola
$(x,y) \rightarrow (x+4, y-1)$
$x^2 - 16y = 0$



e) ellipse
$(x,y) \rightarrow (x+1, y-3)$
$9x^2 + y^2 = 9$

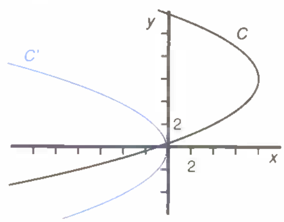


f) hyperbola
$(x,y) \rightarrow (x+1, y+2)$
$4x^2 - 4y^2 = -25$



g) circle
$(x,y) \rightarrow (x+2, y+3)$
$2x^2 + 2y^2 = 25$
C: circle centre $(-2,-3)$
radius $\sqrt{\frac{25}{2}}$
C': circle centre $(0,0)$
radius $\sqrt{\frac{25}{2}}$

h) parabola
$(x,y) \rightarrow (x-8, y-6)$
$y^2 + 4x = 0$



4. a) $(x,y) \rightarrow (x-1, y+2)$
b) $4x^2 - y^2 = 0$;
$2x - y = 0,\ 2x + y = 0$
c) $2x + y = 0,\ 2x - y - 4 = 0$
d) $ab < 0$, but graph is two lines

6. b) $g^2 + f^2 < c$
c) $g^2 + f^2 = c$

8. a) parabola

8.4 Exercises, page 364

1. a) $[46 \quad 52]$
b) $\begin{bmatrix} 14 \\ 23 \end{bmatrix}$

2. a) $7x^2 + 16xy + 9y^2 = k$
b) $6x^2 - y^2 = k$
c) $4x^2 - 10xy + 2y^2 = k$
d) $-2x^2 - 8xy + 3y^2 = k$
e) $12x^2 + 20xy + 2y^2 = k$
f) $-2x^2 + 18xy + y^2 = k$

3. a) $\begin{bmatrix} 4 & 3 \\ 3 & 5 \end{bmatrix}, [3]$
b) $\begin{bmatrix} 7 & -4 \\ -4 & 3 \end{bmatrix}, [1]$
c) $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, [9]$
d) $\begin{bmatrix} 5 & -1.5 \\ -1.5 & -7 \end{bmatrix}, [8]$
e) $\begin{bmatrix} 9 & 0 \\ 0 & -11 \end{bmatrix}, [-5]$
f) $\begin{bmatrix} 1 & -0.5 \\ -0.5 & 5 \end{bmatrix}, [3]$

4. a) $[x \quad y]\begin{bmatrix} 4 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [3]$
b) $[x \quad y]\begin{bmatrix} 7 & -4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [1]$
c) $[x \quad y]\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [9]$
d) $[x \quad y]\begin{bmatrix} 5 & -1.5 \\ -1.5 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [8]$
e) $[x \quad y]\begin{bmatrix} 9 & 0 \\ 0 & -11 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [-5]$
f) $[x \quad y]\begin{bmatrix} 1 & -0.5 \\ -0.5 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [3]$

5. a) $\begin{bmatrix} 7 & 8 \\ 8 & 9 \end{bmatrix}$
b) $\begin{bmatrix} 6 & 0 \\ 7 & -1 \end{bmatrix}$
c) $\begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix}$
d) $\begin{bmatrix} -2 & -4 \\ -4 & 3 \end{bmatrix}$
e) $\begin{bmatrix} 12 & 10 \\ 0 & 2 \end{bmatrix}$
f) $\begin{bmatrix} 1 & 7 \\ 4 & 1 \\ 3 & 2 \end{bmatrix}$

6. a) $\begin{bmatrix} 11 & 4 \\ 19 & 8 \end{bmatrix}$
b) $\begin{bmatrix} 11 & 19 \\ 4 & 8 \end{bmatrix}$
c) $\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 11 & 19 \\ 4 & 8 \end{bmatrix}$

7. a) $(AB)^t = B^tA^t = \begin{bmatrix} 3 & 4 \\ 10 & 17 \end{bmatrix}$
b) $(AB)^t = B^tA^t = \begin{bmatrix} -15 & -2 \\ -8 & 8 \end{bmatrix}$
c) $(AB)^t = B^tA^t = \begin{bmatrix} 14 & 8 \\ 21 & 8 \end{bmatrix}$

8.4 Exercises, page 364, continued

8. a) $\begin{bmatrix} 0.87 & -0.50 \\ 0.50 & 0.87 \end{bmatrix}$

b) $\begin{bmatrix} 0.94 & -0.34 \\ 0.34 & 0.94 \end{bmatrix}$

c) $\begin{bmatrix} -0.77 & -0.64 \\ 0.64 & -0.77 \end{bmatrix}$

d) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

9. a) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

b) $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

c) $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

10. $\begin{bmatrix} 3.06 & -5.79 \\ 2.57 & 6.89 \end{bmatrix}$

11. a) $\begin{bmatrix} 2.10 & -1.63 \\ 2.37 & 4.83 \end{bmatrix}$

b) $\begin{bmatrix} -0.10 & -0.23 \\ 5.83 & 3.60 \end{bmatrix}$

12. a) $\begin{bmatrix} \frac{3\sqrt{3}}{2} - \frac{1}{2} & \frac{\sqrt{3}}{2} - \frac{5}{2} \\ \frac{3}{2} + \frac{\sqrt{3}}{2} & \frac{1}{2} + \frac{5\sqrt{3}}{2} \end{bmatrix}$

b) $\begin{bmatrix} \frac{5}{2} - \frac{3\sqrt{3}}{2} & \frac{3}{2} - \sqrt{3} \\ \frac{5\sqrt{3}}{2} + \frac{3}{2} & \frac{3\sqrt{3}}{2} + 1 \end{bmatrix}$

13. b) $R^tR = R^{-1}R = I$

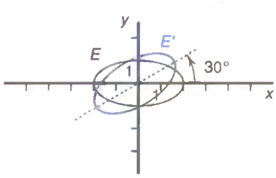
8.5 Exercises, page 369

1. a) $[x \quad y]\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [4]$

b) $[x \quad y]\begin{bmatrix} 1.8 & -1.3 \\ -1.3 & 3.3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [4]$

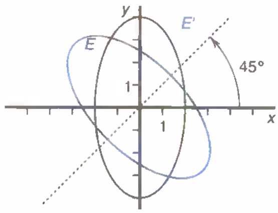
c) $1.8x^2 - 2.6xy + 3.3y^2 = 4$

d)

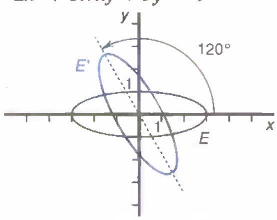


2. a) $2.5x^2 + 3xy + 2.5y^2 = 16$

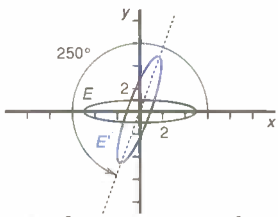
b)



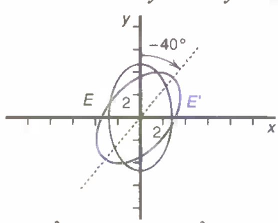
3. a) $2x^2 + 6.9xy + 3y^2 = 9$



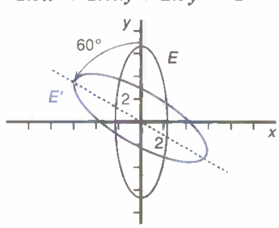
b) $22.5x^2 - 13.5xy + 6.5y^2 = 100$



c) $18.4x^2 - 15.8xy + 15.6y^2 = 200$



d) $1.5x^2 + 1.7xy + 2.5y^2 = 4$



5. a) $\frac{5}{2}x^2 + 3xy + \frac{5}{2}y^2 = 16$

See 2b) for sketch.

b) $3x^2 - 4\sqrt{3}xy + 7y^2 = 9$

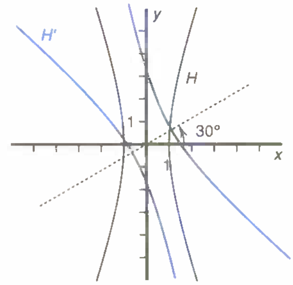
See 3a) for sketch.

c) $\frac{3}{2}x^2 + \sqrt{3}xy + \frac{5}{2}y^2 = 4$

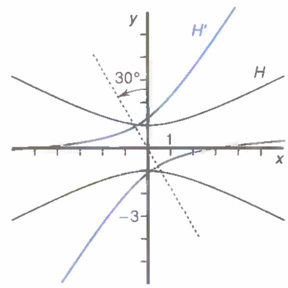
See 3d) for sketch.

6. a) $6.5x^2 + 8.6xy - 1.5y^2 = 9$

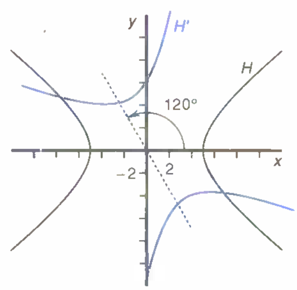
b)



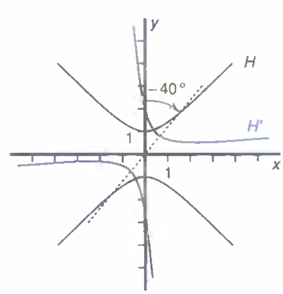
7. a) $-0.3x^2 + 4.3xy - 2.8y^2 = -4$



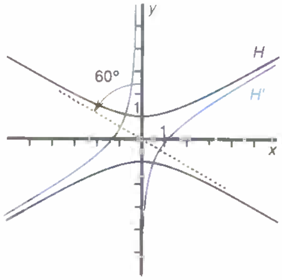
b) $-14.8x^2 - 35.5xy + 5.8y^2 = 400$



c) $-0.2x^2 - 2.0xy + 0.2y^2 = -1$



d) $-2x^2 + 3.5xy = -3$



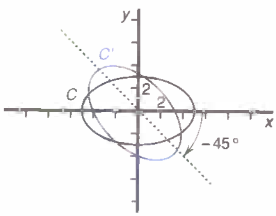
9. a) $\frac{13}{2}x^2 + 5\sqrt{3}xy - \frac{3}{2}y^2 = 9$
See 6b) for sketch.

b) $-\frac{59}{4}x^2 - \frac{41\sqrt{3}}{2}xy + \frac{23}{4}y^2 = 400$
See 7b) for sketch.

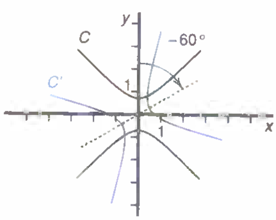
c) $-2x^2 + 2\sqrt{3}xy = -3$
See 7d) for sketch.

10. a) $9x^2 + 4y^2 = 36$
b) $-9x^2 + y^2 = 9$
c) $x^2 + 4y^2 = 16$

12. a) ellipse
b) $9x^2 + 25y^2 = 225$
c)



13. a) hyperbola
b) $2x^2 - 2y^2 = -1$
c)



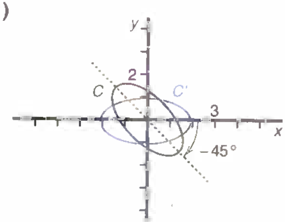
14. Since $f \neq 0$ and $g \neq 0$, the equation cannot be written $ax^2 + 2hxy + by^2 = k$

8.6 Exercises, page 378

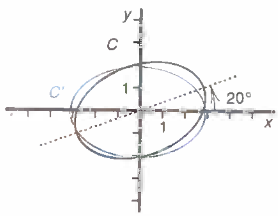
1. a) ellipse
b) 67.5°

2. a) hyperbola, 13.3°
b) ellipse, 45°
c) ellipse, −38.0°
d) hyperbola, −41.8°
e) hyperbola, 39.2°
f) ellipse, 35.8°

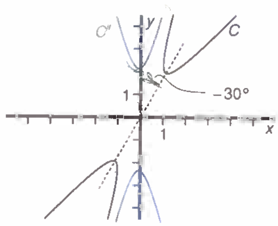
3. a) ellipse
b) 45°
c) $x^2 + 4y^2 = 4$
d)



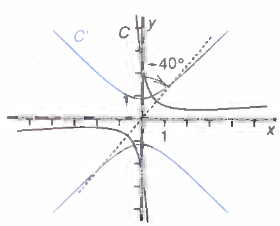
4. a) ellipse, −20°
$4x^2 + 9y^2 = 36$



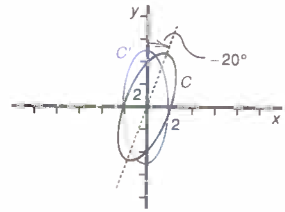
b) hyperbola, 30.0°
$16x^2 - y^2 = -5$



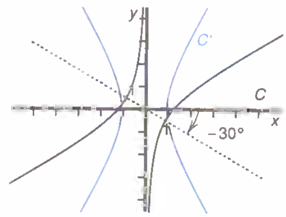
c) hyperbola, 40.1°
$x^2 - y^2 = -1$



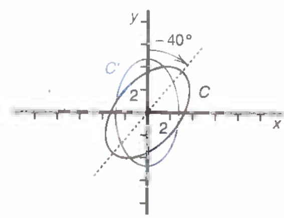
d) ellipse, 20.1°
$25x^2 + 4y^2 = 100$



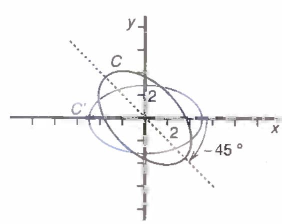
e) hyperbola, 30.0°
$3x^2 - y^2 = 3$



f) ellipse, 40.0°
$25x^2 + 9y^2 = 200$

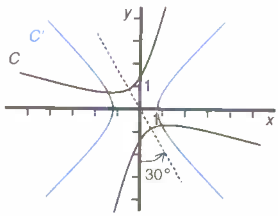


5. a) ellipse
b) 45°
c) $9x^2 + 25y^2 = 225$
d)

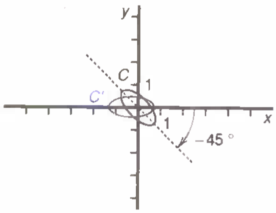


8.6 Exercises, page 378, continued

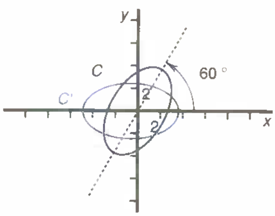
6. a) hyperbola 30°
$x^2 - y^2 = 1$



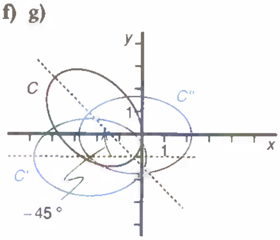
b) ellipse, 45°
$4x^2 + 9y^2 = 4$



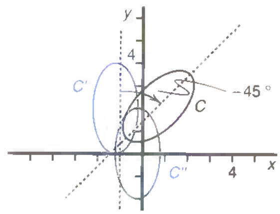
c) ellipse, −60°
$x^2 + 3y^2 = 18$



8. a) ellipse
b) 45°
c) $4x^2 + 9y^2 + 16x + 18y = 0$
d) $(x,y) \rightarrow (x + 2, y + 1)$
$4x^2 + 9y^2 = 25$
e) f) g)



9. a) ellipse
b) 45°
c) $4x^2 + y^2 + 8x - 4y + 4 = 0$
d) $(x,y) \rightarrow (x + 1, y - 2)$
$4x^2 + y^2 = 4$
e) f) g)



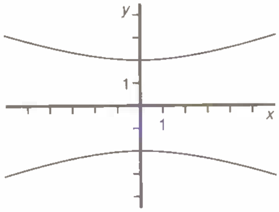
10. same answers as 9

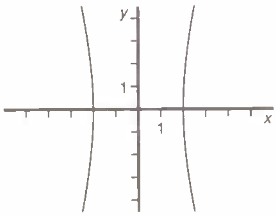
11.
```
100 REM PROGRAM TO ELIMINATE XY TERM FROM AX∧2+2XY+BY∧2=K
110 PRINT "STATE THE VALUE OF A"
120 INPUT A
130 PRINT "STATE VALUE OF H"
140 INPUT H
150 PRINT "STATE VALUE OF B"
160 INPUT B
170 PRINT "STATE VALUE OF K"
180 INPUT K
184 IF A< >B THEN GOTO 190
185 IF A=B GOTO 186
186 T=3.1415925/4
188 GOTO 200
190 T=0.5* ATN (2*H/(B-A))
200 PRINT "ANGLE OF ROTATION IS "T*180/3.14158265" DEGREES"
210 A1=(A+B)/2+(A-B)/2* COS(2*T)-H* SIN(2*T)
220 B1=(A+B)/2-(A-B)/2* COS(2*T)+H* SIN(2*T)
230 H1=(A-B)/2* SIN(2*T) +H* COS(2*T)
240 PRINT "H="H1
250 PRINT "AN EQUATION OF THE CONIC IN STANDARD POSITION IS"
260 PRINT A1"X∧2+"B1"Y∧2="K
270 STOP
```

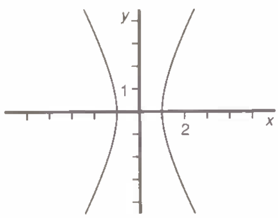
Inventory, page 381

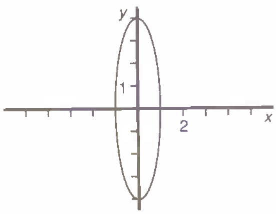
1. circle; centre; radius
2. ellipse; centre; ±2; ±3
3. hyperbola; centre; ±2; not real
4. hyperbola; centre; not real; ±3
5. parabola; vertex; (positive) y-axis
6. parabola; vertex; (positive) x-axis
7. translation; square; $6x$; $9x^2$
8. 3; 9
9. $(x - 2, y - 3)$
10. $ab - h^2$; greater than; $ab - h^2$; less than
11. rotation; xy
12. 2θ; $\frac{2h}{b - a}$; not equal to; equals; 135°
13. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$; $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$
14. 4; 5; 3; $\begin{bmatrix} x \\ y \end{bmatrix}$; $\begin{bmatrix} 4 & 3 \\ 3 & 5 \end{bmatrix}$; [7]
15. $\begin{bmatrix} \cos 20^\circ & -\sin 20^\circ \\ \sin 20^\circ & \cos 20^\circ \end{bmatrix}$, $\begin{bmatrix} 5 & 3 \\ 3 & 7 \end{bmatrix}$

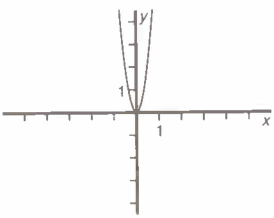
Review Exercises, pages 382–383

1. a) circle centre (0,0) radius 2
   b)


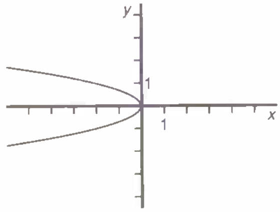
   c)


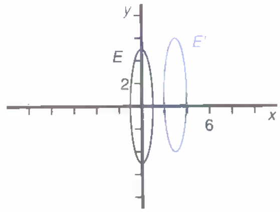
   d)


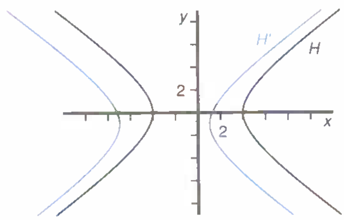
   e)


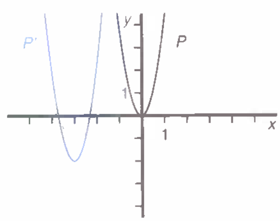
   f)


   g) circle centre (0,0) radius $\frac{5}{3}$

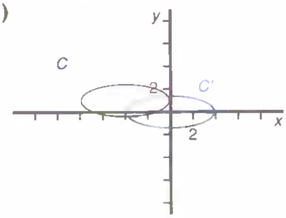
   h)


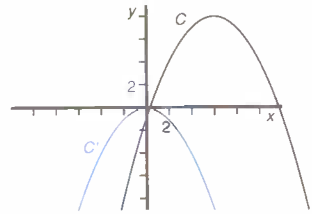
2. a) $25x^2 + y^2 - 150x - 2y + 201 = 0$
   b)


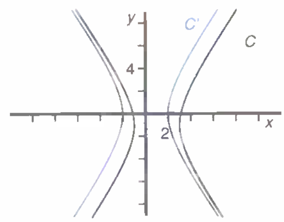
3. a) $9x^2 - 16y^2 + 54x - 32y - 79 = 0$


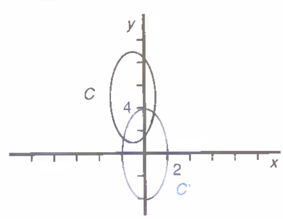
   b) $x^2 + y^2 + 4x - 6y - 3 = 0$
   C: circle centre (0,0) radius 4
   C': circle centre (−2,3) radius 4
   c) $4x^2 + 24x - y + 34 = 0$


4. a) hyperbola
   b) circle
   c) parabola
   d) ellipse
   e) hyperbola

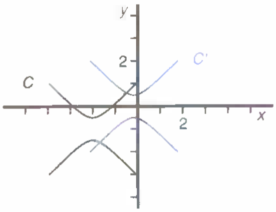
5. a) ellipse
   b) $(x,y) \rightarrow (x + 4, y - 1)$
   c) $x^2 + 8y^2 = 15$
   d)


6. a) circle
   $(x,y) \rightarrow (x + 3, y + 6)$
   $x^2 + y^2 = 47$
   C: circle centre (−3,−6) radius $\sqrt{47}$
   C': circle centre (0,0) radius $\sqrt{47}$
   b) parabola
   $(x,y) \rightarrow (x - 6, y - 8)$
   $x^2 + 4y = 0$


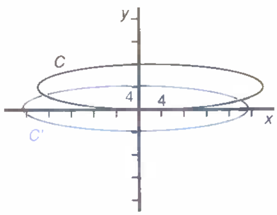
   c) hyperbola
   $(x,y) \rightarrow (x - 1, y + 1)$
   $9x^2 - 4y^2 = 36$


   d) ellipse
   $(x,y) \rightarrow (x + 1, y - 5)$
   $4x^2 + y^2 = 16$


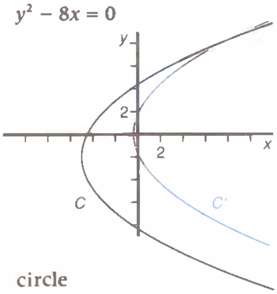
e) hyperbola
$(x,y) \rightarrow (x+2, y+1)$
$4x^2 - 4y^2 = -1$



f) ellipse
$(x,y) \rightarrow (x-2, y-4)$
$x^2 + 25y^2 = 400$



g) parabola
$(x,y) \rightarrow (x+5, y+2)$
$y^2 - 8x = 0$



h) circle
$(x,y) \rightarrow (x-4, y+3)$
$x^2 + y^2 = 26$
C: circle centre $(4,-3)$
radius $\sqrt{26}$
C': circle centre $(0,0)$
radius $\sqrt{26}$

7. a) $M = \begin{bmatrix} 5 & -3 \\ -3 & 4 \end{bmatrix}, K = [7]$

b) $M = \begin{bmatrix} 3 & -3 \\ -3 & -1 \end{bmatrix}, K = [1]$

c) $M = \begin{bmatrix} 2 & 1.5 \\ 1.5 & -1 \end{bmatrix}, K = [9]$

d) $M = \begin{bmatrix} 3 & 1 \\ 1 & -5 \end{bmatrix}, K = [8]$

e) $M = \begin{bmatrix} 4 & 0 \\ 0 & -12 \end{bmatrix}, K = [15]$

f) $M = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}, K = [13]$

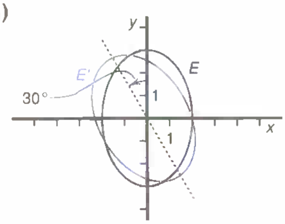
8. $[x \quad y]\, M \begin{bmatrix} x \\ y \end{bmatrix} = K,$
using M, K from question 7.

9. a) $4x^2 - 6xy + 9y^2 = k$
b) $4x^2 - y^2 = k$
c) $-7x^2 - 8xy + 3y^2 = k$
d) $17x^2 + 20xy + 7y^2 = k$

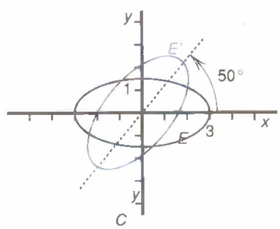
10. a) $\begin{bmatrix} 4 & -3 \\ 5 & 9 \end{bmatrix}$
b) $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$

11. a) $\begin{bmatrix} 2 & 0 \\ x & y \end{bmatrix}$
b) $x = 0, y \in \mathbb{R}$

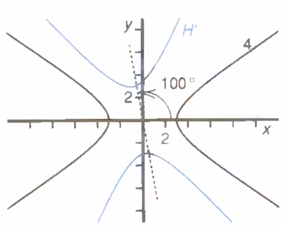
12. a) $[x \quad y]\begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [36]$
b) $\begin{bmatrix} 7.8 & 2.2 \\ 2.2 & 5.3 \end{bmatrix}$
c) $7.8x^2 + 4.4xy + 5.3y^2 = 36$
d)



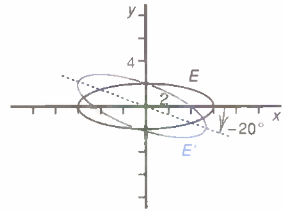
13. a) $2.8x^2 - 3.0xy + 2.2y^2 = 9$



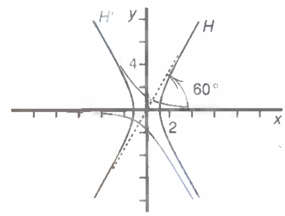
b) $-8.6x^2 - 4.4xy + 3.6y^2 = 36$



c) $1.9x^2 - 5.1xy - 8.0y^2 = 36$



d) $3.5xy + 2y^2 = 4$



14. a) $[x \quad y]\begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [36]$

$\begin{bmatrix} \frac{31}{4} & \frac{5\sqrt{3}}{4} \\ \frac{5\sqrt{3}}{4} & \frac{21}{4} \end{bmatrix}$

$\frac{31}{4}x^2 + \frac{5\sqrt{3}}{2}xy + \frac{21}{4}y^2 = 36$

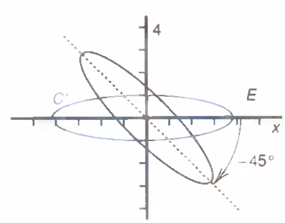
b) See sketch for 12d)

15. a) The circle maps to itself.
b) no

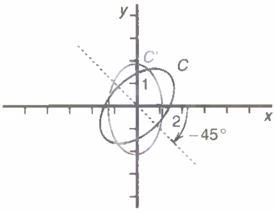
17. a) ellipse
b) 13.3°

18. a) hyperbola, −7.0°
b) ellipse, 22.5°
c) ellipse, −22.5°
d) ellipse, 45°
e) hyperbola, −41.4°
f) ellipse, 18.4°

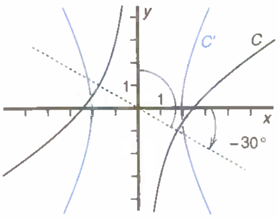
19. a) ellipse
b) 45°
c) $x^2 + 16y^2 = 16$
d)



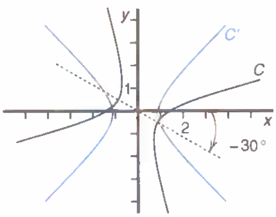
20. a) ellipse, 45°
$3x^2 + y^2 = 4$



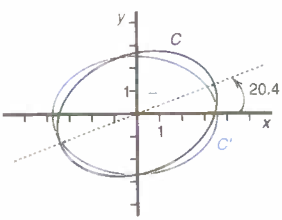
b) hyperbola, 30.0°
$149.9x^2 - 49.9y^2 = 600$



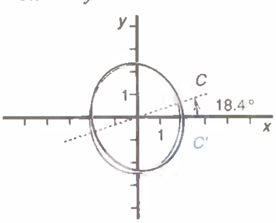
c) hyperbola, 29.9°
$39.9x^2 - 39.9y^2 = 40$



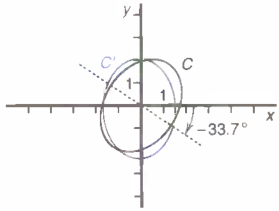
d) ellipse, −20.4°
$26.8x^2 + 53.2y^2 = 360$



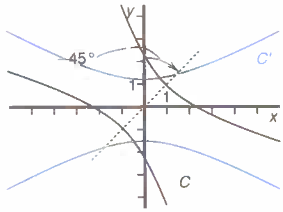
e) ellipse, −18.4°
$3x^2 + 2y^2 = 12$



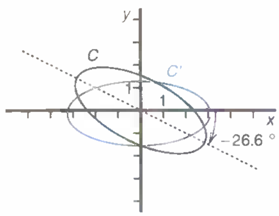
f) ellipse, 33.7°
$2x^2 + y^2 = 5$



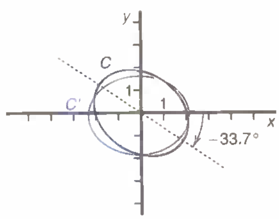
g) hyperbola, 45°
$-4.0x^2 + 14y^2 = 26$



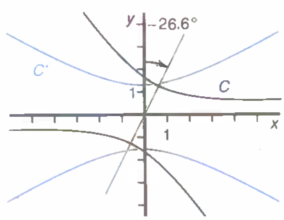
h) ellipse, 26.6°
$x^2 + 5y^2 = 10$



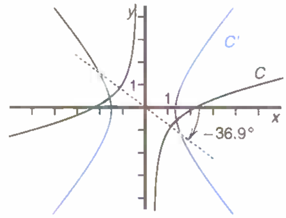
i) ellipse, 33.7°
$3x^2 + 5y^2 = 15$



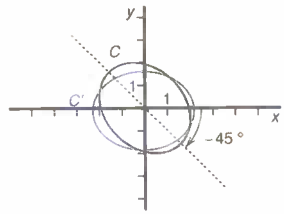
j) hyperbola, 26.6°
$x^2 - 3y^2 = -6$



k) hyperbola, 36.9°
$3x^2 - 2y^2 = 6$

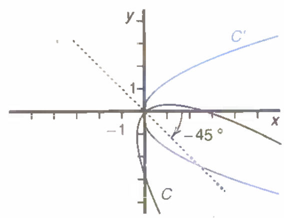


l) ellipse, 45°
$3x^2 + 5y^2 = 15$



21. a) See answer 19.
b) See answer 4b, 8.6 Exercises.

22. a) 45°
b) $y^2 = 2x$
c) parabola
d)



Chapter Nine Mathematical Induction

9.1 Exercises pages 390–391

1. $\frac{1}{2^n}$
2. a) $2^{n+1} - 2$
   b) $2^n - 1$
3. a) 1, 3, 6, 10
   b) 2, 3, 4, 5
   c) $\frac{n(n+1)}{2}$
4. $\frac{n}{2n+1}$
5. a) $\frac{n}{4n+1}$
   b) $\frac{n}{2n+4}$
   c) $1 - \frac{1}{2^n}$
6. a) 3
   b) 4
   c) 5
   d) 6
   e) $n + 1$
7. a) $\frac{n+2}{2(n+1)}$
   b) $(n+1)^2$
8. a) 2, 8, 20, 40
   b) all $n \in \mathbb{N}$
9. all $n \in \mathbb{N}$
10. a) all
    b) all
    c) all
    d) all
    e) even $n \in \mathbb{W}$
    f) even $n \in \mathbb{W}$
11. $n \geqslant 4$
12. a) $n \geqslant 7$
    b) $n \geqslant 2$
    c) all n
13. a) 1, 9, 36, 100, 225, 441, 784
    b) 1, 3, 6, 10, 15, 21, 28
    c) $\left[\frac{n(n+1)}{2}\right]^2$
14. a) 3
    b) 6
    c) 10
    d) $\frac{n(n-1)}{2}$
15. a) 2
    b) 5
    c) 9, 14
    d) $\frac{n(n-3)}{2}$
16. a) 4
    b) 7
    c) 11
    d) $\frac{n^2+n+2}{2}$

9.2 Exercises, page 395

1. *Step 1* Show the statement is true for $n = 1$.
   Step 2 Assume the statement is true for $n = k$.
   Step 3 Prove the statement is true for $n = k + 1$, using result of step 2.
5. a) 43, 47, 53, 61
   b) $41^2 + 41 + 41 = 41(41 + 1 + 1) = 41 \times 43$
   c) steps 2 and 3
6. a) step 1
   b) no

9.3 Exercises, page 398

No answers provided.

9.4 Exercises, page 405

1. a)

```
            1 1
           1 2 1
          1 3 3 1
         1 4 6 4 1
       1 5 10 10 5 1
      1 6 15 20 15 6 1
    1 7 21 35 35 21 7 1
  1 8 28 56 70 56 28 8 1
```

   b) 24, 720
   c) 35
   d) 1, 5, 10, 10, 5, 1
2. a) $C(4,0)a^4x^0 + C(4,1)a^3x^1 + C(4,2)a^2x^2 + C(4,3)a^1x^3 + C(4,4)a^0x^4$
   b) $C(5,0)a^5x^0 + C(5,1)a^4x^1 + C(5,2)a^3x^2 + C(5,3)a^2x^3 + C(5,4)a^1x^4 + C(5,5)a^0x^5 +$
   c) $C(6,0)a^6x^0 + C(6,1)a^5x^1 + C(6,2)a^4x^2 + C(6,3)a^3x^3 + C(6,4)a^2x^4 + C(6,5)a^1x^5 + C(6,6)a^0x^6$
   d) $C(7,0)a^7x^0 + C(7,1)a^6x^1 + C(7,2)a^5x^2 + C(7,3)a^4x^3 + C(7,4)a^3x^4 + C(7,5)a^2x^5 + C(7,6)a^1x^6 + C(7,7)a^0x^7$
   e) $C(8,0)a^8x^0 + C(8,1)a^7x^1 + C(8,2)a^6x^2 + C(8,3)a^5x^3 + C(8,4)a^4x^4 + C(8,5)a^3x^5 + C(8,6)a^2x^6 + C(8,7)a^1x^7 + C(8,8)a^0x^8$
   f) $C(9,0)a^9x^0 + C(9,1)a^8x^1 + C(9,2)a^7x^2 + C(9,3)a^6x^3 + C(9,4)a^5x^4 + C(9,5)a^4x^5 + C(9,6)a^3x^6 + C(9,7)a^2x^7 + C(9,8)a^1x^8 + C(9,9)a^0x^9$
3. a) $a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$
   b) $a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$
   c) $a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6$
   d) $a^7 + 7a^6x + 21a^5x^2 + 35a^4x^3 + 35a^3x^4 + 21a^2x^5 + 7ax^6 + x^7$
   e) $a^8 + 8a^7x + 28a^6x^2 + 56a^5x^3 + 70a^4x^4 + 56a^3x^5 + 28a^2x^6 + 8ax^7 + x^8$
   f) $a^9 + 9a^8x + 36a^7x^2 + 84a^6x^3 + 126a^5x^4 + 126a^4x^5 + 84a^3x^6 + 36a^2x^7 + 9ax^8 + x^9$
4. a) $a^4 + 4a^3y + 6a^2y^2 + 4ay^3 + y^4$
   b) $b^4 - 4b^3c + 6b^2c^2 - 4bc^3 + c^4$
   c) $m^3 + 3m^2z + 3mz^2 + z^3$
   d) $32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$
   e) $a^8 + 8a^7 + 28a^6 + 56a^5 + 70a^4 + 56a^3 + 28a^2 + 8a + 1$
   f) $81 - 108b + 54b^2 - 12b^3 + b^4$
5. a) $a^4 + 8a^3b + 24a^2b^2 + 32ab^3 + 16b^4$
   b) $81a^4 + 432a^3b + 864a^2b^2 + 768ab^3 + 256b^4$
   c) $27 - 54m + 36m^2 - 8m^3$
   d) $64a^3 - 240a^2 + 300a - 125$
   e) $32x^5 + 240x^4a + 720x^3a^2 + 1080x^2a^3 + 810xa^4 + 243a^5$
   f) $1 - 6m^2 + 15m^4 - 20m^6 + 15m^8 - 6m^{10} + m^{12}$
6. a) $C(40,0)a^{40}b^0 + C(40,1)a^{39}b^1 + C(40,2)a^{38}b^2 + C(40,3)a^{37}b^3$
   b) $C(34,0)m^{34}(-k)^0 + C(34,1)m^{33}(-k)^1 + C(34,2)m^{32}(-k)^2 + C(34,3)m^{31}(-k)^3$
   c) $C(23,0)3^{23}x^0 + C(23,1)3^{22}x^1 + C(23,2)3^{21}x^2 + C(23,3)3^{20}x^3$

d) $C(85,0)4^{85}(2a)^0 +$
$C(85,1)4^{84}(2a)^1 +$
$C(85,2)4^{83}(2a)^2 +$
$C(85,3)4^{82}(2a)^3$

e) $C(25,0)(2m)^{25}(-3t)^0 +$
$C(25,1)(2m)^{24}(-3t)^1 +$
$C(25,2)(2m)^{23}(-3t)^2 +$
$C(25,3)(2m)^{22}(-3t)^3$

f) $C(36,0)1^{36}(b^2)^0 +$
$C(36,1)1^{35}(b^2)^1 +$
$C(36,2)1^{34}(b^2)^2 +$
$C(36,3)1^{33}(b^2)^3$

7. a) $x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$

b) $x^5 - 10x^2 + \frac{40}{x} - \frac{80}{x^4} + \frac{80}{x^7} - \frac{32}{x^{10}}$

8. a) $C(6,k)x^{12-3k}$
b) $6x^9$
c) 15

11. 4

14. a) $1 - 2x + 3x^2 - 4x^3$
b) $1 + \frac{x}{2} - \frac{x^3}{8} + \frac{x^4}{4}$

Inventory, page 409

1. 1
2. k; in step 3 prove; $k + 1$
3. a) sometimes
b) sometimes
4. a) 1; 1
b) $1 + 3 + 5 + \ldots + (2k - 1)$; k^2
c) $1 + 3 + 5 + \ldots + (2k + 1)$; $(k + 1)^2$
5. a) 2; 2
b) $(1 + 1)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\ldots\left(1 + \frac{1}{k}\right)$; $k + 1$
c) $(1 + 1)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\ldots\left(1 + \frac{1}{k+1}\right)$; $k + 2$
6. a) $f(1) = \frac{6}{3} = 2$
is a natural number
b) $f(k) = \frac{k^3 + 3k^2 + 2k}{3}$
is a natural number
c) $f(k + 1) = \frac{(k + 1)^3 + 3(k + 1)^2 + 2(k + 1)}{3}$
is a natural number

7. a) $2^4 < 4!$
b) $2^k < k!$
c) $2^{k+1} < (k + 1)!$

8. 6;
$C(5,0)a^5x^0 + C(5,1)a^4x^1 +$
$C(5,2)a^3x^2 + C(5,3)a^2x^3 +$
$C(5,4)a^1x^4 + C(5,5)a^0x^5$

9. 1, 9, 36, 84, 126, 84, 36, 9, 1

Review Exercises, pages 410–411

1. *Step 1* Show the statement is true for $n = 1$.
Step 2 Assume the statement is true for $n = k$.
Step 3 Prove the statement is true for $n = k + 1$, using result of step 2.

3. n^3

10. step 1

13. b) not true for $n = 1$

15. a) $a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$
b) $243 + 405b + 270b^2 + 90b^3 + 15b^4 + b^5$
c) $8 + 12x^2 + 6x^2 + x^3$
d) $64k^6 - 960k^5m + 6000k^4m^2 - 20\,000k^3m^3 + 37\,500k^2m^4 - 37\,500km^5 + 15\,625m^6$

16. not true for $n = 4$

21. 7

23. i) $\frac{1}{2}n(n + 1)$
iii) $\frac{1}{6}(n + 1)(n + 2)(2n + 3) - 5$

Chapter Ten
Complex Numbers

10.1 Exercises, page 418

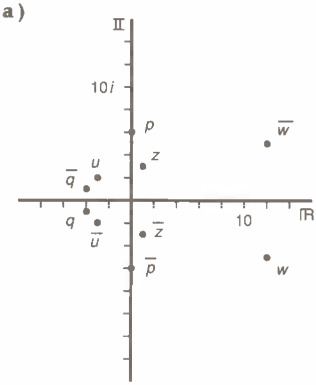
1. a) $9 + 3i$
   b) 1
   c) $17 + 7i$
   d) 10
   e) $-3 + 4i$
   f) $-3 - 4i$
   g) $-4 + 6i$
   h) $-11 - 2i$
2. $\frac{7 \pm 5i}{2}$
3. $2 \pm i$
4. $-\frac{1}{2} \pm i\frac{\sqrt{7}}{2}$
5. a) $4 \pm 3i$
   b) $-2 \pm i$
   c) $\frac{5 \pm i\sqrt{59}}{6}$
7. $(4 + 3i) + (4 - 3i) = 8$
   $(4 + 3i)(4 - 3i) = 25$
   order does not matter
8. sum $-\frac{b}{a}$; product $\frac{c}{a}$
9. a) 0 or $4i$
   b) $4i$ or $-i$
   c) $i\left(\frac{1 \pm \sqrt{13}}{2}\right)$
   d) $2i$ or $1 - i$
10. a) 4,6
    b) 0,7
    c) 9,7
    d) 26,0
    e) 0,2
    f) 7,−24
11. $z + w = (a + c) + i(b + d)$
    $= w + z$
    $\Rightarrow$ commutative
12. $(z + w) + u = (a + c + e) + i(b + d + f)$
    $= z + (w + u)$
    $\Rightarrow$ associative
13. $zw = (ac - bd) + i(ad + bc)$
    $= wz$
    $\Rightarrow$ commutative
14. $(zw)u = (ace - bde - adf - bcf) + i(ade + bce + acf - bdf)$
    $= z(wu)$
    $\Rightarrow$ associative
15. $zw + zu = (ac - bd + ae - bf) + i(ad - bc + af - be)$
    $= z(w + u)$
    $\Rightarrow$ distributive over addition

10.2 Exercises, page 422

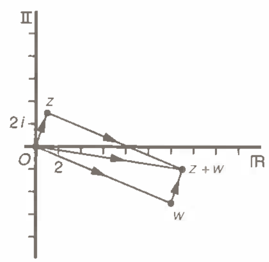
1. a) $-i$
   b) 1
   c) i
   d) -1
   e) $-i$
   f) -1
   g) 1
   h) i
   j) -1
2. a) $1 - 4i$
   b) $-2 + 2i$
   c) $6i$
   d) $-2 + 16i$
   e) $52 - 8i$
3. a) $4 - i$
   b) $\frac{6}{5} + \frac{7}{5}i$
   c) $3 + 7i$
   d) $\frac{21}{29} + \frac{20}{29}i$
   e) $\frac{2}{13} - \frac{3}{13}i$
   f) $\frac{4}{25} + \frac{3}{25}i$
   g) $\frac{6}{25}$
   h) $\frac{355}{3721} - \frac{245}{3721}i$
5. c) Yes.
8. b) $\frac{42}{13} - \frac{24}{13}i$
9. $-\frac{b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a}$
   $\Rightarrow$ conjugates
10. a) $x = 17, y = 0$
    b) $x = \frac{4}{17}, y = -\frac{18}{17}$
11. a) $\text{Re}(z) = \frac{7}{13}$
    $\text{Im}(z) = 1$
    b) $\text{Re}(z) = -4$
    $\text{Im}(z) = 0$
12. $\text{Re}(z) = 3, \text{Im}(z) = 3$
    $\text{Re}(z^2) = 0, \text{Im}(z^2) = 18$
13. 10 000
14. $-\frac{3\sqrt{5}}{2} + 5i$ or $\frac{3\sqrt{5}}{2} + 7i$
15. $\cos\alpha \pm i \sin\alpha$

10.3 Exercises, page 429

1. a)



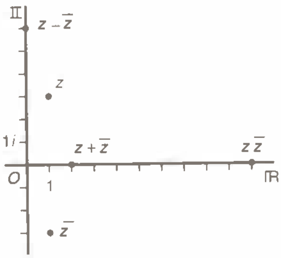
   b) $\bar{z} = 1 - 3i$
   $\bar{w} = 12 + 5i$
   $\bar{p} = -6i$
   $\bar{q} = -4 + i$
   $\bar{u} = -3 - 2i$
2. a) $|z| = \sqrt{10}$
   $|w| = 13$
   $|p| = 6$
   $|q| = \sqrt{17}$
   $|u| = \sqrt{13}$
   b) $\arg z = 72°$
   $\arg w = -23°$
   $\arg p = 90°$
   $\arg q = -166°$
   $\arg u = 146°$
3. a) $|\bar{z}| = \sqrt{10}, |\bar{w}| = 13$
   b) $\arg \bar{z} = -72°, \arg \bar{w} = 23°$
   c) same modulus, opposite argument,
4. a) impossible
   b) $|z| < |u| < |q| < |p| < |w|$
5. a) $z + w = 13 - 2i$
   b)



   c) numbers represented by vectors, as shown

6. a) $z + \bar{z} = 2$
$z - \bar{z} = 6i$
$z\bar{z} = 10$

b)

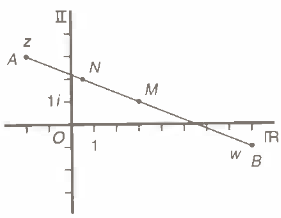


7. a) If $a \in \mathbb{R}$,
$|a| = \sqrt{a^2} = \pm a$
$\arg a = 0°$ or $180°$
$\bar{a} = a$

b) If $b \in \mathbb{I}$,
$|b| = \sqrt{-b^2}$
$\arg b = \pm 90°$
$\bar{b} = -b$

8. $|z| = 1, |\bar{w}| = 3$

9. a)



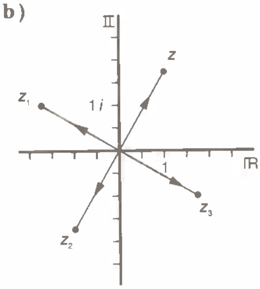
b) $3 + i$

c) $\frac{1}{2} + 2i$

d) M midpoint of AB,
N divides AB in ratio 1 : 3

10. a) $z_1 = -\sqrt{3} + i$
$z_2 = -1 - i\sqrt{3}$
$z_3 = \sqrt{3} - i$

b)

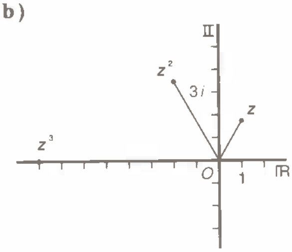


c) $|z| = |z_1| = |z_2| = |z_3| = 2$
$\arg z = 60°$
$\arg z_1 = 150°$
$\arg z_2 = 240°$
$\arg z_3 = 330°$

d) rotates counterclockwise about O, by 90°

11. a) $z^2 = -2 + 2i\sqrt{3}$
$z^3 = -8$

b)



c) It is true that
$(1 + i\sqrt{3})^3 = -8$, but
$(-2)^3 = -8$ also!

12. a) $|z| = 3\sqrt{2}$
$\arg z = 45°$

10.4 Exercises, page 433

1. a) 2 or −3
b) $1 \pm i$
c) $\frac{1}{4}$ or $\pm i$
d) 0 or $-2i$ or $\frac{3}{2} + 2i$

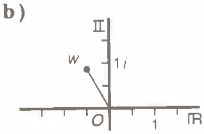
2. a) b) c)

3. a) $z^2 - (2 + 5i)z - 4 + 8i = 0$
b) $z^2 - 2pz + p^2 + q^2 = 0$

5. a) $z^3 - (3 + 2i)z^2 + (13 + 7i)z - 20(1 + i) = 0$
b) $z^3 - 2pz^2 + (p^2 + q^2)z = 0$

7. a) $w^3 = 1$

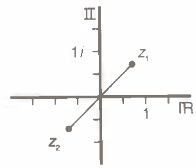
b)



8. $x = \frac{7}{34}, y = -\frac{11}{34}$

9. $x = -\frac{(ac + bd)}{a^2 + b^2}, y = \frac{bc - ad}{a^2 + b^2}$;
yes

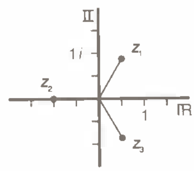
10. $z_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ or
$z_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$



11. $\frac{3 + 3\sqrt{2}}{2} + i\frac{(1 + 3\sqrt{2})}{2}$ or
$\frac{3 - 3\sqrt{2}}{2} + i\frac{(1 - 3\sqrt{2})}{2}$

12. a) $z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$
$z_2 = -1$
$z_3 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

b)

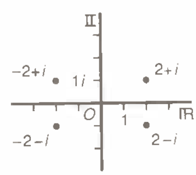


13. a) true in $\mathbb{R}$, but in $\mathbb{C}$, $z = \pm iw$
b) true in $\mathbb{R}$ only; in $\mathbb{C}$, $z = w$ or
$z = \frac{-w \pm iw\sqrt{3}}{2}$

14. b) $\frac{2}{3}$ or $-\frac{1}{2}$ or $2 \pm i$

15. a) real coefficients, so
root $2 + i \Rightarrow$ root $2 - i$
root $-2 + i \Rightarrow$ root $-2 - i$

b)

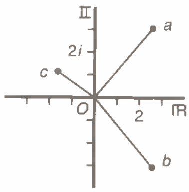


16. $z = r$ is the only real root.

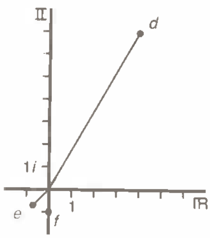
17. a) $r = 4, s = -5$
b) 5

10.5 Exercises, page 439

1. a) $a = 2.57 + 3.06i$
$b = 2.57 - 3.06i$
$c = -1.64 + 1.15i$



2. $d = 4 + 4i\sqrt{3}$
$e = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$
$f = -i$

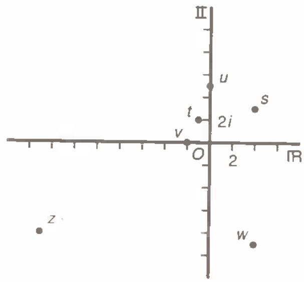


3. a) $|z| = 3, \arg z = \frac{\pi}{2}$
b) $|z| = 4, \arg z = 0$
c) $|z| = 17, \arg z = \pi$
d) $|z| = 1, \arg z = -\frac{\pi}{2}$

4. a) $|\bar{z}| = 3, \arg \bar{z} = -\frac{\pi}{2}$
b) $|\bar{z}| = 4, \arg \bar{z} = 0$
c) $|\bar{z}| = 17, \arg \bar{z} = \pi$
d) $|\bar{z}| = 1, \arg \bar{z} = \frac{\pi}{2}$

5. a) 115°
b) 65°
c) $-\frac{2\pi}{3}$
d) $-\frac{\pi}{6}$

6. $s = 5(\cos 37° + i \sin 37°)$
$t = \sqrt{5}(\cos 117° + i \sin 117°)$
$u = 5(\cos 90° + i \sin 90°)$
$v = 2(\cos 180° + i \sin 180°)$
$z = 17(\cos 152° - i \sin 152°)$
$w = \sqrt{97}(\cos 66° - i \sin 66°)$



7. a) $\sqrt{2}\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$
b) $2\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$
c) $4\sqrt{3}\left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3}\right)$
d) $2\sqrt{3}\left(\cos \frac{5\pi}{6} - i \sin \frac{5\pi}{6}\right)$

8. 0 or π

9. a) $r = 5, \theta = 30°$
b) $r = 6, \theta = -32°$
c) $r = 1, \theta = -\frac{\pi}{8}$

11. a) $50(\cos 105° + i \sin 105°)$
b) $2(\cos 37° + i \sin 37°)$
c) $\frac{1}{2}(\cos 37° - i \sin 37°)$

12. a) $z = \sqrt{41}(\cos 51° - i \sin 51°)$
$w = \sqrt{13}(\cos 124° + i \sin 124°)$
b) $zw = \sqrt{533}(\cos 73° - i \sin 73°)$
$\frac{z}{w} = \sqrt{\frac{41}{13}}(\cos 175° - i \sin 175°)$
$\frac{w}{z} = \sqrt{\frac{13}{41}}(\cos 175° + i \sin 175°)$

13. $|z| = 2, \arg z = \frac{2\pi}{3}$
$|w| = 4, \arg w = \frac{\pi}{6}$

14. a) $4\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$
b) $16\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$
c) $8\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$
d) $2\left(\cos \frac{\pi}{2} - i \sin \frac{\pi}{2}\right)$

15. a) $|i| = 1, \arg i = \frac{\pi}{2}$
b) length unchanged;
rotates about origin, by $\frac{\pi}{2}$

16. a) $\arg(z^2) = 2\theta$

10.6 Exercises, page 443

1. a) $-0.766 - 0.643i$
b) $-365 - 631i$
c) $-928 - 433i$

2. a) 65 536
b) 4096
c) -1
d) $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$

3. a) $-0.766 + 0.643i$
b) $-0.000\,686 + 0.001\,19i$
c) $-0.000\,885 + 0.000\,413i$

4. a) $\frac{1}{2^{16}}$
b) $\frac{1}{2^{12}}$
c) -1
d) $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$

5. a) -1024
b) $-32i$

6. a) -1
b) 1

7. $\cos 4\theta =$
$\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta$
$\sin 4\theta =$
$4 \cos \theta \sin \theta(\cos^2 \theta - \sin^2 \theta)$

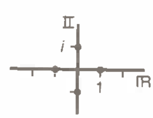
8. d) $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$

9. b) $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$

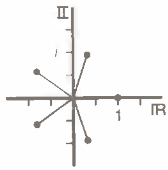
10. c) $\cos 4\theta =$
$8 \cos^4 \theta - 8 \cos^2 \theta + 1$

10.7 Exercises, page 448

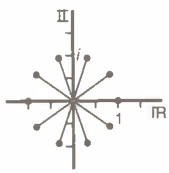
2. a) $w_0 = 1$
$w_1 = i$
$w_2 = -1$
$w_3 = -i$



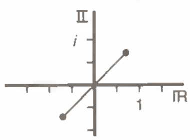
b) $w_0 = 1$
$w_1 = 0.309 + 0.951i$
$w_2 = -0.809 + 0.588i$
$w_3 = -0.809 - 0.588i$
$w_4 = 0.309 - 0.951i$



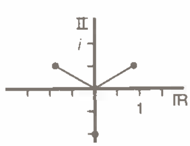
c) $w_0 = 1$
$w_1 = 0.809 + 0.588i$
$w_2 = 0.309 + 0.951i$
$w_3 = -0.309 + 0.951i$
$w_4 = -0.809 + 0.588i$
$w_5 = -1$
$w_6 = -0.809 - 0.588i$
$w_7 = -0.309 - 0.951i$
$w_8 = 0.309 - 0.951i$
$w_9 = 0.809 - 0.588i$



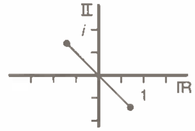
3. a) $w_0 = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$
$w_1 = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$



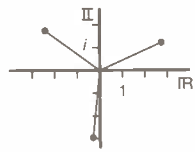
b) $w_0 = \frac{\sqrt{3}}{2} + \frac{i}{2}$
$w_1 = -\frac{\sqrt{3}}{2} + \frac{i}{2}$
$w_2 = -i$



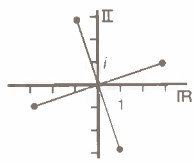
c) $w_0 = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$
$w_1 = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$



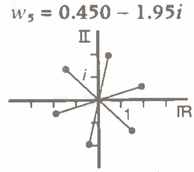
d) $w_0 = 2.74 + 1.22i$
$w_1 = -2.43 + 1.76i$
$w_2 = -0.314 - 2.98i$



e) $w_0 = 2.85 + 0.927i$
$w_1 = -0.927 + 2.85i$
$w_2 = -2.85 - 0.927i$
$w_3 = 0.927 - 2.85i$



f) $w_0 = 1.91 - 0.585i$
$w_1 = 1.46 + 1.36i$
$w_2 = 0.450 + 1.95i$
$w_3 = -1.91 + 0.585i$
$w_4 = -1.46 - 1.36i$
$w_5 = 0.450 - 1.95i$



4. $2\left(\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}\right)$ or $2\left(\cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}\right)$ or 2

5. a) $z \in \left\{1, \cos\frac{2\pi}{5} \pm i\sin\frac{2\pi}{5}, \cos\frac{4\pi}{5} \pm i\sin\frac{4\pi}{5}\right\}$

b) $z^5 - 1 =$
$$(z-1)\left(z - \left[\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right]\right)\left(z - \left[\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}\right]\right)\left(z - \left[\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right]\right)\left(z - \left[\cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}\right]\right)$$

c) $z^5 - 1 =$
$$(z-1)\left(z^2 - 2\cos\frac{2\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5}z + 1\right)$$

6. a) $(z+1)\left(z^2 - 2\cos\frac{\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{3\pi}{5}z + 1\right)$

b) $(z-1)\left(z^2 - 2\cos\frac{2\pi}{7}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{7}z + 1\right)\left(z^2 - 2\cos\frac{6\pi}{7}z + 1\right)$

c) $(z+1)(z-1)\left(z^2 - 2\cos\frac{\pi}{3}z + 1\right)\left(z^2 - 2\cos\frac{2\pi}{3}z + 1\right)$

7. b) other roots are $\sqrt{3} + i, -\sqrt{3} + i, -\sqrt{3} - i$

8. a) The arguments of the non-real roots of unity are the multiples of $\frac{2\pi}{7}$, with coefficients in $S = \{1,2,3,4,5,6\}$ Taking any one of these, and multiplying by the numbers 1-6 yields results as follows. (This is known as multiplication modulo 7.)

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

Each row yields all the members of S.

10.7 Exercises, page 448, continued

b) Let
$1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = z$,
If w is a non-real 7th root,
$w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 = zw$,
or
$w + w^2 + w^3 + w^4 + w^5 + w^6 + 1 = zw$
$\Rightarrow z = zw$
$\Rightarrow z(1 - w) = 0$
$\Rightarrow z = 0$
(since $w \neq 1$)

c) No: a multiplication table as above holds only if n is prime.

9. a) $\frac{w_{k+1}}{w_k} = \cos 72° + i \sin 72°$

b) Each root is the previous root, rotated by 72°.

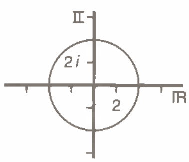
10. a) $(z + 1)\left(z^2 - 2\cos\frac{\pi}{7}z + 1\right)\left(z^2 - 2\cos\frac{3\pi}{7}z + 1\right)\left(z^2 - 2\cos\frac{5\pi}{7}z + 1\right) = 0$

b) On expansion all powers of z have zero coefficients, except z^7; the coefficient of z is $\left(1 - 2\cos\frac{\pi}{7} - 2\cos\frac{3\pi}{7} - 2\cos\frac{5\pi}{7}\right)$, which equals zero.
Thus,
$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}$

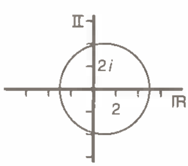
11. $z = \sqrt[3]{2}(\cos k\pi \pm i \sin k\pi)$, $k = \frac{1}{9}, \frac{5}{9}, \frac{7}{9}$

10.8 Exercises, page 453

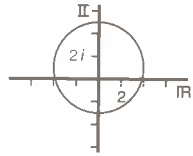
1. a)



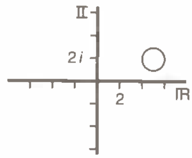
b)



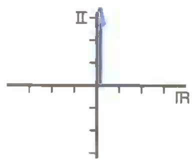
c)



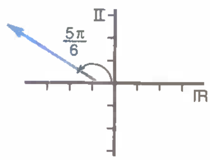
d)



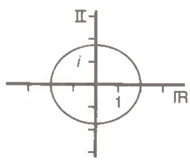
e)



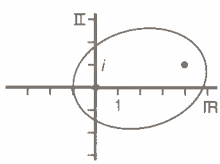
f)



g)



h)



2. a) $|z| = 6$
b) $|z + 1 - 3i| = 5$
c) $|z - u| = a$

3. a) $(x + 4)^2 + (y + 3)^2 = 4$
b) $x^2 + \left(y + \frac{5}{4}\right)^2 = \frac{9}{16}$

5. $z = -(2 + i) + pi$

6. $x^2 + y^2 = 1$ or $x = 0$

7. $y = -\sqrt{3}x + 2 + 4\sqrt{3}$ and $x < 4$

8. $x = 0$ or $y = 0$

9. $y = 0$ or $x^2 + y^2 = 2$

10. a) $|z - 2| = |z + 6|$
b) $|z - 2 - i| = |z - 3 + 2i|$

12. a) $x = \frac{9}{2}$
b) $\left(x - \frac{8}{3}\right)^2 + \left(y - \frac{4}{3}\right)^2 = \frac{80}{9}$
c) $(x + 1)^2 + (y - 1)^2 = 2$
d) $\left(x + \frac{1}{2}\right)^2 + (y - 1)^2 = \frac{9}{4}$

13. $xy = 1$

14. a) interior of circle: centre O, radius 5
b) interior of circle and circumference: centre $5 - 3i$, radius 3
c) 'exterior' of hyperbola $x^2 - y^2 = 2$
d) annulus: centre $2i$, radii 2 and 3

15. a) parabola
b) $y^2 = 4x$

16. a) circle: centre $2 + 3i$ radius 4
b) line segment

17. a) $|z| \doteq 7.61$
b) $|z| = 4$

10.9 Exercises, page 461

1. a) $r^2e^{2i\theta}$
b) $r^3e^{3i\theta}$
c) $\frac{1}{r}e^{-i\theta}$
d) $re^{-i\theta}$

2. a) $z = 2.24e^{0.46i}$
b) $w = 3.16e^{-1.89i}$

3. a) $u = 10e^{-\frac{i\pi}{3}}$
b) $v = 3\sqrt{2}e^{\frac{3i\pi}{4}}$

4. a) $4e^{\frac{4i\pi}{5}}$
b) 32
c) $\frac{1}{2}e^{-\frac{2i\pi}{5}}$
d) $\frac{1}{4}e^{-\frac{4i\pi}{5}}$
e) $\sqrt{2}e^{\frac{i\pi}{5}}$
f) $-4\sqrt{2}$

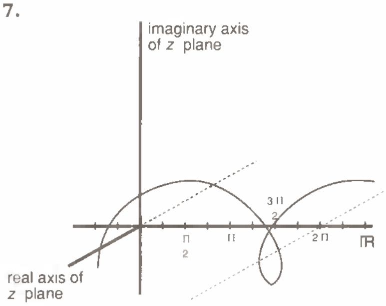
6. $e^{2ki\pi} = 1, k \in \mathbb{Z}$
An infinite number of different arguments give the same number.

10. a) $k\pi$
b) $\left(k + \frac{1}{2}\right)\pi$

11. $z^{\frac{1}{2}} = r^{\frac{1}{2}} e^{\frac{i\theta}{2}} = w_0$, or
$z^{\frac{1}{2}} = -r^{\frac{1}{2}} e^{\frac{i\theta}{2}} = w_1$

16. $f'(\theta) = ie^{i\theta}$
$= i(\cos\theta + i\sin\theta)$
$= i\cos\theta - \sin\theta$
or
$f'(\theta) = -\sin\theta + i\cos\theta$

17.



Inventory, page 466

1. -1
2. imaginary
3. $a + ib$ or $a + bi$
4. $\mathbb{R} \subset \mathbb{C}$
5. real; imaginary
6. non-real
7. $a = c$; $b = d$
8. a; b; $a - bi$;
$\sqrt{a^2 + b^2}$;
$\tan(\arg z) = \frac{b}{a}$
9. reflections;
real axis
10. z
11. Complex
12. $r(\cos\theta + i\sin\theta)$
13. equal;
any multiple of 2Π
14. product
15. difference
16. $r^n(\cos n\theta + i\sin n\theta)$
17. $z^5 - 1 = 0$
18. n
19. two
20. $|z - w|$ or $|w - z|$
21. e^x; y
22. -1

Review Exercises, pages 467–471

1. a) 10
b) $25 - 8i$
c) $120 - 22i$
d) $-2 + 2i$
e) $-i$
f) 1
g) i
h) $-2 + 6i$
i) $-7 + i$
j) $24i$
k) $-6 - 11i$

2. a) $-3 - i$
b) $-\frac{4}{5} - \frac{3}{5}i$
c) $-5 + 8i$
d) $\frac{9}{10} + \frac{7}{10}i$
e) $\frac{3}{25} + \frac{4}{25}i$
f) $\frac{1}{5} + \frac{2}{5}i$
g) $\frac{267}{962} + \frac{24}{481}i$
h) $-\frac{688}{7225} - \frac{134}{7225}i$

3. a) $4bi$
b) $\frac{2b}{a^2 + b^2}i$
c) $\frac{a(a^2 + b^2 + 1)}{a^2 + b^2} + \frac{ib(a^2 + b^2 - 1)}{a^2 + b^2}$

4. $5 \pm 2i$

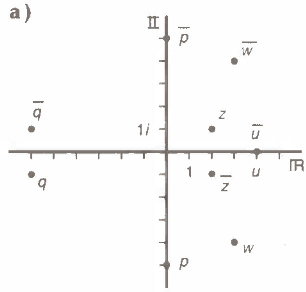
5. a) $6 \pm i$
b) $-2 \pm 4i$
c) $\frac{3 \pm i\sqrt{11}}{2}$

6. 4 or i

8. 3 570 125

9. $k = \pm\sqrt{46}$

10. a)



b) $\bar{z} = 2 - i$
$\bar{w} = 3 + 4i$
$\bar{p} = 5i$
$\bar{q} = -6 + i$
$\bar{u} = 4$

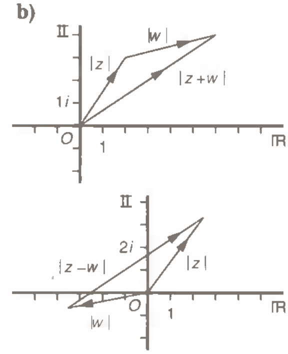
c) $|z| = \sqrt{5}$
$|w| = 5$
$|p| = 5$
$|q| = \sqrt{37}$
$|u| = 4$

d) $\arg z = 27°$
$\arg w = -53°$
$\arg p = -90°$
$\arg q = -171°$
$\arg u = 0°$

11. a) $\bar{z} = a - bi$
b) $z + \bar{z} = 2a \in \mathbb{R}$
c) $z\bar{z} = a^2 + b^2 \in \mathbb{R}$

12. a) 180°
b) rotate by 180°; yes

13. a) b)

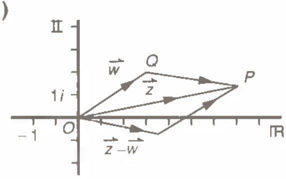


One side of a triangle is shorter than the sum of the other two sides.

c) One side of a triangle is greater than the difference between the other two sides.

14. a) midpoint between z and w
b) divides segment from z to w, in ratio $n : m$

15. b)

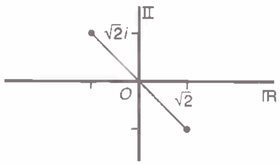


Review Exercises, pages 467–471, continued

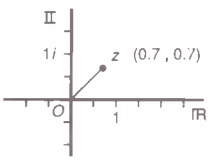
16. **a)** $z^2 - (5 - 2i)z + (-1 - 5i) = 0$
b) $z^2 - [a + c + (b - d)i]$
$+ [ac + bd + i(bc - ad)] = 0$

17. **b)** 0, 2, 4. . .or n

18. $-\sqrt{2} + i\sqrt{2}$ or $\sqrt{2} - i\sqrt{2}$



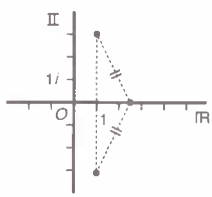
19. **a)** $z^2 = 1$
b)



c) It is true that
$$\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^2 = i,$$
but there is another 'root of i'.

20. $x^2 + y^2 = (x + iy)(x - iy)$ (only in $\mathbb{C}$)

21. **a)** $1 \pm 3i, \frac{5}{2}$
b)



22. **a)** $w^3 = 1$
b) $w^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$
$= \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$
$(w^2)^3 = 1$

24. Yes; e.g. $z = 1$ or $z = i$ are roots of $z^2 - (1 + i)z + i = 0$

25. **a)** i
b) i
c) $\cos 45° + i\sin 45°$ or $\cos 135° - i\sin 135°$
d) -1

26. **b)** $z = 2(\cos\theta + i\sin\theta)$ or $z = \cos\theta - i\sin\theta$

28. **a)** $|z| = y$, $\arg z = 90°$
b) $|z| = x$, $\arg z = 180°$

29. **a)** $\bar{z} = 3(\cos 67° - i\sin 67°)$
b) $\bar{w} = 2(\cos 123° + i\sin 123°)$
c) $zw = 6(\cos 56° - i\sin 56°)$
d) $\bar{z}\,\bar{w} = 6(\cos 56° + i\sin 56°)$
e) $\frac{3}{2}(\cos 190° + i\sin 190°)$ or $\frac{3}{2}(\cos 170° - i\sin 170°)$
f) $\frac{2}{3}(\cos 190° - i\sin 190°)$ or $\frac{2}{3}(\cos 170° + i\sin 170°)$

30. **a)** $z = \sqrt{101}(\cos 6° + i\sin 6°)$
b) $w = \sqrt{65}(\cos 60° - i\sin 60°)$
c) $zw = \sqrt{6565}(\cos 54° - i\sin 54°)$
d) $\frac{z}{w} = \sqrt{\frac{101}{65}}(\cos 66° + i\sin 66°)$
e) $\frac{w}{z} = \sqrt{\frac{65}{101}}(\cos 66° - i\sin 66°)$

31. **a)** $|z| = 2$, $\arg z = 120°$
$|w| = \sqrt{2}$, $\arg w = -135°$
b) $z^3 = 8$
$w^4 = -4$

32. **a)** $2\sqrt{2}(\cos 15° - i\sin 15°)$
b) $\sqrt{2}(\cos 255° + i\sin 255°)$ or $\sqrt{2}(\cos 105° - i\sin 105°)$
c) $\frac{1}{\sqrt{2}}(\cos 105° + i\sin 105°)$

33. **a)** 3θ
b) $\cos 3\theta =$
$\cos^3\theta - 3\cos\theta\sin^2\theta$
$\sin 3\theta =$
$3\cos^2\theta\sin\theta - \sin^3\theta$

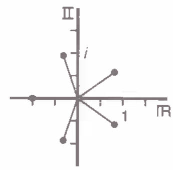
34. **a)** 1
b) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$
c) $32i$
d) $-\frac{1}{8} - \frac{\sqrt{3}}{8}i$
e) $-\frac{\sqrt{3}}{2} + \frac{i}{2}$
f) i

35. $\cos\theta + i\sin\theta$

36. modulus 1
argument 2θ

38. **b)** $\cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6}$,
$k \in \{0,1,2,3,4,5\}$
That is,
$1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$
$-1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$

39. $\cos\left(\pi + \frac{2k\pi}{5}\right) + i\sin\left(\pi + \frac{2k\pi}{5}\right)$,
$k \in \{0,1,2,3,4\}$
that is,
$-1, -0.31 \pm 0.95i,$
$0.81 \pm 0.59i$



40. **a)** $|z - 3 - 4i| = 5$
b) If $z = 0$,
$L.S. = |-3 - 4i| = 5 = R.S.$

41. **a)** $(x - 1)^2 + (y + 3)^2 = 1$
b) $x^2 + \left(y - \frac{9}{7}\right)^2 = \frac{25}{49}$

42. circle: centre $\frac{1}{3} - \frac{2}{3}i$
radius $\frac{\sqrt{20}}{3} \doteq 1.49$

43. **a)** $4e^{-\frac{2\pi i}{3}}$
b) $5\sqrt{2}\,e^{-\frac{\pi i}{4}}$

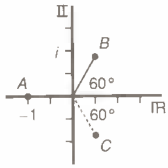
45. **i)** $z = 1 + i, 1 - i, -\frac{1}{2}$
ii) centre $-4 + 3i$, radius 5
$3x - 4y + 24 = 0$

46. **i)** $z = \sqrt{2} + i\sqrt{3}, -2 - i\sqrt{3},$
$\sqrt{2} - i\sqrt{3}, -\sqrt{2} + i\sqrt{3}$
ii) $\theta = \pm 48.6°$ or $\theta = \pm 131.4°$

48. **a)** $z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$
b) $\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}$

d) All three roots have modulus 1.

Arguments are $\pi, \frac{\pi}{3}, -\frac{\pi}{3}$



49. **i)** $z = 2, -1 + 3i, -1 - 3i$

$\measuredangle A = 90°, \measuredangle B = 45°, \measuredangle C = 45°$

arg of $-1 + 3i$ is 1.89

arg of $-1 - 3i$ is -1.89

ii) $\theta = 15°, 75°, 195°, 255°$

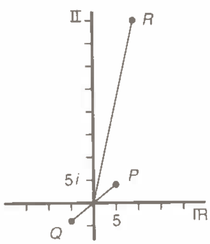
iii) $\theta = 167°$ or $300°$

50. **a)** $z^2 = (x^2 - y^2) + 2i\,xy$

b) $x = \pm 5, y = \pm 4;$

$z = 5 + 4i$ or $z = -5 - 4i$

c)



d) $OP = OQ = \sqrt{41}$

e) $\measuredangle POX \doteq 39°$

f) $\measuredangle POR \doteq 39°$

51. **i)** $z = 1 + 2i$

$w = 3 - i$

ii) $\frac{z_1}{z_2}$ has modulus $\frac{1}{\sqrt{2}}$,

argument $\frac{5}{12}\pi$

$\frac{1}{z_2^6}$ has modulus $\frac{1}{64}$,

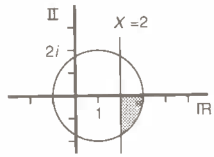
argument π

52. **i)** $z =$

$2\left[\cos\left(-\frac{13\pi}{18}\right) + i\sin\left(-\frac{13\pi}{18}\right)\right]$

$z = 1 + 2i, z = 1 - 2i$

ii)



53. **i)** $\left(\frac{5}{3}, 0\right)$ radius $\frac{4}{3}$

ii) $\cos 54° = \frac{1}{4}\sqrt{10 - 2\sqrt{5}}$

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4. **a)** no; not coplanar
b) yes; coplanar
c) no; not coplanar

5. **b)** $\vec{a} - 0.5\vec{b}$

6. $\vec{b} + 2\vec{c} + 3\vec{d}$

7. 8.5

8. **a)** $\frac{5}{8}\overrightarrow{OA} + \frac{3}{8}\overrightarrow{OB}$
b) $-\frac{5}{6}\overrightarrow{OA} + \frac{11}{6}\overrightarrow{OB}$

9. **a)** R, K, M collinear; R divides KM externally in ratio 5 : 3; the four points are coplanar
b) (11,4.5,−1)

10. $\overrightarrow{AD} = 3\overrightarrow{AB} + 5\overrightarrow{AC}$

11. 13 : 5

12. **b)** any vector $k\vec{a} + m\vec{b}$, where $21k - 69m = 0$

13. **b)** (3,0,1)

14. $k = -\frac{46}{17}, m = \frac{67}{17}$

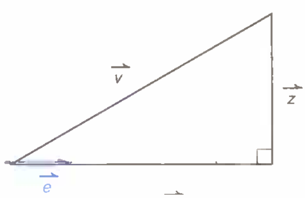
16. $2\vec{a} - \vec{b} + \vec{c} - 3\vec{d}$

20. $\overrightarrow{OD} = m(\vec{b} - \vec{c}) + k(\vec{c} - \vec{a})$

21. $\overrightarrow{BM} = \frac{1}{2}(\vec{a} + \vec{c})$
$\overrightarrow{AM} = \frac{1}{2}(\vec{c} - \vec{a})$
$\overrightarrow{AM}^2 = \overrightarrow{BM}^2$

23. $\overrightarrow{AB} = \vec{b} - \vec{a}$
$\overrightarrow{BC} = \vec{c} - \vec{b}$
$\overrightarrow{OB} \cdot \overrightarrow{AC} = 0$

25.



26. $\overrightarrow{(-2,3)} = \frac{1}{2}\overrightarrow{(1,1)} + \frac{5}{2}\overrightarrow{(-1,1)}$

27. $u_1v_1 + u_2v_2 + u_3v_3 \leqslant \sqrt{(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2)}$

29. 5 km/h

30. **a)** $\overrightarrow{(6,6,0)} \cdot \overrightarrow{(1,-1,2)} = 0$
⇒ diagonals perpendicular
b) $\frac{\sqrt{78}}{2} \doteq 4.42$
$\alpha = 32°, \beta = 148°$

33. $|\vec{F}| = 70.7$ N
$\theta = 135°$

34. **a)** 3.91 km/h; bearing 040°
b) 208 m
c) 5 min

35. **a)** bearing 304°
b) 1.66 km/h
c) 9 min

36. **a)** $|\vec{v}| = 375$ km/h
b) bearing 173°

37. **a)** $\vec{r} = \overrightarrow{(2,5,3)} + k\overrightarrow{(2,3,6)}$

$$\begin{cases} x = 2 + 2k \\ y = 5 + 3k \\ z = 3 + 6k \end{cases}$$

$$\frac{x-2}{2} = \frac{y-5}{3} = \frac{z-3}{6}$$

b) $\vec{r} = \overrightarrow{(-2,-1,4)} + t\overrightarrow{(5,3,-2)}$

$$\begin{cases} x = -2 + 5t \\ y = -1 + 3t \\ z = 4 - 2t \end{cases}$$

$$\frac{x+2}{5} = \frac{y+1}{3} = \frac{z-4}{-2}$$

38. **a)** skew
b) $\frac{2}{\sqrt{222}}$

39. $\vec{r} = \overrightarrow{(8,-6,7)} + t\overrightarrow{(8,11,-10)}$

40. $\vec{r} = \overrightarrow{(6,-1)} + k\overrightarrow{(3,5)}$

41. $\left(1 \pm \frac{2}{\sqrt{3}}, 3 \pm \frac{1}{\sqrt{3}}, 2 \mp \frac{2}{\sqrt{3}}\right)$

42. **a)** $\frac{8}{3}$

43. a) c)

44. **a)** 45°
b) 45°; 90°

45. **a)** 3
b) (−1,−6,3)

46. **a)** Line can be on either side of $\overrightarrow{AQ} = \vec{u}$.
b) $\vec{r} = \overrightarrow{(2,1)} + k\overrightarrow{(1,-7)}$
$\vec{r} = \overrightarrow{(2,1)} + t\overrightarrow{(7,1)}$

47. $\vec{r} = \overrightarrow{(-1,2,1)} + k\overrightarrow{(2,0,3)} + s\overrightarrow{(0,1,0)}$

48. $\vec{r} = \overrightarrow{(1,2,3)} + k\overrightarrow{(1,-2,2)} + s\overrightarrow{(1,2,-3)}$

49. (1,1,3)

50. $11x + 28y + 9z = 10$

51. $14x - 13y + 8z = -18$

52. (1,2,−3)

54. **a)** $k \neq -3, k \neq -1$
b) $k = -3$
c) $k = -1$

55. $x - y + 3z = 3$

56. **a)** $\frac{14}{\sqrt{26}}$
b) $\pm\sqrt{\frac{2511}{46}}$

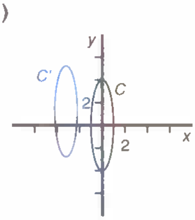
58. One matrix being $m \times n$, the other is $n \times m$.

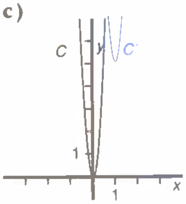
59. **a)** $(0,1) \rightarrow (0,1)$
$(0,b) \rightarrow (0,b)$
Points on y-axis remain invariant.
b) $(1,0) \rightarrow (1,k)$
$(1,b) \rightarrow (1,b+k)$
Points on $x = 1$ move upward by k.
c) $(a,0) \rightarrow (a,ka)$
Points on x-axis 'dilated' by k, upward.
d) det(s) = 1
same area,
same orientation

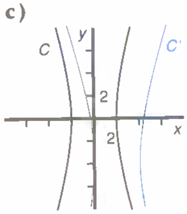
60. **a)** $\vec{i} \rightarrow \begin{bmatrix} \cos 60° \\ \sin 60° \end{bmatrix}$
b) $\vec{j} \rightarrow \begin{bmatrix} \sin 60° \\ -\cos 60° \end{bmatrix}$
c) $R = \begin{bmatrix} \cos 60° & \sin 60° \\ \sin 60° & -\cos 60° \end{bmatrix}$
d) $R = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

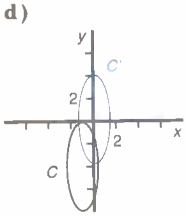
61. $M_{60} = M_{240}$ (same line)

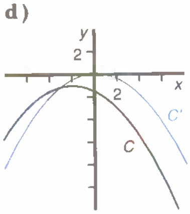
62. **a)** ellipse
b) $16x^2 + y^2 + 96x - 2y + 129 = 0$
c)

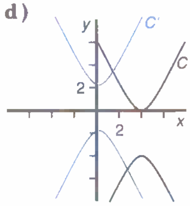


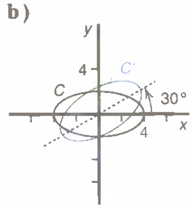
63. a) parabola
b) $16x^2 - 32x - y + 21 = 0$
c)


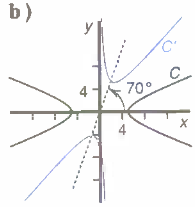
64. a) hyperbola
b) $9x^2 - y^2 - 36x - 8y - 34 = 0$
c)


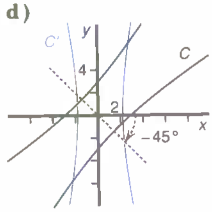
65. a) ellipse
b) $(x,y) \to (x + 1, y + 4)$
c) $8x^2 + y^2 = 15$
d)


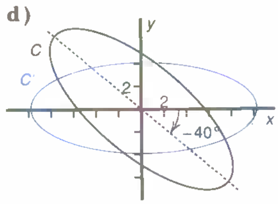
66. a) parabola
b) $(x,y) \to (x + 2, y + 1)$
c) $x^2 + 8y = 0$
d)


67. a) hyperbola
b) $(x,y) \to (x - 4, y + 2)$
c) $4x^2 - y^2 = -4$
d)


68. a) $1.7x^2 - 2.6xy + 3.3y^2 = 16$
b)


69. a) $-21.6x^2 + 18.6xy - 0.6y^2 = 100$
b)


70. a) hyperbola
b) 45°
c) $16x^2 - y^2 = 64$
d)


71. a) ellipse
b) 40.0°
c) $x^2 + 25y^2 = 100$
d)


73. b) 323 160

79. a) $a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$
b) $256a^4 - 768a^3 + 864a^2 - 432a + 18$

80. a) $\cos\frac{2k\pi}{9} + i\sin\frac{2k\pi}{9}$, $k \in \{1,2,4,5,7,8\}$
b) $\theta = \frac{2k\pi}{9}$, $k \in \{2,4,5,7,8\}$

82. a) 0 or 3
b) $4w$; or $4w^2$

83. a) $A^{-1} = \frac{1}{9}\begin{bmatrix}1 & -1\\ 7 & 2\end{bmatrix}$
$x = 2,\ y = 3$
b) $\vec{v} = \begin{bmatrix}3t_1 - 2t_2\\ t_1\end{bmatrix}$
$\lambda = 1$ or 2
c) $\vec{v} \cdot \vec{w} \neq 0$
$t_1 = 1,\ t_2 = -2$
d) $\vec{z} = \begin{bmatrix}-1\\ 8\end{bmatrix}$

84. i) $\vec{b} = \vec{a} + \vec{c}$
iii) $\measuredangle ABC \doteq 48°$
$\measuredangle ABO \doteq 23°$
Adjacent sides have length $\sqrt{14}$ and 3,
$\Rightarrow$ *OABC* not a rhombus

85. a) midpoint of
$[EF] = \frac{1}{4}(\vec{a} + \vec{b} + \vec{c} + \vec{d})$;
midpoints of $[GH]$ and $[LM]$
have same position vector;
the lines are concurrent

86. a) $P_1(3,0,2)$; $P_2(1,-3,-4)$
c) $2x + 3y + 6z = 18$

87. i) a) $\left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$
b) $p = k$;
$|k|$ is the perpendicular distance from O to plane Π_k; Π_k and Π_{-k} are on different sides of the origin
d) $(-1,-1,-3)$
e) $\sqrt{11}$ units
ii) b) $\overrightarrow{(x,y,z)} = \overrightarrow{(3,0,0)} + s\overrightarrow{(0,1,1)}$

88. a) $\lambda = \frac{1}{\sqrt{2}}$

$\mu = -\frac{1}{\sqrt{2}}$

$\vec{u'} = \pm\left(\frac{1}{\sqrt{5}}\vec{j} + \frac{2}{\sqrt{5}}\vec{k}\right)$

b) $t' = 2$

$t = 5$

c) $H(1,0,5)$

$K(0,2,4)$

89. b) $\vec{OA} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \vec{OB} = \begin{bmatrix} 4 \\ -2 \end{bmatrix},$

$\vec{OC} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$k = 3, l = 4$

c) $n = -1$

d) $p = -1$

90. a) $x + 4y = 18$

b) $Q(-6,6)$

d) $R'(3b,3b)$, if $R(b,b)$

e) T is a one-way stretch, by a factor of 3, in the direction $x = y$

91. $S = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

92. a) F b) T c) F d) F

93. a) $\vec{w} = \begin{bmatrix} x + 4y \\ 2x - y \end{bmatrix}$

b) $(1 - \lambda)x + 4y = 0$

$2x - (1 + \lambda)y = 0$

For $\lambda = 3$: $\begin{bmatrix} 2y \\ y \end{bmatrix}$ and $\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

For $\lambda = -3$: $\begin{bmatrix} x \\ -x \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

c) $\vec{u} = \begin{bmatrix} x + 4y \\ -2x - y \end{bmatrix}$

No.

94. a) $\vec{u} = \vec{(2,1)}$

$\vec{v} = \vec{(-1,2)}$

b) corresponding to $\vec{u}$: $y = \frac{1}{2}x$

corresponding to $\vec{v}$: $y = -2x$

c) T is the reflection in the line $y = \frac{1}{2}x$.

95. b) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

c) $M^2 = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

$M^4 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

$M^6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

d) counterclockwise rotations about the orgin, of 120°, 240°, and 0° (identity)

96. $M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

97. $\lambda_1 = 2, \lambda_2 = 11$

$\vec{e_1} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \vec{e_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

a) $3x + 2y = 0$ and $y = 3x$

b) $P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 11 \end{bmatrix}$

98. $M = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

99. i) $L\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

$M\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin 2\alpha \\ -\cos 2\alpha \end{bmatrix}$

ii) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$B = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\sin 2\alpha \end{bmatrix}$

iii) $A^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$

i.e. a rotation of 2θ

$B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

i.e. the identity matrix

100. b) For $a = 0$:

$\vec{(x,y,z)} = k\vec{(1,-1,-1)}$

Two coincident planes meet the plane $x + y = 0$ in a line.

for $a = -1$:

$\vec{(x,y,z)} = \vec{(-1,1,0)} + s\vec{(0,1,1)}$

Three planes intersect in a line.

101. a) $-4d^2 + 26d - 36$

b) $d = 2$ or $d = \frac{9}{2}$

c) For $d = 2$:

$\vec{(-10,4,0)} + s(13,-5,1)$

For $d = 0$: $\left(-\frac{1}{2},\frac{1}{6},\frac{5}{6}\right)$

For $d = \frac{9}{2}$: no solution

102. a) -1

b) $\begin{bmatrix} -5 & 4 \\ 4 & -3 \end{bmatrix}$

c) $x = 20, y = -12$

d) 4 unit²

e) $\frac{1}{5}\vec{(4,-3)}$

103. a) $\vec{OC} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}, \vec{OD} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$

b) $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\vec{OE} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \vec{OF} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$

$\det(N) = -4$

$OEF = 4(OAB)$

c) $R = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$S = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$

$S^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$

$\measuredangle GOH \doteq 42°$

APPENDIX

NOTATION

General

$\{\ldots\}$	set notation
$\{x \mid \ldots\}$	set of all x such that
$\varnothing$	empty set
$\in$	belongs to
$\subset$	is a subset of
$\mathbb{N}$	natural numbers
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathbb{V}_2$	two-dimensional vector space
$\mathbb{V}_3$	three-dimensional vector space
$\mathbb{V}_{2\times 2}$	vector space of 2×2 matrices
$\Rightarrow$	implies *or* therefore
$\Leftrightarrow$	is equivalent to *or* if and only if
$\parallel$	is parallel to
$\perp$	is perpendicular to

Arithmetic

$\doteq$	equals approximately *
3.204...	(more decimals exist)

Algebra

$C(n,r)$	$(n \text{ choose } r) = \dfrac{n!}{(n-r)!r!}$
$\sum_{r=1}^{n} t_r$	series $t_1 + t_2 + t_3 + \ldots + t_n$
$\lvert x \rvert$	absolute value of x
$f: x \rightarrow y$ or $\{(x,y) \mid y = f(x)\}$	function f maps x onto y

Matrices

a_{ij}	element in the ith row, jth column
I	unit matrix
$0_{2\times 2}$	zero matrix (of dimension 2 by 2)
M^{-1}	inverse of M
$\det(M)$	determinant of M
M^t	transpose of M

*within given restrictions, $\doteq$ may be written $\fallingdotseq$

Vectors

$\vec{v}$	vector
$\overrightarrow{AB}$	vector defined by two points
$\lvert\vec{v}\rvert, \lvert\overrightarrow{AB}\rvert$	length or magnitude of a vector
$\overrightarrow{(x,y)}$ or $\begin{bmatrix} x \\ y \end{bmatrix}$	vector as ordered pair
$\overrightarrow{(x,y,z)}$ or $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$	vector as ordered triple
$\vec{i}, \vec{j}, \vec{k}$	standard basis vectors
$\vec{e}_v$	unit vector in the direction of $\vec{v}$
$\vec{u} \cdot \vec{v}$	dot product
$\vec{u} \times \vec{v}$	cross product
$\vec{v}_{PQ}$	velocity of P relative to Q

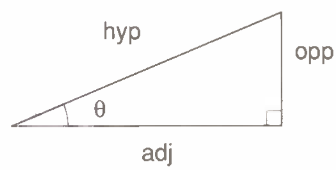
Complex Numbers

$\mathrm{Re}(z)$	real part of z
$\mathrm{Im}(z)$	imaginary part of z
$\bar{z}$	conjugate of z
$\lvert z\rvert = r$	modulus or absolute value of z
$\arg z$	argument of z

The notation used in the International Baccalaureate questions is not always the same as the notation given above. The context of any 'different' notation should make the meaning obvious.

FORMULAS IN TRIGONOMETRY

$\sin\theta = \dfrac{opp}{hyp}$ $\quad\cos\theta = \dfrac{adj}{hyp}$ $\quad\tan\theta = \dfrac{opp}{adj}$



$\cos(90^\circ - \theta) = \sin\theta$
$\sin(90^\circ - \theta) = \cos\theta.$

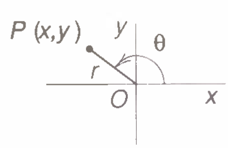
Angles in All Quadrants

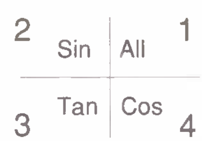
The position vector $\overrightarrow{OP} = \overrightarrow{(x,y)}$, where $\lvert\overrightarrow{OP}\rvert = r$, defines an angle θ with the positive x-axis such that

$\sin\theta = \dfrac{y}{r}$

$\cos\theta = \dfrac{x}{r}$

$\tan\theta = \dfrac{y}{x}$





The diagram on the right indicates where the ratios are positive.

Special Angles

$\sin 45^\circ = \dfrac{1}{\sqrt{2}} = \cos 45^\circ$

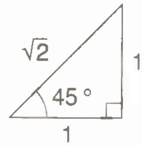
$\tan 45^\circ = 1$

$\sin 30^\circ = \dfrac{1}{2} = \cos 60^\circ$

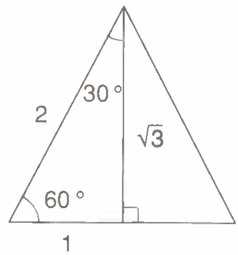
$\sin 60^\circ = \dfrac{\sqrt{3}}{2} = \cos 30^\circ$

$\tan 30^\circ = \dfrac{1}{\sqrt{3}}$

$\tan 60^\circ = \sqrt{3}$



isosceles right triangle



semi-equilateral triangle

Pythagorean Formulas

$\sin^2 A + \cos^2 A = 1$

$\sec^2 A = 1 + \tan^2 A$

$\csc^2 A = 1 + \cot^2 A$

Compound Angles

$\sin(A + B) = \sin A \cos B + \sin B \cos A$

$\sin(A - B) = \sin A \cos B - \sin B \cos A$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

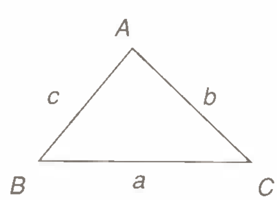
Solution of Triangles

The cosine law

$a^2 = b^2 + c^2 - 2bc \cos A$

The sine law

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



θ		$\sin\theta$	$\cos\theta$	$\tan\theta$
degree	radian/approx.value			
30	$\frac{\pi}{6} \doteq 0.52$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45	$\frac{\pi}{4} \doteq 0.79$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60	$\frac{\pi}{3} \doteq 1.0$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2} \doteq 1.6$	1	0	undefined
120	$\frac{2\pi}{3} \doteq 2.1$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
135	$\frac{3\pi}{4} \doteq 2.4$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1
150	$\frac{5\pi}{6} \doteq 2.6$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
180	$\pi \doteq 3.1$	0	-1	0
210	$\frac{7\pi}{6} \doteq 3.7$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
225	$\frac{5\pi}{4} \doteq 3.9$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1
240	$\frac{4\pi}{3} \doteq 4.2$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
270	$\frac{3\pi}{2} \doteq 4.7$	-1	0	undefined
300	$\frac{5\pi}{3} \doteq 5.2$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
315	$\frac{7\pi}{4} \doteq 5.5$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1
330	$\frac{11\pi}{6} \doteq 5.8$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
360	$2\pi \doteq 6.3$	0	1	0

The values of the other ratios can be obtained from the identities
$\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, and $\cot\theta = \frac{1}{\tan\theta}$

GLOSSARY

This abbreviated glossary provides definitions for mathematical terms as they are used in this text. Consult a mathematics dictionary for more complete or alternative definitions.

Acceleration. The rate at which a speed or velocity is changing.

Angle between a line and a plane. The acute angle between a line and its perpendicular projection in the plane.

Angle between planes. Either the angle between normals to the planes, or the supplement of that angle.

Angle between vectors. The angle between two vectors is the angle between them when they are drawn with a common tail.

Argand diagram. See **Complex plane.**

Argument. If P is the point representing the complex number z in the complex plane, then the argument of z is the angle between OP and the positive real axis.

Associativity. The operation $*$ in the set S has the associative property if, for all $a, b, c \in S$, $a*(b*c) = (a*b)*c$.

Basis. A set of vectors forms a basis for a vector space $\mathbb{V}$ if:
1. it is a linearly independent set and
2. it generates the space; that is, every vector of $\mathbb{V}$ can be expressed as a linear combination of the vectors of the set.

Basis vectors for $\mathbb{V}_2$. Any two linearly independent vectors $\vec{a}$ and $\vec{b}$ form a basis for $\mathbb{V}_2$.

Basis vectors for $\mathbb{V}_3$. Any three linearly independent vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ form a basis for $\mathbb{V}_3$.

Bearing. Direction expressed in degrees measured clockwise from north.

Binary operation. Given a set S, a binary operation $*$ in S combines any two elements of S to give an element of S; that is, if $a, b \in S$, then $a*b \in S$.

Commutativity. The operation $*$ in the set S has the commutative property if, for all $a, b \in S$, $a*b = b*a$.

Complex plane. Complex numbers can be represented geometrically in a complex plane, determined by a real axis (representing all the real numbers) and an imaginary axis. (It is also known as an Argand diagram.)

Complex number. A number that can be expressed as $z = x + iy$, where $x, y \in \mathbb{R}$, and $i^2 = -1$. The real numbers form a subset of complex numbers.

Component. The component of the vector $\vec{u}$ in the direction of $\vec{v}$, where the angle between $\vec{u}$ and $\vec{v}$ is θ, is the scalar $|\vec{u}|\cos\theta$.

Components. If a vector $\vec{v}$ is expressed as $\vec{v} = x\vec{e}_1 + y\vec{e}_2$, where $\vec{e}_1$ and $\vec{e}_2$ are unit vectors, then the scalars x and y are the components of $\vec{v}$ in the directions of $\vec{e}_1$ and $\vec{e}_2$ respectively.

Composition of transformations. The composition of two transformations is the result of applying one transformation after the other.

Conic. A curve such as a circle, parabola, ellipse, or hyperbola that can be obtained from the intersection of a cone and a plane.

Conic, general form.
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$
For a central conic $g = f = 0$.
If axes are parallel to x-axis and to y-axis, then $h = 0$.

Coordinates. Numbers that locate a point in 2-space or 3-space.

Coplanar vectors. Three vectors are coplanar if directed line segments that represent them can be translated so that all three segments lie in the same plane.

Cross product. The cross product of vectors $\vec{u}$ and $\vec{v}$, making an angle θ, is the vector $|\vec{u}||\vec{v}|\sin\theta\vec{e}$, where $\vec{e}$ is a unit vector normal to $\vec{u}$ and $\vec{v}$, such that $\vec{u}$, $\vec{v}$, $\vec{e}$ form a right-handed system.

Degenerate conic. A point, a straight line, or part of a line formed by the intersection of a cone and a plane.

Determinant. The determinant of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the real number $ad - bc$.

The value of the 2×2 determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

The value of the 3×3 determinant

$$\begin{vmatrix} m & n & p \\ q & r & s \\ t & u & v \end{vmatrix} = m(rv - su) - n(qv - st) + p(qu - rt).$$

Direction. A set of parallel lines with arrows pointing the same way.

Direction cosines of a line. The numbers $\cos\alpha = \frac{m_1}{|\vec{m}|}$, $\cos\beta = \frac{m_2}{|\vec{m}|}$, $\cos\gamma = \frac{m_3}{|\vec{m}|}$, where α, β, and γ are the angles made with the x, y, and z axes respectively and $\vec{m} = (m_1, m_2, m_3)$ is parallel to the line.

Direction vector of a line. A vector that is collinear with, or parallel to, a line.

Displacement. A directed distance (which can hence be represented by a vector).

Distributivity. In the set S, the operation * is distributive over the operation # if, for all $a, b, c \in S$, $a*(b\#c) = (a*b)\#(a*c)$ and $(b\#c)*a = (b*a)\#(c*a)$.

Dot product. The dot product of vectors $\vec{u}$ and $\vec{v}$, making an angle θ, is the scalar $|\vec{u}||\vec{v}|\cos\theta$.

Dynamics. The study of how objects change their motion under the action of forces.

Equilibrant. Given that $\vec{R}$ is the resultant of a number of force vectors, then the equilibrant of the force vectors is $-\vec{R}$.

Equilibrium. A particle is said to be in equilibrium when the vector sum of all forces acting upon it is zero.

Gravitational force. A particle of mass m kilograms has a gravitational force of mg newtons acting upon it, where g is the acceleration due to gravity. (On earth, $g \doteq 9.8\ m/s^2$.)

Identity. See **Neutral element.**

Image. Given a transformation T and a vector $\vec{v}$, the vector $\vec{v'} = T\vec{v}$ is the image of $\vec{v}$.

Imaginary number. The name describing any number ki where $k \in \mathbb{R}$, and $i^2 = -1$. Imaginary numbers form a subset of complex numbers.

Invariant lines. Lines that are their own images under a particular transformation are known as invariant for that transformation.

Inverse element. $a' \in S$ is the inverse of element $a \in S$ for the operation * if $a'*a = a*a' = e$, where $e \in S$ is the neutral element for *.

Isometry. A transformation that maps a line segment into a congruent line segment.

Linear dependence of two vectors. Two vectors $\vec{a}$ and $\vec{b}$ are linearly dependent if and only if $\vec{a}$ and $\vec{b}$ are collinear.

Linear dependence of three vectors. Three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent if and only if m, k, and p exist, *not all equal to* 0, such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$, $m, k, p \in \mathbb{R}$.

Linear independence of vectors. Vectors that are not linearly dependent are linearly independent.

Linear transformation. A linear transformation T of a vector space is such that, for any vectors $\vec{u}$, $\vec{v}$, and any scalar k:

1. $T(\vec{u} + \vec{v}) = T\vec{u} + T\vec{v}$
2. $T(k\vec{u}) = k(T\vec{u})$

Magnitude. The length or norm of a vector.

Matrix. A rectangular array of numbers. A square matrix can be used as an operator to effect transformations of a vector space.

Modulus. The modulus of the complex number $z = x + iy$ is $\sqrt{x^2 + y^2}$. (It can also be called the length, magnitude, or absolute value of z.)

Natural measure. Angles are commonly measured in degrees. However, the "natural measure" of an angle is defined as the ratio of the arc subtended by the angle and the radius of the circle. (Natural measure is in radians.)

Neutral element. The set S has an neutral element or identity element e with respect to the operation * if for all $x \in S$, $e*x = x*e = x$.

Norm. See **Magnitude.**

Normal vector. A normal vector to a plane (or a line) is a vector that is perpendicular to the plane (or the line).

Orthogonal. Perpendicular.

Orthonormal set. A set of unit vectors that are mutually orthogonal.

Parameter. An arbitrary constant or a variable in a mathematical expression that distinguishes various specific cases. In the parametric equation of a line a parameter determines a point in the line.

Parametric equations of a line.

in 2-space: $x = x_0 + km_1$
$y = y_0 + km_2$

where (x_0, y_0) is a point on the line, $\overrightarrow{(m_1, m_2)}$ is parallel to the line, and k is a parameter.

in 3-space: $x = x_0 + km_1$
$y = y_0 + km_2$
$z = z_0 + km_3$

where (x_0, y_0, z_0) is a point on the line, $\overrightarrow{(m_1, m_2, m_3)}$ is parallel to the line, and k is a parameter.

Parametric equations of a plane.

in 3-space: $x = x_0 + km_1 + su_1$
$y = y_0 + km_2 + su_2$
$z = z_0 + km_3 + su_3$

where (x_0, y_0, z_0) is a point on the plane, $\vec{m} = \overrightarrow{(m_1, m_2, m_3)}$ and $\vec{u} = \overrightarrow{(u_1, u_2, u_3)}$ are vectors parallel to the plane, $\vec{m} \nparallel \vec{u}$, and k and s are parameters.

Particle. An object that is modelled by a single point.

Position vector. If O is the origin and A is any point, then $\overrightarrow{OA}$ is called the position vector of A.

Projection. The projection of the vector $\vec{u}$ in the direction of $\vec{v}$, where the angle between $\vec{u}$ and $\vec{v}$ is θ, is the vector $(|\vec{u}|\cos\theta)\frac{\vec{v}}{|\vec{v}|}$

Projections. If a vector $\vec{v}$ is expressed as $\vec{v} = x\vec{e}_1 + y\vec{e}_2$, where $\vec{e}_1$ and $\vec{e}_2$ are unit vectors, then the vectors $x\vec{e}_1$ and $y\vec{e}_2$ are the projections of $\vec{v}$ in the directions of $\vec{e}_1$ and $\vec{e}_2$.

Radian. See **Natural measure.**

Relative velocity. The relative velocity of A from B is the velocity of A as perceived by the observer B.

Resultant. Given two vectors $\vec{a}$ and $\vec{b}$, then the vector $\vec{a} + \vec{b}$ is called the resultant of $\vec{a}$ and $\vec{b}$.

Right-handed system. Three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ form a right-handed system if their directions are such that they could be represented respectively by the thumb, the first finger, and the second finger of a right hand. (See diagram, page 11.)

Roots. The roots of an equation in a single variable are the values of the variable that satisfy the equation.

Scalar. A term used to describe real numbers, to distinguish them from vectors.

Singularity. A singular transformation is one that destroys one or more dimensions by its action. Its matrix is also known as singular, and the determinant of this matrix is zero.

Skew lines. Lines in 3-space that are neither parallel nor intersecting.

Statics. The study of forces acting upon objects that are stationary in a given frame of reference.

Supplement. The supplement of an angle of θ^0 is $180^0 - \theta^0$.

Tension. The pulling force in a taut string.

Thrust. The pushing force exerted by a strut.

Transformation. Any action that changes vectors, points, or figures is a transformation of those vectors, points, or figures.

Translation. A transformation in which the image of a figure is obtained by sliding or displacing the original figure without rotation. It can be defined by the vector $\overrightarrow{(h,k)}$ which maps the point (x,y) to the point $(x + h, y + k)$.

Triangle law of vector addition. $\overrightarrow{OS} + \overrightarrow{ST} = \overrightarrow{OT}$

Triangle law of vector subtraction. $\overrightarrow{ST} = \overrightarrow{OT} - \overrightarrow{OS}$

Triangle inequality. Given any three points P, Q, and R, the lengths $PQ + QR \geqslant PR$.

Unit vector. A vector whose length is 1.

Vector. A mathematical entity that can be represented by a directed line segment, or by an ordered n-tuple of numbers, and that obeys a law of addition.

Vector equation of a line. $\vec{r} = \vec{r}_0 + k\vec{m}$, where $\vec{r}$ is the position vector of any point on the line, $\vec{r}_0$ is the position vector of a given point on the line, $\vec{m}$ is a vector parallel to the line, and k is a parameter.

Vector equation of a plane. $\vec{r} = \vec{r}_0 + k\vec{m} + s\vec{u}$, where $\vec{r}$ is the direction vector of any point on the plane, $\vec{r}_0$ is the direction vector of a given point on the plane, $\vec{m}$ and $\vec{u}$ are vectors parallel to the plane, $\vec{m} \nparallel \vec{u}$, and k and s are parameters.

Vector space. A set of vectors, together with the operations of vector addition and multiplication by a scalar.

Work. The product of the component of a force along a distance moved and of the distance moved.

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