VECTORS, MATRICES
and
COMPLEX NUMBERS

with
International Baccalaureate
questions

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CHAPTER ONE

INTRODUCTION TO VECTORS

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ISBN of Gage Edition 0-7715-3503-1

CHAPTER ONE

Introduction to Vectors

In the first seven chapters of this book, you will be studying the algebra of mathematical objects called **vectors**.

Vectors are among the most recent inventions in mathematics that you will encounter in high school. Indeed, vector analysis was only fully developed at the turn of the 20th century. Unfortunately, very few properties and theorems in vector analysis are named after their originators. To compensate for this, a short history of the development of vectors is presented here.

In 1843, the famous Irish mathematician Sir William Hamilton invented an algebraic system that he thought could model any physical situation. He called the elements of his algebraic system *quaternions*. (Complex numbers, which you will study in chapter 10, form a subset of quaternions.) Hamilton, with his followers, attempted to apply the theory of quaternions to many areas in mathematics, and to use these to describe physical phenomena. A movement was created, which kept quaternion theory in the forefront of mathematical research until the end of the century.

Around the same time, the German mathematician Hermann Grassmann published a treatise, called *Die Ausdehnungslehre*, discussing much more general extensions of the number system than Hamilton's quaternions. (These extensions include complex numbers, quaternions, vectors, and matrices. The general name for these objects is *hypernumbers* or *holors*.) However, Grassmann's work was considered incomprehensible at the time of its publication, in 1844.

In 1881, the American mathematician Josiah Willard Gibbs, who was the first professor of mathematical physics at Yale, used ideas based on the works of Hamilton and Grassmann to publish a pamphlet called *Vector Analysis*. At first, Gibbs' ideas were rejected strongly by the supporters of quaternions, who maintained that their theory was 'complete', and that vectors were unpalatable 'hermaphrodites'.

The self-taught English scientist, Oliver Heaviside, like Gibbs, found quaternions unsatisfactory for the description of many physical phenomena in mechanics and electromagnetism. Despite his lack of a formal education beyond the age of 16, he started producing original and entertaining scientific papers in 1872, at the age of 22. The works that he published between 1893 and 1912 firmly established the superiority of vectors over quaternions to explain electromagnetic theory.

Today, vector analysis is an important part of mathematics. Furthermore, any physicist or engineer must have a firm grasp of the methods and symbolism of vector analysis.

HAMILTON was portrayed by a contemporary artist with his mace of office as President of the Royal Academy of Ireland. He was Royal Astronomer of Ireland from 1826 to 1865.

1.1 What is a Vector?

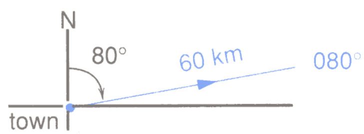
In mathematics, you often deal with numbers. However, many of the concepts used are not just numbers, or numbers alone. For example, lines, points, sets, matrices are not numbers.

Vectors are objects that generally need more than one number, to be defined. In this way, they are similar to points. However, vectors have other qualities that make them radically different from points. The nature of vectors should become apparent through the examples of statements below, as you shall see presently.
Note: Each statement makes use of a single number.

1. She is 60 km out of town.
2. The wind is blowing from the northeast (that is, from bearing 045°).
3. The elevator is stuck on the 12th floor.
4. The temperature in the office is 25°C.
5. The truck drove 5 blocks away from the Post Office.
6. The rocket moved in a straight line away from the earth with a speed of 5000 km/h.

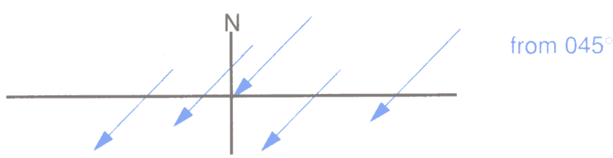
Some of these statements require more information to be complete, whereas others do tell you everything you want to know with the single number contained. You will see the idea of a vector emerging from those statements which appear incomplete.

In 1, you need to know in *which direction* the 60 km is; thus you need to know *another number*, the *bearing*. (The bearing of a object is its direction measured clockwise from north in degrees, and expressed as three digits.)



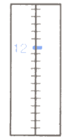
The diagram indicates that she is 60 km out of town, *on a bearing of 080°*. You need the two numbers, 60 and 080. The 60 km describes her distance from the town. The *two* numbers 60 and 080 describe her **displacement** from the town.

In 2, you know where the wind is coming from, but you do not know its *speed*. The statement is incomplete. You need another *number*.

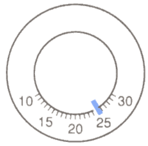


The diagram indicates a speed of *15 km/h* as well as the bearing 045°.
1 cm represents 10 km/h.

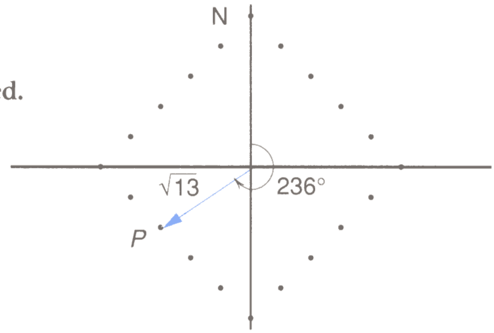
In 3, you *do* have complete information. The single number 12 tells you exactly where the elevator is.



In 4, you again have enough information. The single number is 25.

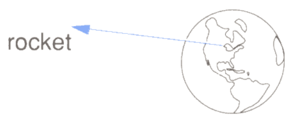


In 5, direction must be considered. After travelling 5 blocks, the truck may be in many different places.



If it is at point P as the diagram indicates, it would be more informative to have, for example, the two numbers 3 and 2. These numbers tell you respectively how far west and south it is. Or, the two numbers could be the length of $OP = \sqrt{3^2 + 2^2} = \sqrt{13}$, and the bearing of OP, namely 236°.

In 6, besides the number 5000 representing the speed, one or two other numbers indicating the direction of travel of the rocket would be needed.

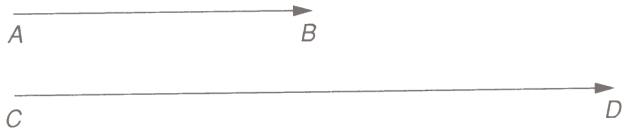


Observe the following about the preceding examples: Whenever *more than one number* was needed, a specific direction was involved, and it was convenient to draw a line segment with an *arrow* on it.

These 'directed line segments' represent *vectors*.

- The *length of the directed line segment* represents the *magnitude of the vector*.

In the diagram, the directed line segment CD is twice as long as the directed line segment AB. If AB represents a magnitude of 5, then CD will represent a magnitude of $2 \times 5 = 10$.



The *direction of the line segment*, as indicated by the arrow, represents the *direction of the vector*.

(Any two *parallel lines* with arrows pointing the same way are said to define the same *direction*.)

Thus,

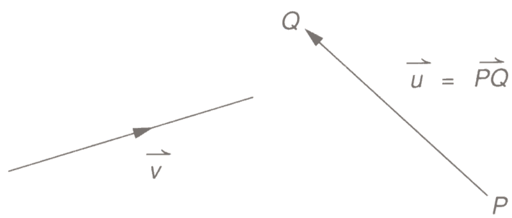
the *magnitude* and *direction* of a vector can be represented by a directed line segment.

A directed line segment gives you a 'picture' of a vector. The vector can also be represented by two (or more) numbers, as you saw in the examples. You will be investigating that type of representation further in section 1.3.

NOTATION

A vector can be represented by a single letter like this: $\vec{v}$. Note that an arrow $\rightharpoonup$ is put over the "v"; this is to indicate that $\vec{v}$ represents a vector.

A similar notation is also used for the vector $\vec{u}$ represented by the directed line segment joining a point P to a point Q: you can write $\vec{u} = \overrightarrow{PQ}$.



Note: In some texts, vectors are represented in bold print, without an arrow, thus: **v**.

The *length* or *magnitude* of a vector is a real number. (Any real number is called a **scalar**, to distinguish it from a vector.) The length of the vector $\vec{v}$ is denoted by $|\vec{v}|$. The length of the vector $\overrightarrow{PQ}$ is denoted by $|\overrightarrow{PQ}|$.

Note: In some texts, the length of the vector $\vec{v}$ is represented by v, and the length of the vector $\overrightarrow{PQ}$ by PQ.

An Important Property of Vectors

Take another look at the directed line segments representing the vectors in the examples 1, 2, 5, and 6 above. In statement 2 about the wind, many directed line segments were drawn, although there is only one 'wind'! This indicated that the wind does not blow on only one point. The directed line segments all represent the *same vector*.

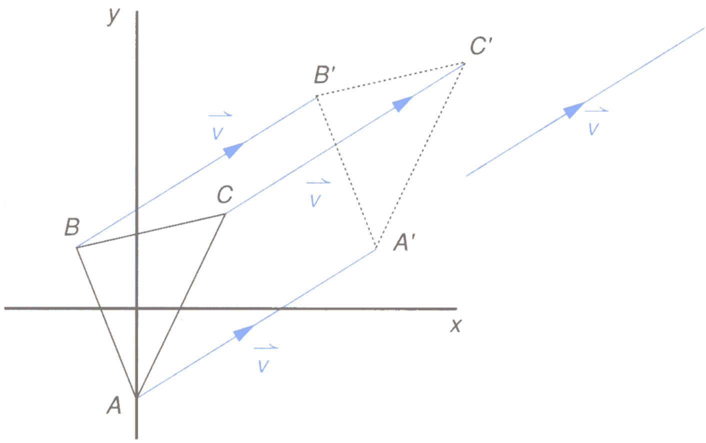
Indeed, one of the most important attributes of a vector is that it has magnitude and direction *only*. It does *not* have a particular *place*. Because a vector can be represented by any one of a family of directed line segments having the same length and direction, the following is true.

A vector is *everywhere*. *A directed line segment representing a vector can be drawn where you want.*

Translations and Vectors

You have studied translations before, and you will be seeing them again in more detail in chapter 8.

A translation is a transformation in which a figure or an object is moved to any other position, without altering its shape or size and *without turning*. (Instead of saying that an object is *translated*, you can say that it is *shifted* or *displaced*.)



For example, the triangle ABC is translated to the triangle $A'B'C'$ in the xy-plane above. The translation is depicted by the vector $\vec{v}$.

Indeed, there is a one-to-one correspondence between translations and vectors. It may help you to understand better that a vector is everywhere if you imagine translating the entire plane with the vector $\vec{v}$, then drawing the infinite number of equal directed line segments showing the translation of every point in the plane.

Here, $\vec{v} = \overrightarrow{AA'} = \overrightarrow{BB'} = \overrightarrow{CC'}$, etc.

Thus, any directed line segment with the appropriate magnitude and direction will represent the vector correctly. This leads to the following definition of the equality of vectors.

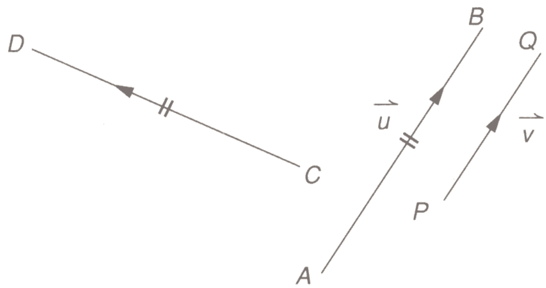
DEFINITION

Two vectors are equal if and only if they have the same magnitude and the same direction.

Alternatively,

two directed line segments that have the same length and the same direction represent the same vector.

Example Given $\overrightarrow{AB} = \vec{u}$, $|\overrightarrow{CD}| = |\vec{u}|$, $\overrightarrow{PQ} = \vec{v}$, with line segment PQ parallel to line segment AB, as shown.



Justify the following: **a)** $\overrightarrow{AB} \neq \overrightarrow{BA}$

b) $\overrightarrow{CD} \neq \vec{u}$

c) $\vec{u} \neq \vec{v}$

Solution

a) Although $|\overrightarrow{AB}| = |\overrightarrow{BA}|$, the direction of $\overrightarrow{AB}$ is opposite to the direction of $\overrightarrow{BA}$. Therefore the directions are not the same. Thus, $\overrightarrow{AB} \neq \overrightarrow{BA}$.

b) $\overrightarrow{CD}$ and $\vec{u}$ have different directions, thus are not equal.

c) $|\vec{u}| \neq |\vec{v}|$, that is, $\vec{u}$ and $\vec{v}$ have different lengths. Thus $\vec{u}$ and $\vec{v}$ are not equal. ■

SUMMARY

Any number of parallel lines with arrows pointing the same way define a particular direction.

A vector is everywhere; it can be represented by any directed line segment which has the correct magnitude and direction.

Equal vectors have the same magnitude and direction.

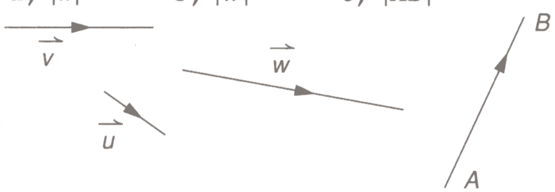
1.1 Exercises

1. Which of the following should be represented by vectors?
   a) weight
   b) frequency
   c) velocity
   d) volume (of a solid)
   e) volume (of a sound)
   f) age
   g) temperature change
   h) distance
   i) displacement
   j) capacity
   k) force
   l) acceleration

2. Most calculations done on a small scale, compared to the size of the earth, assume that the earth is flat.
   With that assumption, which of the following can be considered *directions*?
   a) east
   b) up
   c) downtown
   d) next door

   In reality, knowing the earth to be approximately spherical, which of the above are directions?

3. If $|\vec{v}| = 2$, state the value of the following.
   a) $|\vec{u}|$
   b) $|\vec{w}|$
   c) $|\overrightarrow{AB}|$

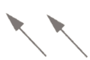
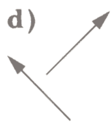
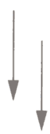


4. A directed line segment of length 0.5 cm pointing directly downwards represents a weight of 10 newtons (N). Use the same scale to draw directed line segments to represent the following.
   a) Juan is pushing upward with a force of 100 N.
   b) Helen is pulling a sled westward with a force of 80 N.
   c) The briefcase weighs 25 N.

5. Given that $\overrightarrow{AB} = \vec{v}$ and $\overrightarrow{PQ} = \vec{v}$, what can you say about the line segments AB and PQ?

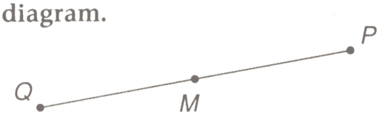
6. a) If $\vec{u} = \vec{v}$, does it follow that $|\vec{u}| = |\vec{v}|$? Explain.
   b) If $|\vec{u}| = |\vec{v}|$, does it follow that $\vec{u} = \vec{v}$? Explain.

7. The vector representing the displacement from Montréal to Québec can be described by its magnitude, namely 245 km, and its bearing, namely 041°. Describe in a similar manner
   a) the displacement from Québec to Montréal,
   b) the displacement from Québec to St Célestin, given that St Célestin is exactly halfway between Montréal and Québec.

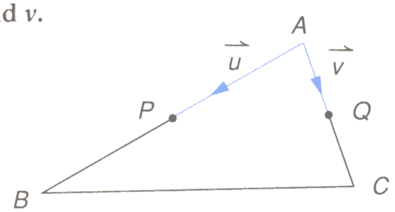
8. State, with reasons, whether or not the following pairs of directed line segments represent the same vector.
   a) 
   b) 
   c) 
   d) 
   e) 
   f) 

9. *ABCD* is a square. State, with reasons, whether or not the following statements are true.
   a) $\overrightarrow{AC} = \overrightarrow{BD}$
   b) $\overrightarrow{AC} = \overrightarrow{DB}$
   c) $|\overrightarrow{AC}| = |\overrightarrow{BD}|$

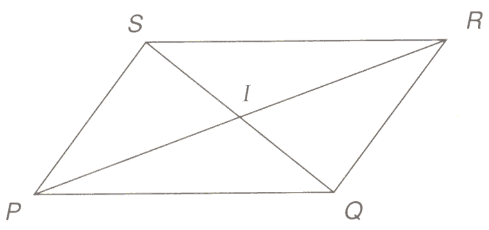
10. Given that *M* is the midpoint of segment *PQ*,
   a) give reasons why $\overrightarrow{QM} = \overrightarrow{MP}$
   b) state any other vector equality from the diagram.



11. In the triangle *ABC*, *P* is the midpoint of *AB*, and *Q* is the midpoint of *AC*. If $\overrightarrow{AP} = \vec{u}$ and $\overrightarrow{AQ} = \vec{v}$, express $\overrightarrow{PB}$ and $\overrightarrow{QC}$ in terms of $\vec{u}$ and $\vec{v}$.

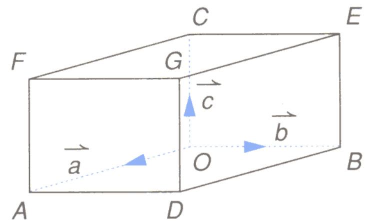


12. *PQRS* is a parallelogram whose diagonals intersect at *I*. Assuming all the properties of a parallelogram, state, where possible, another vector equal to
   a) $\overrightarrow{PQ}$
   b) $\overrightarrow{PR}$
   c) $\overrightarrow{PI}$
   d) $\overrightarrow{RQ}$
   e) $\overrightarrow{IQ}$
   f) $\overrightarrow{SQ}$

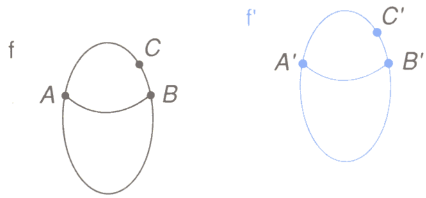


13. In question 12, if $|\overrightarrow{PQ}| = 5, |\overrightarrow{PS}| = 4$, and $|\overrightarrow{PI}| = 3$, state the value of each of the following.
   a) $|\overrightarrow{QP}|$ b) $|\overrightarrow{QR}|$ c) $|\overrightarrow{RI}|$

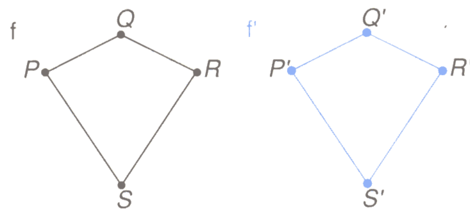
14. *OADBECFG* is a rectangular solid where $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$, and $\overrightarrow{OC} = \vec{c}$. Name all the other vectors equal to $\vec{a}, \vec{b}$, and $\vec{c}$.



15. For each of the following, name and draw three representatives of the vector depicting the translation from figure f to figure f′, where $A \to A', B \to B'$, etc.
   a)



   b)

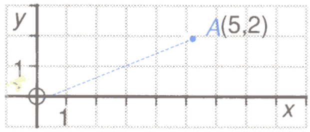


16. In question 15 a), what would the diagram look like if *all* the possible representatives of the translation vector where drawn?

1.2 Three-dimensional Space

Vectors can be represented by directed line segments. In two dimensions, a directed line segment drawn on a page represents a vector. In three dimensions, a vector could be modelled by a pencil held above your desk, with the point indicating the direction of the vector. To understand vectors better, you need familiarity with three-dimensional space, and with diagrams on a plane surface that represent three-dimensional objects. You must learn how to represent three-dimensional objects on paper, so as to produce a visual impression of the third dimension.

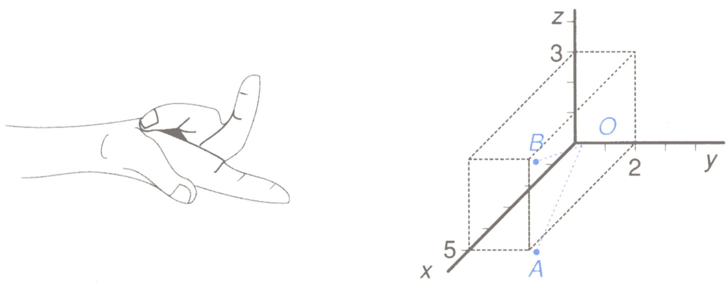
In two-dimensional coordinate geometry, or 2-space, you are accustomed to the *xy*-plane. The figure shows how the point $A = (5,2)$ is represented.



The *x* and *y* axes are both real number lines. Thus, the set of points in this plane can be called $\mathbb{R} \times \mathbb{R}$ or $\mathbb{R}^2$, that is, the set of all ordered pairs of real numbers.

In three-dimensional coordinate geometry, or 3-space, you use three mutually perpendicular axes *x*, *y*, and *z*. The *xy*-plane becomes part of the *xyz*-space. By convention, you put the *y* and *z* axes on the paper, and you try to give the impression of the *x*-axis rising out of the page at right angles to the paper.

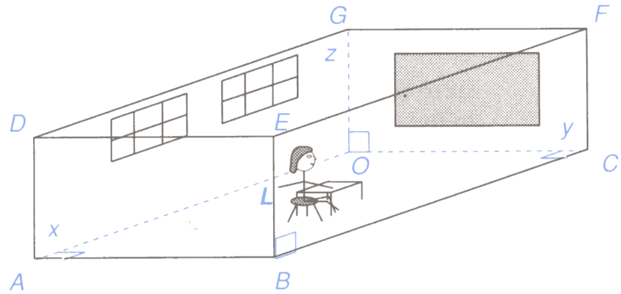
The coordinate system using the triple (x,y,z) is said to form a **right-handed system.** Use your right hand as a model. Stretch out the thumb and the first two fingers. Let the positive *x*-axis be represented by your thumb, and the positive *y*-axis by your first finger. Then the direction of the positive *z*-axis will be represented by your second finger.



If you followed the same instructions, using your left hand instead of the right, you would find the *z*-axis pointing in the *opposite* direction. That is why it is important to define the triple (x,y,z) in a foolproof way. If you use an (x,y,z) system, instead of saying that the point *A* of the *xy*-plane has coordinates $(x,y) = (5,2)$, you say that its coordinates are $(x,y,z) = (5,2,0)$. The point *B*, shown on the same diagram, has coordinates $(5,2,3)$.

3-space is not difficult to imagine, since it is all around you. In fact, any rectangular room, such as your classroom, gives you a good framework for imagining a 3-space coordinate system.

In the drawing shown, Maria is sitting at her desk, facing the blackboard. There are windows on the wall to her left. The corners are represented by the letters as shown. If you choose the point O as **origin**, then you could represent the x-axis by OA, the y-axis by OC, and the z-axis by OG.



In this figure, the floor $OABC$ is the xy-plane; here, every point has coordinates $(x,y,0)$.

The blackboard wall $OCFG$ is the yz-plane; points here have coordinates $(0,y,z)$.

The window wall $OADG$ is the zx-plane; points here have coordinates $(x,0,z)$.

Once you have specified a unit of measurement along the axes, any point in the room can now be described by its three coordinates.

For example, suppose you choose the unit as 1 metre. Maria's left hand, L, is 3 m from the blackboard, 2 m from the window wall, and 1 m above the floor. Hence the coordinates of L are $(x,y,z) = (3,2,1)$.

The drawing of the classroom above illustrates the three essential rules of mathematical 3-dimensional drawing.

For a 3-dimensional mathematical drawing:

1. do not overlap distinct lines
2. keep verticals vertical
3. keep parallels parallel

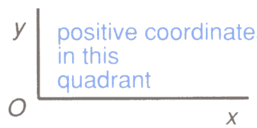
It is especially important to mark right-angles according to these rules. Observe how some of the right angles at O, A, B and C are marked in the figure. Incorrectly drawn right-angle markers can lead to very confusing 3-D drawings. Producing a good drawing is more of a challenge in 3-D than in 2-D. It requires practice, and sometimes more than one attempt.

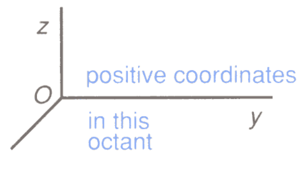
So far you have only looked at *positive values* along the axes of your 3-space. In the diagram above of a classroom, any point in the room could be specified by its three positive number coordinates.

How can you represent the points *outside* the room?

If you use negative numbers too, a 3-space coordinate system will allow you to specify *any* point in space, in the same way that a 2-space coordinate system allows you to specify *any* point in its plane.

The two-dimensional plane is divided by the axes into *four quadrants*. Points whose coordinates are all positive are found in only *one* of these quadrants.



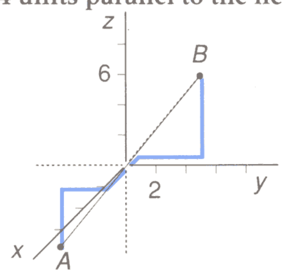


Three-dimensional space is divided into *eight octants* (from the Greek "octo", meaning eight). Points whose coordinates are all positive are found in only *one* of these octants.

The set of all points in 3-space, regardless of the signs of their coordinates, is called $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ or $\mathbb{R}^3$.

Example Plot the points $A = (2,-3,-4)$ and $B = (-1,5,6)$ in a 3-space coordinate system, and draw the segment AB.

Solution To locate A, proceed 2 units along the positive x-axis, then 3 units parallel to the negative y-axis, then 4 units parallel to the negative z-axis.



B is located in a similar fashion.
The points A and B can be joined to obtain the segment AB. ■

Note: From the diagram, it looks as if AB passes through O. However, this is *not* the case; but this does show that you can sometimes have difficulty in interpreting a 3-dimensional drawing. You must beware of the pitfalls of a drawing in perspective.

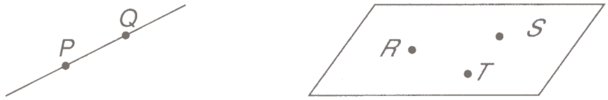
You cannot yet prove that AB does not pass through O. However, the methods shown in chapter 5 will allow you to do that.

General 3-space Concepts

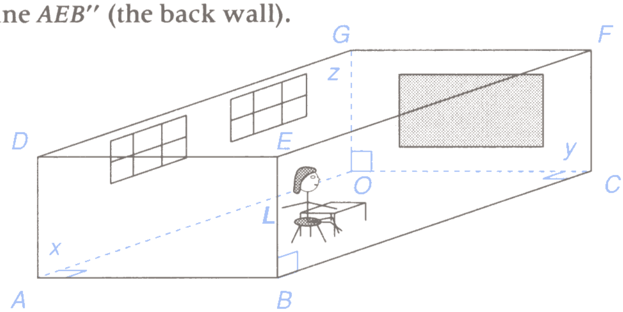
Look again at your classroom. Without making any specific reference to the 'origin' and the 'axes', you can discover other important facts about lines and planes in 3-space.

Points

You know that any two distinct points determine a straight line. Similarly, any *three distinct points*, not all on a single straight line, determine a *plane*.



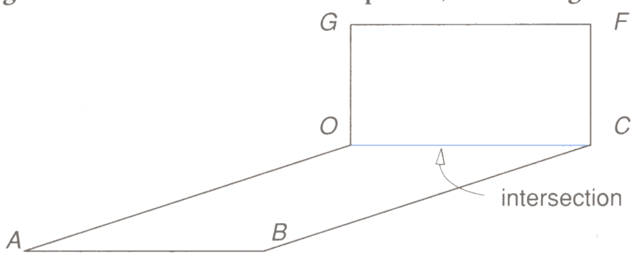
In the classroom drawn below, whenever you name any two points, you are identifying a *unique line* containing those two points:
"the line *AE*" (a diagonal across the back wall).
Whenever you mention three points, you are identifying a *unique plane* containing those three points:
"the plane *AEB*" (the back wall).



Intersections of Planes

You can see that the plane of the ceiling (*EFG* or *DEFG*) and the plane of the floor (*OABC*) *never meet*. These are *parallel planes*.

However, the floor and the front wall (*OCFG*) **do** intersect. The points of the straight line *OC* lie in both of these planes, as the diagram shows.



Two planes are either parallel, or they intersect in a straight line.

Since any straight line can be the intersection of two planes, the following converse is true.

Any straight line is contained in an infinite number of planes.

Intersections of Lines

In the drawing of the classroom, the lines *BC* and *CF* meet at the point *C*. Note that they are in the same plane *BCFE*.

The lines *BC* and *EF* are parallel—they never meet.

Although the lines *BC* and *OG* never meet, they do not seem to fit the notion of 'parallelism'. Indeed, they are not parallel, and they do not intersect. Such lines in space are called **skew lines**. You can draw skew lines as follows.

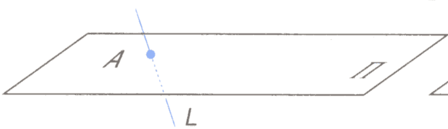
Two distinct *parallel* lines in 3-space *never meet, but are in the same plane*. This is not the case for *BC* and *OG*, so they are skew. On the other hand, the lines *BC* and *EF*, which also never meet, *are* in the same plane *BCFE*, so they are parallel.
Given two distinct lines, L_1 and L_2, in 3-space, there are three possibilities.

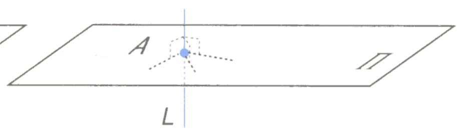
1. L_1, L_2 meet; therefore they define a plane. They are coplanar.
2. L_1, L_2 are parallel; therefore they do not meet, but they are coplanar.
3. L_1, L_2 are skew; they do not meet, and they are not coplanar.

Intersections of Lines and Planes

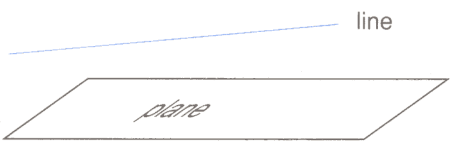
In the drawing of the classroom, the line *CF* meets the floor *OABC* at the point *C*.

In general, a line intersects a plane in a single point. The diagram shows the line *L* crossing the plane Π at the point *A*. (The dotted line indicates that part of *L* which is *behind* Π.) A line perpendicular to a plane is perpendicular to all the lines in that plane.





Some lines and planes never meet. For example, the lines *EF*, *ED*, *DF* never meet the plane of the floor. In these cases, the line is said to be *parallel* to the plane.

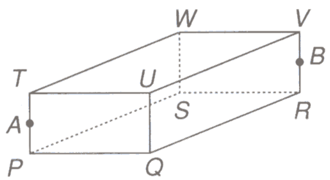


SUMMARY

Three distinct points, not on the same straight line, determine a plane.
Two planes are either parallel, or intersect in a straight line.
Two lines which are neither parallel nor intersecting are skew.
A line is either parallel to a plane, contained in the plane, or intersects the plane in a single point.

1.2 Exercises

(The diagram is to be used to answer questions 1-4.)



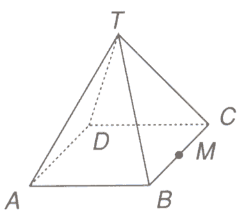
1. Above is a drawing of a rectangular box *PQRSWTUV*. *A* is the midpoint of segment *PT*, and *B* is the midpoint of segment *RV*. State two lines parallel to each of the following.
   a) *TW* b) *WS* c) *PR*

2. State two lines perpendicular to each of the following.
   a) *TW* b) *WS* c) *AB*

3. State whether the following pairs of lines are parallel, intersecting, or skew.
   a) *TQ* and *WR*
   b) *TQ* and *SV*
   c) *PW* and *QW*
   d) *PV* and *QS*
   e) *PV* and *QW*
   f) *AB* and *SQ*

4. Copy the above drawing and join the line *AB*. Does the line *AB* really pass through *S*? What could you do to enhance the three-dimensional drawing, indicating precisely whether or not *S* is on the line *AB*?

5. *ABCDT* is a right square pyramid. It is called "square" because it has a square base *ABCD*, and "right" because its apex, *T*, is vertically above the centre of the base.



   a) Draw the pyramid, and locate the centre *O* of its base (the intersection of the diagonals *AC* and *DB*).
   b) Join *TO*, and mark the right angles *TOA* and *TOD*.
   c) Given that *M* is the midpoint of *BC*, join *OM*, and *MT*. Mark the right angle in the triangle *TOM*.

6. In question 5, if $AB = 6$ cm, and $TO = 4$ cm, use the theorem of Pythagoras to calculate the exact value of the lengths of *TM* and *TB*.

7. In question 5, name the following.
   a) two skew lines
   b) the three planes intersecting at point *B*

8. Plot the following points in $\mathbb{R}^3$.

   $A(1,1,1)$
   $B(2,0,0)$
   $C(0,3,0)$
   $D(0,0,-1)$
   $E(2,0,-3)$
   $F(0,-2,5)$
   $G(-3,-3,-3)$
   $H(1,2,-5)$
   $J(-1,2,5)$

9. $\mathbb{R}^3$ is divided into eight octants. State the signs of the coordinates (x,y,z) of a point in each octant.

10. State the condition that must be satisfied by the coordinates (x,y,z) of a point positioned as follows.
    a) on the x-axis
    b) on the y-axis
    c) on the z-axis

11. State the condition that must be satisfied by the coordinates (x,y,z) of a point positioned as follows.
    a) in the xy-plane
    b) in the yz-plane
    c) in the zx-plane

12. How would you describe the set of points $P(x,y,z)$, given that $x = y$? Draw this set in $\mathbb{R}^3$.

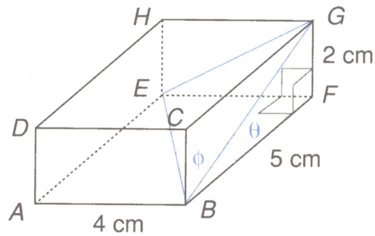
In Search of Trigonometry as an Aid To Visualization of 3-Space

The use of trigonometry to solve triangles that occur in three-dimensional situations should help you to visualize and familiarize yourself with 3-space concepts. To solve triangles that are not right-angled, you can use the sine law and the cosine law. You will find these described on page 542.

Examine the following examples and their diagrams very carefully. The examples should help you to grasp some intuitive notions about angles in 3-space.

Example 1 Given the rectangular box shown, find, correct to the nearest degree, the angle between

a) the lines *BG* and *BF*
b) the lines *BG* and *BE*
c) the line *BH* and the plane *ABFE*
d) the planes *ABGH* and *ABFE*.



Solution **a)** Since *F* is a right angle, you can apply trigonometry to the right triangle *BFG*. The angle required is θ where $\tan\theta = \frac{GF}{BF} = \frac{2}{5} = 0.4$ hence $\theta \doteq 22°$.

b) The angle ϕ between *BG* and *BE* is in triangle *BGE* which is not a right triangle. You can use the cosine law to find ϕ, if you can determine the sides *BG*, *BE* and *GE*.

BG is the hypotenuse of the triangle you used in part a), that is, triangle *BFG*, thus $BG^2 = BF^2 + FG^2 = 5^2 + 2^2 = 29$, so $BG = \sqrt{29}$.

Also, the triangle *BFE* (on the 'floor') is right-angled at *F*, as is the triangle *EFG* (on the 'front wall'). You can find the lengths *BE* and *EG* in the same way.

$BE^2 = BF^2 + FE^2 = 5^2 + 4^2 = 41$, so $BE = \sqrt{41}$
$GE^2 = GF^2 + FE^2 = 2^2 + 4^2 = 20$, so $GE = \sqrt{20}$.

Finally, you can apply the cosine law to the triangle *BGE* to find the angle ϕ.

$$GE^2 = BE^2 + BG^2 - (2)(BE)(BG)\cos\phi$$

$$20 = 41 + 29 - 2\sqrt{41}\,\sqrt{29}\cos\phi$$

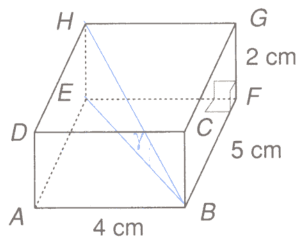
hence $$\cos\phi = \frac{(41 + 29 - 20)}{2\sqrt{41}\,\sqrt{29}} = 0.725\,018\ldots,$$

giving $$\phi \doteq 44°.$$

c) If you hold your pencil against the paper you are writing on, you will notice that there are many different angles between the pencil and the paper. Similarly, there are many different possible angles between *BH* and the 'floor' *ABFE*. It depends on which line from *B* you choose in the floor!

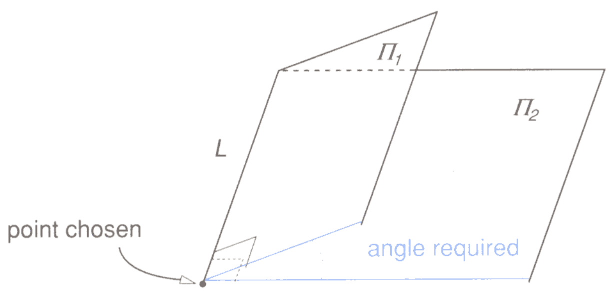
The *angle between a line and a plane* means *the smallest possible angle* between the line and the plane. This angle will be found between the line and its perpendicular projection on the plane. In our example, it is the angle γ between *BH* and *BE*. Note that *BEH* is a right triangle. Thus

$\tan\gamma = \frac{EH}{BE} = \frac{2}{\sqrt{41}} = 0.3123\ldots$, so $\gamma \doteq 17°$.



(In the activities you will be calculating the angle between *BH* and *BF*, and the angle between *BH* and *BA*. You will find that both of these are greater than 17°.)

d) The angle between two planes Π_1 and Π_2 can be found as follows.
1 Find the line of intersection, *L*, of the two planes.
2 Choose a point on *L*, that you will use to find a line perpendicular to *L*, in Π_1, and a line perpendicular to *L*, in Π_2.
3 The angle between those two lines is the required angle.

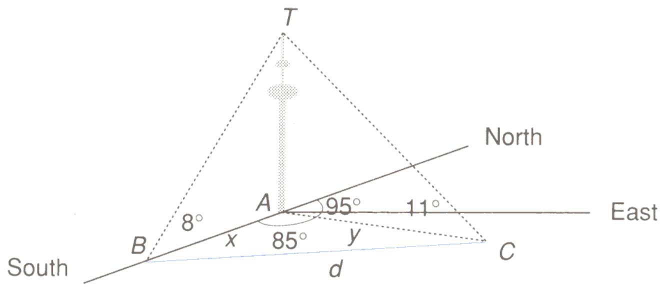


In Example 1, the line of intersection of the planes is *AB*. If you choose *B* as the point on this line, *BF* and *BG* are lines which obey the above criteria. Thus the angle required is θ, calculated in a) as 22°. ■

Example 2 The CN tower in Toronto is 553 m high. A person in a boat on Lake Ontario, at a point B due south of the tower, observes the top T of the tower at an angle of elevation of 8°. At the same time, another person, at a point C on a bearing 095° from the tower, observes the angle of elevation of T to be 11°.

Calculate the distance BC between the two people, correct to 3 significant figures.

Solution Your diagram must indicate bearings at ground level, as well as a vertical tower. Look carefully at the features of the diagram.



Denote the distance of the first person from the base of the tower by x, and the distance of the second person from the base of the tower by y.

From right triangles BAT and CAT respectively,

$\tan 8° = \dfrac{553}{x}$ and $\tan 11° = \dfrac{553}{y}$

thus $x = \dfrac{553}{0.1405\ldots} \doteq 3935$ m, and $y = \dfrac{553}{0.1943\ldots} \doteq 2945$ m.

Note: The distances x and y are written here to 4 significant digits for clarity of reading. You should retain the actual values using the full accuracy of your calculator for subsequent use of x and y.

You can now find the distance d between the people by using the cosine law in the surface triangle BAC. Note that the angle opposite d is $180° - 95° = 85°$.

$$d^2 = x^2 + y^2 - (2)(x)(y)\cos 85°$$

$= 21\ 625\ 032.67$ using full calculator accuracy

so $d \doteq 4650$ correct to 3 significant digits

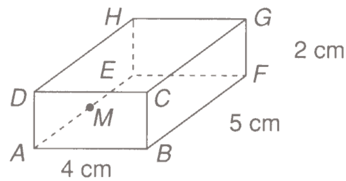
Thus the distance between the two people is about 4650 m. ■

SUMMARY

The angle between a line and a plane is the angle between the line and its perpendicular projection on the plane.

The angle between two planes Π_1 and Π_2 is the angle between a line in Π_1 and a line in Π_2, each line chosen to be perpendicular to the line of intersection of the planes.

Activities



Give all angles correct to nearest degree, and all lengths correct to 3 significant digits.

1. Given the rectangular box shown, where M is the midpoint of AE, find the following.
   a) the angle α between the lines BH and BF
   b) the angle β between the lines BH and BA
   c) the angle γ between the lines BH and BM

Note: All your answers should be greater than 17°, which is the value of the angle between BH and BE, that is, the angle between BH and the plane $ABFE$.

2. Given a right pyramid $ABCDT$, on a square base $ABCD$, with $AB = 6$ cm, and height $TO = 4$ cm, calculate the following.
   a) the angle TAB
   b) the angle θ between a slant edge (such as TB) and the base $ABCD$
   c) the angle ϕ between a slant face (such as TBC) and the base $ABCD$.

3. A plane sheet of plywood measuring 1.2 m by 2.4 m is inclined with its shorter edge at an angle of 30° to the horizontal.
   a) How high is the top edge of the plywood?
   b) Calculate the angle between a diagonal of the sheet of plywood and the horizontal.

4. A woman on a frozen lake observes the top of a radio tower, due north of her, at an angle of elevation of 21°. She then skis for 500 m on a bearing 065°, and finds herself due east of the tower. Calculate the following.
   a) the height of the radio tower
   b) the angle of elevation of the top of the tower from the second point of observation

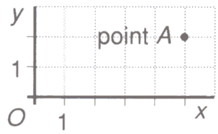
5. A flagpole is placed at one corner of a courtyard 25 m long and 20 m wide. The angle of elevation of the top of the flagpole from the opposite corner of the courtyard is 18°. Calculate the height of the flagpole, and the angles of elevation of the top of the flagpole from the other two corners of the courtyard.

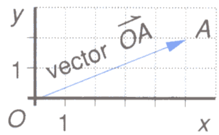
6. The angles of elevation of the top T of a vertical post TO are observed to be α and β from points A and B due west and due north of the post. If the distance $AB = d$, show that the height of the post is

$$\frac{d}{\sqrt{(\cot^2 \alpha + \cot^2 \beta)}}$$

1.3 Vectors as Ordered Pairs or Triples

A point in a plane coordinate system can be represented by an ordered pair of numbers. In the diagram the point A has coordinates (5,2).





Draw the vector $\overrightarrow{OA}$.

This vector, represented by a directed line segment joining the origin O to a point A, is called the **position vector** of point A. Recall from section 1.1 that vectors can also be represented by two (or more) numbers; here, you could represent vector $\overrightarrow{OA}$ by the same ordered pair, (5,2), as point A. *But beware*: it is very important to distinguish between *vectors* and *points*. When writing vectors as ordered pairs, an **arrow notation** will therefore be used, as follows: vector $\overrightarrow{OA} = \overrightarrow{(5,2)}$, as distinct from the point $A = (5,2)$.

The entries for the vector $\overrightarrow{OA}$ are called *components*.

The entries for the point A are called *coordinates*.

In some texts, vectors as ordered pairs are written as columns, in order to be distinguished from points, as follows: $\overrightarrow{OA} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

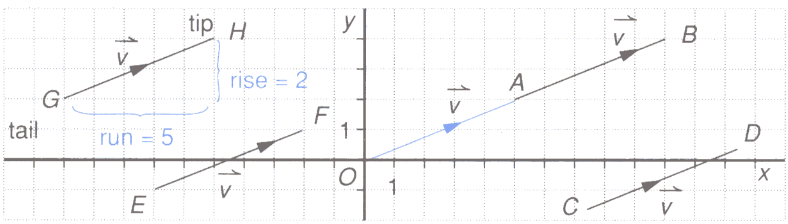
Indeed, you will be using this notation for vectors in chapters 7 and 8.

Other texts use square brackets for vectors as ordered pairs. Yet others make no distinction between representing points or vectors. In these texts, the reader must be vigilant as to which is which, noting the distinction from the context.

Unfortunately, there is no standard notation. As a student of mathematics, you should familiarize yourself with the different representations in use.

The vector $\vec{v}$, equal to the position vector $\overrightarrow{OA}$, can be drawn wherever you like. In other words, it can be represented by any directed line segment parallel to OA, pointing the same way as OA, and congruent to OA.

$\vec{v} = \overrightarrow{OA} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{GH} = \overrightarrow{AB} = \overrightarrow{(5,2)}$



In each of these cases, the run from the tail of the vector to the tip is 5, and the corresponding rise is 2.

Because vectors can be drawn anywhere, you can draw a vector expressed as an ordered pair using grid lines only, without an *x*-axis, a *y*-axis, or an origin. The diagram shows the following vectors:

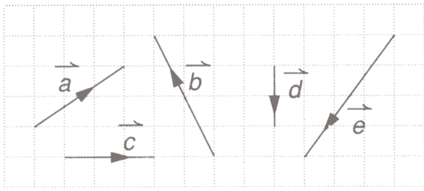
$\vec{a} = \overrightarrow{(3,2)}$

$\vec{b} = \overrightarrow{(-2,4)}$

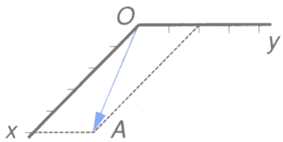
$\vec{c} = \overrightarrow{(3,0)}$

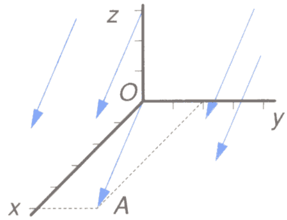
$\vec{d} = \overrightarrow{(0,-2)}$

$\vec{e} = \overrightarrow{(-3,-4)}$



A corresponding result holds in 3-space. Draw the *xy*-plane in perspective, with the *x*-axis coming out of the page, at right angles to it. Again, draw the vector $\vec{v} = \overrightarrow{OA}$.





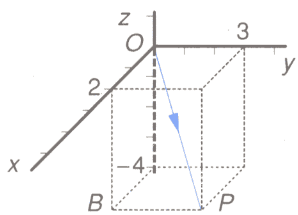
Because a vector can be drawn where you want, as long as it has the correct magnitude and direction, the vector $\vec{v}$ can be represented by directed line segments hovering *above* or *below* the plane, as you can see from the second diagram.

If you now think of *A* as a point in 3-space, you would call *A* by its *three* coordinates (x,y,z), thus: $A = (5,2,0)$.

The vector $\overrightarrow{OA}$, which is the position vector of the point *A*, would be represented by an *arrowed ordered triple*, thus: $\overrightarrow{OA} = \overrightarrow{(5,2,0)}$

The coordinates of point $P(2,3,-4)$ in the figure are the numbers 2, 3, and -4.

The components of vector $\overrightarrow{OP} = \overrightarrow{(2,3,-4)}$ are also the numbers 2, 3 and -4.



In general, the point $P(a,b,c)$, defined in a 3-space coordinate system with origin *O*, has position vector $\overrightarrow{OP} = \overrightarrow{(a,b,c,)}$.

Similarly, the point $P(a,b)$, defined in a 2-space coordinate system with origin *O*, has position vector $\overrightarrow{OP} = \overrightarrow{(a,b)}$.

The abbreviations $\overrightarrow{OP} = \vec{p}$, $\overrightarrow{OQ} = \vec{q}$ etc., are often used in problems that involve the position vectors of different points.

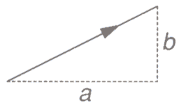
The set of two-dimensional vectors, or vectors in 2-space, will be designated by $\mathbb{V}_2$.

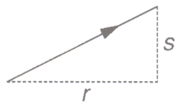
The set of three-dimensional vectors, or vectors in 3-space, will be designated by $\mathbb{V}_3$.

Equality of Vectors

You saw in section 1.1 that vectors are equal if and only if they have the same magnitude and direction. Thus, two equal vectors can be represented by a directed line segment from the origin to the same point, and hence by the same ordered pair. This leads to the following.

In $\mathbb{V}_2$, $\overrightarrow{(a,b)} = \overrightarrow{(r,s)}$ if and only if $a = r$ and $b = s$





Similarly, in $\mathbb{V}_3$,

$$\overrightarrow{(a,b,c)} = \overrightarrow{(r,s,t)} \text{ if and only if } a = r \text{ and } b = s \text{ and } c = t$$

Length of a Vector

The length of a vector is defined as the length of a directed line segment which represents the vector.

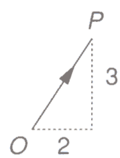
Note: The length of a vector is sometimes called its "magnitude" or its "norm."

First, look at an example in 2-space.

Example 1 Find the length of the vector $\vec{v} = \overrightarrow{(2,3)}$.

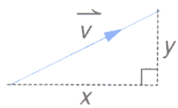
Solution $\vec{v}$ can be represented by $\overrightarrow{OP}$, the position vector of the point $P(2,3)$.

You need to find $|\overrightarrow{OP}|$ = the length of line segment OP.
By using the theorem of Pythagoras,
$|\vec{v}|^2 = |\overrightarrow{OP}|^2 = 2^2 + 3^2$, so
$|\vec{v}| = |\overrightarrow{OP}| = \sqrt{2^2 + 3^2} = \sqrt{13}$.



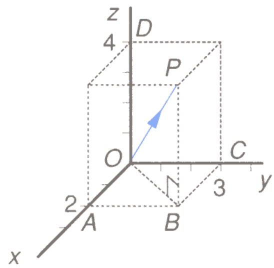
■

In general, in $\mathbb{V}_2$,
if $\vec{v} = \overrightarrow{(x,y)}$, then $|\vec{v}| = \sqrt{x^2 + y^2}$



A similar result is true for the length of a vector in 3-space.

Example 2 Find the length of the vector $\vec{p} = \overrightarrow{(2,3,4)}$.



Solution $\vec{p}$ can be represented by $\overrightarrow{OP}$, the position vector of the point $P(2,3,4)$.

In the diagram, $OA = CB = 2$, $OC = AB = 3$, and $OD = BP = 4$. You need to find $|\vec{p}| = |\overrightarrow{OP}| =$ length of segment OP.

From the right triangle OBP, $OP^2 = OB^2 + BP^2$

but from the right triangle OAB, $OB^2 = OA^2 + AB^2$

$\Rightarrow$ $OP^2 = OA^2 + AB^2 + BP^2$

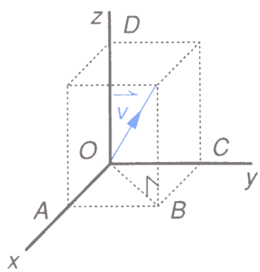
and since $BP = OD$, $AB = OC$, then $OP^2 = OA^2 + OC^2 + OD^2$

or $|\overrightarrow{OP}|^2 = 2^2 + 3^2 + 4^2 = 29$

thus $|\vec{p}| = |\overrightarrow{OP}| = \sqrt{29}$. ■

In general, if $\overrightarrow{OP} = \overrightarrow{(x,y,z)}$, then $OA = x$, $OB = y$, and $OC = z$; you have the following result in $\mathbb{V}_3$.

If $\vec{v} = \overrightarrow{(x,y,z)}$, then $|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$



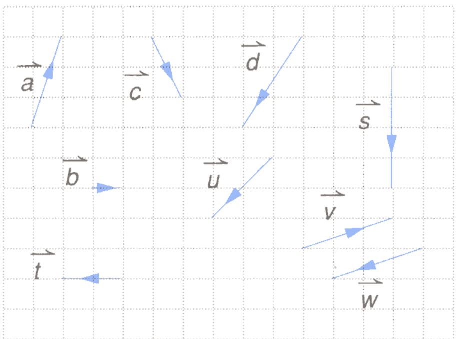
SUMMARY

If P is a point in a coordinate system of origin O, then $\overrightarrow{OP}$ is called the *position vector* of P.

in 2-space	*in 3-space*
If $P = (a,b)$, then $\overrightarrow{OP} = \overrightarrow{(a,b)}$	If $P = (a,b,c,)$, then $\overrightarrow{OP} = \overrightarrow{(a,b,c)}$
Vectors $\overrightarrow{(a,b)} = \overrightarrow{(r,s)}$ if and only if the numbers $a = r$ and $b = s$	Vectors $\overrightarrow{(a,b,c,)} = \overrightarrow{(r,s,t)}$ if and only if the numbers $a = r$, $b = s$ and $c = t$
The length of $\vec{v} = \overrightarrow{(x,y)}$ is $|\vec{v}| = \sqrt{x^2 + y^2}$	The length of $\vec{v} = \overrightarrow{(x,y,z)}$ is $|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$

1.3 Exercises

1. Write the vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{s}, \vec{t}, \vec{u}, \vec{v}, \vec{w}$ represented below, as ordered pairs.



2. On a grid, draw six directed line segments representing the vector $\vec{u} = \overrightarrow{(6,2)}$.

3. Repeat question 2 for the vector $\vec{v} = \overrightarrow{(-3,-5)}$.

4. Use a grid to draw the vectors $\vec{a} = \overrightarrow{(3,4)}$, $\vec{b} = \overrightarrow{(4,-1)}$, $\vec{c} = \overrightarrow{(-2,5)}$, and $\vec{d} = \overrightarrow{(-1,-1)}$.

5. Calculate the lengths of the vectors $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$ of question 4.

6. **a)** Given the point $P(4,-3)$, draw the position vector $\overrightarrow{OP}$.

b) If $\overrightarrow{OQ} = \overrightarrow{(-1,7)}$, state the coordinates of the point Q.

c) If $\overrightarrow{OR} = \overrightarrow{(0,2,-2)}$, state the coordinates of the point R.

7. In a 3-space coordinate system, draw the position vector $\vec{p}$ of the point $P(2,3,5)$.

8. **a)** On a grid, draw points A, B, and C such that $\overrightarrow{AB} = \overrightarrow{(4,3)}$ and $\overrightarrow{BC} = \overrightarrow{(1,-5)}$.

b) Use your drawing to express $\overrightarrow{AC}$ as an ordered pair.

c) Conjecture a rule by which the components of $\overrightarrow{AC}$ can be obtained from the components of $\overrightarrow{AB}$ and of $\overrightarrow{BC}$.

9. A point P, whose position vector is $\overrightarrow{OP} = \overrightarrow{(1,2)}$, is translated to position P' according to the vector $\vec{v} = \overrightarrow{(4,1)}$. What are the coordinates of P'?

10. Show that the length of $\vec{v} = \overrightarrow{\left(\frac{3}{5}, -\frac{4}{5}\right)}$ is 1 unit. ($\vec{v}$ is known as a **unit vector**.)

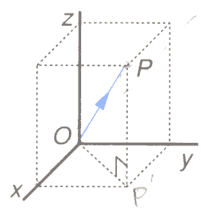
11. Given that $\overrightarrow{OP} = \overrightarrow{(2,x)}$ and that $|\overrightarrow{OP}| = 5$, calculate x.

12. Given that $\vec{u} = \overrightarrow{(2,3)}$ and $\vec{v} = \overrightarrow{(n,-1)}$, calculate the following.

a) $|\vec{u}|$

b) n, given that $|\vec{u}| = |\vec{v}|$

13. The position vector of P is $\overrightarrow{OP} = \overrightarrow{(x,y,z)}$. Show that $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$.



14. In a certain city the blocks are 100 m square. You walk, from a point O, 4 blocks east, then 3 blocks north, to arrive at a point P.

a) How far have you walked?

b) What is the (direct) distance from O to P?

15. The vectors $\vec{p} = \overrightarrow{(3,x)}$ and $\vec{q} = \overrightarrow{(w,-6)}$ are equal. State the values of x and w.

16. The vectors $\vec{a} = \overrightarrow{(2,-1,k)}$ and $\vec{b} = \overrightarrow{(m,n,7)}$ are equal. State the values of k, m and n.

17. $\vec{u} = \overrightarrow{(2h-k,-3)}$, and $\vec{v} = \overrightarrow{(4,h+k)}$. Given that $\vec{u} = \vec{v}$, calculate the values of h and k.

18. Use a 2-space coordinate system to find the vector $\overrightarrow{PQ}$ given the following points.

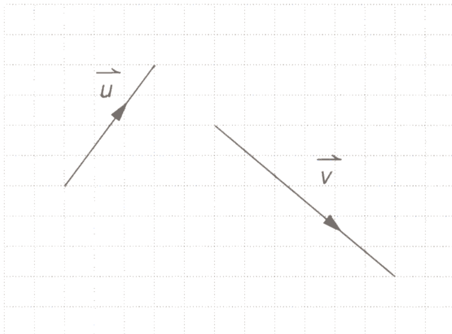
a) $P(0,0)$, $Q(3,2)$,
b) $P(3,2)$, $Q(0,0)$,
c) $P(3,2)$, $Q(4,7)$
d) $P(3,2)$, $Q(-1,2)$
e) $P(5,-2)$, $Q(-1,-3)$
f) $P(a,b)$, $Q(c,d)$

1.4 Vector Addition

You know that vectors can be represented by directed line segments, or by ordered pairs or ordered triples of numbers. Vectors will be more useful to you once you can combine them according to certain operations.

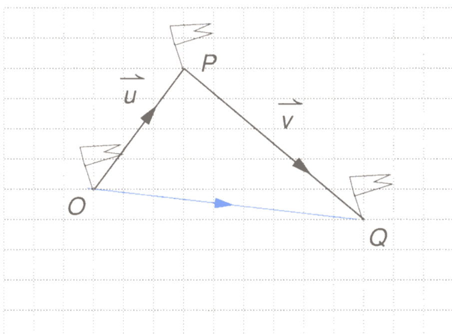
The first operation you will learn will be vector addition. Recall that there is a one-to-one correspondence between translations and vectors.

To find a definition for the addition of two vectors, consider two translations, represented by vectors $\vec{u}$ and $\vec{v}$, performed in succession. (This is sometimes called the **composition** of two translations.)



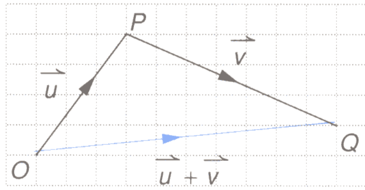
Recall that a vector can be drawn anywhere. To find a vector that represents the result of performing these translations in succession, let $\vec{u} = \overrightarrow{OP}$ and $\vec{v} = \overrightarrow{PQ}$. Then

the translation from O to P (vector $\vec{u} = \overrightarrow{OP}$)
followed by
the translation from P to Q (vector $\vec{v} = \overrightarrow{PQ}$)
gives
a "resultant" translation from O to Q (vector $\overrightarrow{OQ}$).



This idea suggests the following definition of **vector addition** for vectors represented by directed line segments.

DEFINITION Given any three points O, P, and Q: $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$



This definition is also called the **triangle law of vector addition.**

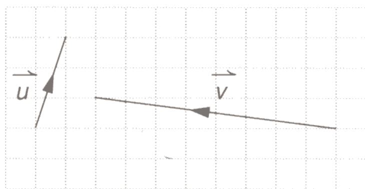
This law states that, if two vectors are represented by directed line segments such that the tail of the second is the same point as the tip of the first, then the directed line segment from the tail of the first to the tip of the second represents their vector sum.

Note 1 Because a triangle is always in a *plane*, the triangle law works in $\mathbb{V}_3$ as well.

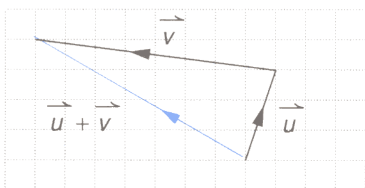
2 If you consider the vectors as displacements, then the resultant or sum vector is the *short cut* from the initial point to the final destination.

Example 1

Given the vectors $\vec{u}$ and $\vec{v}$ shown, draw a directed line segment representing the vector $\vec{u} + \vec{v}$.



Solution You must draw at least one of the vectors again, so that the tip of $\vec{u}$ is coincident with the tail of $\vec{v}$. (It *does not matter where* you draw them).



■

The following example will suggest a rule for vector addition for vectors represented by ordered pairs.

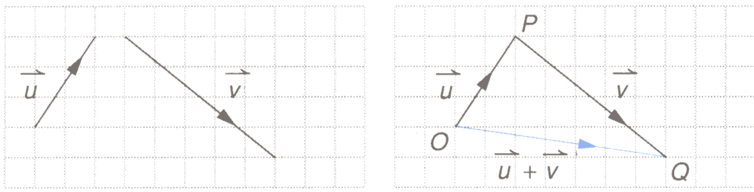
Example 2 Given the vectors $\vec{u} = \overrightarrow{(2,3)}$ and $\vec{v} = \overrightarrow{(5,-4)}$.

a) Use a grid to draw $\vec{u}$ and $\vec{u}$ wherever you wish; then draw them again so that $\overrightarrow{OP} = \vec{u}$ and $\overrightarrow{PQ} = \vec{v}$.

b) Express $\overrightarrow{OQ} = \vec{u} + \vec{v}$ as an ordered pair.

c) State an algebraic relationship among $\vec{u}$, $\vec{v}$, and $\vec{u} + \vec{v}$.

Solution a)



b) $\overrightarrow{OQ} = \vec{u} + \vec{v} = \overrightarrow{(7,-1)}$

c) $\vec{u} + \vec{v} = \overrightarrow{(2,3)} + \overrightarrow{(5,-4)} = \overrightarrow{(2+5, 3+(-4))} = \overrightarrow{(7,-1)}$. ■

It appears that the resultant vector can be obtained in component form by adding the first components of $\vec{u}$ and $\vec{v}$, then the second components of $\vec{u}$ and $\vec{v}$.

This leads to the following definition of vector addition in $\mathbb{V}_2$, for vectors represented by ordered pairs.

DEFINITION $\overrightarrow{(a,b)} + \overrightarrow{(p,q)} = \overrightarrow{(a+p, b+q)}$

Similarly, the definition of vector addition in $\mathbb{V}_3$ for vectors in component form is as follows.

DEFINITION $\overrightarrow{(a,b,c)} + \overrightarrow{(p,q,r)} = \overrightarrow{(a+p, b+q, c+r)}$

Example 3 Given $\vec{p} = \overrightarrow{(-5,6)}$, $\vec{q} = \overrightarrow{(4,3)}$, and $\vec{u} = \overrightarrow{(2,3,4)}$, $\vec{w} = \overrightarrow{(-1,1,5)}$ find

a) $\vec{p} + \vec{q}$,

b) $\vec{u} + \vec{w}$.

Solution

a) $\vec{p} + \vec{q} = \overrightarrow{(-5,6)} + \overrightarrow{(4,3)}$
$= \overrightarrow{(-5+4, 6+3)}$
$= \overrightarrow{(-1,9)}$

b) $\vec{u} + \vec{w} = \overrightarrow{(2,3,4)} + \overrightarrow{(-1,1,5)}$
$= \overrightarrow{(2+(-1), 3+1, 4+5)}$
$= \overrightarrow{(1,4,9)}$ ■

Example 4 An object is displaced according to the vector $\vec{u} = \overrightarrow{(4,7)}$, then according to the vector $\vec{v} = \overrightarrow{(1,-6)}$. What is the resultant displacement of the object?

Solution The resultant displacement is given by $\vec{u} + \vec{v}$

$\vec{u} + \vec{v} = \overrightarrow{(4,7)} + \overrightarrow{(1,-6)} = \overrightarrow{(4+1, 7-6)} = \overrightarrow{(5,1)}$. ■

Example 5 Given $\vec{u} = \overrightarrow{(2,-1,-6)}$, $\vec{v} = \overrightarrow{(2,5,x)}$, and $\vec{u} + \vec{v} = \overrightarrow{(y,4,-3)}$, calculate the values of x and y.

Solution $\vec{u} + \vec{v} = \overrightarrow{(2,-1,-6)} + \overrightarrow{(2,5,x)} = \overrightarrow{(2+2,-1+5,-6+x)} = \overrightarrow{(4,4,-6+x)}$
but $\vec{u} + \vec{v} = \overrightarrow{(y,4,-3)}$,
so $\overrightarrow{(4,4,-6+x)} = \overrightarrow{(y,4,-3)}$.

Thus, from the definition of equality of vectors,
$y = 4$, and $-6 + x = -3$ so that $x = 3$. ■

Note: The calculation of the middle component, 4, confirms that the arithmetic is correct.

A mathematical object that can be represented by directed line segments, or by ordered pairs or ordered triples of numbers, is a vector provided that it obeys the laws of addition defined above.

Such a vector is the ordered triple used to describe the monthly sales of a realtor, for example,
$\vec{v} = \overrightarrow{(h,c,b)}$, where h = number of houses sold
c = number of condominiums sold
b = number of business locations sold.

Example 6 Suppose a real estate agent, Jessie LaRue, made the following sales in the winter of 1989:
in January, $\vec{j} = \overrightarrow{(2,1,1,)}$,
in February, $\vec{f} = \overrightarrow{(1,3,0)}$.

She then left town for her annual holiday for the next three months, so she made no further sales. Indicate, by means of a vector, how many properties of each type she sold during the winter period.

Solution $\vec{w} = \overrightarrow{(2+1,1+3,1+0)} = \overrightarrow{(3,4,1)}$
Thus she sold 3 houses, 4 condominiums and 1 business location. ■

SUMMARY

Component laws of vector addition:

in $\mathbb{V}_2$
$\overrightarrow{(a,b)} + \overrightarrow{(p,q)} = \overrightarrow{(a+p,b+q)}$

in $\mathbb{V}_3$
$\overrightarrow{(a,b,c)} + \overrightarrow{(p,q,r)} = \overrightarrow{(a+p,b+q,c+r)}$

Geometric law of vector addition (the triangle law):
$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$

Vectors are mathematical objects that may be represented either

by directed line segments that combine according to the triangle law of addition, or

by ordered pairs or ordered triples of numbers, that combine by the addition of components.

1.4 Exercises

1. Given the parallelogram $PQRS$, use the triangle law of vector addition to simplify

a) $\overrightarrow{PQ} + \overrightarrow{QR}$, **b)** $\overrightarrow{PS} + \overrightarrow{SR}$.

2. In the parallelogram $PQRS$, $\overrightarrow{PQ} = \overrightarrow{SR} = \vec{u}$, and $\overrightarrow{PS} = \overrightarrow{QR} = \vec{v}$. What can you conclude about the sums $\vec{u} + \vec{v}$ and $\vec{v} + \vec{u}$?

3. Given the regular hexagon $PQRSTU$ shown, where $\overrightarrow{PQ} = \vec{a}$, $\overrightarrow{QR} = \vec{b}$, and $\overrightarrow{RS} = \vec{c}$, find the following vectors in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

a) $\overrightarrow{TS}$

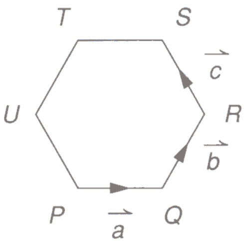
b) $\overrightarrow{UT}$

c) $\overrightarrow{PR}$

d) $\overrightarrow{US}$

e) $\overrightarrow{PT}$

f) $\overrightarrow{QS}$



4. A parallelepiped is a prism whose opposite sides are congruent parallelograms. Given the parallelepiped shown, find a single vector equal to each of the following.

a) $\overrightarrow{AB} + \overrightarrow{BC}$

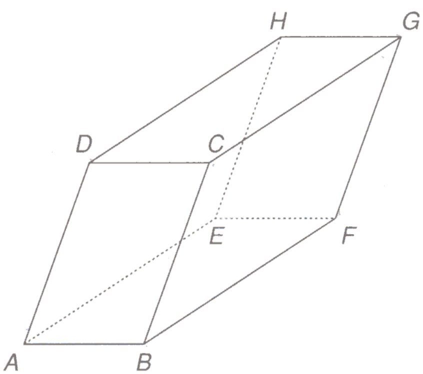
b) $\overrightarrow{AB} + \overrightarrow{FG}$

c) $\overrightarrow{CG} + \overrightarrow{FG}$

d) $\overrightarrow{AB} + (\overrightarrow{CG} + \overrightarrow{FG})$

e) $\overrightarrow{DH} + \overrightarrow{CB}$

f) $\overrightarrow{HC} + \overrightarrow{BF}$



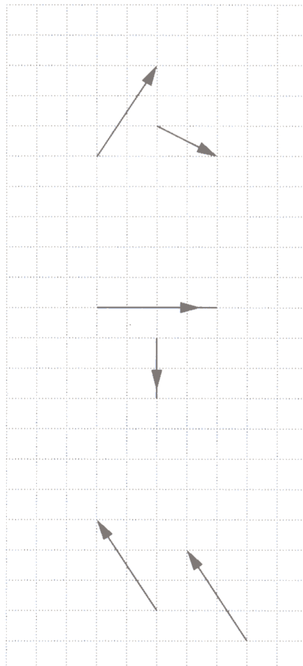
5. In question 4, if $\overrightarrow{AB} = \vec{u}$, $\overrightarrow{AE} = \vec{v}$, and $\overrightarrow{AD} = \vec{w}$, express each vector sum of parts a)–d) in terms of $\vec{u}$, $\vec{v}$, $\vec{w}$.

6. Redraw the following vectors appropriately to find a directed line segment representing their sum.

a)

b)

c)



7. If $\vec{a} = \overrightarrow{(3,5)}$, $\vec{b} = \overrightarrow{(2,-7)}$, $\vec{c} = \overrightarrow{(5,-2)}$, show that $\vec{a} + \vec{b} = \vec{c}$ by drawing the three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ on a grid.

8. An object is displaced in a plane according to the vector $\vec{u} = \overrightarrow{(-1,4)}$, then displaced again according to the vector $\vec{v} = \overrightarrow{(-1,-4)}$.

a) Calculate the resultant displacement, $\vec{w}$.

b) Draw the three displacements $\vec{u}$, $\vec{v}$, $\vec{w}$, on a grid.

1.4 Exercises

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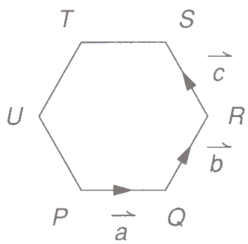
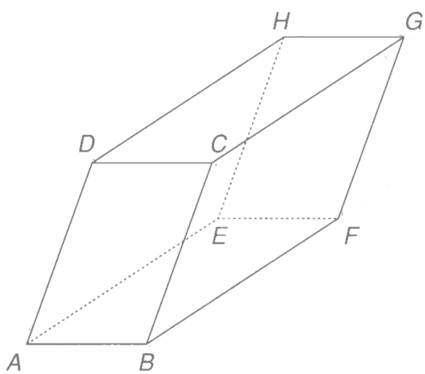
   a) $\overrightarrow{TS}$
   b) $\overrightarrow{UT}$
   c) $\overrightarrow{PR}$
   d) $\overrightarrow{US}$
   e) $\overrightarrow{PT}$
   f) $\overrightarrow{QS}$



4. A parallelepiped is a prism whose opposite sides are congruent parallelograms. Given the parallelepiped shown, find a single vector equal to each of the following.
   a) $\overrightarrow{AB} + \overrightarrow{BC}$ d) $\overrightarrow{AB} + (\overrightarrow{CG} + \overrightarrow{FG})$
   b) $\overrightarrow{AB} + \overrightarrow{FG}$ e) $\overrightarrow{DH} + \overrightarrow{CB}$
   c) $\overrightarrow{CG} + \overrightarrow{FG}$ f) $\overrightarrow{HC} + \overrightarrow{BF}$



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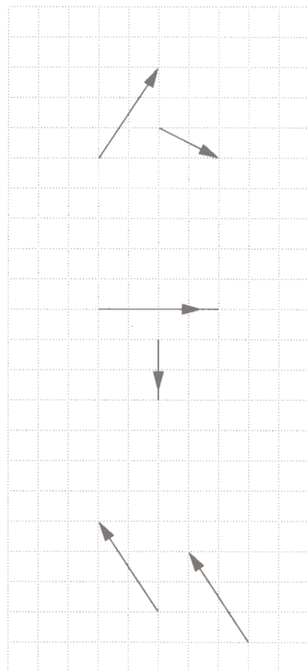
6. Redraw the following vectors appropriately to find a directed line segment representing their sum.

a)

b)

c)



7. If $\vec{a} = \overrightarrow{(3,5)}$, $\vec{b} = \overrightarrow{(2,-7)}$, $\vec{c} = \overrightarrow{(5,-2)}$, show that $\vec{a} + \vec{b} = \vec{c}$ by drawing the three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ on a grid.

8. An object is displaced in a plane according to the vector $\vec{u} = \overrightarrow{(-1,4)}$, then displaced again according to the vector $\vec{v} = \overrightarrow{(-1,-4)}$.
   a) Calculate the resultant displacement, $\vec{w}$.
   b) Draw the three displacements $\vec{u}$, $\vec{v}$, $\vec{w}$, on a grid.

9. Given $\vec{p} = \overrightarrow{(2,3)}, \vec{q} = \overrightarrow{(-1,5)}, \vec{r} = \overrightarrow{(3,-4)}$, find the following vector sums.
   a) $\vec{p} + \vec{q}$
   b) $\vec{p} + \vec{r}$
   c) $\vec{q} + \vec{r}$
   d) $\vec{r} + \vec{q}$

10. Given $\vec{u} = \overrightarrow{(0,1,-2)}, \vec{v} = \overrightarrow{(3,-3,7)}$, $\vec{w} = \overrightarrow{(-4,-5,1)}$, find the following vector sums.
   a) $\vec{u} + \vec{v}$
   b) $\vec{u} + \vec{w}$
   c) $\vec{v} + \vec{w}$
   d) $\vec{w} + \vec{v}$

11. What conclusion can you draw from parts c) and d) of questions 9 and 10?

12. Given that $\vec{u} = \overrightarrow{(2,5)}$ and $\vec{v} = \overrightarrow{(4,3)}$,
   a) calculate $\vec{u} + \vec{v}$ as an ordered pair
   b) draw $\vec{u}$, $\vec{v}$, and $\vec{u} + \vec{v}$ on a grid
   c) calculate $|\vec{u}|$, $|\vec{v}|$, and $|\vec{u} + \vec{v}|$.
   d) Does $|\vec{u}| + |\vec{v}| = |\vec{u} + \vec{v}|$? Explain.

13. Repeat question 12 for $\vec{u} = \overrightarrow{(-1,3)}$, $\vec{v} = \overrightarrow{(-2,6)}$. Do you get the same answer for part d)? Explain.

14. Given $\vec{u} = \overrightarrow{(2,5)}, \vec{v} = \overrightarrow{(4,3)}, \vec{w} = \overrightarrow{(1,-2)}$, $\vec{p} = \vec{u} + \vec{v}$, and $\vec{q} = \vec{v} + \vec{w}$
   a) calculate $\vec{p} + \vec{w}$ as an ordered pair
   b) calculate $\vec{u} + \vec{q}$ as an ordered pair.

   What conclusion can you draw from your results?

15. Given $\vec{u} = \overrightarrow{(3,p,q)}, \vec{v} = \overrightarrow{(-1,p,7)}$, and $\vec{u} + \vec{v} = \overrightarrow{(2,8,-2)}$, find the values of p and q.

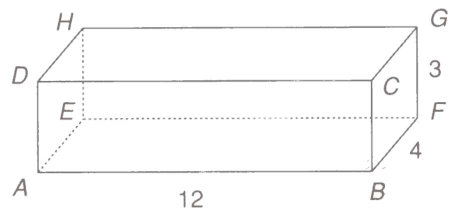
16. The realtor Jessie LaRue, back in town for July and August, notes $\vec{j} = \overrightarrow{(1,2,1)}$ according to the number of houses, condominiums, and business locations she sold during July. Her summer sales, for the two months July and August, are represented by the vector $\vec{s} = \overrightarrow{(2,2,1)}$. Write the vector $\vec{a}$, representing her sales in August, as an ordered triple.

17. A skater at a point A on a frozen lake goes 240 m towards the north, then 100 m towards the east to arrive at point B. Draw a vector diagram to represent the resultant displacement $\overrightarrow{AB}$. What is the magnitude of this displacement? What is the bearing of B from A?

18. A particle is displaced 5 cm along bearing 295°, then 8 cm along bearing 190°. Find the magnitude, correct to 2 decimal places, and direction of the resultant displacement.

19. Two vectors $\vec{p}$ and $\vec{q}$ are drawn so that they have a common tail and form an angle of 110°. If $|\vec{p}| = 7$, $|\vec{q}| = 3$, calculate the following.
   a) $|\vec{p} + \vec{q}|$, correct to 3 significant digits
   b) the angle θ between $\vec{p}$ and $(\vec{p} + \vec{q})$, to the nearest degree

20. Given the rectangular box shown, where $|\overrightarrow{AB}| = 12$, $|\overrightarrow{BF}| = 4$, and $|\overrightarrow{FG}| = 3$, calculate the following.
   a) $|\overrightarrow{AB} + \overrightarrow{BF}|$
   b) $|(\overrightarrow{AB} + \overrightarrow{BF}) + \overrightarrow{FG}|$
   c) Show that $|\overrightarrow{AB} + \overrightarrow{BF}| \leqslant |\overrightarrow{AB}| + |\overrightarrow{BF}|$



21. The relationship in question 20 part c) is known as the *triangle inequality*. Why? There is a special case of three points A, B and F which make the *equality* hold true. How then are the points A, B, and F positioned?

Between Pigeons and Problem Solving

If six people are in a room, prove that at least three of them are mutual acquaintances or at least three are mutual strangers.

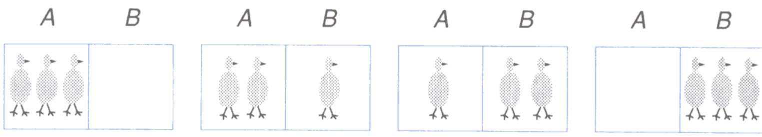
What does this problem have to do with pigeons? There is a very simple idea in discrete mathematics called "the pigeonhole principle" that will enable you to solve this problem.

In its simplest form the pigeonhole principle is stated as follows.

If m pigeons are placed in k pigeonholes and $m > k$, then at least one of the holes must contain at least two pigeons.

To understand the principle, consider the case of $m = 3$ and $k = 2$, that is, three pigeons and two pigeon holes.

The diagrams show all possible situations.



At least two pigeons are in one of the holes in each of the four cases.

You can also see the truth of the principle for three pigeons and two pigeonholes in another way.

Suppose you try to distribute the pigeons as evenly as possible in an attempt to avoid two pigeons in one hole. Then you would put one pigeon in A and a second pigeon in B. This still leaves one pigeon to be pigeonholed, so that one of A or B must contain two pigeons.

The following problems can be solved using the pigeonhole principle.

You have 10 identical red socks and 12 identical yellow socks in a drawer. It is so dark that you cannot see. How many socks must you remove from the drawer so that you will have at least two socks of the same colour?

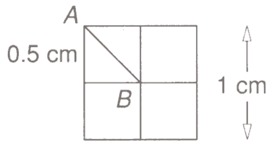
Think of the colours as pigeonholes and the socks as pigeons. With only two pigeonholes (colours), if you select three pigeons (socks) you must have at least two pigeons in the same hole. This means you must have two socks of the same colour.

You select, at random, 5 points in the xy-plane whose coordinates are integers. The points can be joined by a total of 10 line segments. Prove that at least one of the line segments contains a third point whose coordinates are integers.

The coordinates of the 5 points can be separated into 4 classes as follows.
(even, even) (even, odd) (odd, even) (odd, odd)

Since there are 5 points (pigeons) and 4 classes (pigeonholes), at least two points must be in the same class. If these points are (a,b) and (c,d), then their midpoint $\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$ must have coordinates that are integers. This is true because a and b will be both even or both odd. Also c and d will be both even or both odd. Since the sum of two odd numbers and the sum of two even numbers are each divisible evenly by two, both $\frac{a+b}{2}$ and $\frac{c+d}{2}$ will be integers.

Five points are randomly selected in a square whose sides are 1 cm. Prove that at least two of the points are at most a distance $\sqrt{0.5}$ cm apart.



Divide the square into four congruent squares (pigeonholes). Because of the pigeonhole principle at least two of the points (pigeons) must be located in the same small square, say in the top left one. Now the furthest apart these two points can be is at the corners of a diagonal, say at A and B. But by the Pythagorean theorem, $AB = \sqrt{0.5^2 + 0.5^2} = \sqrt{0.25 + 0.25} = \sqrt{0.5}$ Hence there must be at least two points that are at most $\sqrt{0.5}$ cm apart.

You may wish to try these problems.

1. Five points are randomly selected in an equilateral triangle whose sides are 2 cm. Prove that at least two of the points are at most a distance 1 cm apart.

2. 26 distinct numbers are selected from among the first 50 natural numbers. Prove that at least two of these numbers must be consecutive.

3. How many playing cards must you draw from a deck of 52 playing cards to be certain that at least two cards are from the same suit?

4. How many students must be in your school to be certain that at least two of them have the same birthday?

The following problems, the second of which is the problem at the beginning, need the generalized pigeonhole principle for their solution. This is stated as follows.

If more than sn pigeons are placed in n pigeonholes, then at least one hole must contain at least $s + 1$ pigeons.

5. How many students must be in your school to be certain that at least four of them have the same birthday?

6. If six people are in a room, prove that at least three of them are mutual acquaintances or at least three are mutual strangers.

1.5 Properties of Vector Addition

Sulu and Mary met downtown and decided to shop separately before having lunch together. Mary walked 3 blocks south, then 4 blocks east. Sulu walked 4 blocks east, then 3 blocks south. Did they then arrive at the same spot to meet for lunch?

Vector addition has two important properties which you will discover in the examples that follow.

Example 1 Given that $\vec{u} = \overrightarrow{(2,3)}$ and $\vec{v} = \overrightarrow{(5,-1)}$, find the following.

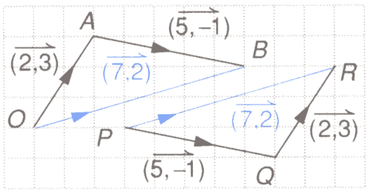
a) $\vec{u} + \vec{v}$, geometrically as a directed line segment, and algebraically as an ordered pair

b) $\vec{v} + \vec{u}$, also geometrically and algebraically.

c) Draw conclusions from your results.

Solution

geometric (a, b):



algebraic:

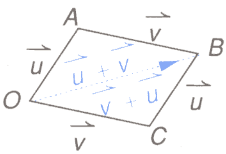
a) $\vec{u} + \vec{v} = \overrightarrow{(2,3)} + \overrightarrow{(5,-1)}$
$= \overrightarrow{(2+5, 3-1)} = \overrightarrow{(7,2)}$

b) $\vec{v} + \vec{u} = \overrightarrow{(5,-1)} + \overrightarrow{(2,3)}$
$= \overrightarrow{(5+2, -1+3)} = \overrightarrow{(7,2)}$

c) $\overrightarrow{OB} = \overrightarrow{PR} = \overrightarrow{(7,2)}$, $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ $\qquad$ $\vec{u} + \vec{v} = \vec{v} + \vec{u} = \overrightarrow{(7,2)}$ ■

Thus, you can add two vectors in either order and obtain the same resultant vector.

This holds true for any vectors. By virtue of the diagram below (diagrams of Example 1 combined) the **triangle law of vector addition** is sometimes known as the **parallelogram law of vector addition.**



Given any vectors $\vec{u}$ and $\vec{v}$, the following property holds.

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$

Vector addition is thus said to be commutative.

Note: The triangle law also shows that commutativity holds, as follows.

$$\vec{u} + \vec{v} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

and

$$\vec{v} + \vec{u} = \overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OB}$$

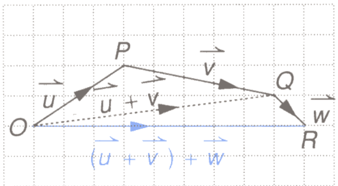
(Observe that this can be checked on the above diagram.)

Example 2 Given $\vec{u} = \overrightarrow{(3,2)}$, $\vec{v} = \overrightarrow{(5,-1)}$, and $\vec{w} = \overrightarrow{(1,-1)}$, find the following.

a) $[\vec{u} + \vec{v}] + \vec{w}$, geometrically as a directed line segment, and algebraically as an ordered pair
b) $\vec{u} + [\vec{v} + \vec{w}]$, also geometrically and algebraically.
c) Draw conclusions from your results.

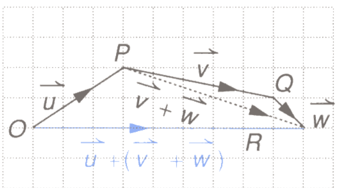
Solution geometric algebraic

a) $[\vec{u} + \vec{v}] + \vec{w} = \overrightarrow{OR}$



$$[\vec{u} + \vec{v}] + \vec{w}$$
$$= [\overrightarrow{(3+5, 2-1)}] + \overrightarrow{(1,-1)}$$
$$= \overrightarrow{(8,1)} + \overrightarrow{(1,-1)}$$
$$= \overrightarrow{(9,0)}$$

b) $\vec{u} + [\vec{v} + \vec{w}] = \overrightarrow{OR}$



$$\vec{u} + [\vec{v} + \vec{w}]$$
$$= \overrightarrow{(3,2)} + [\overrightarrow{(5+1, -1-1)}]$$
$$= \overrightarrow{(3,2)} + \overrightarrow{(6,-2)}$$
$$= \overrightarrow{(9,0)}$$

c) $[\vec{u} + \vec{v}] + \vec{w} = \vec{u} + [\vec{v} + \vec{w}]$ $\quad$ $[\vec{u} + \vec{v}] + \vec{w} = \vec{u} + [\vec{v} + \vec{w}]$ ■

The following property holds true for any vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$.

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

Vector addition is thus said to be associative.

The Associative Property of Vector Addition

Given any three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, draw the vectors so that $\vec{u} = \overrightarrow{OP}$, $\vec{v} = \overrightarrow{PQ}$, and $\vec{w} = \overrightarrow{QR}$. The triangle law shows that associativity holds as follows.

$$(\overrightarrow{OP} + \overrightarrow{PQ}) + \overrightarrow{QR} = \overrightarrow{OQ} + \overrightarrow{QR} = \overrightarrow{OR}$$

and
$$\overrightarrow{OP} + (\overrightarrow{PQ} + \overrightarrow{QR}) = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}$$

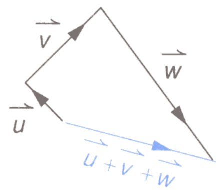
(Observe that this can be checked on the above diagram.)
Since the operation of vector addition is associative, you do not *need* the brackets. You have a perfectly clear meaning for $\vec{u} + \vec{v} + \vec{w}$,

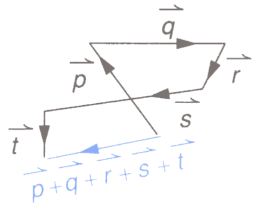
namely, $\vec{u} + \vec{v} + \vec{w} = (\vec{u} + \vec{v}) + \vec{w}$.

This means the additions are performed in order one after the other. Similarly, $\vec{p} + \vec{q} + \vec{r} + \vec{s} + \ldots$ also indicates that the additions are to be performed in order, one after the other.

Thus, you can add any number of vectors (as long as they have the same dimensions), as follows.

Geometrically, use the triangle law repeatedly, with each vector's tip joined to the tail of the next one, as shown in the diagrams below. This law, for adding more than two vectors represented by directed line segments, is known as the **polygon law of vector addition.**





Algebraically, add the respective components, for example,
$\overrightarrow{(3,5)} + \overrightarrow{(4,-6)} + \overrightarrow{(-2,8)} = \overrightarrow{(3+4-2, 5-6+8)} = \overrightarrow{(5,7)}$

The polygon law of vector addition can also be expressed as follows.

For any points A, B, C, D, E: $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{AE}$

Note: The polygon is not necessarily in one plane; a polygon that does not lie in a plane is known as a **skew polygon.**

Example 3

Given the vectors $\vec{u} = \overrightarrow{(5,2,0)}$, $\vec{v} = \overrightarrow{(0,6,0)}$, $\vec{w} = \overrightarrow{(0,0,-4)}$,

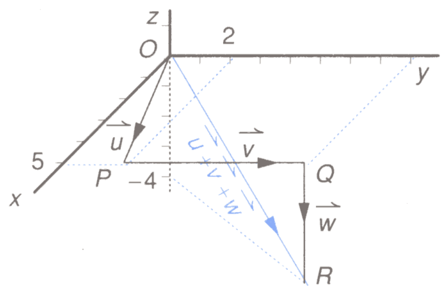
a) find $\vec{u}+\vec{v}+\vec{w}$

b) show a geometrical interpretation of this sum by drawing the tail of the vector $\vec{u}$ at the origin O of a 3-space coordinate system.

Solution

a) $\vec{u}+\vec{v}+\vec{w}$
$= \overrightarrow{(5,2,0)} + \overrightarrow{(0,6,0)} + \overrightarrow{(0,0,-4)}$
$= \overrightarrow{(5+0+0, 2+6+0, 0+0-4)}$
$= \overrightarrow{(5,8,-4)}$

b) Using the polygon law,
$\overrightarrow{OP} + \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{OR}$



This is a good example of a skew polygon (*OPQR* is a skew quadrilateral). ■

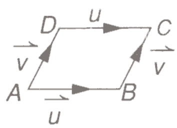
SUMMARY

Vector addition is commutative: $\vec{u}+\vec{v} = \vec{v}+\vec{u}$

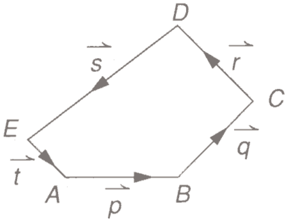
Vector addition is associative: $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$
(thus brackets are not required for multiple additions.)

1.5 Exercises

1. Name a single vector equal to each of the following sums.
   a) $\vec{u} + \vec{v}$
   b) $\vec{v} + \vec{u}$



2. Repeat question 1 for the following sums.
   a) $\vec{p} + \vec{q} + \vec{r} + \vec{s} + \vec{t}$
   b) $\vec{p} + \vec{q}$
   c) $\vec{p} + \vec{q} + \vec{r}$
   d) $\vec{p} + \vec{q} + \vec{r} + \vec{s}$
   e) $\vec{q} + \vec{p}$



3. a) State the commutative law for vector addition.
   b) Use any two vectors $\vec{u}$ and $\vec{v}$ to illustrate geometrically that vector addition is commutative.
   c) Use the vectors $\vec{u} = \overrightarrow{(-3,5)}$ and $\vec{v} = \overrightarrow{(4,1)}$ to illustrate algebraically that vector addition is commutative.

4. a) State the associative law for vector addition.
   b) Use any three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ to illustrate geometrically that vector addition is associative.
   c) Use the vectors $\vec{u} = \overrightarrow{(-3,5)}$, $\vec{v} = \overrightarrow{(4,1)}$, and $\vec{w} = \overrightarrow{(2,-7)}$ to illustrate algebraically that vector addition is associative.

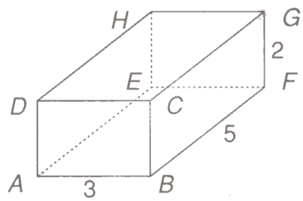
5. State which of the four usual operations in $\mathbb{R}$, namely addition, multiplication, subtraction and division, are commutative and associative.
   If an operation does *not* have either the commutative property or the associative property, give an example to demonstrate this. (Such examples, used to prove that a property does not hold true, are known as counterexamples. *One* counterexample is sufficient for disproof.)

6. a) State whether or not the operation of exponentiation is associative in $\mathbb{R}$. That is, state whether or not it is true that, for any three real numbers a, b, c, $[a^b]^c = a^{[b^c]}$
   b) If exponentiation is associative, prove it. If it is not, disprove it by using a counterexample.

7. Simplify the following.
   a) $\overrightarrow{AP} + \overrightarrow{PC}$
   b) $\overrightarrow{YZ} + \overrightarrow{XY}$
   c) $\overrightarrow{AD} + \overrightarrow{DC} + \overrightarrow{CB}$
   d) $\overrightarrow{QR} + \overrightarrow{RQ}$

8. Given any five points in space P, Q, R, S, T,
   a) explain why $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{ST} = \overrightarrow{PT}$
   b) simplify $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{ST} + \overrightarrow{TP}$.
   c) What is the magnitude of your answer to b)?
   d) Are there any special cases in which any of the answers to a) or b) do not hold true?

9. Given the rectangular box shown, calculate the following
   a) $|\overrightarrow{AB} + \overrightarrow{BF} + \overrightarrow{FG}|$
   b) $|\overrightarrow{BF} + \overrightarrow{FE} + \overrightarrow{EH}|$
   c) $|\overrightarrow{BF} + \overrightarrow{FG} + \overrightarrow{GH}|$
   d) $|\overrightarrow{AB} + \overrightarrow{BF} + \overrightarrow{FE}|$



10. Find $\vec{u} + \vec{v} + \vec{w}$ in the following cases.
    a) $\vec{u} = \overrightarrow{(2,-5)}$, $\vec{v} = \overrightarrow{(1,3)}$, $\vec{w} = \overrightarrow{(-6,1)}$,
    b) $\vec{u} = \overrightarrow{(0,0,3)}$, $\vec{v} = \overrightarrow{(4,4,-1)}$, $\vec{w} = \overrightarrow{(1,1,-2)}$,

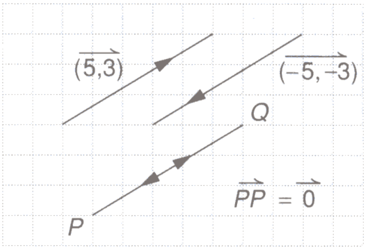
11. a) Show a geometric interpretation of each of the sums in question 10.
    b) Which of the above cases, if any, give an example of a skew polygon?

12. Given the skew quadrilateral $OPQR$ where $\overrightarrow{OP} = \overrightarrow{(5,2,0)}$, $\overrightarrow{PQ} = \overrightarrow{(0,6,0)}$, and $\overrightarrow{QR} = \overrightarrow{(0,0,-4)}$, calculate
    a) $|\overrightarrow{OQ}|$,
    b) $|\overrightarrow{OR}|$,
    c) the angle QOR.

1.6 Vector Subtraction

The Zero Vector

In $\mathbb{V}_2$, suppose $\vec{u} = \overrightarrow{(5,3)}$ and $\vec{v} = \overrightarrow{(-5,-3)}$,
then $\vec{u} + \vec{v} = \overrightarrow{(5,3)} + \overrightarrow{(-5,-3)} = \overrightarrow{(5-5,3-3)} = \overrightarrow{(0,0)}$.
The vector $\overrightarrow{(0,0)}$ is called the **zero vector** of $\mathbb{V}_2$: $\overrightarrow{(0,0)} = \vec{0}$.



The vector $\vec{v} = \overrightarrow{(-5,-3)}$ is written $-\vec{u}$ and is called the **opposite** of $\vec{u}$.

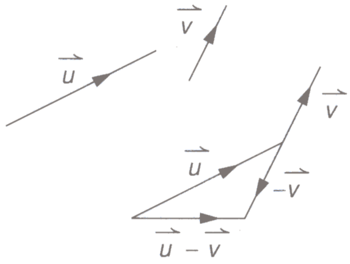
Geometrically, if the triangle law is used:
$\overrightarrow{PQ} + \overrightarrow{QP} = \overrightarrow{PP}$,
thus $\overrightarrow{PP} = \vec{0} = \overrightarrow{(0,0)}$

In general, $\vec{v} = \overrightarrow{(a,b)}$ and $-\vec{v} = \overrightarrow{(-a,-b)}$ are called **opposite vectors**, and $\overrightarrow{(a,b)} + \overrightarrow{(-a,-b)} = \vec{0}$.

These concepts hold in a similar fashion in $\mathbb{V}_3$: $-\vec{v} = \overrightarrow{(-a,-b,-c)}$ is the opposite of vector $\vec{v} = \overrightarrow{(a,b,c)}$, and the zero vector is $\vec{0} = \overrightarrow{(0,0,0)}$.

The vectors $\vec{v}$ and $-\vec{v}$ are said to have **opposite directions**.

Subtraction of Vectors



The subtraction of vectors is defined by 'adding the opposite' vector.

Thus

$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$$

The geometric interpretation of this is shown in the diagram.

Now the zero vector can be defined by

$$\vec{u} - \vec{u} = \vec{0}$$

As an example of vector subtraction, if $\vec{u} = \overrightarrow{(8,6)}$ and $\vec{v} = \overrightarrow{(3,2)}$,
$\vec{u} - \vec{v} = \vec{u} + (-\vec{v}) = \overrightarrow{(8,6)} + \overrightarrow{(-3,-2)} = \overrightarrow{(8-3,6-2)} = \overrightarrow{(5,4)}$.

In general, $\overrightarrow{(a,b)} - \overrightarrow{(p,q)} = \overrightarrow{(a,b)} + \overrightarrow{(-p,-q)} = \overrightarrow{(a-p,b-q)}$.

Thus vector subtraction in component form is carried out as follows.

in $\mathbb{V}_2$,
$$\overrightarrow{(a,b)} - \overrightarrow{(p,q)} = \overrightarrow{(a-p,b-q)}$$

in $\mathbb{V}_3$,
$$\overrightarrow{(a,b,c)} - \overrightarrow{(p,q,r)} = \overrightarrow{(a-p,b-q,c-r)}$$

Example 1 a) Calculate the vector $\overrightarrow{(5,6)} - \overrightarrow{(1,4)}$.

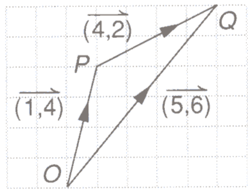
b) Demonstrate this subtraction by using a diagram.

Solution a) $\overrightarrow{(5,6)} - \overrightarrow{(1,4)}$

$= \overrightarrow{(5-1, 6-4)}$

$= \overrightarrow{(4,2)}$

b)



Notice that the diagram indicates that $\overrightarrow{OQ} - \overrightarrow{OP} = \overrightarrow{PQ}$. ■

This is a universal result, and is known as the **subtraction form of the triangle law.**

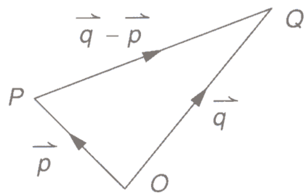
(Watch the order of the letters carefully.)

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

Note: When the law is written this way, any other single letter could be substituted for O. This very important property allows you to *choose* an origin, and this can help you through some tricky vector problems which you will encounter later.

The subtraction form of the triangle law can also be written as follows, using the abbreviations $\overrightarrow{OQ} = \vec{q}$, $\overrightarrow{OP} = \vec{p}$.

$\overrightarrow{PQ} = \vec{q} - \vec{p}$



Example 2 Use the subtraction form of the triangle law to simplify the following.

a) $\overrightarrow{ED} - \overrightarrow{EF}$

b) $\overrightarrow{CD} + \overrightarrow{BC} + \overrightarrow{AB}$

Solution a) Using E as origin, and applying the law directly,

$\overrightarrow{ED} - \overrightarrow{EF} = \overrightarrow{FD}$

b) Use any origin O, and the abbreviations $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$.

Then $\overrightarrow{CD} + \overrightarrow{BC} + \overrightarrow{AB}$

$= \vec{d} - \vec{c} + \vec{c} - \vec{b} + \vec{b} - \vec{a}$

$= \vec{d} - \vec{a}$

$= \overrightarrow{AD}$ ■

Example 3 a) Given the points in 2-space $A = (2,3)$ and $B = (7,4)$, find the vector $\overrightarrow{AB}$.

b) Given the points in 3-space $P = (2,0,-1)$ and $Q(-3,4,1)$, find the vector $\overrightarrow{PQ}$.

Solution a) You can use the points to determine the following position vectors.

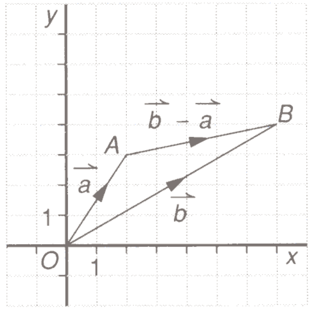
Since $A = (2,3)$, $\overrightarrow{OA} = \vec{a} = \overrightarrow{(2,3)}$

Since $B = (7,4)$, $\overrightarrow{OB} = \vec{b} = \overrightarrow{(7,4)}$

Now $\overrightarrow{AB} = \vec{b} - \vec{a}$

$= \overrightarrow{(7,4)} - \overrightarrow{(2,3)}$

$= \overrightarrow{(5,1)}$



b) In a similar manner, the position vectors of P and Q are respectively $\vec{p} = \overrightarrow{(2,0,1)}$ and $\vec{q} = \overrightarrow{(-3,4,1)}$.

Thus $\overrightarrow{PQ} = \vec{q} - \vec{p} = \overrightarrow{(-3,4,1)} - \overrightarrow{(2,0,1)} = \overrightarrow{(-5,4,0)}$. ■

In this example, you have used vector subtraction. Be careful not to attempt to 'subtract points'. Such an operation has *not* been defined.

The example leads to the following rules.

RULES

In $\mathbb{V}_2$, given points $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$, the vector $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \overrightarrow{(x_2 - x_1, y_2 - y_1)}$.
Thus $|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

In $\mathbb{V}_3$, given points $P_1 = (x_1,y_1,z_1)$ and $P_2 = (x_2,y_2,z_2)$, the vector $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \overrightarrow{(x_2 - x_1, y_2 - y_1, z_2 - z_1)}$.
Thus $|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Example 4 If vector $\overrightarrow{RS} = \overrightarrow{(-1,4,6)}$, and the point $S = (1,2,6)$, find the coordinates of R.

Solution Let $R = (x,y,z)$, so that $\vec{r} = \overrightarrow{(x,y,z)}$.

Since $\overrightarrow{RS} = \vec{s} - \vec{r}$

$\overrightarrow{(-1,4,6)} = \overrightarrow{(1,2,6)} - \overrightarrow{(x,y,z)}$

thus $\overrightarrow{(x,y,z)} = \overrightarrow{(1,2,6)} - \overrightarrow{(-1,4,6)}$

$= \overrightarrow{(2,-2,0)}$

Hence the coordinates of R are $(2,-2,0)$. ■

You might note that examples involving vector subtraction can be resolved using vector addition. For instance, in Example 3a), you could state that $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{(-2,-3)} + \overrightarrow{(7,4)} = \overrightarrow{(5,1)}$, as before.

However, you will find that handling subtraction and addition equally well gives you a lot more flexibility.

Further Properties of Vector Addition

Observe that, given any vector $\vec{v}$,

$$\vec{v} + \vec{0} = \vec{v} \text{ and } \vec{0} + \vec{v} = \vec{v}.$$

Because of this property, the zero vector is called the **neutral element**, or the **identity element** for vector addition.

Observe also that, given any vector $\vec{v}$,

$$\vec{v} + (-\vec{v}) = \vec{0} \text{ and } (-\vec{v}) + \vec{v} = \vec{0}.$$

Because of this property, $(-\vec{v})$ and $\vec{v}$ are called **inverses** of each other, for vector addition.

SUMMARY

$\vec{u} = \overrightarrow{(a,b)}$ and $-\vec{u} = \overrightarrow{(-a,-b)}$ are opposite vectors in $\mathbb{V}_2$

$\vec{v} = \overrightarrow{(a,b,c)}$ and $-\vec{v} = \overrightarrow{(-a,-b,-c)}$ are opposite vectors in $\mathbb{V}_3$

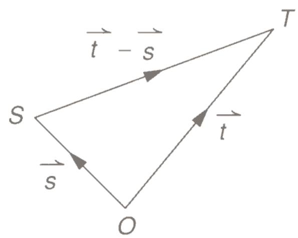
The sum of two opposite vectors is the zero vector.

Subtraction of Vectors

$\overrightarrow{ST} = \overrightarrow{OT} - \overrightarrow{OS} = \vec{t} - \vec{s}$

$\overrightarrow{(a,b)} - \overrightarrow{(p,q)} = \overrightarrow{(a-p,b-q)}$ in $\mathbb{V}_2$

$\overrightarrow{(a,b,c)} - \overrightarrow{(p,q,r)} = \overrightarrow{(a-p,b-q,c-r)}$ in $\mathbb{V}_3$



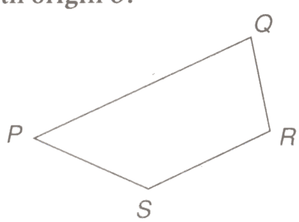
1.6 Exercises

1. Determine whether the following are true or false.
   a) $\vec{AX} - \vec{XC} = \vec{AC}$
   b) $\vec{AX} - \vec{AC} = \vec{CX}$
   c) $\vec{CA} - \vec{XA} = \vec{CX}$
   d) $-\vec{AX} - \vec{XC} = \vec{AC}$

2. If Q, R and S are any points, express the vector $\vec{RS}$ in terms of $\vec{QR}$ and $\vec{QS}$.

3. Simplify by using the subtraction form of the triangle law.
   a) $\vec{OP} - \vec{OR}$
   b) $\vec{QZ} - \vec{QX}$
   c) $\vec{AD} + \vec{CB} + \vec{DC}$
   d) $\vec{AB} + \vec{CA} + \vec{BC}$
   e) $\vec{OB} - \vec{OA} + \vec{BC}$

4. $PQRS$ is a plane quadrilateral, and O is *any* other point in space. Express the following vectors in subtraction form, using position vectors with origin O.
   a) $\vec{PQ}$
   b) $\vec{QR}$
   c) $\vec{RS}$
   d) $\vec{RP}$



5. $ABCD$ is a square and O is *any* point. If $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$, and $\vec{OC} = \vec{c}$, express the vector $\vec{OD} = \vec{d}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$. (Hint: Recall that in a square, $\vec{AB} = \vec{DC}$; write these vectors as subtractions.)

6. Use the properties of vector addition to prove that $(-\vec{q}) + (\vec{p} + \vec{q}) = \vec{p}$

7. Given $\vec{p} = \overrightarrow{(2,3)}$, $\vec{q} = \overrightarrow{(-1,5)}$, $\vec{r} = \overrightarrow{(3,-4)}$, find the following.
   a) $\vec{p} - \vec{q}$
   b) $\vec{q} - \vec{r}$
   c) $\vec{r} - \vec{p}$
   d) $\vec{p} - \vec{r}$

8. Given $\vec{u} = \overrightarrow{(0,1,-2)}$, $\vec{v} = \overrightarrow{(3,-3,7)}$, $\vec{w} = \overrightarrow{(-4,-5,1)}$, find the following.
   a) $\vec{u} - \vec{v}$
   b) $\vec{v} - \vec{w}$
   c) $\vec{w} - \vec{u}$
   d) $\vec{u} - \vec{w}$

9. What conclusion can you draw from parts c) and d) of questions 7 and 8?

10. Given the vectors of questions 7 and 8, simplify the following.
    a) $\vec{p} + \vec{q} - \vec{r}$
    b) $\vec{p} - \vec{q} + \vec{r}$
    c) $\vec{w} - \vec{v} + \vec{u}$
    d) $-\vec{u} - \vec{v} - \vec{w}$

11. Given $\vec{a} = \overrightarrow{(11,-2,k)}$, $\vec{b} = \overrightarrow{(m,n,8)}$, and $\vec{a} - \vec{b} = \overrightarrow{(5,0,1)}$, find the values of the real numbers k, m, and n.

12. Given the points $P(2,3)$, $Q(7,4)$, and $R(-1,1)$, use vector subtraction to determine the following vectors in component form.
    a) $\vec{PQ}$
    b) $\vec{QR}$
    c) $\vec{RP}$

13. Use the answers of question 12 to calculate the sum $\vec{PQ} + \vec{QR} + \vec{RP}$. Explain your result.

14. Given the points $L(1,0,-7)$, $M(2,2,9)$ and $N(-3,-4,6)$, use vector subtraction to determine the following vectors in component form.
    a) $\vec{LM}$
    b) $\vec{MN}$
    c) $\vec{NL}$

15. Given that R is at $(12,10)$, S is at $(15,11)$, and $\vec{OP} = \vec{RS}$, find the coordinates of the point P.

16. Repeat question 15 for $R(6,-1,3)$ and $S(2,-5,-1)$.

17. Given the points $A(2,5)$, $B(3,-1)$, $C(-4,6)$, $D(0,2)$, find the coordinates of the point M such that $\vec{OM} = \vec{AB} - \vec{CD}$.

18. Given two perpendicular vectors $\vec{u}$ and $\vec{v}$, whose magnitudes are not necessarily the same,
    a) find $|\vec{u} + \vec{v}|$ in terms of $|\vec{u}|$ and $|\vec{v}|$,
    b) show that $|\vec{u} + \vec{v}| = |\vec{u} - \vec{v}|$.

1.7 Multiplication of a Vector by a Scalar

Consider the vectors $\vec{u} = \overrightarrow{(1,2)}$, $\vec{v} = \overrightarrow{(2,4)}$, $\vec{w} = \overrightarrow{(5,10)}$.

Is there any relationship among them?

By vector addition, you can see that

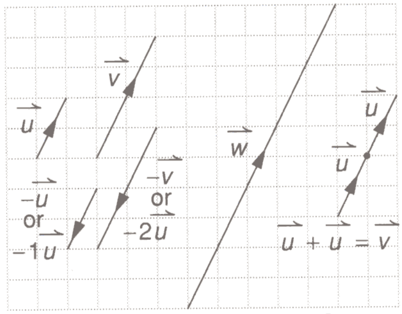
$$\vec{v} = \vec{u} + \vec{u}$$

and $\vec{w} = \vec{u} + \vec{u} + \vec{u} + \vec{u} + \vec{u}$.

It seems natural to write

$$\vec{v} = 2\vec{u}$$

and $\vec{w} = 5\vec{u}$



Note: $\vec{u}$, $\vec{v}$, and $\vec{w}$ are all in the same direction (that is, parallel, and pointing the same way).

In the last section, you saw that the vector $-\vec{u}$ had the same length as $\vec{u}$, but the opposite direction. You can think of $-\vec{u}$ as being the same as $-1\vec{u}$.

Similarly, $-\vec{v}$ has the same length as $\vec{v}$, but the opposite direction. Thus $-\vec{v} = -2\vec{u}$.

Although these give rise to an unusual combination of symbols

$$\vec{w} = 5\vec{u}$$

vector — real number — vector

this operation is accepted. It is called multiplication of a vector by a scalar or **scalar multiplication of a vector.**

If k is a *positive* scalar and $\vec{u}$ is a vector, then $k\vec{u}$ is a vector in the *same direction* as $\vec{u}$, with length k times the length of $\vec{u}$.

If k is *negative*, then the direction of $k\vec{u}$ is reversed (that is, the direction of $k\vec{u}$ is *opposite to the direction* of $\vec{u}$), and the *length* of $k\vec{u}$ is $|k|$ times *the length* of $\vec{u}$.

In general,

the vectors $\vec{u}$ and $k\vec{u}$ are parallel;

the length of vector $k\vec{u}$ is $|k||\vec{u}|$.

Note: $|k|$ means the absolute value of the real number k, whereas $|\vec{u}|$ means the length of the vector $\vec{u}$.

This should not lead to any confusion, because both length and absolute value are always numbers (scalars) greater than or equal to zero.

In some texts, a distinction is made by writing the length of the vector $\vec{u}$ as $\|\vec{u}\|$.

Scalar multiplication can also be defined for vectors in component form, as shown in the following example.

Example 1 Calculate the following by using vector addition, given $\vec{u} = \overrightarrow{(1,3)}$.

a) $\vec{v} = 2\vec{u}$ b) $\vec{w} = 5\vec{u}$

Solution a) $\vec{v} = \vec{u} + \vec{u} = \overrightarrow{(1,3)} + \overrightarrow{(1,3)}$
$= \overrightarrow{(2,6)} = \overrightarrow{(2 \times 1,2 \times 3)}$ or
$2\overrightarrow{(1,3)} = \overrightarrow{(2 \times 1,2 \times 3)}$

b) $\vec{w} = \vec{u} + \vec{u} + \vec{u} + \vec{u} + \vec{u} = \overrightarrow{(1,3)} + \overrightarrow{(1,3)} + \overrightarrow{(1,3)} + \overrightarrow{(1,3)} + \overrightarrow{(1,3)}$
$= \overrightarrow{(1 + 1 + 1 + 1 + 1,3 + 3 + 3 + 3 + 3)}$
$= \overrightarrow{(5,15)} = \overrightarrow{(5 \times 1,5 \times 3)}$ or
$5\overrightarrow{(1,3)} = \overrightarrow{(5 \times 1,5 \times 3)}$. ■

This leads to the following definition for the multiplication of a vector in component form by a scalar.

DEFINITION

$$k\overrightarrow{(x,y)} = \overrightarrow{(kx,ky)} \text{ in } \mathbb{V}_2 \text{ or } k\overrightarrow{(x,y,z)} = \overrightarrow{(kx,ky,kz)} \text{ in } \mathbb{V}_3$$

Example 2 Find the lengths of vectors $\vec{u} = \overrightarrow{(1,3)}$, $\vec{v} = \overrightarrow{(2,6)}$ and $\vec{w} = \overrightarrow{(5,15)}$ of Example 1, to verify that the length of a vector $k\vec{a}$ equals $|k|$ times the length of $\vec{a}$.

Solution $|\vec{u}| = \sqrt{1^2 + 3^2} = \sqrt{10}$
$|\vec{v}| = \sqrt{2^2 + 6^2} = \sqrt{40} = 2\sqrt{10} = 2|\vec{u}|$, as expected.
$|\vec{w}| = \sqrt{5^2 + 15^2} = \sqrt{250} = 5\sqrt{10} = 5|\vec{u}|$, as expected.
An alternate way of calculating $|\vec{w}|$ is to use some factoring:
$|\vec{w}| = \sqrt{5^2 + (3 \times 5)^2} = \sqrt{5^2(1^2 + 3^2)} = 5\sqrt{10}$, as before. ■

You can now combine the operations of vector addition and multiplication by a scalar.

Example 3 Express as a single vector $3\overrightarrow{(1,-6,4)} + 2\overrightarrow{(5,0,2)} - 4\overrightarrow{(1,1,1)}$.

Solution

$3\overrightarrow{(1,-6,4)} + 2\overrightarrow{(5,0,2)} - 4\overrightarrow{(1,1,1)}$
$= \overrightarrow{(3,-18,12)} + \overrightarrow{(10,0,4)} - \overrightarrow{(4,4,4)}$
$= \overrightarrow{(3 + 10 - 4,-18 + 0 - 4,12 + 4 - 4)}$
$= \overrightarrow{(9,-22,12)}$ ■

At this point, you will look at an example which will illustrate the various properties of the operation of multiplication of a vector by a scalar.

Example 4 Given the vectors $\vec{u} = \overrightarrow{(2,0,-3)}$ and $\vec{v} = \overrightarrow{(1,-6,-2)}$, calculate the following.

a) $5(4\vec{u})$ b) $5(\vec{u} + \vec{v})$ c) $(2 + 3)\vec{u}$

Solution a) $5(4\vec{u}) = 5\overrightarrow{(4 \times 2, 4 \times 0, 4 \times [-3])} = 5\overrightarrow{(8,0,-12)} = \overrightarrow{(40,0,-60)}$.
Note that this result is the same as
$(5 \times 4)\vec{u} = 20\overrightarrow{(2,0,-3)} = \overrightarrow{(20 \times 2, 20 \times 0, 20 \times [-3])} = \overrightarrow{(40,0,-60)}$.

b) $5(\vec{u} + \vec{v}) = 5\overrightarrow{(2 + 1, 0 - 6, -3 - 2)} = 5\overrightarrow{(3,-6,-5)} = \overrightarrow{(15,-30,-25)}$.
Note that this result is the same as
$5\vec{u} + 5\vec{v} = 5\overrightarrow{(2,0,-3)} + 5\overrightarrow{(1,-6,-2)}$
$= \overrightarrow{(10,0,-15)} + \overrightarrow{(5,-30,-10)} = \overrightarrow{(15,-30,-25)}$.

c) $(2 + 3)\vec{u} = 5\vec{u}$,
but also $2\vec{u} + 3\vec{u} = \vec{u} + \vec{u} + \vec{u} + \vec{u} + \vec{u} = 5\vec{u}$,
because vector addition is associative.
$5\vec{u} = 5\overrightarrow{(2,0,-3)} = \overrightarrow{(10,0,-15)}$ ■

The above example suggests the following properties of multiplication of a vector by a scalar.

PROPERTIES

For any vectors $\vec{u}$, $\vec{v}$, and scalars k, m:

1. $k(m\vec{u}) = (km)\vec{u}$
2. $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$
3. $(k + m)\vec{u} = k\vec{u} + m\vec{u}$

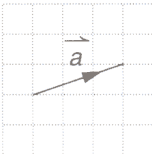
Example 5 Simplify $\overrightarrow{BD} - \overrightarrow{AD} + \overrightarrow{AB} - \overrightarrow{CB}$.

Solution Using vector subtraction from a common origin O, and the abbreviations $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$, $\overrightarrow{OD} = \vec{d}$,

$$\begin{aligned}\overrightarrow{BD} - \overrightarrow{AD} + \overrightarrow{AB} - \overrightarrow{CB} &= \vec{d} - \vec{b} - (\vec{d} - \vec{a}) + \vec{b} - \vec{a} - (\vec{b} - \vec{c}) \\ &= \vec{d} - \vec{b} - \vec{d} + \vec{a} + \vec{b} - \vec{a} - \vec{b} + \vec{c} \\ &= \vec{c} - \vec{b} \\ &= \overrightarrow{BC}\end{aligned}$$ ■

1.7 Exercises

1. Copy the vector $\vec{a}$ shown onto a grid.



On the same grid, draw representatives of the following vectors.

$2\vec{a}$
$3\vec{a}$
$5\vec{a}$
$-2\vec{a}$
$-\vec{a}$

2. Given $\vec{u} \| \vec{v}$, $\vec{p} \| \vec{q}$, but $\vec{u} \nparallel \vec{p}$, state which of the following are parallel to $\vec{u}$.
   a) $2\vec{u}$
   b) $-3\vec{v}$
   c) $4\vec{p}$
   d) $-5\vec{q}$

3. *ABCD* is a parallelogram and *M* is the midpoint of *BC*. If $\overrightarrow{AB} = 2\vec{u}$ and $\overrightarrow{BM} = \vec{v}$, express the following vectors in terms of $\vec{u}$ and $\vec{v}$.
   a) $\overrightarrow{AM}$
   b) $\overrightarrow{BC}$
   c) $\overrightarrow{AD}$
   d) $\overrightarrow{DB}$
   e) $\overrightarrow{CA}$

4. Draw on a grid the vectors $\vec{p} = \overrightarrow{(2,-5)}$, $\vec{q} = \overrightarrow{(2,5)}$, $\vec{r} = \overrightarrow{(6,-15)}$, $\vec{s} = \overrightarrow{(-4,10)}$. Which of these vectors are parallel? Which are in the same direction?

5. The points *A*, *B*, *C* are such that $\overrightarrow{AB} = \overrightarrow{(2,-3)}$ and $\overrightarrow{BC} = \overrightarrow{(4,-6)}$.
   a) Express $\overrightarrow{BC}$ in terms of $\overrightarrow{AB}$.
   b) Express $\overrightarrow{AB}$ in terms of $\overrightarrow{BC}$.
   c) Show that the points *A*, *B*, and *C* must lie on the same straight line (that is, points *A*, *B*, *C* are collinear).

6. Using the points of question 5, calculate the following vectors in component form.
   a) $\overrightarrow{BA}$ b) $\overrightarrow{AC}$ c) $\overrightarrow{CA}$

7. Express each of the following as a single vector.
   a) $5\overrightarrow{(2,-1)} + 2\overrightarrow{(3,2)} + \overrightarrow{(-7,1)}$
   b) $\overrightarrow{(1,-4,-3)} - 4\overrightarrow{(2,3,-5)} + 2\overrightarrow{(2,2,2)}$
   c) $\frac{1}{2}\overrightarrow{(2,-7,1)} + \frac{5}{2}\overrightarrow{(3,2,1)}$

8. Given the vectors $\vec{u} = \overrightarrow{(2,1,-3)}$, $\vec{v} = \overrightarrow{(1,0,4)}$, $\vec{w} = \overrightarrow{(4,1,5)}$, express each of the following as a single vector.
   a) $3\vec{u} - 4\vec{v} + 2\vec{w}$,
   b) $\vec{u} + 2\vec{v} - \vec{w}$.
   c) What does your result in b) indicate about the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$?

9. Given the vectors of question 8, calculate the following.
   a) $|3\vec{u} - 4\vec{v} + 2\vec{w}|$
   b) $|\vec{u} + 2\vec{v} - \vec{w}|$

10. Given the vectors $\vec{u} = \overrightarrow{(-1,3)}$ and $\vec{v} = \overrightarrow{(2,1)}$, illustrate, both algebraically and geometrically, that
    a) $3(2\vec{u}) = 6\vec{u}$,
    b) $4(\vec{u} + \vec{v}) = 4\vec{u} + 4\vec{v}$
    c) $(3 + 1)\vec{u} = 3\vec{u} + \vec{u}$

11. Simplify each of the following.
    a) $3\vec{v} + 4\vec{v} - 6\vec{v}$
    b) $2(\vec{u} - 2\vec{v}) + 4\vec{v}$
    c) $-5(\vec{u} + 2\vec{v}) + 5\vec{u} + 9\vec{v}$
    d) $3\vec{u} - \vec{v} + 2\vec{w} - (\vec{u} - \vec{v}) - 2(\vec{u} + \vec{w})$

12. Simplify the following, by using a common origin O and vector subtraction.

a) $\overrightarrow{AC} - \overrightarrow{DC} + \overrightarrow{DB}$

b) $\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{BC} - \overrightarrow{AD}$

c) $\overrightarrow{AD} - \overrightarrow{BC} + \overrightarrow{DC} - 2\overrightarrow{AB}$

13. Given that $\vec{b} = k\vec{a}$, where $k \in \mathbb{R}$ is not zero, use the properties of scalar multiplication to prove that $\vec{a} = \frac{1}{k}\vec{b}$.

14. Prove the properties

$k(m\vec{u}) = (km)\vec{u}$

$k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

$(k + m)\vec{u} = k\vec{u} + m\vec{u}$,

for any scalars k, m and vectors $\vec{u}$, $\vec{v}$.

15. Given any non-zero vector $\vec{v}$, calculate the length of $\frac{1}{|\vec{v}|}\vec{v}$.

16. a) If A is the point (2,3) and B is the point (8,1), calculate in component form the vector $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}$.

b) Plot the points O, A, B, M in a 2-space coordinate system of origin O.

c) Show that M is the midpoint of AB.

17. Given any three points O, A, and B, a point M is positioned such that $\overrightarrow{MA} + \overrightarrow{MB} = \vec{0}$.

a) Express $\overrightarrow{MA}$ in terms of $\overrightarrow{MB}$.

b) How are the points A, B, and M related geometrically?

c) Use vector subtraction, with origin O, to find $\overrightarrow{OM}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

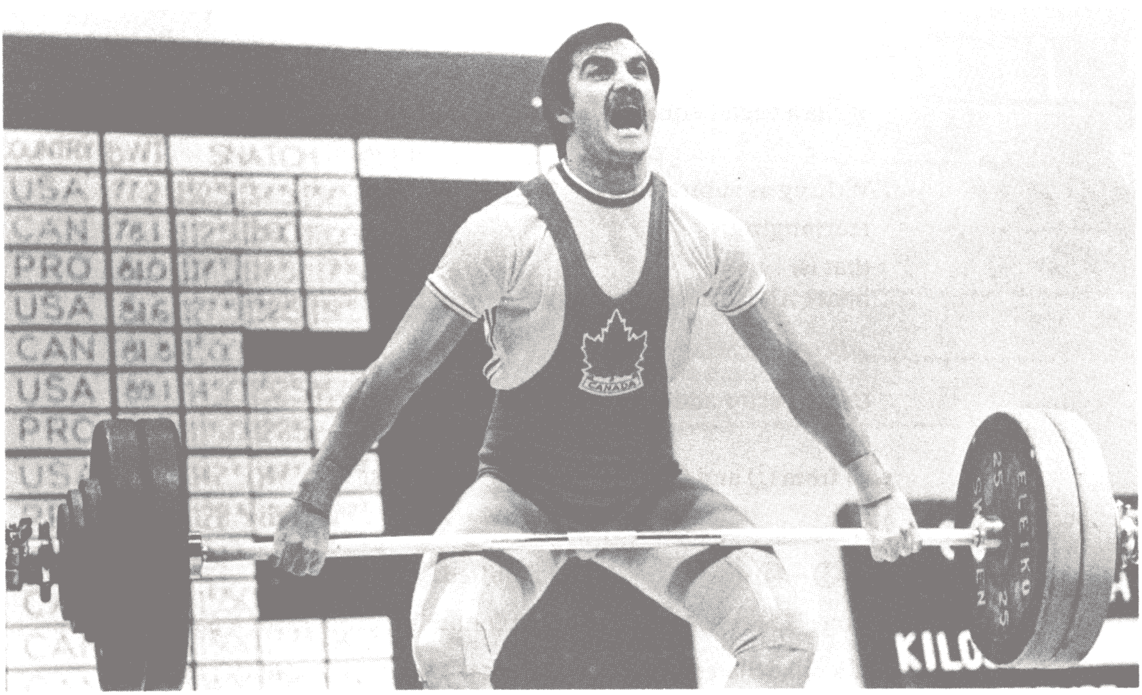
18. Given the two vectors $\vec{u}$ and $\vec{v}$, and $|\vec{u}| = 4$. Find the value of $|\vec{u} + \vec{v}|$ in each of the following cases.

a) $\vec{v} = 3\vec{u}$

b) $\vec{v} = -5\vec{u}$

c) $\vec{u}$ and $\vec{v}$ perpendicular, $|\vec{v}| = 3$

d) $\vec{u}$ perpendicular to $(\vec{u} + \vec{v})$, $|\vec{v}| = 9.6$



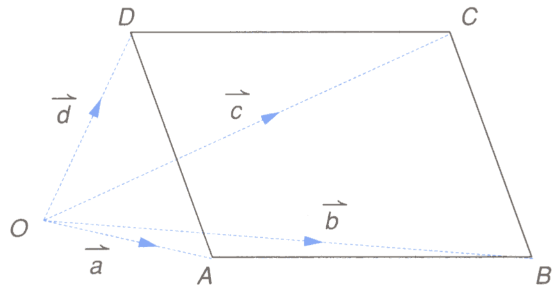
1.8 Applications of Vector Subtraction and Multiplication by a Scalar

You can now apply the properties you have learned, to solve certain geometric problems by using vectors.

Most problems can be solved by using vector addition. However, the technique of using the subtraction form of the triangle law, together with a common origin, unravels many geometric problems quite neatly. For clarity, the abbreviations $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}, \ldots$ are used for the position vectors of $A, B, \ldots$

Example 1 $ABCD$ is a quadrilateral in which AB is parallel to and congruent to DC. Prove that sides AD and BC are also parallel and congruent.

Solution Let O be any point. Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$, and $\overrightarrow{OD} = \vec{d}$.



Write a vector equality from what is given in the problem.

$$\overrightarrow{AB} = \overrightarrow{DC}$$

Writing as subtractions, $\quad \vec{b} - \vec{a} = \vec{c} - \vec{d}$

rearranging, $\quad \vec{d} - \vec{a} = \vec{c} - \vec{b}$

that is, $\quad \overrightarrow{AD} = \overrightarrow{BC}$

hence AD is parallel to and congruent to BC, as required.

Alternative Solution

Using vector addition,	$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$	①
and	$\overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC}$	②
so from ① and ②,	$\overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AB} + \overrightarrow{BC}$	③
but	$\overrightarrow{DC} = \overrightarrow{AB}$	④ [given]

so ③ − ④ gives $\overrightarrow{AD} = \overrightarrow{BC}$, as required. ■

Note: This example proves the following property of a parallelogram.

PROPERTY

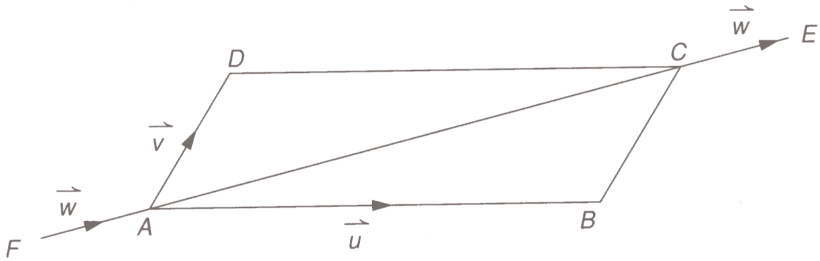
If a quadrilateral $ABCD$ is such that $AB = DC$ and $AB \| DC$, then that quadrilateral is a parallelogram.

The next example uses vector addition, and the above property.

Example 2 A parallelogram $ABCD$ is such that $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{AD} = \vec{v}$. AC is produced to E, and CA is produced to F such that $\overrightarrow{FA} = \overrightarrow{CE} = \vec{w}$.

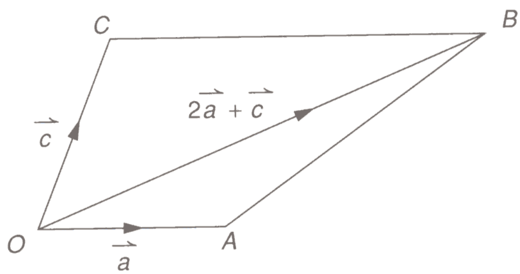
a) Express the vectors $\overrightarrow{FB}$ and $\overrightarrow{DE}$ in terms of $\vec{u}$, $\vec{v}$, and $\vec{w}$.
b) Hence show that $FBED$ is a parallelogram.

Solution



a) $\overrightarrow{FB} = \overrightarrow{FA} + \overrightarrow{AB} = \vec{w} + \vec{u}$
$\overrightarrow{DE} = \overrightarrow{DC} + \overrightarrow{CE} = \vec{u} + \vec{w}$
b) $\overrightarrow{FB} = \overrightarrow{DE}$ from a)
Hence, $FB = DE$ and $FB \| DE$.
Thus, by the above property, $FBED$ is a parallelogram. ■

Example 3 $OABC$ is a quadrilateral with $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OC} = \vec{c}$, and $\overrightarrow{OB} = 2\vec{a} + \vec{c}$.



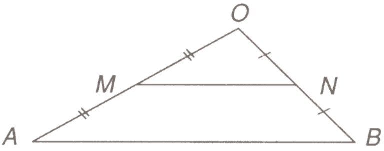
a) Express the vectors $\overrightarrow{AB}$ and $\overrightarrow{CB}$ in terms of $\vec{a}$ and $\vec{c}$.
b) Draw geometric conclusions about the quadrilateral.

Solution a) Using the subtraction form of the triangle law, with O as origin,
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2\vec{a} + \vec{c}) - \vec{a} = \vec{a} + \vec{c}$ and
$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC} = (2\vec{a} + \vec{c}) - \vec{c} = 2\vec{a}$

b) Thus $\overrightarrow{CB}$ is in the *same direction* as $\vec{a}$ and *twice the length* of $\vec{a}$. In geometrical terms, that means that CB is parallel to OA, thus the quadrilateral is a *trapezoid*. ■

Example 4 In triangle OAB, M is the midpoint of side OA and N is the midpoint of side OB. Prove that the side AB is parallel to segment MN, and that the length of AB is twice the length of MN.

Solution Choose O as origin, and use the abbreviations $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OM} = \vec{m}$, $\overrightarrow{ON} = \vec{n}$.



Write vector statements from what is given in the problem.

① $\vec{m} = \frac{1}{2}\vec{a}$ and ② $\vec{n} = \frac{1}{2}\vec{b}$

Subtracting the equations ② − ①,

$$\vec{n} - \vec{m} = \frac{1}{2}\vec{b} - \frac{1}{2}\vec{a}$$

$$= \frac{1}{2}(\vec{b} - \vec{a})$$

thus $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AB}$

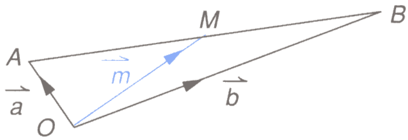
which proves that AB is parallel to MN, and is twice the length of MN, as required. ■

Note: In this example, you have proved the midpoint theorem.

THEOREM In any triangle, the line segment joining the midpoints of any two sides is parallel to, and half the length of, the third side.

Example 5 If M is the midpoint of AB, and O is any point, find an expression for $\overrightarrow{OM}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Solution



Use the abbreviations $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, and $\overrightarrow{OM} = \vec{m}$. Since M is the midpoint of AB, the lengths $AM = MB$, and $AM \| MB$,

thus

$$\overrightarrow{AM} = \overrightarrow{MB}$$

$$\vec{m} - \vec{a} = \vec{b} - \vec{m}$$

$$2\vec{m} = \vec{a} + \vec{b}$$

$$\vec{m} = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}$$ ■

FORMULA Note: This example gives the following mid-point formula. If M is the midpoint of AB, and O is any point, then $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}$

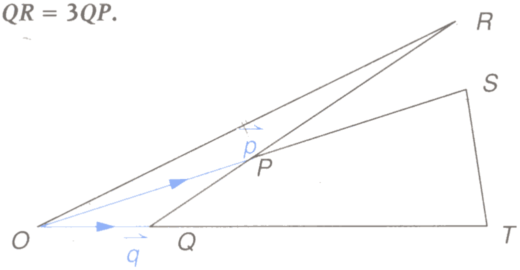
1.8 Exercises

1. $OABC$ is a quadrilateral in which $\overrightarrow{OA} = 2\vec{v} - \vec{u}$, $\overrightarrow{OC} = \vec{u}$, and $\overrightarrow{OB} = 2\vec{v}$. Express $\overrightarrow{AB}$ and $\overrightarrow{CB}$ in terms of $\vec{u}$ and $\vec{v}$, and thus describe the quadrilateral.

2. Describe the quadrilateral of question 1, given that $|2\vec{v} - \vec{u}| = |\vec{u}|$.

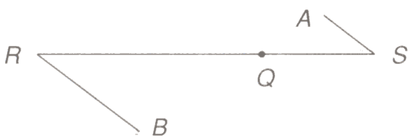
3. $OABC$ is a quadrilateral in which $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OC} = \vec{c}$, and $\overrightarrow{OB} = \vec{a} + \frac{1}{2}\vec{c}$. Express $\overrightarrow{AB}$ and $\overrightarrow{CB}$ in terms of $\vec{a}$ and $\vec{c}$, and thus describe the quadrilateral.

4. In the diagram, $\overrightarrow{OP} = \vec{p}$ and $\overrightarrow{OQ} = \vec{q}$. It is also known that $\overrightarrow{OS} = 2\overrightarrow{OP}$, $\overrightarrow{OT} = 4\overrightarrow{OQ}$, and $\overrightarrow{QR} = 3\overrightarrow{QP}$.



   a) Express the following vectors in terms of $\vec{p}$ and $\vec{q}$: $\overrightarrow{OS}$, $\overrightarrow{OT}$, $\overrightarrow{QP}$, $\overrightarrow{TS}$, $\overrightarrow{QR}$, $\overrightarrow{OR}$, $\overrightarrow{TR}$.
   b) Hence show that the points T, S and R are on straight line.

5. Given that $\overrightarrow{RQ} = 2\overrightarrow{QS}$, and $\overrightarrow{RB} = 2\overrightarrow{AS}$, use vectors to show that AQB is a straight line.



6. Assuming that the opposite sides of a parallelogram are parallel and congruent, prove that vector addition has the commutative property. That is, prove that for any vectors $\vec{u}$ and $\vec{v}$, $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

7. If $k\vec{a} + m\vec{b} - (k + m)\vec{c} = \vec{0}$, prove that $(\vec{a} - \vec{c})$ and $(\vec{b} - \vec{c})$ are parallel vectors.

8. Points A and B are the midpoints of sides PQ and SR respectively of parallelogram $PQRS$. Prove that $PBRA$ is a parallelogram.

9. In parallelogram $PQRS$, prove that $\overrightarrow{QS} + \overrightarrow{RP} = 2\overrightarrow{RS}$.

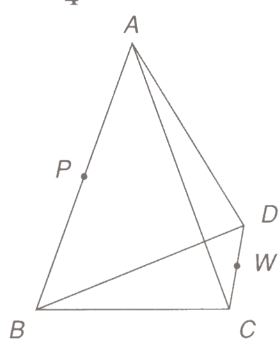
10. D, E and F are the midpoints of sides AB, BC and CA respectively of a triangle ABC. If O is any point, prove that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF}$.

11. Given any quadrilateral $ABCD$ (which may be skew), let K, L, M, and N be the midpoints of sides AB, BC, CD, and DA respectively. Prove that $KLMN$ is a parallelogram.

12. Given a quadrilateral $ABCD$ (which may be skew), let M be the midpoint of AC and N be the midpoint of BD.
   a) Show that $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 4\overrightarrow{MN}$.
   b) If M and N coincide, show that $\overrightarrow{AB} = \overrightarrow{DC}$.
   c) What kind of a quadrilateral is $ABCD$ if M and N coincide?

13. Given a tetrahedron (a triangular pyramid) $ABCD$, let P be the midpoint of AB and W be the midpoint of CD. M is the midpoint of PW. The position vectors of A, B, C, D, M from some origin O are $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{m}$ respectively.

   Show that $\vec{m} = \frac{1}{4}(\vec{a} + \vec{b} + \vec{c} + \vec{d})$.



1.9 Unit Vectors— Standard Basis of a Vector Space

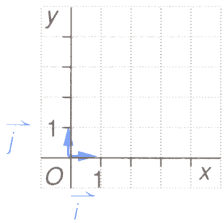
In this section, you will learn how to express vectors of $\mathbb{V}_2$ and $\mathbb{V}_3$ in terms of certain vectors whose length is 1 unit.

Any vector whose length is 1 is called a **unit vector**.

Example 1 Find the lengths of $\vec{i} = \overrightarrow{(1,0)}$ and $\vec{j} = \overrightarrow{(0,1)}$.

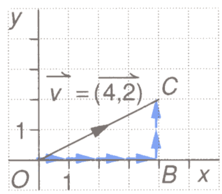
Solution $|\vec{i}| = \sqrt{1^2 + 0^2} = 1$ and similarly,
$|\vec{j}| = 1$.
$\vec{i}$ and $\vec{j}$ are thus unit vectors. ■

If you draw these vectors with their tails at the origin, O, you can see that they are unit vectors along the x and y axes respectively.



Example 2 Write the vector $\vec{v} = \overrightarrow{(4,2)}$ in terms of $\vec{i}$ and $\vec{j}$, using the operations of vector addition, and multiplication by a scalar.

Solution $\vec{v} = \overrightarrow{(4,2)}$ is the position vector of point $C(4,2)$.
$OB = 4$, so $\overrightarrow{OB} = 4\vec{i}$, and
$BC = 2$, so $\overrightarrow{BC} = 2\vec{j}$.
But $\vec{v} = \overrightarrow{OB} + \overrightarrow{BC}$
so $\overrightarrow{(4,2)} = 4\vec{i} + 2\vec{j}$. ■



A similar property holds true for any vector.

PROPERTY

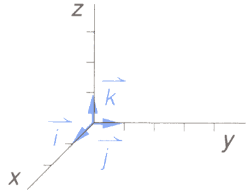
If $P = (x,y)$ is any point in 2-space, then
$\overrightarrow{OP} = x\vec{i} + y\vec{j}$ is the position vector of P or
$\overrightarrow{(x,y)} = x\vec{i} + y\vec{j}$

Every vector of $\mathbb{V}_2$ is equal to a position vector. Hence, every vector of $\mathbb{V}_2$ can be expressed in terms of the vectors $\vec{i}$ and $\vec{j}$. For this reason, $\vec{i}$ and $\vec{j}$ are called the **standard basis vectors of $\mathbb{V}_2$**.

In $\mathbb{V}_3$, the unit vectors along the x-axis, and y-axis, and z-axis are called $\vec{i}$, $\vec{j}$ and $\vec{k}$ respectively, where
$\vec{i} = \overrightarrow{(1,0,0)}$, $\vec{j} = \overrightarrow{(0,1,0)}$, and $\vec{k} = \overrightarrow{(0,0,1)}$.
Similarly, every vector of $\mathbb{V}_3$ can be expressed in terms of these three vectors according to the following property.

PROPERTY

If $P = (x,y,z)$ is any point in 3-space, then
$\overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}$ is the position vector of P or
$\overrightarrow{(x,y,z)} = x\vec{i} + y\vec{j} + z\vec{k}$



For this reason, $\vec{i}$, $\vec{j}$ and $\vec{k}$ are called the **standard basis vectors of** $\mathbb{V}_3$.

These basis vectors, whether in $\mathbb{V}_2$ or in $\mathbb{V}_3$, are all perpendicular or **orthogonal** to each other.
Furthermore, because they are all unit vectors, they are called **normed** vectors.

These words are combined and used to describe the set of vectors $\{\vec{i},\vec{j}\}$ as an **orthonormal basis of** $\mathbb{V}_2$.
Similarly, the set of vectors $\{\vec{i},\vec{j},\vec{k}\}$ is an **orthonormal basis of** $\mathbb{V}_3$.

You will study other bases of $\mathbb{V}_2$ and $\mathbb{V}_3$ further in chapter 2.

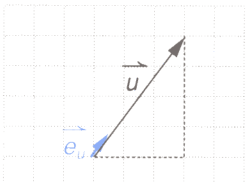
Other Unit Vectors

Example 3

a) Find a unit vector in the direction of $\vec{u} = \overrightarrow{(3,4)}$.
b) Verify that your solution is a unit vector.

Solution

a) The length $|\vec{u}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$. You must find a vector that has the same direction as $\vec{u}$, but whose length is only $\frac{1}{5}$ of $|\vec{u}|$. Call this vector $\vec{e}_u$: then $\vec{e}_u = \frac{1}{5}\vec{u} = \frac{1}{5}\overrightarrow{(3,4)} = \overrightarrow{\left(\frac{3}{5},\frac{4}{5}\right)}$.



b) $|\vec{e}_u| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{(9 + 16)}{25}} = \sqrt{1} = 1$, as required. ■

In general, given a vector $\vec{v}$,

the unit vector in the direction of $\vec{v}$ is $\vec{e}_v = \frac{1}{|\vec{v}|}\vec{v}$

Replacing $\vec{v}$ by its unit counterpart $\vec{e}_v$ is called **normalizing** $\vec{v}$.

Example 4 Normalize the following.

a) $\vec{v} = \overrightarrow{(-2,5)}$

b) $\vec{w} = \overrightarrow{(0.2,0.2,0.1)}$

Solution

a) Let $\vec{e}_v$ be a unit vector in the direction of $\vec{v}$. Because

$|\vec{v}| = \sqrt{(-2)^2 + 5^2} = \sqrt{29}$,

$$\vec{e}_v = \frac{1}{\sqrt{29}}\vec{v} = \frac{1}{\sqrt{29}}\overrightarrow{(-2,5)} = \overrightarrow{\left(-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)}.$$

b) Let $\vec{e}_w$ be a unit vector in the direction of $\vec{w}$. Because

$|\vec{w}| = \sqrt{0.2^2 + 0.2^2 + 0.1^2} = \sqrt{0.09} = 0.3$,

$$\vec{e}_w = \frac{1}{0.3}\vec{w} = \frac{10}{3}\overrightarrow{(0.2,0.2,0.1)} = \overrightarrow{\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)}. \quad ■$$

Example 5 Normalize $4\vec{v}$, where $\vec{v}$ is any vector.

Solution Let the unit vector in the direction of $4\vec{v}$ be $\vec{e}_{4v}$

$$\begin{aligned} \vec{e}_{4v} &= \frac{1}{|4\vec{v}|}(4\vec{v}) \\ &= \frac{1}{|4\vec{v}|}4\vec{v} \\ &= \frac{\vec{v}}{|\vec{v}|} \\ &= \vec{e}_v. \quad ■ \end{aligned}$$

This last example shows that there is only *one* unit vector in the direction of $\vec{v}$. In other words, the unit vector in the direction of $\vec{v}$ does *not* depend on the length of $\vec{v}$.

Vector Spaces

The properties of vectors that you have learned so far will now allow you to define the following.

DEFINITION

A vector space $\mathbb{V}$ is a set of mathematical objects called vectors, together with two operations, called vector addition and multiplication by a scalar, having the following properties.

PROPERTIES

Vector Addition

A1. $\mathbb{V}$ is closed under addition: $\vec{u}, \vec{v} \in \mathbb{V}$ implies $\vec{u} + \vec{v} \in \mathbb{V}$

A2. Addition is associative: $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

A3. There is a $\vec{0} \in \mathbb{V}$ such that for all $\vec{u} \in \mathbb{V}$, $\vec{u} + \vec{0} = \vec{u}$

A4. If $\vec{u} \in \mathbb{V}$, then there exists $-\vec{u} \in \mathbb{V}$ such that $\vec{u} + (-\vec{u}) = \vec{0}$

A5. Addition is commutative: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(These properties mean that $\mathbb{V}$ is a **commutative group** with respect to addition.)

Multiplication of a Vector by a Scalar

M1. If $\vec{u} \in \mathbb{V}$, $k \in \mathbb{R}$, then $k\vec{u} \in \mathbb{V}$

M2. $(km)\,\vec{u} = k(m\vec{u})$, $k, m \in \mathbb{R}$

M3. $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

M4. $(k + m)\,\vec{u} = k\vec{u} + m\vec{u}$

M5. There exists $1 \in \mathbb{R}$ such that $1\vec{u} = \vec{u}$

Whenever you work with vectors, you are working in a "vector space". You have been doing just that in this entire chapter. What you have above is a summary of all the properties of vectors, valid for $\mathbb{V}_2$, and $\mathbb{V}_3$, and any other vector space. You may need to refer back to these properties in the chapters to come.

Note: Just as a vector is different from a point, a vector space is *not* a set of points such as $\mathbb{R}^2$ or $\mathbb{R}^3$. (There are no *points* in a vector space.)

SUMMARY

A vector whose length is 1 is a unit vector (or a normed vector).

$\{\vec{i},\vec{j}\}$, where $\vec{i} = \overrightarrow{(1,0)}$ and $\vec{j} = \overrightarrow{(0,1)}$,
is the standard basis for the vector space $\mathbb{V}_2$
$\{\vec{i},\vec{j},\vec{k}\}$, where $\vec{i} = \overrightarrow{(1,0,0)}$, $\vec{j} = \overrightarrow{(0,1,0)}$, and $\vec{k} = \overrightarrow{(0,0,1)}$,
is the standard basis for the vector space $\mathbb{V}_3$
These sets of vectors are called orthonormal bases.

In $\mathbb{V}_2$, $\overrightarrow{(p,q)} = p\vec{i} + q\vec{j}$
In $\mathbb{V}_3$, $\overrightarrow{(p,q,r)} = p\vec{i} + q\vec{j} + r\vec{k}$

To normalize $\vec{v}$ is to find the unit vector $\vec{e}_v$ in the direction of $\vec{v}$:

$$\vec{e}_v = \frac{1}{|\vec{v}|}\vec{v}$$

1.9 Exercises

1. Express the following vectors as ordered pairs.
   a) $\vec{i}+\vec{j}$
   b) $-5\vec{i}+\vec{j}$
   c) $-2\vec{j}$

2. Express the following vectors of $\mathbb{V}_2$ in terms of $\vec{i}$ and $\vec{j}$.
   a) $\vec{u}=\overrightarrow{(2,-7)}$
   b) $\vec{v}=\overrightarrow{(6,1)}$
   c) $\vec{w}=\overrightarrow{(-3,0)}$

3. Express the following vectors as ordered triples.
   a) $\vec{i}+2\vec{j}+3\vec{k}$
   b) $4\vec{i}-\vec{k}$
   c) $-\vec{j}-\vec{k}$

4. Express the following vectors of $\mathbb{V}_3$ in terms of $\vec{i}$, $\vec{j}$ and $\vec{k}$.
   a) $\vec{u}=\overrightarrow{(2,-4,6)}$
   b) $\vec{v}=\overrightarrow{(0,-1,-1)}$
   c) $\vec{w}=\overrightarrow{(0,10,0)}$

5. Simplify the following.
   a) $6(3\vec{i}-\vec{j})$
   b) $-2(-\vec{i}+\vec{j}-\vec{k})$
   c) $5(3\vec{i}-\vec{k})$
   d) $\sqrt{2}(\sqrt{2}\,\vec{i}+\sqrt{8}\,\vec{j})$

6. Simplify the following.
   a) $5(\vec{i}+\vec{j})-3(2\vec{i}-\vec{j})$
   b) $-(4\vec{i}-2\vec{j}+\vec{k})+2(\vec{i}-5\vec{k})-2\vec{j}$
   c) $\frac{1}{2}(3\vec{i}+5\vec{j}-\vec{k})+\frac{5}{2}(-\vec{i}-\vec{j}-3\vec{k})+\vec{i}$

7. Which of the following are unit vectors?
   a) $\vec{a}=\overrightarrow{\left(\frac{1}{2},\frac{1}{2}\right)}$
   b) $\vec{b}=\overrightarrow{\left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right)}$
   c) $\vec{c}=\overrightarrow{\left(\frac{1}{\sqrt{5}},0,-\frac{2}{\sqrt{5}}\right)}$
   d) $\vec{d}=\overrightarrow{\left(\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right)}$
   e) $\vec{e}=\overrightarrow{(0.2,0.4,0.4)}$
   f) $\vec{f}=\overrightarrow{\left(\frac{3}{13},-\frac{4}{13},\frac{12}{13}\right)}$

8. Normalize the following vectors.
   a) $\vec{u}=\overrightarrow{(3,-4)}$
   b) $\vec{v}=\overrightarrow{(0.1,0,0)}$
   c) $\vec{w}=\overrightarrow{(2,6,-1)}$
   d) $\vec{z}=\overrightarrow{(1,1,1)}$
   e) $\vec{p}=\overrightarrow{(\sqrt{7},3)}$
   f) $\vec{q}=\overrightarrow{(5,-4)}$

9. Given the non-zero vector $\vec{r}=\overrightarrow{(x,y,z)}$, prove that the unit vector in the direction of $\vec{r}$ is $\vec{e}_r=\overrightarrow{\left(\frac{x}{|\vec{r}|},\frac{y}{|\vec{r}|},\frac{z}{|\vec{r}|}\right)}$

10. Find the unit vector in the direction of $\overrightarrow{PQ}$ in each of the following cases.
    a) P is the point $(-3,6)$ and Q is the point $(4,1)$
    b) P is the point $(2,-3,5)$ and Q is the point $(1,-1,0)$

11. Given that $\vec{u}=\overrightarrow{(-5,12)}$, $\vec{v}=\overrightarrow{\left(\frac{3}{2},2,-6\right)}$ and $\vec{w}=\overrightarrow{(2,0,7)}$, find the unit vectors $\vec{e}_u$, $\vec{e}_v$, $\vec{e}_w$ in the directions of $\vec{u}$, $\vec{v}$, $\vec{w}$ respectively.

12. $\vec{e}_1$ and $\vec{e}_2$ are two perpendicular unit vectors. Calculate the following lengths.
    a) $|\vec{e}_1+\vec{e}_2|$
    b) $|\vec{e}_1+2\vec{e}_2|$.

13. If $\vec{u}=\overrightarrow{(4,1,-2)}$ and $\vec{v}=\overrightarrow{(0,3,3)}$, find the unit vector in the direction of each of the following.
    a) $\vec{u}+\vec{v}$
    b) $3\vec{u}-\vec{v}$
    c) $\vec{u}-4\vec{i}-\vec{j}$

14. For any $\vec{v}\in\mathbb{V}_3$ and any $k\in\mathbb{R}$, prove that the unit vector in the direction of $\vec{v}$ is equal to the unit vector in the direction of $k\vec{v}$.

15. If $\overrightarrow{OA}=2\vec{i}-3\vec{j}-\vec{k}$ and $\overrightarrow{OB}=\vec{i}+\vec{j}-\vec{k}$, show that the vector $\overrightarrow{AB}$ is parallel to the xy-plane.

16. If $\vec{u}=\overrightarrow{\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)}$ and $\vec{v}=\overrightarrow{\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)}$, prove that the vectors $\vec{u}$ and $\vec{v}$ form an orthonormal set.

In Search of Vectors as Classes of an Equivalence Relation

This section attempts to show how vectors as particular mathematical objects can be manufactured from elementary mathematical building blocks.

Relations and their Graphs

When you link certain elements between two sets, you say that you are setting up a **relation** between the two sets.

A relation is often described by a sentence.

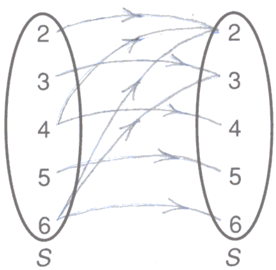
Example

"....is exactly divisible by...." between the set $S = \{2,3,4,5,6\}$ and itself creates the following relation.
$2 \to 2,\ 3 \to 3,\ 4 \to 2$, etc.,
where the arrow replaces the words "is exactly divisible by".

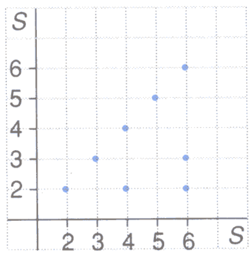
The set of all ordered pairs satisfying the sentence defines the relation, as follows.

$R = \{(2,2), (3,3), (4,2), (4,4), (5,5), (6,2), (6,3), (6,6)\}$

The relation can be graphed, or represented pictorially, in different ways. Here are two examples.

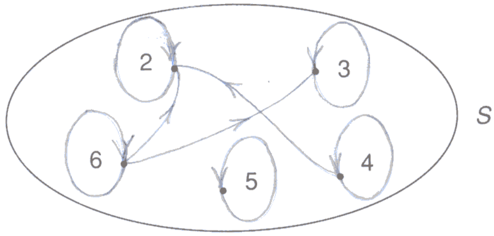


arrowgraph



Cartesian graph

However, a relation "from S to S" can also be called a relation "in S". The above relation in S can be graphed with the following special type of arrowgraph.



Equivalence Relation in a Set

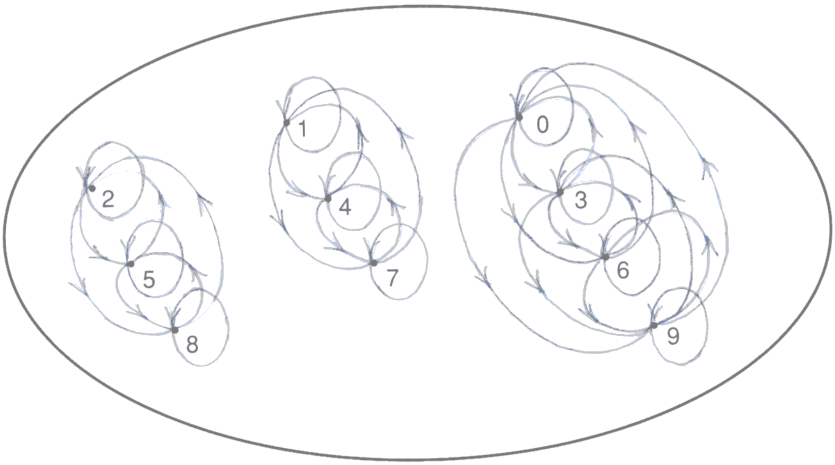
An equivalence relation in a set S has the following properties. It is

1. reflexive (for all $x \in S, x \rightarrow x$)
2. symmetric ($x \rightarrow y \Rightarrow y \rightarrow x$)
3. transitive ($x \rightarrow y$ and $y \rightarrow z \Rightarrow x \rightarrow z$)

Example

Consider the relation in the set of whole numbers, $\mathbb{W}$, defined by "...has the same remainder on division by 3 as...".

Observe what happens as the arrows are placed in a diagram.



- A set of subsets called a **partition** naturally forms.
- Each subset (or element of the partition) is called an **equivalence class** for the relation.

Good names for the equivalence classes here would be

{0,3,6,9,...} = "the class of 0" = *0*,

{1,4,7,...} = "the class of 1" = *1*, and

{2,5,8,...} = "the class of 2" = *2*.

You could call these classes "remainders".

Notice that *0* = *3*, *1* = *4*, etc., and that these classes are infinite sets.

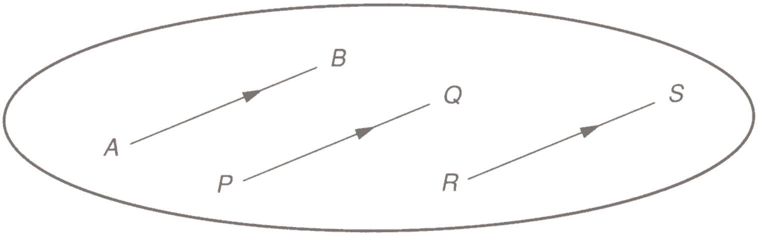
- The partition is the set of classes $\mathbb{W}_3 = \{0, 1, 2\}$.

(The classes in this example are sometimes called the "integers modulo 3")

Vectors

Consider the set D of all directed line segments in a plane. (Denote the directed line segment from A to B by $\underline{AB}$.)
In this set, consider the relation defined by
"$\underline{AB} \rightarrow \underline{PQ}$ if and only if $ABQP$ forms a parallelogram."

You will have the opportunity to verify that this is an equivalence relation.



The diagram shows that the directed line segments $\underline{AB}$, $\underline{PQ}$, $\underline{RS}$ satisfy the relation. There are many others! The "class of $\underline{AB}$" = $\{\underline{AB}, \underline{PQ}, \underline{RS}, \ldots\}$
Another name for this class is the vector $\overrightarrow{AB}$.

From this precise definition of a vector in a plane, you can see that 'drawing' a vector would actually cover the entire plane. Thus the vector is effectively *everywhere*. That is what allows you to draw a directed line segment representing the vector wherever you choose.

Activities

1. Verify that the relation defined by "$\underline{AB} \rightarrow \underline{PQ}$ if and only if $ABQP$ is a parallelogram", in the set of all directed line segments of a plane, is an equivalence relation.
   That is, verify that this relation is
   reflexive ($\underline{AB} \rightarrow \underline{AB}$),
   symmetric ($\underline{AB} \rightarrow \underline{PQ} \Rightarrow \underline{PQ} \rightarrow \underline{AB}$), and
   transitive ($\underline{AB} \rightarrow \underline{PQ}$ and $\underline{PQ} \rightarrow \underline{RS} \Rightarrow \underline{AB} \rightarrow \underline{RS}$)

2. Describe the partition created by each of the following equivalence relations.
   a) "is similar to" in the set of all triangles
   b) "has the same mother as" in the set of all Canadians
   c) "is in the same class as" in the set of children attending a public school
   d) "is parallel to" in the set of all straight lines
   e) "is congruent to" in the set of all line segments in a plane
   f) "$(p,q) \rightarrow (p',q')$ if and only if $p + q' = q + p'$" in the set of ordered pairs of whole numbers $\mathbb{W} \times \mathbb{W}$
   g) "$(p,q) \rightarrow (p',q')$ if and only if $pq' = qp'$" in the set of ordered pairs of integers $\mathbb{Z} \times [\mathbb{Z} - \{0\}]$ (that is, the ordered pairs (p,q) where p, q are integers and $q \neq 0$).

Summary

General Concepts

- Any number of parallel lines with arrows pointing the same way define a particular direction.
- Any real number is called a scalar, to distinguish it from a vector.
- A vector is everywhere: it can be represented by any directed line segment that has the correct magnitude and direction.

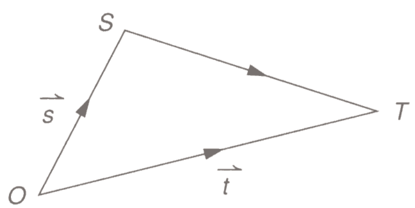
Equality and Length

- Equal vectors have the same magnitude and the same direction.
- If P is a point in a coordinate system of origin O, then $\overrightarrow{OP}$ is called the position vector of P.

in 2-space	*in 3-space*
• If $P = (a,b)$, then $\overrightarrow{OP} = \overrightarrow{(a,b)}$	• If $P = (a,b,c)$, then $\overrightarrow{OP} = \overrightarrow{(a,b,c)}$
• The length of $\vec{v} = \overrightarrow{(x,y)}$ is $\lvert\vec{v}\rvert = \sqrt{x^2 + y^2}$	• The length of $\vec{v} = \overrightarrow{(x,y,z)}$ is $\lvert\vec{v}\rvert = \sqrt{x^2 + y^2 + z^2}$

Vector Addition and Subtraction

- geometric law of vector addition (the triangle law)
  $\overrightarrow{OS} + \overrightarrow{ST} = \overrightarrow{OT}$



- geometric law of vector subtraction (the triangle law)
  $\overrightarrow{ST} = \overrightarrow{OT} - \overrightarrow{OS} = \vec{t} - \vec{s}$
- component laws of vector addition and vector subtraction

in $\mathbb{V}_2$	in $\mathbb{V}_3$
$\overrightarrow{(a,b)} + \overrightarrow{(p,q)} = \overrightarrow{(a + p,b + q)}$	$\overrightarrow{(a,b,c)} + \overrightarrow{(p,q,r)} = \overrightarrow{(a + p,b + q,c + r)}$
$\overrightarrow{(a,b)} - \overrightarrow{(p,q)} = \overrightarrow{(a - p,b - q)}$	$\overrightarrow{(a,b,c)} - \overrightarrow{(p,q,r)} = \overrightarrow{(a - p,b - q,c - r)}$

Multiplication of a Vector by a Scalar

- If k is a scalar and $\vec{u}$ is a vector, then $k\vec{u}$ is a vector parallel to $\vec{u}$, whose length is $|k||\vec{u}|$
  ($\vec{u}$ and $k\vec{u}$ have the same direction if $k > 0$, but have opposite directions if $k < 0$.)
- In component form, $k\overrightarrow{(x,y)} = \overrightarrow{(kx,ky)}$ in $\mathbb{V}_2$ or $k\overrightarrow{(x,y,z)} = \overrightarrow{(kx,ky,kz)}$ in $\mathbb{V}_3$.

Unit Vectors

- Any vector whose length is 1 is a unit vector (or a normed vector).
- To normalize $\vec{v}$ is to find the unit vector $\vec{e}_v$ in the direction of $\vec{v}$:
  $$\vec{e}_v = \frac{1}{|\vec{v}|}\vec{v}$$
- Given the standard basis vectors
  $\vec{i} = \overrightarrow{(1,0)}$ and $\vec{j} = \overrightarrow{(0,1)}$ of $\mathbb{V}_2$, then $\overrightarrow{(p,q)} = p\vec{i} + q\vec{j}$

  $\vec{i} = \overrightarrow{(1,0,0)}$, $\vec{j} = \overrightarrow{(0,1,0)}$ and $\vec{k} = \overrightarrow{(0,0,1)}$ of $\mathbb{V}_3$, then $\overrightarrow{(p,q,r)} = p\vec{i} + q\vec{j} + r\vec{k}$

Vector Spaces

A vector space $\mathbb{V}$ is a set of mathematical objects called vectors, together with two operations, called vector addition and multiplication by a scalar, having the following properties.

Vector Addition

A1. $\mathbb{V}$ is closed under addition: $\vec{u}, \vec{v} \in \mathbb{V}$ implies $\vec{u} + \vec{v} \in \mathbb{V}$

A2. Addition is associative: $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

A3. There is a $\vec{0} \in \mathbb{V}$ such that for all $\vec{u} \in \mathbb{V}$, $\vec{u} + \vec{0} = \vec{u}$

A4. If $\vec{u} \in \mathbb{V}$, then there exists $-\vec{u} \in \mathbb{V}$ such that $\vec{u} + (-\vec{u}) = \vec{0}$

A5. Addition is commutative: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(These properties mean that $\mathbb{V}$ is a commutative group with respect to addition.)

Multiplication of a Vector by a Scalar

M1. If $\vec{u} \in \mathbb{V}$, $k \in \mathbb{R}$, then $k\vec{u} \in \mathbb{V}$

M2. $(km)\vec{u} = k(m\vec{u})$, $k, m \in \mathbb{R}$

M3. $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

M4. $(k + m)\vec{u} = k\vec{u} + m\vec{u}$

M5. There exists $1 \in \mathbb{R}$ such that $1\vec{u} = \vec{u}$

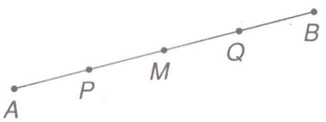
Inventory

Complete each of the following statements.

1. A number of arrowed parallel lines with the arrows pointing the same way is said to define a ______.
2. A ______ has both length and direction.
3. A real number is called a ______, to distinguish it from a vector.
4. A vector can be represented by a ______ line segment. A vector can also be represented by an ______ or an ______ of numbers.
5. Two directed line segments represent the same vector if they have the same ______ and the same ______.
6. If $\overrightarrow{(x,2)} = \overrightarrow{(-3,y)}$, then $x =$ ______ and $y =$ ______.
7. A plane can be determined by three distinct non-collinear ______, or by two distinct ______ lines.
8. In a three-dimensional mathematical drawing, you must keep ______ vertical, and ______ parallel.
9. Skew lines are lines that are neither ______ nor ______.
10. Given $P(2,-3,4)$, then the position vector $\overrightarrow{OP} =$ ______.
11. If $\vec{v} = \overrightarrow{(2,2,-1)}$, then the length of the vector $\vec{v}$, written ______, is equal to ______.
12. The vector sum $\overrightarrow{(1,-7)} + \overrightarrow{(-1,9)} =$ ______.
13. The vector sum $\overrightarrow{FG} + \overrightarrow{GH} =$ ______.
14. The vector difference $\overrightarrow{NK} - \overrightarrow{NL} =$ ______.
15. The product of the scalar 5 and the vector $\overrightarrow{(2,1,4)}$ is ______.
16. If $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{CD} = k\vec{u}$, where k is a scalar, then the lines AB and CD are ______. The length of the vector $k\vec{u}$ is ______.
17. Given the points $P(3,8)$ and $Q(1,6)$, the vector $\overrightarrow{PQ}$ in component form is ______.
18. A vector of length one is called a ______ vector.
19. The vector $\vec{v} = \overrightarrow{(4,-3)}$ can be expressed in terms of the standard basis vectors $\vec{i}$ and $\vec{j}$ as follows. $\vec{v} =$ ______.
20. Normalizing $\vec{v}$ means finding the ______ vector $\vec{e}_v$ in the direction of $\vec{v}$. If $\vec{v} = \overrightarrow{(4,-3)}$, then $\vec{e}_v =$ ______.

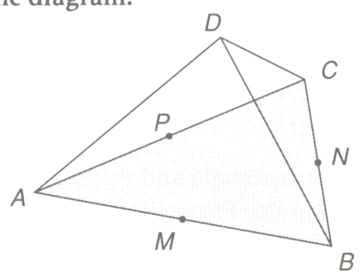
Review Exercises

1. M is the midpoint of segment AB, P is the midpoint of AM, Q is the midpoint of MB.
   a) Give reasons why $\overrightarrow{AP} = \overrightarrow{PM}$.
   b) State all other vectors in the diagram equal to $\overrightarrow{AP}$.
   c) State all vectors in the diagram equal to $\overrightarrow{BM}$.



2. In the square $ABCD$, P, Q, R and S are the midpoints of AB, BC, CD, and DA respectively. If $\overrightarrow{AP} = \vec{u}$ and $\overrightarrow{BQ} = \vec{v}$, express the following in terms of $\vec{u}$ and $\vec{v}$.
   a) $\overrightarrow{PB}$ b) $\overrightarrow{RC}$ c) $\overrightarrow{DR}$ d) $\overrightarrow{QC}$ e) $\overrightarrow{AS}$

3. $ABCD$ is a **regular tetrahedron**, that is, a solid made up of four congruent equilateral triangular faces. Its apex, D, is vertically above the centre of the base. The midpoints of AB, BC, and CA are M, N, and P respectively.
   a) Draw the tetrahedron, and locate the centroid O of its base (the intersection of the medians AN, BP, and CM of the triangular base).
   b) Join DO, and mark the right angles DOA, DOB, and DOC.
   c) List the other right angles defined in the diagram.



4. In the tetrahedron of question 3, calculate the following.
   a) the angle between two edges (such as angle DAB)
   b) the angle between an edge and a face (such as angle DAN)

5. In the tetrahedron of question 3, name the following.
   a) a pair of skew lines
   b) the three planes intersecting at point D

6. In a 3-space coordinate system, draw the position vector $\vec{p}$ of the point $P(-1,2,3)$.

7. A point P, whose position vector is $\overrightarrow{OP} = \overrightarrow{(5,-2,-4)}$, is translated to position P' by the vector $\vec{v} = \overrightarrow{(1,2,3)}$. What are the coordinates of P'?

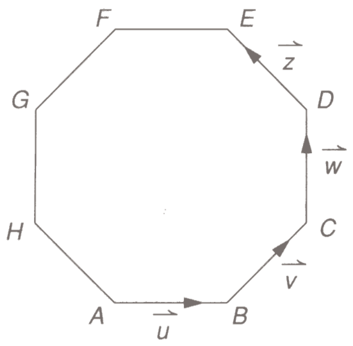
8. Find k so that the length of the vector $\vec{v} = \overrightarrow{\left(\frac{1}{2},-\frac{2}{3},k\right)}$ is one unit.

9. Given that $\vec{u} = \overrightarrow{(-2,2,5)}$ and $\vec{v} = \overrightarrow{(k,-1,-3)}$, calculate k, where $2|\vec{u}| = |\vec{v}|$.

10. An object is displaced in a plane according to the vector $\vec{u} = \overrightarrow{(2,-3)}$, then displaced again according to the vector $\vec{v} = \overrightarrow{(6,1)}$.
    a) Calculate the resultant displacement, $\vec{w}$.
    b) Draw the three displacements $\vec{u}$, $\vec{v}$, $\vec{w}$ on a grid.
    c) Calculate $|\vec{w}|$.

11. Given the regular octogon $ABCDEFGH$ shown, where $\overrightarrow{AB} = \vec{u}$, $\overrightarrow{BC} = \vec{v}$, $\overrightarrow{CD} = \vec{w}$, and $\overrightarrow{DE} = \vec{z}$, find the following vectors in terms of $\vec{u}$, $\vec{v}$, $\vec{w}$, and $\vec{z}$.

a) $\overrightarrow{FE}$
b) $\overrightarrow{GF}$
c) $\overrightarrow{AF}$
d) $\overrightarrow{GE}$
e) $\overrightarrow{AE}$



12. Given $\vec{u} = \overrightarrow{(k,4,m)}$, $\vec{v} = \overrightarrow{(1,n,-8)}$, and $\vec{u} + \vec{v} = \overrightarrow{(-3,4,-2)}$, find the values of k, m and n.

13. A boat starts at a point A and sails 600 m on bearing 045°, then 400 m towards the east to arrive at point B. Draw a vector diagram to represent the resultant displacement $\overrightarrow{AB}$. What is the magnitude of this displacement? What is the bearing of B from A?

14. Simplify the following by using the subtraction form of the triangle law.

a) $\overrightarrow{QP} - \overrightarrow{QR}$
b) $\overrightarrow{XA} - \overrightarrow{XY} + \overrightarrow{AY}$

15. Given the points $L(5,6,-3)$ and $M(-8,11,4)$, use vector subtraction to determine the vector $\overrightarrow{LM}$ in component form.

16. Given any four points A, B, C, and D in space, express the following vectors in subtraction form, using position vectors with origin A.

a) $\overrightarrow{BC}$ b) $\overrightarrow{CD}$ c) $\overrightarrow{BA}$

17. $PQRS$ is a parallelogram and O is any point. If $\overrightarrow{OP} = \vec{p}$, $\overrightarrow{OR} = \vec{r}$, and $\overrightarrow{OS} = \vec{s}$, express the vector $\overrightarrow{QO}$ in terms of $\vec{p}$, $\vec{r}$, and $\vec{s}$.

18. Given $\vec{u} = \overrightarrow{(5,-6,-3)}$, $\vec{v} = \overrightarrow{(2,0,4)}$, $\vec{w} = \overrightarrow{(-1,2,-5)}$, find the following.

a) $\vec{u} - \vec{v}$ d) $2\vec{u} - 3\vec{v}$
b) $\vec{v} - \vec{w}$ e) $\vec{u} + 2\vec{v} + 3\vec{w}$
c) $\vec{w} - \vec{u}$ f) $\frac{1}{2}\vec{u} - \vec{v} - 2\vec{w}$

19. The points P, Q, R are such that $\overrightarrow{PQ} = \overrightarrow{(4,-1,2)}$ and $\overrightarrow{PR} = \overrightarrow{(12,-3,6)}$.

a) Express $\overrightarrow{QR}$ in terms of $\overrightarrow{PQ}$.
b) What can you say about the points P, Q and R?

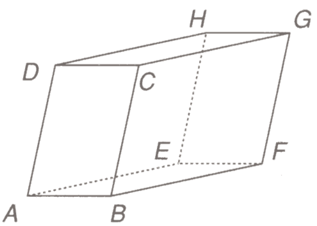
20. Find the coordinates of the points P, Q, and R, of question 19, given that $\overrightarrow{OP} = \overrightarrow{(3,0,-1)}$.

21. Given the two vectors $\vec{u}$ and $\vec{v}$, and $|\vec{v}| = 5$, find the value of $|\vec{u} + \vec{v}|$ in each of the following cases.

a) $3\vec{v} = 2\vec{u}$
b) $\vec{v} = -\vec{u}$
c) $\vec{u}$ and $\vec{v}$ are perpendicular, and $|\vec{u}| = 2$

22. Given the parallelepiped shown, state a single vector equal to each of the following.

a) $\overrightarrow{AC} - \overrightarrow{AD}$
b) $\overrightarrow{AB} - \overrightarrow{AD}$
c) $\overrightarrow{BF} - \overrightarrow{CG}$
d) $\overrightarrow{AC} - \overrightarrow{EH}$
e) $\overrightarrow{HD} - \overrightarrow{GB}$



23. The diagonals of a quadrilateral bisect each other. Prove that the quadrilateral is parallelogram.

24. $ABCD$ is a rectangle and $PQCD$ is a parallelogram. Prove that $ABQP$ is a parallelogram.

25. Simplify the following.

a) $6(-\vec{i} + 2\vec{j}) - 3(3\vec{i} - 4\vec{j})$

b) $10(4\vec{i} + 2\vec{j} - 5\vec{k})$
$- \frac{1}{2}(16\vec{i} - 4\vec{j})$
$- 6\vec{j} + 3\vec{k}$

26. Find the unit vector in the direction of $\overrightarrow{PQ}$ in each of the following cases.

a) P is the point $(-1,-2)$ and Q is the point $(1,3)$

b) P is the point $(4,3,-3)$ and Q is the point $(6,-5,-1)$

c) P is the point (a,b,c) and Q is the point (d,e,f)

27. $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ form an orthonormal set. Calculate the following lengths.

a) $|\vec{e}_1 + \vec{e}_3|$

b) $|\vec{e}_1 + \vec{e}_2 + \vec{e}_3|$

c) $|\vec{e}_2 + 3\vec{e}_3|$

d) $|\vec{e}_1 - 2\vec{e}_2 - 3\vec{e}_3|$

28. In triangle OAB, points P and Q divide the side AB into three equal segments with P closer to vertex A than to B.

a) Show that $\overrightarrow{AQ} = \overrightarrow{PB}$.

b) Prove that $\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{OB}$.

29. A vector having the same magnitude as $-6\vec{i} + 8\vec{k}$ is

A. $3\vec{i} + 4\vec{j} - 5\vec{k}$

B. $-3\vec{i} + 4\vec{k}$

C. $-6\vec{i} + 4\vec{j} + 4\vec{k}$

D. $10\vec{i} + 10\vec{j} + 10\vec{k}$

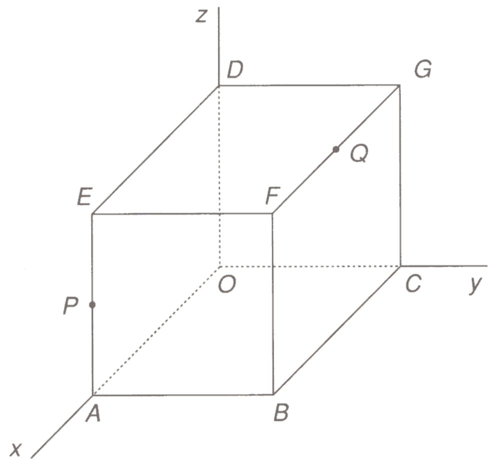
E. $-10\vec{j}$

*(83 H)

*For information about these questions, see the introductory pages of this book.

30. In a three dimensional rectangular Cartesian co-ordinate system, the points O, A, C and D have co-ordinates $(0,0,0)$, $(6,0,0)$, $(0,6,0)$ and $(0,0,6)$ respectively. $OABCGFED$ is a cube, as shown in the figure.

The points P and Q are the mid-points of $[AE]$ and $[FG]$ respectively.



a) Write, in column vector form, each of the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.

b) i. Find the square of the length of each of the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.

ii. Show that $PQ = 3\sqrt{6}$ units.

iii. Examine whether OPQ is a right angled triangle, giving a reason for your result.

c) Given that the points R and S are the midpoints of $[BP]$ and $[CQ]$ respectively,

i. prove that $\overrightarrow{OR} = \frac{1}{2}\overrightarrow{OP} + \frac{1}{2}\overrightarrow{OB}$ and

ii. find the length of RS.

d) An ant walks from P to Q along the surface of the cube. By considering the net of the cube, or otherwise, find how far the ant walks, given that the distance travelled must be as small as possible.

(87 SMS)