VECTORS, MATRICES
and
COMPLEX NUMBERS

with
International Baccalaureate
questions

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CHAPTER TEN

COMPLEX NUMBERS

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CHAPTER TEN

Complex Numbers

Is $\sqrt{-1}$ a number?

You know that $\sqrt{25} = 5$, $\sqrt{1} = 1$, $\sqrt{0} = 0$, $\sqrt{0.49} = 0.7$, etc. Also, you know that $\sqrt{3}$ cannot be written as a terminating or periodic decimal, but it is quite close to 1.732 050 808. Where could $\sqrt{-1}$ possibly fit in? You know that both positive and negative real numbers square to positive numbers. (For example, $2^2 = 4$, which is positive, and $(-1)^2 = 1$, which is positive.) Zero squares to zero. So what could possibly square to -1?? These considerations indicate that no place can be found for $\sqrt{-1}$ on the real number line.

The mathematicians who first encountered "$\sqrt{-1}$" quite naturally called it "imaginary". And so the name continues to this day, although these types of numbers are frequently used in mathematics and physics. Such numbers are now considered to be no more imaginary than irrational numbers, points, vectors, or any other mathematical object with which you are familiar.

This change came about very slowly. In general, the time-lag is such that the first person to make an important discovery does not see it fully accepted in his or her lifetime. As for many new ideas in mathematics, the introduction of numbers containing an "imaginary" component, now called **complex numbers**, had to go through at least three stages.

1. The ground had to be prepared for the discovery to take place. (In this case, simple "negative numbers" had to be accepted first.)
2. The discovery itself had to occur and be used. This meant going beyond writing "$\sqrt{-1}$", and actually attempting to 'work' with it. (This took a lot of courage, because it required going against accepted practice at the time.)
3. The new discovery had to be applied by other mathematicians—preferably well-respected mathematicians—before it could be fully accepted.

It may surprise you to learn that negative numbers did not gain a foothold in Europe until 1544, through the German mathematician Michael Stifel (1486–1567). The theory of negative numbers had in fact been completely developed more than 800 years before that, in India. However, in Europe, until the 1500s, the difference $a - b$ was deemed meaningful only for a greater than b. A first-degree equation such as $x + 3 = 7$ could be solved, but $x + 7 = 3$ was avoided because a solution was considered to be 'impossible'. And amazingly, this belief was not eradicated until the 1800s!

The theory of second-degree equations (that is, quadratic equations) was even more muddled. Writing such equations in the form $ax^2 + bx + c = 0$ did not appear until 1631, through a posthumous publication by Thomas Harriott (1560–1621). Indeed, 0 was not really considered a number that could be used like the others.

Before Harriott, quadratic equations were broken down into 'types' as follows, each with its own set of rules.
I $ax^2 = bx + c$ II $ax^2 + c = bx$ III $ax^2 + bx = c$.
Writing these equations in the form $ax^2 + bx + c = 0$ takes care of all possibilities as well as simplifying the theory of quadratic equations. This is a good example of the way that the discovery of a new entity can sometimes lead to a simplification of an entire theory.

As well as extending the concept of 'number', the acceptance of negative numbers led mathematicians to try 'solving' equations for which no real solutions could be found, such as $x^2 + 1 = x$. You will see that the solution to such an equation contains "$\sqrt{-1}$".

The first appearance of an imaginary number was in a publication by Girolamo Cardano, in his quest for a solution to a cubic equation. The time span between this first appearance, in 1545, until the full acceptance of complex numbers by the mathematical community in the mid-1800s, exceeded 300 years.

As you work through this chapter, you will have an opportunity to understand how these 'non-real' numbers grew into full acceptance. At the same time, you will extend your concept of number, and unify some of your theories in algebra.

10.1 What is a Complex Number?

The discovery of complex numbers was likely linked to the analysis of the problem of finding two numbers, knowing their sum and their product. The following will guide you through this discovery process.

Example 1 Find two numbers whose sum is 4 and whose product is 3.

Solution Let one number be x, then the other number must be $4 - x$. The product of the numbers is 3, thus

$$\begin{aligned} x(4 - x) &= 3 \\ 4x - x^2 &= 3 \\ x^2 - 4x + 3 &= 0 \\ (x - 1)(x - 3) &= 0 \\ x - 1 = 0 \text{ or } x - 3 &= 0 \\ x = 1 \text{ or } \quad x &= 3. \end{aligned}$$

If $x = 1$, then $(4 - x) = 3$, and
if $x = 3$, then $(4 - x) = 1$.
Thus the required numbers are 1 and 3.

A very important part of this discovery process is the following.
Check: the sum is $1 + 3 = 4$, and
the product is $(1)(3) = 3$, as required. ■

The next example yields a more complicated solution.

Example 2 Find two numbers whose sum is 6 and whose product is 3.

Solution Let one number be x, then the other must be $6 - x$.
Thus

$$\begin{aligned} x(6 - x) &= 3 \\ 6x - x^2 &= 3 \\ x^2 - 6x + 3 &= 0. \end{aligned}$$

But this quadratic expression will not factor over the integers. By 'completing the square' (as in section 8.3, page 356),

$$\begin{aligned} x^2 - 6x + 9 - 9 + 3 &= 0 \\ (x - 3)^2 - 6 &= 0 \\ (x - 3)^2 &= 6 \\ (x - 3) = \sqrt{6} \quad \text{or } (x - 3) &= -\sqrt{6} \\ x = 3 + \sqrt{6} \text{ or } \quad x &= 3 - \sqrt{6}. \end{aligned}$$

Thus the numbers required are $(3 + \sqrt{6})$ and $(3 - \sqrt{6})$, or approximately 5.44948...and 0.55051....Once again, although the numbers are irrational, check the result, using the exact values.

Check: the sum is $(3 + \sqrt{6}) + (3 - \sqrt{6}) = 6$, as required, and

the product is $(3 + \sqrt{6})(3 - \sqrt{6}) = 3^2 - 3\sqrt{6} + 3\sqrt{6} - \sqrt{6}^2 = 9 - 6 = 3$, as required. ■

In the next example, it will appear that no numbers exist to give the required sum and product. Most mathematicians did not attempt to go beyond the seemingly insurmountable difficulty encountered, but simply classified the problem as unsolvable. Then the Italian mathematician Girolamo Cardano (1501–1576) made a breakthrough in 1572, as you shall see presently.

Example 3 Find two numbers whose sum is 6 and whose product is 10.

Solution Let one number be x, then the other is $6 - x$.

Thus

$$\begin{aligned} x(6 - x) &= 10 \\ 6x - x^2 &= 10 \\ x^2 - 6x + 10 &= 0. \end{aligned}$$

Completing the square,

$$\begin{aligned} x^2 - 6x + 9 - 9 + 10 &= 0 \\ (x - 3)^2 + 1 &= 0 \\ (x - 3)^2 &= -1. \end{aligned}$$

This appears to be 'impossible'. Indeed, the square of any real number, whether positive, negative, or zero, is always greater than or equal to zero. Thus, no real number has a square of −1. Now Cardano had the insight and the courage to simply 'carry on regardless', as follows.

$$x - 3 = \sqrt{-1} \text{ or } x - 3 = -\sqrt{-1}.$$

Thus the numbers 'are' $(3 + \sqrt{-1})$ and $(3 - \sqrt{-1})$. ■

Although these numbers may appear to be meaningless, attempt the following check. Assume that $\sqrt{-1}$ is a number with the usual algebraic properties, including $(\sqrt{-1})^2 = -1$.

Check: the sum is $(3 + \sqrt{-1}) + (3 - \sqrt{-1}) = 6$, and

the product is $(3 + \sqrt{-1})(3 - \sqrt{-1})$

$$\begin{aligned} &= 9 - 3\sqrt{-1} + 3\sqrt{-1} - (\sqrt{-1})^2 \\ &= 9 - (-1) = 10. \end{aligned}$$

These 'numbers' seem to work!

As pointed out in the introduction, this breakthrough did not have immediate results. The world had to wait more than 200 years before these inventions were fully accepted. Cardano's ideas were finally formalized by Jean Argand, Leonhard Euler, Karl Friedrich Gauss and other mathematicians towards the end of the 18th century.

The Acceptance Phase

To simplify matters, the symbol i will be used to represent a number (which does not belong to $\mathbb{R}$) that has the property $i^2 = -1$.

It is now tempting to write "$i = \sqrt{-1}$", and by analogy, "$\sqrt{-4} = \sqrt{-1}\sqrt{4} = (i)(2) = 2i$", "$\sqrt{-3} = \sqrt{-1}\sqrt{3} = i\sqrt{3}$", etc. However, great care must be exercised in the use of the symbol "$\sqrt{\ }$" when dealing with roots of negative numbers. Examine the paradox illustrated by the two following 'simplifications'.

$$\sqrt{-4}\sqrt{-1} = \sqrt{(-4)(-1)} = \sqrt{4} = 2 \qquad ①$$

or $\sqrt{-4}\sqrt{-1} = (2i)(i) = 2i^2 = 2(-1) = -2$ ②

Two different results are obtained. Which of these is correct?

① uses the familiar algebraic property of real positive numbers $\sqrt{a}\sqrt{b} = \sqrt{ab}$

② makes use of the symbol i, where $i^2 = -1$.

The paradox is resolved as follows.

The algebraic property $\sqrt{a}\sqrt{b} = \sqrt{ab}$ is *true only for non-negative real numbers a, b.*

Thus, because $a = -4$ and $b = -1$, ① is *false*, but ② is *correct*.

All positive numbers have two square roots. For example, the two square roots of 9 are $\sqrt{9} = 3$ and $-\sqrt{9} = -3$.

Negative numbers also have two square roots. One must accept that i can represent either $\sqrt{-1}$ or $-\sqrt{-1}$. You will have an opportunity to verify this allegation in the exercises.

DEFINITIONS

Imaginary numbers

- i, and the scalar multiples of i, that is $3i$, $i\sqrt{2}$, $-4i$, etc, shall retain their original name of **imaginary numbers**.
- The set of imaginary numbers will be denoted by $\mathbb{I}$.

Complex numbers

- The sum of a real number and an imaginary number, that is, $z = a + bi$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$, will be called a **complex number**.
- The set of complex numbers will be denoted by $\mathbb{C}$.

Real and imaginary parts

Given the complex number $z = a + bi$, where $a \in \mathbb{R}$, $b \in \mathbb{R}$,

- a is called the **real part** of z, or $a = \text{Re}(z)$
- b is called the **imaginary part** of z, or $b = \text{Im}(z)$

If $b = 0$, z is real. If $b \neq 0$, z is non-real.

Thus it appears that the real numbers form a subset of the complex numbers.

Equality

- Two complex numbers $z = a + bi$ and $w = c + di$, where a, b, c, and d are real, are equal if and only if $a = c$ and $b = d$.

In the following example, you will save time by obtaining the solutions with the quadratic formula, instead of completing the square.

You will use the fact that the solutions of $az^2 + bz + c = 0$ are given by the formula $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example 4 Solve **a)** $2z^2 + 2z + 5 = 0$ **b)** $z^2 - 2z + 3 = 0$.

Solution **a)** Here $a = 2, b = 2, c = 5$.
Substituting in the quadratic formula gives the roots

$$z = \frac{-2 \pm \sqrt{2^2 - (4)(2)(5)}}{2(2)} = \frac{-2 \pm \sqrt{-36}}{4} = \frac{-2 \pm 6i}{4} = \frac{-1 \pm 3i}{2}$$

Thus the solutions are $z = -\frac{1}{2} + \frac{3}{2}i$ or $z = -\frac{1}{2} - \frac{3}{2}i$.

b) Here, $a = 1, b = -2, c = 3$. Using the formula, the roots are

$$z = \frac{2 \pm \sqrt{2^2 - (4)(1)(3)}}{2(1)} = \frac{2 \pm \sqrt{-8}}{2} = \frac{2 \pm i\sqrt{8}}{2} = 1 \pm \frac{i\sqrt{8}}{2}$$

Note that, although these solutions are correct, they can be simplified by writing $\sqrt{8}$ as a mixed radical, as follows.

$$\frac{2 \pm i\sqrt{8}}{2} = \frac{2 \pm i\sqrt{(4)(2)}}{2} = \frac{2 \pm 2i\sqrt{2}}{2} = \frac{2(1 \pm i\sqrt{2})}{2} = 1 \pm i\sqrt{2}.$$

Thus the solutions are $z = 1 + i\sqrt{2}$ or $z = 1 - i\sqrt{2}$. ■

The previous discussion leads to the following formulas for the *addition* and *multiplication* of complex numbers.

- Complex numbers can be added as follows:
  $(a + bi) + (c + di) = (a + c) + (b + d)i$
- Complex numbers can be multiplied as follows:
  $(a + bi)(c + di) = ac + adi + bci + bdi^2$
  but recall that $i^2 = -1$, thus
  $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

Note: that it is easier to use the processes that employ the usual rules of algebra, together with the fact that $i^2 = -1$, than to learn the above formulas.

Thus, when complex numbers are added, or multiplied, other complex numbers are produced. This can be verified in the following example.

Example 5 Given $z = 3 - 5i$ and $w = 1 + i$, calculate the following.
a) $z + w$
b) zw

Solution **a)** $z + w = 3 - 5i + 1 + i = 4 - 4i$.
b)
$$\begin{aligned} zw &= (3 - 5i)(1 + i) \\ &= (3)(1) + (3)(i) - (5i)(1) - (5i)(i) \\ &= 3 + 3i - 5i - 5i^2 \\ &= 3 - 2i - (5)(-1) \\ &= 8 - 2i. \end{aligned}$$
■

You will be looking at operations in $\mathbb{C}$ more formally, and in more detail, in the next section.

10.1 Exercises

1. Simplify the following.
- **a)** $(4 + i) + (5 + 2i)$
- **b)** $(4 - 3i) + (-3 + 3i)$
- **c)** $(3 + 2i)(5 - i)$
- **d)** $(3 + i)(3 - i)$
- **e)** $(1 + 2i)^2$
- **f)** $(1 - 2i)^2$
- **g)** $i(6 + 4i)$
- **h)** $(1 + 2i)^3$

2. Find two numbers whose sum is 7 and whose product is $18\frac{1}{2}$.

3. Find two numbers whose sum is 4 and whose product is 5.

4. Find two numbers whose sum is -1 and whose product is 2.

5. Find the roots of the following equations.
- **a)** $z^2 - 8z + 25 = 0$
- **b)** $z^2 + 4z + 5 = 0$
- **c)** $3z^2 = 5z - 7$

6. The roots of an equation satisfy that equation. By substitution, verify that each of the following is a root of the given equation.
- **a)** $z = 2; z^2 + 3z - 10 = 0$
- **b)** $z = -5; z^2 + 3z - 10 = 0$
- **c)** $z = -i; z^2 + 1 = 0$
- **d)** $z = 2 - \sqrt{3}; z^2 - 4z + 1 = 0$
- **e)** $z = 4 + 3i; z^2 - 8z + 25 = 0$
- **f)** $z = 2i; z^2 + (1 - 2i)z - 2i = 0$
- **g)** $z = \frac{5}{3} - \frac{i}{3}; 3z^2 - 5z + iz = 0$

7. Consider the expressions $z = 4 + \sqrt{-9}$ and $w = 4 - \sqrt{-9}$. Show that $z + w = 8$ and $zw = 25$ with either of the following interpretations.
- **a)** let $\sqrt{-9} = 3i$
- **b)** let $\sqrt{-9} = -3i$

8. The equation $az^2 + bz + c = 0$ is such that $b^2 - 4ac < 0$, where a, b, and c are real. Find the sum and product of the roots of this equation in terms of a, b, and c.

9. Find the roots of the following equations.
- **a)** $z^2 - 4iz = 0$
- **b)** $z^2 - 3iz + 4 = 0$
- **c)** $z^2 = iz - 3$
- **d)** $z^2 - (1 + i)z + 2 + 2i = 0$

10. Solve the following for the real numbers x and y.
- **a)** $x + yi = 4 + 6i$
- **b)** $x + yi = 7i$
- **c)** $x + yi = (3 - i)(2 + 3i)$
- **d)** $x + yi = (5 + i)(5 - i)$
- **e)** $x + yi = (1 + i)^2$
- **f)** $(x + yi) = (4 - 3i)^2$

In the remaining questions of this exercise, use $z = a + ib$, $w = c + id$, and $u = e + if$, where a, b, c, d, e, f are all real numbers.

11. a) Calculate $z + w$ and $w + z$.
- **b)** Draw a conclusion concerning the commutativity of the addition of complex numbers.

12. a) Calculate $(z + w) + u$ and $z + (w + u)$.
- **b)** Draw a conclusion concerning the associativity of the addition of complex numbers.

13. a) Calculate zw and wz.
- **b)** Draw a conclusion concerning the commutativity of the multiplication of complex numbers.

14. a) Calculate $(zw)u$ and $z(wu)$.
- **b)** Draw a conclusion concerning the associativity of the multiplication of complex numbers.

15. a) Calculate $zw + zu$
- **b)** Calculate $z(w + u)$
- **c)** Draw a conclusion concerning the distributivity of multiplication over addition of complex numbers.

10.2 Operations in $\mathbb{C}$

Through the discovery of complex numbers in section 10.1, you learned that complex numbers could be added and multiplied. In questions 11–15 of 10.1 Exercises, you proved certain properties of these operations in $\mathbb{C}$.

In particular, multiplication in $\mathbb{C}$ is associative. That is, for any complex numbers z, w, u,

$$(zw)u = z(wu)$$

This means that a product such as zwu can be calculated without worrying about the order of the operations. Powers can be calculated in a similar fashion, as in the following example.

Example 1 Calculate **a)** i^8 **b)** $(-i)^3$

Solution **a)** $i^8 = i \times i \times i \times i \times i \times i \times i \times i = i^2 \times i^2 \times i^2 \times i^2 = (-1)^4 = 1$.
b) $(-i)^3 = (-i)(-i)(-i) = -(i \times i \times i) = -(i^2)i = -(-1)i = i$. ■

You will now see that other operations can be defined in $\mathbb{C}$. The first person to use the four operations of addition, subtraction, multiplication, and division of complex numbers was Raffaello Bombelli. A contemporary of Cardano, Bombelli published his work in Bologna, Italy, in 1572.

The Subtraction of Complex Numbers

The usual rules of algebra are applied to define the subtraction of complex numbers as follows.

$$(a + bi) - (c + di) = a + bi - c - di = (a - c) + (b - d)i.$$

Example 2 Calculate $(3 + 4i) - (5 - 2i)$.

Solution As before, it is easier to go through the process than to learn the formula.
$(3 + 4i) - (5 - 2i) = 3 + 4i - 5 + 2i = -2 + 6i$. ■

The Division of Complex Numbers

One operation that has not yet been mentioned in $\mathbb{C}$ is division.

Attempting to 'divide', say, $6 + 2i$ by i might yield $\frac{6 + 2i}{i} = \frac{6}{i} + \frac{2i}{i} = \frac{6}{i} + 2$.

This answer is not in the form $a + bi$. Is it a complex number, or something new?
Observe the following strategy.

$$\frac{6}{i} = \frac{6}{i} \times \frac{i}{i} = \frac{6i}{-1} = -6i.$$

Thus, division by i *does* yield a complex number!

Hence, $\frac{6 + 2i}{i} = -6i + 2$, or $2 - 6i$.

A similar trick is used to divide a more general complex number, as in the following example. Observe the solution carefully.

Example 3 Divide $(3 - 8i)$ by $(1 - 2i)$.

Solution

$$\frac{3-8i}{1-2i} = \frac{(3-8i)(1+2i)}{(1-2i)(1+2i)} = \frac{3+6i-8i-16i^2}{1-(-4)} = \frac{3-2i+16}{1+4} = \frac{19-2i}{5}$$

$$= \frac{19}{5} - \frac{2}{5}i \quad ■$$

The numbers $(1 + 2i)$ and $(1 - 2i)$ are known as **complex conjugates**, or simply **conjugates**.

DEFINITION The complex conjugate of $z = a + bi$ is $\bar{z} = a - bi$.

The quotient of two complex numbers $z = a + bi$ and $w = c + di$ is obtained as follows.

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)+(bc-ad)i}{c^2-d^2(-1)} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i.$$

Again, the above formula represents the definition of the division of two complex numbers, but it is much easier to learn the process rather than the formula. In the work above, you have used one of the important properties of conjugates. That is, given $z = c + di$, then $z\bar{z} = c^2 + d^2$ is real.

In the exercises, you will have an opportunity to prove the other properties of conjugates that are listed at the end of this section.

Example 4 Simplify the following. a) $\dfrac{2+i}{41i} + \dfrac{1-3i}{4-5i}$ b) $\dfrac{2}{1-i} - \dfrac{3i}{2+3i}$

Solution

a)
$$\frac{2+i}{41i} + \frac{1-3i}{4-5i} = \frac{(2+i)(i)}{41i(i)} + \frac{(1-3i)(4+5i)}{(4-5i)(4+5i)}$$

$$= \frac{2i-1}{-41} + \frac{4+5i-12i-15(-1)}{16-(-25)}$$

$$= \frac{2i-1}{41} + \frac{19-7i}{41} = \frac{18}{41} - \frac{5}{41}i$$

b)
$$\frac{2}{1-i} - \frac{3i}{2+3i} = \frac{2(1+i)}{(1-i)(1+i)} - \frac{3i(2-3i)}{(2+3i)(2-3i)}$$

$$= \frac{2+2i}{1+1} - \frac{6i+9}{4+9}$$

$$= 1 + i - \frac{(9+6i)}{13}$$

$$= \frac{13+13i-9-6i}{13} = \frac{4}{13} + \frac{7}{13}i \quad ■$$

Here is a summary of the essential properties of $\mathbb{C}$ with respect to the operations of addition and multiplication.

SUMMARY

E.	Equality	$a + bi = c + di$ if and only if $a = c$ and $b = d$
S.	Sum	$(a + bi) + (c + di) = (a + c) + (b + d)i$
P.	Product	$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

Given any numbers z, w and u of $\mathbb{C}$,

1.	Closure	$z + w$ and zw belong to $\mathbb{C}$
2.	Commutativity	$z + w = w + z$ and $zw = wz$
3.	Associativity	$(z + w) + u = z + (w + u)$ and $(zw)u = z(wu)$
4.	Distributivity	$z(w + u) = zw + zu$
5.	Neutral elements	$z + 0 = 0 + z = z$ and $(z)(1) = (1)(z) = z$
6.	Inverse elements	$z + (-z) = (-z) + z = 0$ and $z\left(\frac{1}{z}\right) = \left(\frac{1}{z}\right)z = 1$, provided that $z \neq 0$

Note 1 The neutral elements of $\mathbb{C}$ are defined as follows.
For addition: $0 = 0 + 0i$ For multiplication: $1 = 1 + 0i$

2 All of these properties apply to real numbers. You can check this by letting the imaginary part of each complex number be zero.

3 By virtue of satisfying all these properties, the set $\mathbb{C}$ is called a *field*.

Properties involving conjugates

Consider two complex numbers z, w, and their conjugates $\overline{z}$, $\overline{w}$.

1. $z + \overline{z} = 2\text{Re}(z)$
2. $z - \overline{z} = 2i\text{Im}(z)$
3. $z\overline{z} = [\text{Re}(z)]^2 + [\text{Im}(z)]^2$
4. $(\overline{z + w}) = \overline{z} + \overline{w}$
5. $(\overline{zw}) = \overline{z}\,\overline{w}$
6. $\overline{\overline{z}} = z$
7. Division: $\dfrac{z}{w} = \dfrac{z\overline{w}}{w\overline{w}} = \dfrac{z\overline{w}}{|w^2|}$

10.2 Exercises

1. Simplify the following ($n \in \mathbb{N}$).

a) i^3 **d)** i^6 **g)** i^{4n}

b) i^4 **e)** $\frac{1}{i}$ **h)** i^{4n+1}

c) i^5 **f)** i^{-2} **i)** i^{4n+2}

2. Simplify the following.

a) $(5 - i) - (4 + 3i)$
b) $(-1 + i) - (1 - i)$
c) $4(3 + 2i) - 2(6 + i)$
d) $(2 + i)^2 - (3 - 2i)^2$
e) $(5 + 3i)(3 - i) + 3(1 + i)(1 - i) - 4(3 + 7i)i$

3. Express in the form $a + ib$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

a) $\frac{1 + 4i}{i}$
b) $\frac{1 + 4i}{2 + i}$
c) $\frac{7 - 3i}{-i}$
d) $\frac{5 + 2i}{5 - 2i}$
e) $\frac{1}{2 + 3i}$
f) $\frac{i}{3 + 4i}$
g) $\frac{1}{3 + 4i} + \frac{1}{3 - 4i}$
h) $\frac{1}{6 + 5i} - \frac{1}{(6 + 5i)^2}$

4. Given $z = \cos\theta + i\sin\theta$ and $w = \cos\theta - i\sin\theta$, prove the following.

(Use the formulas on page 542.)

a) $z + w = 2\cos\theta$
b) $z - w = 2i\sin\theta$
c) $zw = 1$
d) $z^2 = \cos 2\theta + i\sin 2\theta$
e) $w^2 = \cos 2\theta - i\sin 2\theta$
f) $\frac{1}{1 + w} = \frac{1}{2} + \frac{i}{2}\tan\frac{1}{2}\theta$

For questions 5 and 6, refer to the properties listed in section 10.2.

5. By making the imaginary part zero, verify that the following properties in $\mathbb{C}$ also hold true in $\mathbb{R}$.

a) properties E, S and P.
b) the properties of conjugates 1, 2, and 3.
c) Is the set $\mathbb{R}$ also a field?

6. Prove the properties of conjugates 1, 2, 4, 5, and 6.

7. Prove that if $zw = 0$, then $z = 0$ or $w = 0$. (Hint: if $z = a + bi$ and $w = c + di$, you must prove $a = b = 0$ or $c = d = 0$.)

8. a) Prove that $z + \frac{1}{z} = \frac{z^2\bar{z} + \bar{z}}{|z|^2}$

b) Simplify $3 - 2i + \frac{1}{3 - 2i}$

9. The equation $az^2 + bz + c = 0$ is such that $b^2 - 4ac < 0$, where a, b, and c are real. Prove that the roots of the equation are complex conjugates.

10. Solve the following for the real numbers x and y.

a) $\frac{x + yi}{4 + i} = 4 - i$
b) $x + yi = \frac{6 - 2i}{3 + 5i}$

11. Find the real and imaginary parts of

a) $\frac{2 + i}{1 + 5i} + \frac{7}{1 - 5i}$
b) $(1 + i)^4$

12. Given $z = \frac{5i - 4}{i} + \frac{3i - 4}{1 - 2i}$, find the real and imaginary parts of z, and of z^2.

13. Simplify $(1 + i)^4(4 - 3i)^2(1 - i)^4(4 + 3i)^2$.

14. Find the number b such that $\left|\frac{2 - 3i\sqrt{5}}{6 + bi}\right| = 2$.

15. Find z in terms of $\cos\alpha$ and $\sin\alpha$, if $z^2 - 2z\cos\alpha + 1 = 0$.

10.3 Geometric Representation of a Complex Number

The previous sections have shown you that there is some validity in working with non-real numbers. However, there is still one major difficulty.

You know how to represent an integer, a rational number, even an irrational number on a number line. Where can i be placed? Where can the multiples of i, and other non-real numbers, be represented? These questions will be answered in this section.

There is a parallel between the history of civilization and the growth of the number sets used. However, the partial list below follows a logical rather than a historical thread.

The simplest number set is the set of natural numbers,
$\mathbb{N} = \{1, 2, 3, \ldots\}$

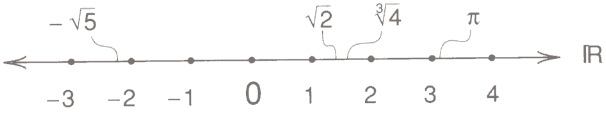
Next is the set of integers,
$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

(The symbol $\mathbb{Z}$ comes from the German "zahlen", to count.)

Then the set of rational numbers,
$\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a \in \mathbb{Z} \text{ and } b \in \mathbb{N} \right\}$

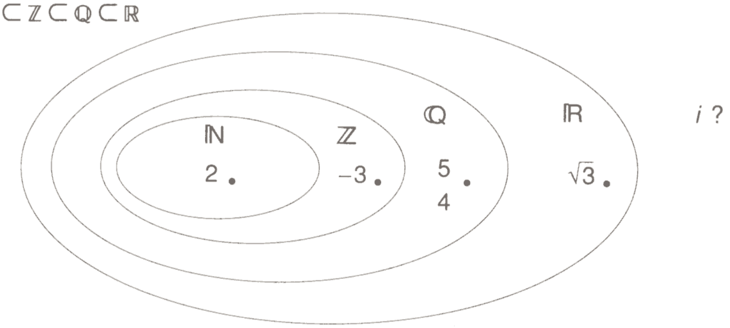
(The symbol $\mathbb{Q}$ comes from the word "quotient".)

It appears that the most complete set is the set of real numbers, $\mathbb{R}$, which is the union of $\mathbb{Q}$ and the set of all irrational numbers.



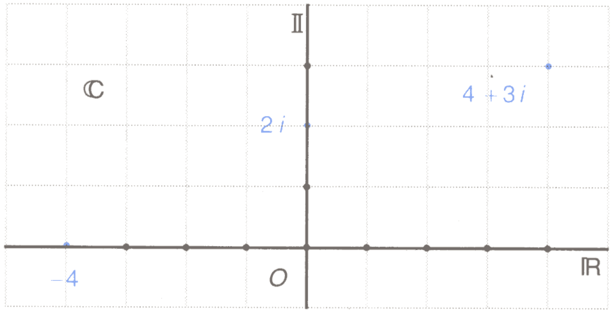
Recall that the representation of the real number line $\mathbb{R}$ is indistinguishable from $\mathbb{Q}$. However, for example, the irrational numbers $\sqrt{2}, -\sqrt{5}, \pi, \sqrt[3]{4}$ *are* elements of $\mathbb{R}$, but they do *not* belong to $\mathbb{Q}$.

Note: Each of the number sets described is a subset of its successor, as follows.
$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$



Venn diagram

As observed in the introduction to this chapter, there is certainly no appropriate spot on the real number line for *i*. However, $\mathbb{R} \subset \mathbb{C}$, the set of complex numbers. Indeed, note that $x \in \mathbb{R}$ can be written $1x + 0i \in \mathbb{C}$. A brilliant idea came from the Swiss mathematician Jean Argand (1768-1822), from a work published in 1806. He simply let the non-real numbers burst out of the real number line $\mathbb{R}$, by drawing another line $\mathbb{I}$ through 0, bearing the purely imaginary numbers.
Hence, any point in the *entire plane* thus created will represent a complex number. The origin, *O*, represents the number 0 (that is, $0 + 0i$).

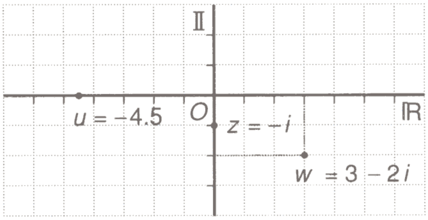


His invention bears the name **complex plane** or **Argand diagram**. Similar methods of picturing complex numbers were invented independently, at about the same time, by a Norwegian surveyor, Caspar Wessel (1745-1818), and by the famous German mathematician Karl Friedrich Gauss (1777-1855).
Note: The real number line, or real axis, is a subset of the complex plane. That is, *all numbers* can be represented by a point in this plane.

- If a number is on the $\mathbb{R}$-axis then it is real. It could be a natural number, an integer, a rational number, or an irrational number. [example: −4]
- If a number is on the $\mathbb{I}$-axis, then it is imaginary. [example: $2i$]
- If a number is *not* on the $\mathbb{R}$-axis, then it is *non-real*. [example: $4 + 3i$]

Example 1 Locate each of the following numbers in the complex plane.
$z = -i$, $w = 3 - 2i$, $u = -4.5$

Solution To plot the point representing $z = -i$, go one unit down from 0 on the $\mathbb{I}$-axis.
To plot the point representing $w = 3 - 2i$, go 3 units to the right of 0 on the $\mathbb{R}$-axis, then 2 units down, parallel to the $\mathbb{I}$-axis.
To plot the point representing $u = -4.5$, go 4.5 units to the left of 0 on the $\mathbb{R}$-axis.



■

Consequences of Representation in the Complex Plane

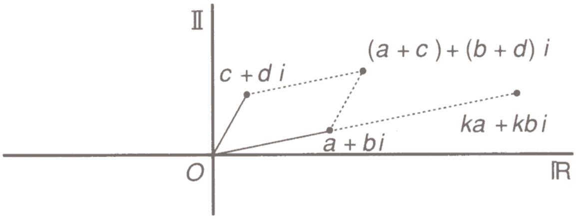
1. *Complex numbers as two-dimensional vectors*
Note that $z = a + bi$ could be written as the ordered pair (a,b). This was first done by Sir William Hamilton in 1835.

Compare the addition of complex numbers with the addition of vectors of $\mathbb{V}_2$.

- $(a + bi) + (c + di) = (a + c) + (b + d)i$
- $\overrightarrow{(a,b)} + \overrightarrow{(c,d)} = \overrightarrow{(a + c, b + d)}$

Similarly, compare "multiplication of a complex number by a real number" with "multiplication by a scalar" in $\mathbb{V}_2$.

- $k(a + bi) = ka + kbi$
- $k\overrightarrow{(a,b)} = \overrightarrow{(ka,kb)}$



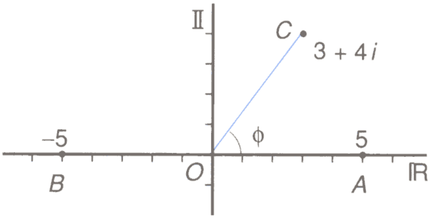
You can see that the results match exactly.
Thus, the set $\mathbb{C}$ of complex numbers can be considered a **vector space**.
All the properties of vectors of $\mathbb{V}_2$ with which you are familiar, including the geometric properties of addition and subtraction, can be applied to complex numbers. (See page 61.)

2. *The modulus of a complex number*

Consider the real number 5. It may be represented in the complex plane either by the point A, or the position vector of A, that is, $\overrightarrow{OA}$.
The absolute value of 5 is the length or magnitude of $\overrightarrow{OA}$, that is, $|\overrightarrow{OA}| = |5| = 5$.
Similarly, if B is the point representing the real number -5, the absolute value of -5 equals $|\overrightarrow{OB}| = |-5| = 5$.
In the same way, if C is the point representing the complex number $w = 3 + 4i$, then

$$|\overrightarrow{OC}| = |w| = \sqrt{3^2 + 4^2} = 5.$$

$|w|$ is called the length, magnitude, absolute value or **modulus** of the complex number w.



3. *The argument of a complex number*

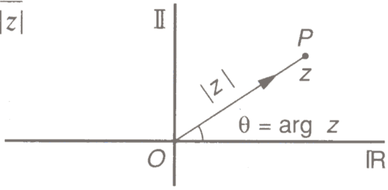
Although you know that $|w| = 5$, this fact is not sufficient to locate w precisely in the complex plane. (The numbers 5 and −5 also have a modulus of 5. Yet all three of these numbers are different, and are represented by different points). However, *w can* be fully determined by its modulus *and* the angle ϕ that it makes with the positive real axis.

In this case, $\tan \phi = \frac{4}{3}$ so $\phi \doteq 53°$ or $\phi \doteq 0.927$ radians.

Alternatively, ϕ can be determined by both $\sin \phi = \frac{4}{5}$ and $\cos \phi = \frac{3}{5}$, giving as before $\phi \doteq 53°$ or 0.927 rad.
ϕ is called an **argument of w**, written **arg w**.

PROPERTY In general, if $z = x + yi$, then the modulus of z, $|z| = \sqrt{x^2 + y^2}$

PROPERTY In general, if $z = x + yi$ is represented by the point P, then $\theta = \arg z$ is the angle that $\overrightarrow{OP}$ makes with the positive real axis; that is, θ is determined by $\sin \theta = \frac{y}{|z|}$ and $\cos \theta = \frac{x}{|z|}$



Note 1 Arguments of complex numbers are frequently expressed in radians. The reasons for this will be made clear in section 10.9. (Recall that π radians = 180°. There is a table of degree-radian equivalences on page 543.)

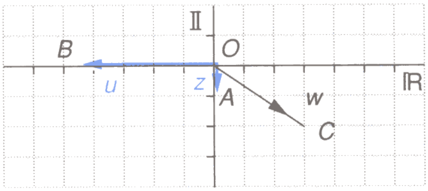
2 Any angle coterminal with θ is also an argument of z. That is, if θ is an argument of z, then so is any other angle $\theta + 2k\pi$ (or $\theta^\circ + 360k^\circ$), $k \in \mathbb{Z}$.

3 The principal argument of z is the angle θ chosen such that $-\pi < \theta \leqslant \pi$ (or $-180^\circ < \theta \leqslant 180^\circ$).
The principal argument is denoted by Arg z.

Example 2 a) Draw the following complex numbers as vectors in the complex plane.
$z = -i$, $u = -4.5$, $w = 3 - 2i$

b) Find the modulus and the principal argument for each of z, u, and w. (Give the arguments correct to the nearest degree.)

Solution a) The numbers z, u, w are represented by the points A, B, C respectively, or by the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ respectively.



b) $|z| = \sqrt{0^2 + (-1)^2} = 1$ Arg $z = -90^\circ$

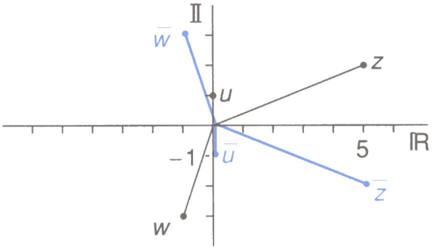
$|u| = 4{\cdot}5$ Arg $u = 180^\circ$

$|w| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$ $\sin(\text{Arg } w) = \dfrac{-2}{\sqrt{13}}$ and $\cos(\text{Arg } w) = \dfrac{3}{\sqrt{13}}$,

so Arg $w = -34^\circ$ ■

4. *Conjugates in the complex plane*

Consider the following complex numbers:
$z = 5 + 2i$, so $\overline{z} = 5 - 2i$; $w = -1 - 3i$, so $\overline{w} = -1 + 3i$; $u = i$, so $\overline{u} = -i$



You can see from the diagram that the conjugate of a complex number is obtained by *reflecting the complex number in the real axis.*

5. *Order in the set* $\mathbb{C}$

You are familiar with the order property of real numbers. That is, given any two distinct real numbers a and b, then either $a > b$ or $b > a$. This is interpreted on the real number line by saying that "greater than" is equivalent to "to the right of". Since $\mathbb{C}$ cannot be represented by a line, *it is impossible to "order" complex numbers*. The task of defining an order relation in $\mathbb{C}$ would be equivalent to that of defining an order relation for points in a plane.

However, since the modulus of a complex number is real, it is possible to say that the modulus of one complex number is greater than the modulus of another.

In the exercises, you will familiarize yourself more with the visual aspects of complex numbers.

SUMMARY

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

The complex plane is determined by a real axis and an imaginary axis, crossing at 0.

Complex numbers have all the properties of vectors of $\mathbb{V}_2$.

If $z = x + yi$ is represented by P in the complex plane:
The modulus of z, $|z| = \sqrt{x^2 + y^2}$
Any argument of z is the angle that $\overrightarrow{OP}$ makes with the positive real axis, that is, an angle satisfying both $\sin(\arg z) = \dfrac{y}{|z|}$ and $\cos(\arg z) = \dfrac{x}{|z|}$

The complex conjugates $z = x + yi$ and $\overline{z} = x - yi$ are reflections of each other in the real axis.

There is no order in $\mathbb{C}$.

10.3 Exercises

In these exercises, where appropriate, calculate all arguments in degrees, correct to the nearest degree.
In questions 1–6, use the numbers $z = 1 + 3i$, $w = 12 - 5i$, $p = 6i$, $q = -4 - i$, $u = -3 + 2i$.

1. a) Plot the points representing numbers z, w, p, q, u in a complex plane.
b) Find the conjugates $\bar{z}, \bar{w}, \bar{p}, \bar{q}$, and $\bar{u}$, and plot them in the same complex plane.

2. a) Find the moduli $|z|, |w|, |p|, |q|$, and $|u|$.
b) Find the arguments Arg z, Arg w, Arg p, Arg q, Arg u.

3. a) Find the moduli of the conjugates, namely $|\bar{z}|$ and $|\bar{w}|$.
b) Find arguments of the conjugates, namely Arg $(\bar{z})$ and Arg$(\bar{w})$.
c) Draw conclusions about the modulus of a conjugate and the argument of a conjugate.

4. a) Attempt to list the numbers z, w, p, q, u in order, from smallest to largest.
b) Attempt to list the moduli of these numbers in order, from smallest to largest.

5. a) Calculate the number $z + w$.
b) Draw z, w, and $z + w$ as vectors in a complex plane.
c) Use the diagram in b) to explain how $z + w$ could be considered an addition of vectors.

6. a) Calculate $z + \bar{z}$, $z - \bar{z}$ and $z\bar{z}$.
b) Plot $z, \bar{z}, z + \bar{z}, z - \bar{z}$, and $z\bar{z}$ in a complex plane.
c) Verify that
$z + \bar{z} = 2\text{Re}(z)$,
$z - \bar{z} = 2i\text{Im}(z)$, and
$z\bar{z} = |z|^2$.

7. Describe the modulus, the argument, and the conjugate of the following.
a) a real number
b) an imaginary number

8. If θ is any angle, calculate the modulus of $z = \cos\theta + i\sin\theta$ and $w = 3\cos\theta - 3i\sin\theta$.

9. a) Plot the points A and B representing the numbers $z = -2 + 3i$ and $w = 8 - i$ respectively in a complex plane.
b) Calculate $u = \frac{1}{2}z + \frac{1}{2}w$ and plot the point M representing u on the same diagram.
c) Calculate $v = \left(\frac{3}{4}\right)z + \frac{1}{4}w$ and plot the point N representing v on the same diagram.
d) Describe M and N geometrically with reference to A and B.

10. Consider the numbers
$z = 1 + i\sqrt{3}$, $z_1 = iz$, $z_2 = iz_1$, and $z_3 = iz_2$.
a) Calculate the numbers z_1, z_2, and z_3.
b) Draw all four numbers as vectors in a complex plane.
c) Calculate the modulus and an argument of all four numbers.
d) Draw conclusions on the effect of i as a multiplier in the complex plane.

11. Given $z = 1 + i\sqrt{3}$,
a) calculate z^2 and z^3,
b) plot z, z^2, and z^3 in a complex plane
c) discuss the statement: "$\sqrt[3]{-8} = 1 + i\sqrt{3}$".

12. a) If $z = 3 + 3i$, find $|z|$ and Arg z.
b) Verify that z could be expressed as $z = 3\sqrt{2}(\cos 45° + i\sin 45°)$
(For an exact solution, use the table on page 543.)

13. If $|z| = r$ and arg $z = \theta$, show that the number z can be represented in the form $z = r(\cos\theta + i\sin\theta)$. (This is known as the **polar form** or **modulus-argument form** of a complex number.)

10.4 Equations in $\mathbb{C}$

Recall the following vocabulary.

- $az^2 + bz + c = 0$, where $a \neq 0$, is a quadratic equation, or a polynomial equation of degree 2.
- $az^3 + bz^2 + cz + d = 0$, where $a \neq 0$, is a cubic equation, or a polynomial equation of degree 3.
- $a_n z^n + a_{n-1} z^{n-1} + \ldots + a_2 z^2 + a_1 z + a_0 = 0$, where $a_n \neq 0$ $\circledast$ is a polynomial equation of degree n.

Consider the quadratic equation $az^2 + bz + c = 0$. Recall that the solutions are given by $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If the variable $z \in \mathbb{R}$, three cases need to be considered.

1. If $b^2 - 4ac > 0$, then z can take two different real values.
2. If $b^2 - 4ac = 0$, then z has a single real value.
3. If $b^2 - 4ac < 0$, then there are no values for z.

Now if you let z take any values in $\mathbb{C}$, roots of $az^2 + bz + c = 0$ will always exist. The three previous cases can be replaced by the following single statement.
All quadratic equations *have two roots*
(which may or may not be real, and may or may not be equal).

This result can be extended to the following general case, which is one version of the **fundamental theorem of algebra.**

THEOREM

A polynomial equation of degree n always has n complex roots.

Note 1 The coefficients $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are not necessarily real.
2 Recall that "complex roots" includes real roots.
3 Some of the roots may be equal.
4 It was not possible to make such a clean statement before the advent of complex numbers. In this way, complex numbers have *simplified* our view of algebra.
5 Given that the roots of the polynomial equation $\circledast$ are $z_1, z_2, \ldots, z_n$, then $\circledast$ is expressible in the factored form
$a_n(z - z_1)(z - z_2)\ldots(z - z_n) = 0$,
or $(z - z_1)(z - z_2)\ldots(z - z_n) = 0$, since $a_n \neq 0$.

The Factor Theorem

Consider the polynomial $p(z) = (z - z_1)(z - z_2)\ldots(z - z_n)$. You can see that $p(z_k) = 0$ where $k \in \{1, \ldots, n\}$. The **factor theorem** is stated as follows.

THEOREM

If $p(z_k) = 0$, then $(z - z_k)$ is a factor of $p(z)$.

This theorem is exactly the same as for $\mathbb{R}$. The factor theorem can be used as an aid in factoring polynomials.

Example 1 Form a quadratic equation whose roots are

a) $3 + 2i$ and $3 - 2i$
b) $3 + 2i$ and $1 - i$

Solution **a)** In factored form, a quadratic equation is

$$(z - [3 + 2i])(z - [3 - 2i]) = 0$$
$$z^2 - [3 + 2i + 3 - 2i]z + [3 + 2i][3 - 2i] = 0$$
$$z^2 - 6z + 13 = 0.$$

Notice that the coefficients of this equation are real.

b) $(z - [3 + 2i])(z - [1 - i]) = 0$

$$z^2 - [3 + 2i + 1 - i]z + [3 + 2i][1 - i]$$
$$z^2 - [4 + i]z + 5 - i = 0$$ ■

Notice that the coefficients of this equation are not all real.

Example 2 By solving the equation $z^3 = 1$, find the three cube roots of 1.

Solution The equation is equivalent to $z^3 - 1 = 0$, a cubic. By the fundamental theorem of algebra, you know that there are three (not necessarily distinct) roots.

To solve the equation, express it in factored form.
[Recall that $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$]

$$z^3 - 1 = 0$$
$$\Rightarrow (z - 1)(z^2 + z + 1) = 0$$

Thus, $z - 1 = 0$ or $z^2 + z + 1 = 0$

$$z = 1$$

$$z = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)}$$

$$z = \frac{-1 \pm i\sqrt{3}}{2}$$

Hence, the cube roots of 1 are 1, $-\frac{1}{2} + \frac{i\sqrt{3}}{2}$, $-\frac{1}{2} - \frac{i\sqrt{3}}{2}$ ■

One of the most useful aspects of working with complex numbers is that each equation in $\mathbb{C}$ incorporates two equations in $\mathbb{R}$, because of the definition of the equality of two complex numbers. This will be illustrated in the following examples.

Example 3 Solve the equation $(2 + i)z - 4i = 0$ by writing $z = x + iy$, where $x, y \in \mathbb{R}$, and solving a system of equations in x and y.

Solution

$$\begin{aligned}(2 + i)z - 4i &= 0\\ (2 + i)(x + iy) - 4i &= 0\\ 2x + 2iy + ix - y - 4i &= 0\\ (2x - y) + i(2y + x - 4) &= 0\end{aligned}$$

That is, $2x - y = 0$ ①

and $x + 2y - 4 = 0$ ②

Thus, the original equation in $\mathbb{C}$ has produced *two* equations in $\mathbb{R}$.

2 × ① + ② gives $5x - 4 = 0$, so $x = \frac{4}{5}$. Substituting this into ① gives $y = \frac{8}{5}$.

Thus, $z = x + iy = \frac{4}{5} + \frac{8}{5}i$. ■

Note: This equation could also be solved by writing

$z = \frac{4i}{2 + i}$ and simplifying, to obtain $z = \frac{4}{5} + \frac{8}{5}i$.

Example 4 Solve the equation $z^2 = 16 - 30i$.

Solution $z \in \mathbb{C}$. Hence, let $z = x + iy$, where x and y are real.

Thus

$$\begin{aligned}(x + iy)^2 &= 16 - 30i\\ x^2 + 2xyi - y^2 &= 16 - 30i\\ (x^2 - y^2) + 2xyi &= 16 - 30i\end{aligned}$$

That is, $x^2 - y^2 = 16$ ①

and $2xy = -30$ ②

Once again, the original equation in $\mathbb{C}$ has produced *two* equations in $\mathbb{R}$.

From ②, $y = -\frac{30}{2x} = -\frac{15}{x}$ ③

Substituting into ①,

$$\begin{aligned}x^2 - \left(-\frac{15}{x}\right)^2 &= 16\\ x^4 - 225 &= 16x^2 \quad \text{multiplying both sides by } x^2\\ x^4 - 16x^2 - 225 &= 0\\ (x^2 - 25)(x^2 + 9) &= 0\\ x^2 &= 25 \quad \text{or } x^2 = -9 \text{ (which is}\\ x = 5 \text{ or } x &= -5 \quad \text{impossible, since } x \text{ is real)}\end{aligned}$$

and so $y = -3$ or $y = 3$, from ③

Thus, $z = 5 - 3i$ or $z = -5 + 3i$. ■

Note: These numbers can be considered the 'square roots' of the number $(16 - 30i)$, since the original equation was $z^2 = 16 - 30i$. However, the notation "$\sqrt{(16 - 30i)} = 5 - 3i$" will be avoided, since there is more than one square root. The term "principal square root" can only be used in relation to a positive real number.

10.4 Exercises

1. State the roots of the following equations.
 a) $(z-2)(z+3)=0$
 b) $(z-1-i)(z-1+i)=0$
 c) $(4z-1)(z+i)(z-i)=0$
 d) $z(z+2i)(2z-3-4i)=0$

2. Which of the equations of question 1 are polynomial equations with real coefficients?

3. Find quadratic equations in the form $az^2+bz+c=0$ with the following roots.
 a) $4i$ and $2+i$
 b) $p+qi$ and $p-qi$

4. The quadratic equation $az^2+bz+c=0$ is such that the coefficients a, b, c, are real, and $b^2-4ac<0$.
 a) Prove that the roots of this quadratic equation must be conjugates.
 b) Use this fact to show that the non-real roots of any polynomial equation with real coefficients must be conjugates, in pairs.

5. Find cubic equations in the form $az^3+bz^2+cz+d=0$ with the following roots.
 a) $4i$, $2+i$, and $1-3i$
 b) 0, $p+qi$, and $p-qi$.

6. Prove that a cubic equation with real coefficients always has at least one real root.

7. a) Verify that $w=-\frac{1}{2}+\frac{i\sqrt{3}}{2}$ is a cube root of 1.
 b) Locate w in the complex plane.

8. Solve the equation $(3-5i)z+1+2i=0$ by writing $z=x+iy$, $x, y \in \mathbb{R}$, and solving a system of equations in x and y.

9. Repeat question 8 for the equation $(a+bi)z+c+di=0$, $a, b, c, d, \in \mathbb{R}$. Does this equation *always* have a unique root?

10. By solving $z^2=i$, find the two square roots of i. Locate these roots in the complex plane.

11. Given that the square roots of i are $\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$, use the quadratic formula to solve the equation $z^2-3z-iz+2=0$. Express the roots in the form $a+bi$, $a, b \in \mathbb{R}$.

12. a) By solving $z^3+1=0$, find the three cube roots of -1.
 b) Locate these roots in the complex plane.

13. Discuss the validity of the following statements.
 a) $z^2+w^2=0 \Rightarrow z=0$ and $w=0$.
 b) $z^3-w^3=0 \Rightarrow z=w$.

14. a) Use the factor theorem to show that $(3z-2)$ and $(2z+1)$ are factors of the polynomial $p(z)=6z^4-25z^3+32z^2+3z-10$.
 b) Hence solve $p(z)=0$.

15. It is given that $2+i$ and $-2+i$ are two of the roots of the equation $z^4-6z^2+25=0$.
 a) Use this information to find all the roots of the equation.
 b) Show that the representations of these roots in a complex plane are the vertices of a rectangle.

16. Show that the equation $z^2-rz-iz+ir=0$, $r \in \mathbb{R}$, has exactly one real root.

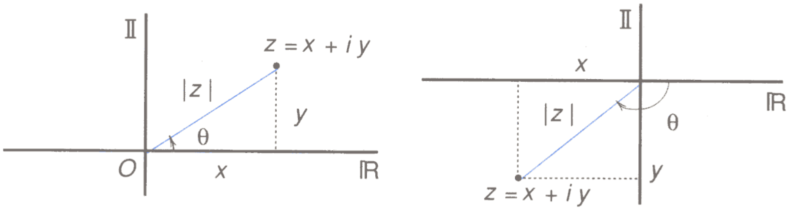
17. $z=\frac{r}{1+i}$, $w=\frac{s}{1+2i}$, where r and s are real, and it is given that $z+w=1$.
 a) Calculate the values of r and s.
 b) Calculate $|z-w|$.

18. Given $z=a+bi$ and $w=c+di$, where a, b, c, and d are real, prove that

$$|z+w|=|z-w| \Rightarrow \frac{iz}{w} \text{ is real.}$$

10.5 Complex Numbers and Trigonometry

Until now, you have used $z = x + yi$ to represent the complex number z. This is called the **Cartesian form** of z.
z can also be expressed by using its modulus $|z| = r$ and its argument θ.



Recall that the number z can be represented by the point with coordinates (x,y) in the complex plane.
The definition of angles in standard position tells you that $x = r\cos\theta$ and $y = r\sin\theta$, no matter what the position of z in the complex plane.
(See page 541.)
Thus $z = x + yi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$.
This is known as the **polar form**, or **modulus-argument form**, of a complex number.

Cartesian form: $z = x + yi$

polar form: $z = r(\cos\theta + i\sin\theta)$,

where $r = |z| = \sqrt{x^2 + y^2}$, $\cos\theta = \dfrac{x}{r}$, and $\sin\theta = \dfrac{y}{r}$

Note: The polar form of representation is not unique. For example,

$$2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left(\cos\frac{13\pi}{6} + i\sin\frac{13\pi}{6}\right), \text{ or}$$
$$2(\cos 30^\circ + i\sin 30^\circ) = 2(\cos 390^\circ + i\sin 390^\circ).$$

If the complex number z is represented in polar form by its modulus r and its argument θ, then any other argument of z, that is, any angle $\theta + 2k\pi$ or $\theta^\circ + 360k^\circ$ (with $k \in \mathbb{Z}$) could be substituted for θ.

In the exercises, you will have an opportunity to prove the equality principle for complex numbers expressed in polar form. That is, you will prove that

$r(\cos\theta + i\sin\theta) = p(\cos\phi + i\sin\phi)$
implies
$r = p$ and $\theta = \phi + 2k\pi$ (or $\theta^\circ = \phi + 360k^\circ$), $k \in \mathbb{Z}$.

Recall that the principal argument θ is such that $-\pi < \theta \leqslant \pi$, or $-180^\circ < \theta^\circ \leqslant 180^\circ$.

Example 1 State the principal arguments of the following complex numbers.

a) $z = 3\left(\cos\dfrac{13\pi}{3} + i\sin\dfrac{13\pi}{3}\right)$

b) $w = 5(\cos[-200°] + i\sin[-200°])$

Solution **a)** Since $\dfrac{13\pi}{3}$ is an argument of z, then $\dfrac{13\pi}{3} + 2k\pi$, $k \in \mathbb{Z}$, are its other arguments.

Since the principal argument θ is such that $-\pi < \theta \leqslant \pi$, you must select $k = -2$.

Thus $\operatorname{Arg} z = \dfrac{13\pi}{3} - 2(2\pi) = \dfrac{13\pi - 12\pi}{3} = \dfrac{\pi}{3}$

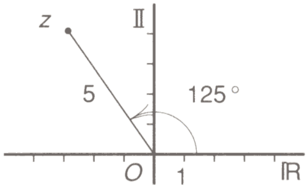
b) The arguments of w are $-200° + 360k°$, $k \in \mathbb{Z}$.
For the principal argument, you must select $k = 1$.

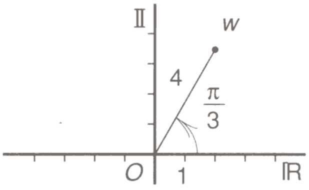
Therefore $\operatorname{Arg} w = -200° + (1)360° = 160°$ ■

For the examples that follow, you may wish to refer to the tables of values of the trigonometric ratios of special angles, and the table of radian and degree equivalences, on page 543.

Example 2 Find the Cartesian form of the following numbers.

a) $z = 5(\cos 125° + i\sin 125°)$ **b)** $w = 4\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$



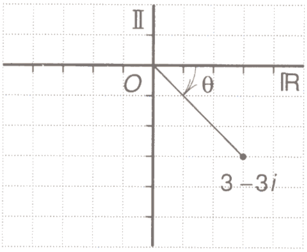


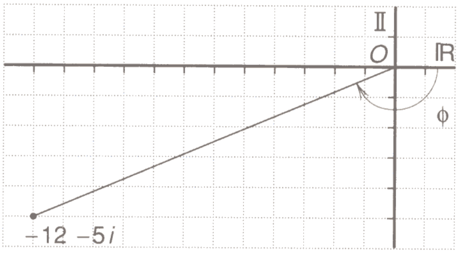
Solution **a)** $z = 5(-0.5735\ldots) + 5i(0.8191\ldots) \doteq -2.9 + 4.1i$, correct to 1 decimal place.

b) $w = 4\left(\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}\right) = 2 + 2i\sqrt{3}$, using exact values. Alternatively, $w = 4(0.5 + [0.8660\ldots]i) \doteq 2 + 3.5i$, correct to 1 decimal place. ■

Example 3 Find the polar form of the following numbers.

a) $z = 3 - 3i$ **b)** $w = -12 - 5i$





Solution **a)** $|z| = \sqrt{3^2 + 3^2} = 3\sqrt{2}$; $\sin\theta = -\frac{3}{3\sqrt{2}} = -\frac{1}{\sqrt{2}}$ and $\cos\theta = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$

Thus, $\theta = -\frac{\pi}{4}$ or $-45°$.

Hence $z = 3\sqrt{2}(\cos[-45°] + i\sin[-45°])$
or $z \doteq 4.24(\cos[-45°] + i\sin[-45°])$.

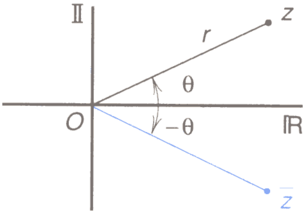
Note: Any angle coterminal with $-45°$ would also be correct.
For example, $-45° + 360° = 315°$ could have been used.

b) $|w| = \sqrt{12^2 + 5^2} = 13$; $\sin\phi = -\frac{5}{\sqrt{13}}$ and $\cos\phi = -\frac{12}{13}$

Thus, $\phi \doteq -157°$, correct to the nearest degree.
Hence $z \doteq 13(\cos[-157°] + i\sin[-157°])$ ■

Conjugates

The reflection in the real axis of
a complex number of modulus r, argument θ, is
a complex number of modulus r, argument $-\theta$.



Thus if $z = r(\cos\theta + i\sin\theta)$, then $\bar{z} = r(\cos[-\theta] + i\sin[-\theta])$
or $\bar{z} = r(\cos\theta - i\sin\theta)$,
since for any angle θ, $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$ (see page 541).

This form is also used routinely for complex numbers with *negative* arguments.
That is, the complex number z of modulus r and argument $-\theta$, where $\theta > 0$, can be written

$$z = r(\cos[-\theta] + i\sin[-\theta]) \quad \text{OR} \quad z = r(\cos\theta - i\sin\theta)$$

Multiplication and Division in Polar Form

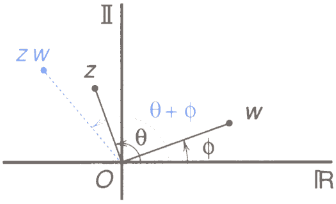
The most useful aspect of the polar form is the stunning result obtained when complex numbers are multiplied or divided. You will observe this presently.

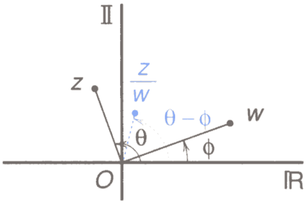
Let $z = p(\cos\theta + i\sin\theta)$ and $w = q(\cos\phi + i\sin\phi)$.

$$\begin{aligned}\text{Then}\quad zw &= pq(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\\ &= pq(\cos\theta\cos\phi + i\cos\theta\sin\phi + i\sin\theta\cos\phi - \sin\theta\sin\phi)\\ &= pq([\cos\theta\cos\phi - \sin\theta\sin\phi] + i[\sin\theta\cos\phi + \sin\phi\cos\theta])\end{aligned}$$

$$\text{so}\quad zw = pq(\cos[\theta + \phi] + i\sin[\theta + \phi])$$

from the formulas for the cosine and sine of compound angles on page 542. The product obtained is a complex number in polar form, whose *modulus is pq, and whose argument is* $\theta + \phi$.





RULE 1

Hence, when two complex numbers are *multiplied, the modulus of their product is the product of their moduli*, and *the argument of their product is the sum of their arguments*.

Similarly, you will have an opportunity to prove in the exercises that

$$\frac{z}{w} = \frac{p}{q}(\cos[\theta - \phi] + i\sin[\theta - \phi])$$

The quotient obtained is a complex number in polar form, whose *modulus is* $\frac{p}{q}$*, and whose argument is* $\theta - \phi$.

RULE 2

Thus, when two complex numbers are *divided, the modulus of their quotient is the quotient of their moduli*, and *the argument of their quotient is the difference of their arguments*.

Example 4 Given $z = 12(\cos 160^\circ + i\sin 160^\circ)$ and $w = 3(\cos 35^\circ + i\sin 35^\circ)$, find zw and $\frac{z}{w}$ in Cartesian form, correct to 1 decimal place.

Solution

$$\begin{aligned}zw &= (12)(3)(\cos[160^\circ + 35^\circ] + i\sin[160^\circ + 35^\circ])\\ &= 36(\cos 195^\circ + i\sin 195^\circ)\\ &= 36([-0.9659\ldots] + i[-0.2588\ldots]) \doteq -34.8 - 9.3i\end{aligned}$$

$$\begin{aligned}\frac{z}{w} &= \frac{12}{3}(\cos[160^\circ - 35^\circ] + i\sin[160^\circ - 35^\circ])\\ &= 4(\cos 125^\circ + i\sin 125^\circ)\\ &= 4([-0.5735\ldots] + i[0.8191\ldots]) \doteq -2.3 + 3.3i \quad ■\end{aligned}$$

Example 5 Find the exact values of zw and $\dfrac{z}{w}$

if $z = 4\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right)$ and $w = 2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

Solution

$$\begin{aligned} zw &= (4)(2)\left(\cos\left[\frac{5\pi}{6} + \frac{\pi}{3}\right] + i\sin\left[\frac{5\pi}{6} + \frac{\pi}{3}\right]\right) \\ &= 8\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) \\ &= 8\left(-\frac{\sqrt{3}}{2} - \frac{1i}{2}\right) \\ &= -4\sqrt{3} - 4i \end{aligned}$$

$$\begin{aligned} \frac{z}{w} &= \frac{4}{2}\left(\cos\left[\frac{5\pi}{6} - \frac{\pi}{3}\right] + i\sin\left[\frac{5\pi}{6} - \frac{\pi}{3}\right]\right) \\ &= 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \\ &= 2(0 + 1i) \\ &= 2i \quad \blacksquare \end{aligned}$$

The next example shows how rules 1 and 2 can be used advantageously in different situations.

Example 6 Given the complex number $z = 3 - 3i$ from Example 3a), calculate the exact values of

a) the modulus and argument of z^2

b) the modulus and argument of $\dfrac{1}{z}$

Solution $z = 3 - 3i = 3\sqrt{2}(\cos[-45°] + i\sin[-45°])$ from Example 3a).

That is, $|z| = 3\sqrt{2}$ and $\arg z = -45°$

a) Using rule 1,
the modulus of z^2 is $(3\sqrt{2})(3\sqrt{2}) = (3\sqrt{2})^2 = 18$
the argument of z^2 is $[-45°] + [-45°] = -90°$.

b) The complex number 1 has modulus 1, argument 0.
Thus, using rule 2,

the modulus of $\dfrac{1}{z}$ is $\dfrac{1}{3\sqrt{2}}$

the argument of $\dfrac{1}{z}$ is $0 - (-45°) = 45°$. $\blacksquare$

10.5 Exercises

1. Plot each of the following numbers in the complex plane and find their Cartesian forms. Use 3 significant digit accuracy.

$a = 4(\cos 50° + i \sin 50°)$

$b = 4(\cos 50° - i \sin 50°)$

$c = 2(\cos 145° + i \sin 145°)$

2. Plot each of the following numbers in the complex plane and find their Cartesian forms. Use exact values.

$d = 8\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$

$e = \left(\cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4}\right)$

$f = -\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$

3. State the modulus and an argument of z in the following cases.

a) $z = 3i$ **b)** $z = 4$ **c)** $z = -17$ **d)** $z = -i$

4. State the modulus and an argument of $\bar{z}$ using the numbers of question 3.

5. State the principal argument of the following numbers.

a) $\cos 115° + i \sin 115°$

b) $\cos 425° + i \sin 425°$

c) $6\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$

d) $2\left(\cos \frac{11\pi}{6} - i \sin \frac{11\pi}{6}\right)$

6. Plot each of the following numbers in the complex plane and find their polar forms. Use degrees.

$s = 4 + 3i$ $\quad v = -2$

$t = -1 + 2i$ $\quad z = -15 - 8i$

$u = 5i$ $\quad w = 4 - 9i$

7. Find the exact polar form of z in the following cases. Use radians.

a) $z = -1 + i$ **c)** $z = 2\sqrt{3} - 6i$

b) $z = \sqrt{3} + i$ **d)** $z = -3 - \sqrt{3}i$

8. Given any complex number z, state the possible values of the argument of $z + \bar{z}$.

9. Find r and a value of θ in the following cases.

a) $r(\cos 30° + i \sin 30°) = 5(\cos \theta + i \sin \theta)$

b) $6 \cos \theta + 6i \sin \theta = r(\cos 328° + i \sin 328°)$

c) $\cos \frac{\pi}{8} - i \sin \frac{\pi}{8} = r \cos \theta + r \sin \theta$

10. Given $z = p(\cos \theta + i \sin \theta)$ and $w = q(\cos \phi + i \sin \phi)$, prove that

$$\frac{z}{w} = \frac{p}{q}(\cos[\theta - \phi] + i \sin[\theta - \phi])$$

$\left[\text{Hint: Recall that } \frac{z}{w} = \frac{z\bar{w}}{w\bar{w}}\right]$

11. Given $z = 10(\cos 71° + i \sin 71°)$ and $w = 5(\cos 34° + i \sin 34°)$, express the following in polar form.

a) zw **b)** $\frac{z}{w}$ **c)** $\frac{w}{z}$

12. Given $z = 4 - 5i$ and $w = -2 + 3i$,

a) express z and w in polar form (use degrees).

b) Hence express zw, $\frac{z}{w}$, and $\frac{w}{z}$ in polar form.

13. Calculate the exact modulus and an exact argument of each of the numbers $z = -1 + \sqrt{3}i$ and $w = 2\sqrt{3} + 2i$.

14. Use the results of question 13 to express the following in polar form.

a) z^2 **b)** w^2 **c)** zw **d)** $\frac{w}{z}$

15. a) State the modulus and argument of i.

b) Describe geometrically what happens to the vector representation of a complex number that is multiplied by i.

16. a) If $z = \cos \theta + i \sin \theta$, state an argument of z^2.

b) Hence show that $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and that $\sin 2\theta = 2 \sin \theta \cos \theta$.

10.6 De Moivre's Theorem

The investigations of the last section lead to the most important theorem concerning complex numbers. This theorem was published by Abraham De Moivre (1667-1754) in 1730, well before the advent of the complex plane.

By the multiplication principle, recall that
$[r(\cos\theta + i\sin\theta)]^2 = r^2(\cos 2\theta + i\sin 2\theta)$.
Similarly,
$[r(\cos\theta + i\sin\theta)][r^2(\cos 2\theta + i\sin 2\theta)] = r^3(\cos 3\theta + i\sin 3\theta)$.

De Moivre's theorem extends this principle as follows.

THEOREM

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

This can be proved by induction for $n \in \mathbb{N}$ as follows.

Step 1: Show the statement is true for $n = 1$.
For $n = 1$, $L.S. = [r(\cos\theta + i\sin\theta)]^1$, $R.S. = r^1(\cos 1\theta + i\sin 1\theta)$.
Since $L.S. = R.S.$, the statement is true for $n = 1$.

Step 2: Assume the statement is true for some $n = k \in \mathbb{N}$. That is, assume $[r(\cos\theta + i\sin\theta)]^k = r^k(\cos k\theta + i\sin k\theta)$ is true.

Step 3: Prove the statement is true for $n = k + 1$. That is, prove $[r(\cos\theta + i\sin\theta)]^{k+1} = r^{k+1}(\cos[k + 1]\theta + i\sin[k + 1]\theta)$.

$$\begin{aligned} L.S. &= [r(\cos\theta + i\sin\theta)]^k[r(\cos\theta + i\sin\theta)] \\ &= [r^k(\cos k\theta + i\sin k\theta)][r(\cos\theta + i\sin\theta)] && \text{from step 2} \\ &= (r^k)(r)(\cos[k\theta + \theta] + i\sin[k\theta + \theta]) && \text{by multiplication property} \\ &= r^{k+1}(\cos[k + 1]\theta + i\sin[k + 1]\theta) = R.S. \end{aligned}$$

Thus, by the principle of mathematical induction,
$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$ is true for all $n \in \mathbb{N}$.

Example 1

Calculate in Cartesian form
a) $(\cos 50^\circ + i\sin 50^\circ)^8$ **b)** $(1 + i)^{24}$

Solution

a) Note that the modulus of $(\cos 50^\circ + i\sin 50^\circ)$ is 1.

$$\begin{aligned} (\cos 50^\circ + i\sin 50^\circ)^8 &= [1(\cos 50^\circ + i\sin 50^\circ)]^8 \\ &= 1^8(\cos[8 \times 50^\circ] + i\sin[8 \times 50^\circ]) && \text{de Moivre} \\ &= 1(\cos 400^\circ + i\sin 400^\circ) \\ &= 0.77 + 0.64i \end{aligned}$$

b) $|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\arg(1 + i) = 45^\circ$

$$\begin{aligned} \text{Thus } (1 + i)^{24} &= [\sqrt{2}(\cos 45^\circ + i\sin 45^\circ)]^{24} \\ &= (\sqrt{2})^{24}(\cos 1080^\circ + i\sin 1080^\circ) && \text{de Moivre} \\ &= 2^{12}(1 + 0i) \\ &= 4096 \end{aligned}$$

■

Negative Exponents

You will now use the division principle and De Moivre's theorem to find the polar form of z^{-n}, where $z = r(\cos\theta + i\sin\theta)$ and $n \in \mathbb{N}$.

$$z^{-n} = \frac{1}{z^n} = \frac{1(\cos 0 + i\sin 0)}{[r(\cos\theta + i\sin\theta)]^n} \qquad \text{since 1 has modulus 1 and argument 0}$$

$$= \frac{1(\cos 0 + i\sin 0)}{r^n(\cos n\theta + i\sin n\,\theta)} \qquad \text{by De Moivre's theorem}$$

$$= \frac{1}{r^n}(\cos[0 - n\theta] + i\sin[0 - n\theta] \qquad \text{by division property}$$

so $\quad z^{-n} = r^{-n}(\cos[-n\theta] + i\sin[-n\theta])$

This last statement is the expression of De Moivre's theorem for a negative integer.

Thus, De Moivre's theorem is true for any $n \in \mathbb{Z}$.

Example 2

Calculate in Cartesian form $(-\sqrt{3} + i)^{-9}$.

Solution

$|-\sqrt{3} + i| = \sqrt{3 + 1} = 2 \quad$ and $\arg(-\sqrt{3} + i) = 150°$

Thus $(-\sqrt{3} + i)^{-9} = [2(\cos 150° + i\sin 150°)]^{-9}$

$$= 2^{-9}[\cos(-1350°) + i\sin(-1350°)]$$

$$= \frac{1}{512}(0 - 1i)$$

$$= -\frac{i}{512} \quad ■$$

De Moivre's theorem can also be used in conjunction with the binomial theorem to establish certain trigonometrical identities. This is one of the applications of complex numbers to other areas of mathematics.

Example 3

Find expressions for $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

Solution

$$\cos 3\,\theta + i\sin 3\,\theta = (\cos\theta + i\sin\theta)^3 \qquad \text{by De Moivre's theorem}$$

$$= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta$$

by the binomial theorem

$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

Thus, by equating real and imaginary parts,

$$\cos 3\,\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

and

$$\sin 3\,\theta = 3\cos^2\theta\sin\theta - \sin^3\theta. \quad ■$$

De Moivre's Theorem for Rational Exponents

De Moivre's theorem is true not only for all positive and negative integer exponents, but also for *all rational exponents* (with a reservation), as the following indicates.

Assume that ϕ is such that
$(\cos\theta + i\sin\theta)^{\frac{p}{q}} = \cos\phi + i\sin\phi$, ①
where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

Then raising each side of ① to the exponent q gives

$$(\cos\theta + i\sin\theta)^p = (\cos\phi + i\sin\phi)^q$$

so $$\cos p\theta + i\sin p\theta = \cos q\phi + i\sin q\phi.$$

Equating real and imaginary parts shows that this is satisfied by

$p\theta = q\phi + 2k\pi$, that is, $\phi = \dfrac{p\theta - 2k\pi}{q}$, $k \in \mathbb{Z}$

If $k = 0$, $\phi = \dfrac{p}{q}\theta$
The statement ① now gives

$$(\cos\theta + i\sin\theta)^{\frac{p}{q}} = \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$$

Thus, De Moivre's theorem appears to be true for a rational exponent. *The reservation* is that, if n is not an integer, then there is *more than one value possible for* z^n, namely

$$z^n = r^n\left(\cos\frac{p\theta - 2k\pi}{q} + i\sin\frac{p\theta - 2k\pi}{q}\right), \; k \in \mathbb{Z}.$$

This will be clarified by Example 1 in section 10.7.

The Norwegian mathematician Niels Henrik Abel (1802-1829) showed that De Moivre's theorem can be extended to include all real, and even all complex exponents.

NOTATION

In some texts, the short form "cis θ" is used as an abbreviation for "$\cos\theta + i\sin\theta$".

SUMMARY

De Moivre's theorem:
$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\,\theta)$, $n \in \mathbb{Q}$.

If n is not an integer, then $(\cos\theta + i\sin\theta)^n$ is *not unique.*

10.6 Exercises

1. Express the following in Cartesian form (use 3 significant digit accuracy).

a) $(\cos 130^\circ + i\sin 130^\circ)^{10}$

b) $[3(\cos 20^\circ - i\sin 20)]^6$

c) $[4(\cos 257^\circ + i\sin 257^\circ)]^5$

2. Express the following in Cartesian form (use exact values).

a) $(1-i)^{32}$

b) $(1+i\sqrt{3})^{12}$

c) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^9$

d) $\left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right)^8$

3. Express the following in Cartesian form (use 3 significant digit accuracy).

a) $(\cos 130^\circ + i\sin 130^\circ)^{-10}$

b) $[3(\cos 20^\circ - i\sin 20^\circ)]^{-6}$

c) $[4(\cos 257^\circ + i\sin 257^\circ)]^{-5}$

4. Express the following in Cartesian form (use exact values).

a) $(1-i)^{-32}$

b) $(1+i\sqrt{3})^{-12}$

c) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{-9}$

d) $\left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right)^{-8}$

5. Calculate the following in Cartesian form.

a) $(1+i)^8(\sqrt{3}-i)^6$

b) $\dfrac{i^{50}(-1+i)^{20}}{(1+i)^{10}}$

6. Simplify the following expressions.

a) $\dfrac{\left(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right)^5}{\left(\cos\frac{\pi}{7} - i\sin\frac{\pi}{7}\right)^2}$

b) $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{100}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{100}$

7. Find expressions for $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

8. Given $z = \cos\theta + i\sin\theta$,

a) use De Moivre's theorem to prove the following.

$$\frac{1}{z} = \cos\theta - i\sin\theta,$$
$$z^3 = \cos 3\theta + i\sin 3\theta,$$
$$\frac{1}{z^3} = \cos 3\theta - i\sin 3\theta$$

b) show that $\left(z + \frac{1}{z}\right)^3 = 8\cos^3\theta$

c) by expanding $\left(z + \frac{1}{z}\right)^3$, prove that

$$2\cos 3\theta + 6\cos\theta = 8\cos^3\theta$$

d) hence find $\cos 3\theta$ in terms of powers of $\cos\theta$.

9. Given $z = \cos\theta + i\sin\theta$,

a) show that $\left(z - \frac{1}{z}\right)^3 = -8i\sin^3\theta$

b) hence find $\sin 3\theta$ in terms of powers of $\sin\theta$.

10. Given $z = \cos\theta + i\sin\theta$,

a) expand and simplify $\left(z + \frac{1}{z}\right)^4$

b) hence prove that

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

c) hence find $\cos 4\theta$ in terms of powers of $\cos\theta$.

11. If $z = -1$, verify that $z^{\frac{1}{3}}$ may take more than one value in $\mathbb{C}$ as follows.

$w = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ is one of the values of $z^{\frac{1}{3}}$,

$u = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ is one of the values of $z^{\frac{1}{3}}$,

$v = -1$ is one of the values of $z^{\frac{1}{3}}$.

12. If $z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$, verify that $z^{\frac{1}{5}}$ may take more than one value in $\mathbb{C}$ as follows.

$p = \cos 9^\circ + i\sin 9^\circ$

is one of the values of $z^{\frac{1}{5}}$,

$q = \cos 153^\circ + i\sin 153^\circ$

is another value of $z^{\frac{1}{5}}$.

10.7 Quest for Roots in $\mathbb{C}$

You know that $1^3 = 1$.
Hence, 1 is a solution of the equation $z^3 = 1$ or $z^3 - 1 = 0$.

According to your experience of mathematics thus far, it may seem that 1 is the only root.

However, recall from the fundamental theorem of algebra (section 10.4) that this equation has *three* roots in $\mathbb{C}$. The roots cannot all be 1, since that would imply that the equation $z^3 - 1 = 0$ could be rewritten in factored form as $(z - 1)^3 = 0$.
(You know that $z^3 - 1 \neq (z - 1)^3$.)

The search for all the roots of this equation will be investigated in the following example.

Example 1 Use De Moivre's theorem to determine all the cube roots of 1 in $\mathbb{C}$.

Solution The cube roots of 1 are the roots of the equation $z^3 - 1 = 0$, or $z^3 = 1$.
You can solve this equation by writing each side in polar form.
Now $|1| = 1$, and $\arg 1 = 0$, thus $1 = 1(\cos 0 + i \sin 0)$.
Let $z = r(\cos \theta + i \sin \theta)$.
Thus you must solve $z^3 = 1$

$$\Rightarrow \quad [r(\cos \theta + i \sin \theta)]^3 = 1(\cos 0 + i \sin 0)$$

$$\Rightarrow \quad r^3(\cos 3\theta + i \sin 3\theta) = 1(\cos 0 + i \sin 0)$$

Recall from section 10.5 that if two complex numbers are equal, then their moduli are equal and their arguments differ by a multiple of 2π.
Thus $r^3 = 1$ and $3\theta = 0 + 2k\pi$, $k \in \mathbb{Z}$,

$$\Rightarrow \quad r = 1 \text{ (since } r \text{ is real)} \quad \text{and} \quad \theta = \frac{2k\pi}{3}, \text{ where } k \text{ is any integer.}$$

That is, $z = 1\left(\cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3}\right) = \cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3}$,
where k is any integer.

You will now see that this expression for z represents *different* complex numbers, depending on the value chosen for k.
Let these numbers be represented by w_k, then, by substituting successively the values 0, 1, 2, 3, . . . , you obtain

$w_0 = \cos 0 + i \sin 0 = 1 + 0i = 1$

$w_1 = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, a value different from w_0

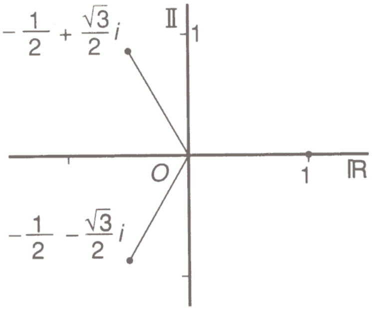
$w_2 = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, a value different from w_0 or w_1

$w_3 = \cos 2\pi + i \sin 2\pi = 1 + 0i = 1 = w_0$

Continuing the process yields the values $w_1, w_2, w_0, w_1, \ldots$etc.

Thus the three cube roots of 1 are 1, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ ■

In the exercises, you will have an opportunity to verify that each of these numbers, when cubed, yields 1.



- Observe from the figure that the roots have a rotational symmetry about the origin, of angle $\frac{2\pi}{3}$. That is, each root, if rotated counterclockwise through $\frac{2\pi}{3}$ about the origin, has for image another root.
- Also observe that $w_1{}^2 = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = w_2$.
- Compare also with the solutions formed earlier, using the quadratic formula (Example 2, section 10.4).

Method to Find the nth Roots of Unity

The above method can be applied to solving the equation $z^n - 1 = 0$, or $z^n = 1$, where n is any natural number. According to the fundamental theorem of algebra, this equation will have n roots in ℂ. These roots are called the **nth roots of 1** or **nth roots of unity**.

Let $z = r(\cos\theta + i\sin\theta)$, then

$$z^n = 1$$

$$\Rightarrow [r(\cos\theta + i\sin\theta)]^n = 1(\cos 0 + i\sin 0)$$

$$\Rightarrow r^n(\cos n\theta + i\sin n\theta) = 1(\cos 0 + i\sin 0)$$

Thus $r^n = 1$ and $n\theta = 0 + 2k\pi$, $k \in \mathbb{Z}$,

$\Rightarrow$ $r = 1$ (since r is real) and $\theta = \frac{2k\pi}{n}$, where k is any integer.

That is, $z = w_k = 1\left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right) = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$,

where k is any integer.

By substituting successively the values 0, 1, 2,..., $n - 1$ of k in $w_k = \left(\cos \dfrac{2k\pi}{n} + i \sin \dfrac{2k\pi}{n}\right)$, you find

$w_0 = \cos 0 + i \sin 0 = 1 + 0i = 1$

$w_1 = \cos \dfrac{2\pi}{n} + i \sin \dfrac{2\pi}{n}$, a value different from w_0

$w_2 = \cos \dfrac{4\pi}{n} + i \sin \dfrac{4\pi}{n}$, a value different from w_0 or w_1

. . . .

$w_{n-1} = \cos \dfrac{2(n-1)\pi}{n} + i \sin \dfrac{2(n-1)\pi}{n}$, a value different from all previous

$w_n = \cos \dfrac{2n\pi}{n} + i \sin \dfrac{2n\pi}{n} = \cos 2\pi + i \sin 2\pi = w_0.$

Successive values of k will again yield solutions equal to $w_1, w_2, \ldots,$ in turn.

Thus the nth roots of unity are given by the n numbers $w_0, w_1, w_2, \ldots, w_{n-1}$.

Once again, observe that $w_1^2 = w_2$.
Furthermore, $w_1^3 = w_3$, $w_1^4 = w_4$, etc.

Rational Powers of z

You are familiar with a result in $\mathbb{R}$ such as "$32^{\frac{1}{5}} = 2$". In $\mathbb{C}$, however, the expression $32^{\frac{1}{5}}$ may take *five different values*. The number 2, which is real and positive, is called the **principal root.** In order to distinguish the principal root from the others when working in $\mathbb{C}$, you can use the notation $\sqrt[5]{32}$ for the principal root. That is,
$32^{\frac{1}{5}}$ may take five different values, including 2, but
$\sqrt[5]{32} = 2$ (a positive real number).

Note: If z is not a positive real number, then there is no principal root of $z^{\frac{1}{n}}$, where $n \in \mathbb{N}$. The ambiguity can occur only if z is a positive real number.

The method of searching for roots can now be extended to any rational power of z, that is, $z^{\frac{p}{q}}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$. This is illustrated in the following example.

Example 2 Find all the values of $z = [16(1 + i\sqrt{3})]^{\frac{2}{5}}$, and sketch them in the complex plane.

Solution The numbers required are the solutions of the equation

$$z^5 = [16(1 + i\sqrt{3})]^2$$

Let $z = r(\cos\theta + i\sin\theta)$, and let $u = 16(1 + i\sqrt{3})$.

Now $|u| = 16\sqrt{1^2 + \sqrt{3}^2} = 16\sqrt{4} = 32$, and $\arg u = \frac{\pi}{3}$ or 60°.

Thus $$z^5 = [32(\cos 60^\circ + i\sin 60^\circ)]^2$$

$\Rightarrow$ $$r^5(\cos 5\theta + i\sin 5\theta) = 32^2\,(\cos[2 \times 60^\circ] + i\sin[2 \times 60^\circ])$$

Thus $r^5 = 32^2$ and $5\theta = 120^\circ + 360k^\circ,\ k \in \mathbb{Z}$

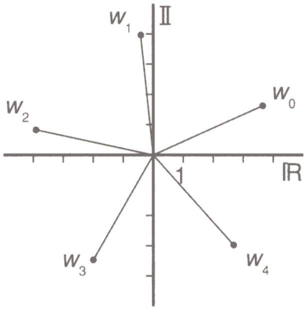
$\Rightarrow$ $r = \sqrt[5]{32^2} = 2^2 = 4$ (since r is real; it is the principal fifth root of 32^2)

and $\theta = \frac{1}{5}(120^\circ + 360k^\circ) = 24^\circ + 72k^\circ$, where k is any integer.

That is, $z = w_k = 4[\cos(24^\circ + 72k^\circ) + i\sin(24^\circ + 72k^\circ)]$

Now substitute the values 0, 1, 2, 3, and 4 of k in w_k.

$w_0 = 4(\cos 24^\circ + i\sin 24^\circ) = 4(0.913\ldots + 0.406\ldots i) \doteq 3.65 + 1.63i$
$w_1 = 4(\cos 96^\circ + i\sin 96^\circ) = 4(-0.104\ldots + 0.994\ldots i) \doteq -0.42 + 3.98i$
$w_2 = 4(\cos 168^\circ + i\sin 168^\circ) = 4(-0.978\ldots + 0.207\ldots i) \doteq -3.91 + 0.83i$
$w_3 = 4(\cos 240^\circ + i\sin 240^\circ) = 4(-0.5 + 0.866\ldots i) \doteq -2 - 3.46i$
$w_4 = 4(\cos 312^\circ + i\sin 312^\circ) \doteq 4(+0.669\ldots - 0.743\ldots i) \doteq 2.68 - 2.97i$



■

Notice again the symmetry of the roots. However, in this case, $w_1{}^2 \neq w_2$. You will investigate this further in the exercises.

10.7 Exercises

In the following, leave numerical answers correct to 3 significant digits, where you cannot find exact values.

1. By finding w_0^3, w_1^3 and w_2^3, verify that each of the following numbers is a cube root of 1.

$$w_0 = 1, \quad w_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad w_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

2. Find the following roots of unity in Cartesian form and represent them in a complex plane.
   - **a)** the fourth roots of unity
   - **b)** the fifth roots of unity
   - **c)** the tenth roots of unity

3. Find the following roots in Cartesian form and represent them in a complex plane.
   - **a)** the square roots of i
   - **b)** the cube roots of i
   - **c)** the square roots of $-i$
   - **d)** the cube roots of $27(\cos 72^\circ + i \sin 72^\circ)$
   - **e)** the fourth roots of $81(\cos 72^\circ + i \sin 72^\circ)$
   - **f)** the sixth roots of $64(\cos 102^\circ - i \sin 102^\circ)$

4. Two of the roots of the equation $z^5 - 32 = 0$ are $2\left(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}\right)$ and $2\left(\cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}\right)$. State the other roots.

5. **a)** Solve the equation $z^5 - 1 = 0$.
   - **b)** Use these solutions to express $z^5 - 1$ in factored form.
   - **c)** Express $z^5 - 1$ in the factored form $(z - 1)p(z)q(z)$, where $p(z)$ and $q(z)$ are quadratic expressions with real coefficients.

6. Find the real factors of
   a) $z^5 + 1$ **b)** $z^7 - 1$ **c)** $z^6 - 1$

7. **a)** Show that $\sqrt{3} - i$ is a fourth root of $-8(1 + i\sqrt{3})$.
   - **b)** Hence solve the equation $z^4 + 8 + 8i\sqrt{3} = 0$.

8. **a)** If w is a non-real seventh root of unity, show that the other roots are w^2, w^3, w^4, w^5, w^6, and 1.
   - **b)** Prove that $1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$
   - **c)** Do similar properties hold for all other nth roots of unity?

9. In section 10.7, Example 2, it was shown that the five values of $[16(1 + i\sqrt{3})]^{\frac{2}{5}}$ could be represented by $w_k = 4[\cos(24^\circ + 72k^\circ) + i \sin(24^\circ + 72k^\circ)]$, $k \in \{0,1,2,3,4\}$.
   - **a)** Show that $\frac{w_{k+1}}{w_k}$ is a constant.
   - **b)** Use your answer to a) to explain the symmetry of the representatives in the complex plane.

10. **a)** Express $z^7 + 1 = 0$ in factored form, using factors with real coefficients.
    - **b)** Hence show that

$$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}$$

11. Solve the equation $z^6 - 2z^3 + 4 = 0$.

10.8 Graphing and Complex Numbers

In this section, you will be describing the set of points representing a complex number z that is subject to certain conditions. Such a set of points is called the **locus** of z in the complex plane. This topic will give you the opportunity to work with complex numbers in a variety of ways.

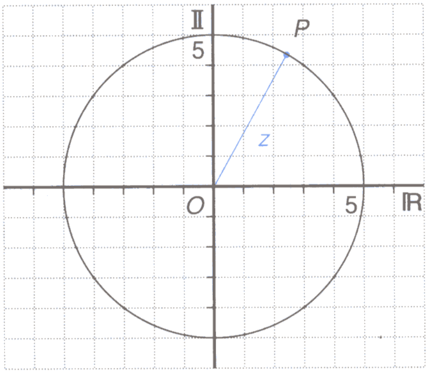
Example 1

a) Describe the locus of the points $z = x + iy$ in the complex plane, given that $|z| = 5$.

b) Find an equation in x and y that represents this locus.

Solution

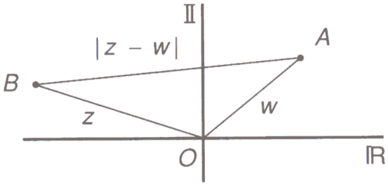
a) The modulus of a complex number is its length, that is, its distance from the origin.
If $|z| = 5$, then z must lie on a circle of centre O and radius 5.



b) $$|z| = 5 \Rightarrow |x + iy| = 5$$
$$\Rightarrow \sqrt{x^2 + y^2} = 5$$
$$\Rightarrow x^2 + y^2 = 25$$ ■

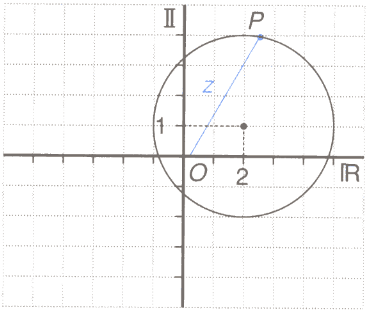
Distance Between Two Points in the Complex Plane

Recall that complex numbers can be represented by vectors. Let the complex numbers w and z be represented in the complex plane by the points A and B respectively.



Then the distance $AB = |\overrightarrow{AB}| = |\overrightarrow{OB} - \overrightarrow{OA}| = |z - w|$.
Thus, the modulus of $(z - w)$ represents the distance between the points representing z and w in the complex plane.

Example 2 a) Describe the locus represented by the equation $|z - 2 - i| = 3$.
b) If $z = x + iy$, find an equation for this locus in terms of x and y.



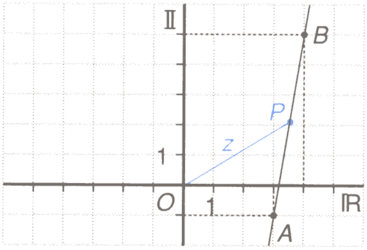
Solution a) $|z - 2 - i| = |z - (2 + i)|$.
Thus, this expression gives the distance between the points representing z and $(2 + i)$. The locus is therefore a circle with centre $(2 + i)$ and radius 3.

b) Since $z = x + iy$,

$$\begin{aligned} |z - 2 - i| &= 3 \text{ becomes} \\ |x + iy - 2 - i| &= 3 \\ |(x - 2) + i(y - 1)| &= 3 \\ \sqrt{(x - 2)^2 + (y - 1)^2} &= 3 \\ (x - 2)^2 + (y - 1)^2 &= 9 \quad \blacksquare \end{aligned}$$

The next example illustrates the link between complex numbers and vectors. Recall that the vector equation of a line is $\vec{r} = \vec{r}_0 + k\vec{m}$,
where $\vec{r}$ is the position vector of any point on the line,
$\vec{r}_0$ is the position vector of a given point on the line,
and $\vec{m}$ is a direction vector of the line.

Example 3 a) Determine a complex number equation for the line passing through the points A and B representing $3 - i$ and $4 + 5i$ respectively.
b) Deduce parametric equations for the line AB.



Solution a) Let z represent any point P on the line AB.

Since $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are collinear, $\overrightarrow{AP} = k\overrightarrow{AB}, k \in \mathbb{R}$

or $$z - (3 - i) = k[(4 + 5i) - (3 - i)]$$

$\Rightarrow$ $$z = (3 - i) + k(1 + 6i) \qquad ①$$

This is the required equation.

b) Let $z = x + iy$, and rewrite ① as follows.

$$x + iy = 3 + k + i(-1 + 6k)$$

Equating real and imaginary parts gives

$x = 3 + k$

$y = -1 + 6k$

These are parametric equations for the line AB. ■

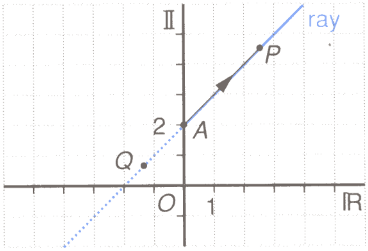
The next example shows that straight lines, or parts thereof, can be described in a totally different way with complex numbers.

Example 4 a) Describe the locus of z if $\arg(z - 2i) = 45°$

b) Find an equation in terms of x and y for this locus, given that $z = x + iy$.

Solution a) Let A be the point representing $2i$, and P be the point representing z. Then the complex number $z - 2i$ is represented by the vector $\overrightarrow{AP}$.

$\arg(z - 2i) = 45°$ means that the vector $\overrightarrow{AP}$ must make an angle of $45°$ with the positive x-axis.



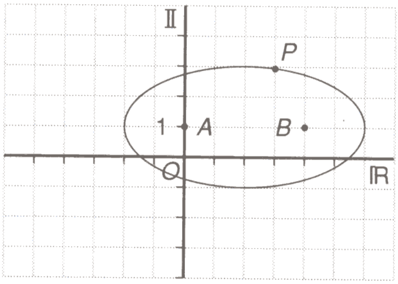
Thus, P is on the part-line, or **ray**, shown in the diagram.

b) This ray has slope 1, and it passes through (0,2). Recall that the equation of a line of slope m passing through the point (x_0,y_0) is $y - y_0 = m(x - x_0)$.

Thus the equation of the ray is $y - 2 = 1(x - 0)$, or $y = x + 2$, with the condition that $x > 0$. ■

Note: The point Q, which is on the line, does *not* satisfy $x > 0$. Indeed, the angle between $\overrightarrow{AQ}$ and the positive x-axis is $135°$, *not* $45°$. Thus points such as Q do not satisfy the original complex equation.

Example 5 An ellipse has a major axis of length 8 and foci at the points A and B, representing i and $4 + i$ respectively. Find a complex equation for this ellipse.



Solution A property of an ellipse is that the sum of the distances from the foci to any point on the ellipse is equal to the length of the major axis.
Let P be any point on the ellipse, represented by the number z.

Thus $$|\overrightarrow{AP}| + |\overrightarrow{BP}| = 8,$$

or $|z - i| + |z - (4 + i)| = 8$ is the required equation. ■

Example 6 Find an equation in x and y for the locus described by $z^2 - (1 + i)^2 = \bar{z}^2 - (1 - i)^2$, where $z = x + iy$.

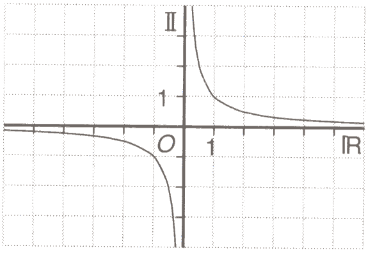
Solution Substituting $z = x + iy$ and $\bar{z} = x - iy$ gives

$$(x + iy)^2 - (1 + i)^2 = (x - iy)^2 - (1 - i)^2 \text{ or}$$
$$(x + iy)^2 - (x - iy)^2 = (1 + i)^2 - (1 - i)^2$$
$$(x + iy + x - iy)(x + iy - x + iy) = (1 + i + 1 - i)(1 + i - 1 + i)$$

factoring as difference of squares

$$(2x)(2iy) = (2)(2i)$$
$$xy = 1$$

This represents a rectangular hyperbola centred at the origin, with the real axis and the imaginary axis as asymptotes.



■

10.8 Exercises

In the following, let $z = x + iy$ wherever appropriate.

1. Sketch the locus of the point P representing the complex number z in the following cases.

a) $|z| = 4$

b) $|z - 1| = 4$

c) $|z - i| = 4$

d) $|z - 5 - 2i| = 1$

e) $\arg z = \frac{\pi}{2}$

f) $\arg(z - 1) = \frac{5\pi}{6}$

g) $|z + 1| + |z - 1| = 4$

h) $|z| + |z - 4 - i| = 6$

2. Find a complex number equation for the following.

a) a circle centre O, radius 6

b) a circle centre $-1 + 3i$, radius 5

c) a circle centre u, radius a, with $u \in \mathbb{C}$, $a \in \mathbb{R}$

3. Find equations in x and y of the loci described by the following.

a) $|z + 4 + 3i| = 2$

b) $|z - i| = 3|z + i|$

4. A point moves so that its distance from the origin is twice its distance from $3 - i$. Show that the locus is a circle, and find its centre and its radius.

5. Determine a complex number equation for the straight line through the points A and B represented respectively by $-2 + 5i$ and $-2 - i$.

6. Describe the locus represented by $\text{Re}\left(z - \frac{1}{z}\right) = 0$.

7. Describe the locus represented by $\arg(z - 4 - 2i) = 120°$.

8. Describe the locus represented by $\text{Im}(z^2) = 0$.

9. Describe the locus represented by $\text{Im}\left(z - 1 + \frac{2}{z}\right) = 0$.

10. Find a complex number equation for the perpendicular bisector of the line segment AB where A and B are represented respectively by the following complex numbers.

a) $2, -6$

b) $2 + \text{i}, 3 - 2\text{i}$

11. Given that $|z - w| = |z + w|$, show that $|\arg z - \arg w| = 90°$.

12. Find an equation in x and y for the following.

a) $\left|\frac{z - 3}{z - 6}\right| = 1$

b) $\left|\frac{z + 4i}{z - 2}\right| = 2$

c) $\arg\left(\frac{z}{z + 2}\right) = \frac{\pi}{4}$

d) $\arg\left(\frac{z - 1 - i}{z + 2 + i}\right) = \frac{\pi}{2}$

13. Describe the locus of z if $\text{Im}(z^2) = 2$.

14. Describe the locus represented by the following.

a) $|z| < 5$

b) $|z - 5 + 3i| \leqslant 3$

c) $\text{Re}(z^2) > 2$

d) $2 \leqslant |z - 2i| \leqslant 3$

e) $|z - 1 - i| + |z + 2 - 4i| < 10$

15. a) Describe the locus represented by $|z - 1| = \text{Re}(z) + 1$.

b) Find an equation in x and y for this locus.

16. Describe the locus represented by each of the following.

a) $|z - 2 - 3i| = 4$

b) $\text{Re}(z) = 2$ and $-\frac{\pi}{3} \leqslant \arg z \leqslant \frac{\pi}{4}$

17. For each locus in question 16, find the greatest value of $|z|$.

In Search of Graphical Representation of Non-real Solutions of Equations

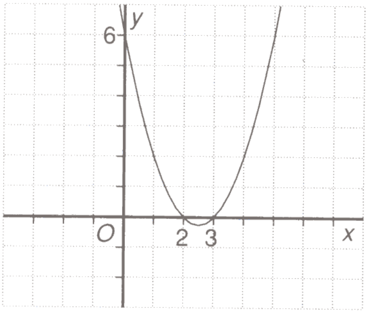
Solutions to quadratic equations in $\mathbb{R}$ can be seen graphically as shown in the examples A and B that follow.

A. $x^2 - 5x + 6 = 0$

$\Rightarrow$ $(x - 3)(x - 2) = 0$

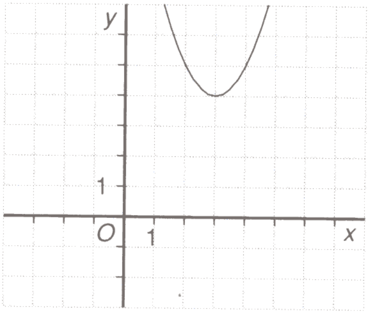
$\Rightarrow$ $x = 3$ or $x = 2$.

Graphically, these solutions can be viewed as the points where the parabola $y = x^2 - 5x + 6$ ① intersects with the line $y = 0$, that is, the x-axis.



B. $x^2 - 6x + 13 = 0$ has no real solutions.
The quadratic formula yields $x = 3 + 2i$ or $x = 3 - 2i$.

The parabola $y = x^2 - 6x + 13$ ② does *not* intersect the x-axis.
Is there any geometric significance in this context for $3 + 2i$ and $3 - 2i$?



Pursuing the question asked in B, proceed as follows. Allow the x-values in the parabola ② to extend into $\mathbb{C}$, that is, let x take the form $a + bi$, with $a, b \in \mathbb{R}$.

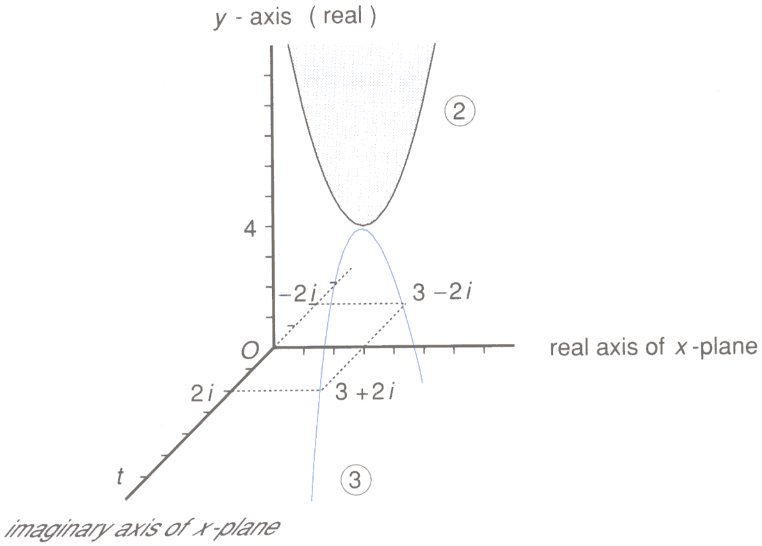
You now have a complex plane, the x-plane, taking the place of the old x-axis. (Note that the old x-axis is contained in this complex plane.)

Unfortunately, y will also now take on non-real values, and a four-dimensional situation is set up.

However, it is still possible to view a part of this, as follows.

You already know that the solutions to $y = 0$ are $x = 3 \pm 2i$. Hence, the real part of each solution is 3. Allow x to take the form $3 + ti$, with $t \in \mathbb{R}$.

$$\text{Then } y = (3 + ti)^2 - 6(3 + ti) + 13$$
$$= 9 + 6ti - t^2 - 18 - 6ti + 13$$
$$\text{or} \quad y = 4 - t^2 \quad ③$$



Now t is the variable along the imaginary axis of the x-plane.

The equation ③ thus represents a parabola whose plane is perpendicular to the plane of the original parabola ②.

Also, this parabola punctures the x-plane at the points $3 + 2i$ and $3 - 2i$.

Thus, you can see that the non-real intersections of a parabola with the x-axis are "somewhere in front of, or behind, the paper"!

A Canadian mathematician, Richard Dewsbury, is presently researching the geometrical aspect of extensions to $\mathbb{C}$ of equations in $\mathbb{R}$.

10.9 Exponential Form of a Complex Number

A geometric series with first term a, and common ratio r, has an 'infinite sum' $S = \frac{a}{1 - r}$, provided $|r| < 1$.

Consider the infinite series $S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots.$

This is a geometric series with $a = 2$, $r = \frac{1}{2}$

Thus

$$S = \frac{2}{1 - \frac{1}{2}} = 4$$

No finite sum of this series has a value 4. However, the sum of a finite part of this series will get as close to 4 as you want, provided that you add a sufficient number of terms.

In chapter 9, you saw that the binomial expansion becomes an infinite series if the exponent is not a natural number, that is,

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots.$$

If $|x| < 1$, the series approaches the value of $(1 + x)^n$ as closely as you like, by taking a sufficient number of terms. The series is said to *converge*.

If $|x| \geq 1$, the series does *not* approximate $(1 + x)^n$. In fact, the series may change value considerably for each extra term added. In that case, the series is said to *diverge*.

The theory of infinite series developed most significantly after the invention of calculus. Around 1700, the mathematicians Brook Taylor (1685-1731) and Colin Maclaurin (1698-1746) developed formulas to find series expansions, or polynomial approximations, to many functions in mathematics.

Three of these series, valid for all $x \in \mathbb{R}$, follow.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots.$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots.$$

For the trigonometrical functions, x is in 'natural measure', that is *radians*, not degrees.

The Swiss mathematician Leonhard Euler (1707-1783) broke from tradition by attempting to use these expansions for $x \in \mathbb{C}$, in 1748.

Using $z = x + iy$, he wrote

$$e^z = e^{x+iy} = e^x e^{iy} = re^{iy}, \text{ where } r = e^x \in \mathbb{R}. \qquad ①$$

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \frac{(iy)^6}{6!} + \ldots. \\ &= 1 + iy - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} + \frac{iy^5}{5!} - \frac{y^6}{6!} - \ldots. \\ &= 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \ldots + i\left[y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots\right] \end{aligned}$$

But these are the series for cos y and for sin y,

FORMULA

so $e^{iy} = \cos y + i \sin y$ ②

or, using ①,

FORMULA

$e^z = re^{iy} = r(\cos y + i \sin y)$, ③

where $z = x + iy$ and $r = e^x$, $x, y \in \mathbb{R}$.

Thus e^{x+iy} is a complex number whose modulus is e^x and whose argument in radians is y, that is,

PROPERTIES

$|e^{x+iy}| = e^x$ and $\arg(e^{x+iy}) = y$

The identities ② and ③ are known as **Euler's formulas**. They show that any complex number can be written in exponential form. The formula ② is the special case where the modulus is 1.

One extraordinary consequence of these formulas is the following identity, obtained by substituting $y = \pi$ in ②.

$e^{i\pi} = \cos \pi + i \sin \pi = -1 + 0$, or

IDENTITY

$e^{i\pi} = -1$

This wonderful relation links π, e, and i, three of the most important numbers that evolved in the history of mathematics. This is another example of the simplification, or rather 'unification', that may result after extended research into new areas.

Recall that

- π is the length of half the circumference of a unit circle (that is, a circle of radius 1). [$\pi \doteq 3.141\,592\,654$]
- e is the base of natural logarithms (the area under the curve $y = \frac{1}{x}$ is a natural logarithm) [$e \doteq 2.718\,281\,828$]
- i is a square root of -1.

Note: Just as the polar form of a complex number is not unique, so the exponential form is also not unique.
That is, since any argument θ can always be replaced by another argument $\theta + 2k\pi$, $k \in \mathbb{Z}$, then $e^{i\theta} = e^{i[\theta+2k\pi]}$

Example 1 Write the complex number $z = 5 + 2i$ in exponential form. (Leave numbers in your answer correct to 2 decimal places.)

Solution $z = re^{iy}$, where $r = |z|$ and $y = \arg z$.

Now $|z| = \sqrt{5^2 + 2^2} = \sqrt{29} \doteq 5.39$,

and $\tan(\arg z) = \frac{2}{5} = 0.4$. Since z is in the first quadrant, $\arg z = 0.38$.

Hence $z \doteq 5.39e^{0.38i}$ ■

Example 2 Write the complex number $w = -\sqrt{3} + i$ in exponential form. (Use exact values.)

Solution $|w| = \sqrt{\sqrt{3}^2 + 1^2} = 2$.

$\tan(\arg w) = -\frac{1}{\sqrt{3}}$ and w is in the second quadrant, so $\arg w = \frac{5\pi}{6}$

Hence $w = 2e^{\frac{5i\pi}{6}}$ ■

De Moivre's Theorem in Exponential Form

For clarity, consider De Moivre's theorem for a complex number of modulus 1.

$$(\cos\theta + i\sin\theta)^n = \cos n\,\theta + i\sin n\,\theta.$$

One immediate result of Euler's formulas is the expression of De Moivre's theorem as follows.

$$(e^{i\theta})^n = e^{in\theta}$$

Thus De Moivre's theorem can be seen as the extension of a normal exponent rule of $\mathbb{R}$ to $\mathbb{C}$!

However, recall that if n is not an integer, then $z = (e^{i\theta})^n$ is *not unique*. In this case, z is called a **multiple-valued function.** One of these values is $e^{in\theta}$, the complex number of modulus 1, with argument $n\theta$.

Example 3 Given $z = e^{\frac{i\pi}{8}}$, find the following in Cartesian form.

a) z^2 b) z^3 c) z^8 d) $z^{\frac{1}{2}}$

Solution

a) $z^2 = (e^{\frac{i\pi}{8}})^2 = e^{\frac{2i\pi}{8}} = e^{\frac{i\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \doteq 0.71 + 0.71i$

b) $z^3 = (e^{\frac{i\pi}{8}})^3 = e^{\frac{3i\pi}{8}} = \cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8} \doteq 0.38 + 0.92i$

c) $z^8 = (e^{\frac{i\pi}{8}})^8 = e^{\frac{8i\pi}{8}} = e^{i\pi} = -1$

d) Since $\frac{1}{2} \notin \mathbb{Z}$, $z^{\frac{1}{2}}$ is not unique.

You must proceed as you did when using De Moivre's theorem to find roots.

If $u = z^{\frac{1}{2}}$, then u is a solution of the equation $u^2 = z$.

Let $u = re^{i\theta}$, then $r^2e^{2i\theta} = e^{\frac{i\pi}{8}}$

Thus $\quad r^2 = 1$ and $2\theta = \frac{\pi}{8} + 2k\pi,\ k \in \mathbb{Z}$

$\Rightarrow \quad r = 1$ (since r is real)

and $\quad \theta = \frac{\pi}{16} + k\pi$, where k is any integer.

That is, $u = w_k = 1e^{i(\frac{\pi}{16}+k\pi)}$

Now substitute successively the values 0 and 1 for k in w_k.

$$w_0 = e^{\frac{i\pi}{16}} = \cos\frac{\pi}{16} + i\sin\frac{\pi}{16} \doteq 0.98 + 0.20i$$

$$w_1 = e^{\frac{17i\pi}{16}} = \cos\frac{17\pi}{16} + i\sin\frac{17\pi}{16} \doteq -0.98 - 0.20i$$

These are the two values of $z^{\frac{1}{2}}$. ■

In Search of Other Results using the Exponential Form

The link that Euler made between complex numbers and the exponential series gave birth to the theory of complex variables, an extensive branch of mathematics that you will have an opportunity to touch upon from the following.

1. Definition of Sine and Cosine using Exponential Forms

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad ①$$

and $$e^{-i\theta} = \cos\theta - i\sin\theta \qquad ②$$

Adding ① and ② yields

FORMULA

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \quad \text{or} \quad \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

Subtracting ① − ② yields

FORMULA

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta \quad \text{or} \quad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

Thus $\cos\theta$ and $\sin\theta$, which are real, can be defined in terms of non-real exponentials.

2. *The Link with Hyperbolic Functions*

The functions cos θ and sin θ are called **circular functions.** One link is that the circle of equation $x^2 + y^2 = 1$ can be represented parametrically by the system of equations $\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}$

The two functions defined as follows, are pronounced "cosh" and "shine".

DEFINITIONS

$\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$ and $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$
are called **hyperbolic functions.**

This name is used because the hyperbola of equation $x^2 - y^2 = 1$ can be represented parametrically by the system of equations

$$\begin{cases} x = \cosh\theta \\ y = \sinh\theta \end{cases}$$

The definitions of cosh θ and sinh θ are deemed to hold also when θ is non-real.

In the exercises you will have an opportunity to verify the following identities.

$\cosh iz = \cos z$ and $\sinh iz = i \sin z$

3. *The Meaning of z^w*

Given $z = x + iy$ and $w = a + ib$, where $x, y, a, b \in \mathbb{R}$, you will have an opportunity to demonstrate in 10.9 Exercises that one value of the complex power of a complex number z^w is

$$z^w = e^{ax - b \arg z}\, e^{i(bx - a \arg z)}$$

4. *Complex Numbers and Calculus*

If the formulas for differentiation are applied to complex numbers, it can be shown that $\frac{d}{d\theta}(e^{i\theta}) = \frac{d}{d\theta}(\cos\theta + i\sin\theta)$.

You will have an opportunity to do this in 10.9 Exercises.

It is hoped that, from this very sketchy introduction to the theory of complex variables, you will be spurred on to study further this area of mathematics in depth in the years to come.

10.9 Exercises

1. If $z = re^{i\theta}$, find the following in exponential form.

a) z^2 **b)** z^3 **c)** $\frac{1}{z}$ **d)** $\bar{z}$

2. Write the following in exponential form (leave numbers correct to 2 decimal places).

a) $z = 2 + i$ **b)** $w = -1 - 3i$

3. Write the following in exponential form, using exact values.

a) $u = 5 - 5i\sqrt{3}$ **b)** $v = -3 + 3i$

4. Given $z = 2e^{\frac{2i\pi}{5}}$, simplify the following

a) z^2 **c)** z^{-1} **e)** $z^{\frac{1}{2}}$

b) z^5 **d)** z^{-2} **f)** $z^{2.5}$

5. If $z = x + iy$ and $w = a + ib$, prove that $e^z e^w = e^{z+w}$.

6. A student claims to have calculated the value of i as follows.

$e^{2i\pi} = 1$ and $e^0 = 1$,

thus $2i\pi = 0$ or $i = 0$.

What is wrong with this demonstration?

7. Express $e^{i\theta} \times e^{i\phi}$, where $\theta, \phi \in \mathbb{R}$, in terms of sines and cosines of θ and ϕ in two different ways, and use your result to prove that

$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$.

8. Use the definitions $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and

$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ to prove the following identities.

a) $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

b) $\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

c) $\cos^2\theta + \sin^2\theta = 1$

9. Using the definitions of $\cos\theta$ and $\sin\theta$ given in question 8, verify that

a) $\sin(-\theta) = -\sin\theta$ **b)** $\cos(-\theta) = \cos\theta$

10. Use the definitions of $\cos\theta$ and $\sin\theta$ given in question 8 to solve the following equations.

a) $\sin\theta = 0$ **b)** $\cos\theta = 0$

11. If z is any complex number, show that $z^{\frac{1}{2}}$ always has two values, w_0 and w_1, such that $w_0 + w_1 = 0$.

12. Given $z = 6e^{\frac{i\pi}{3}}$, prove the following.

a) $iz = -3\sqrt{3} + 3i$

b) $|e^{iz}| = e^{-3\sqrt{3}}$

13. Using the definitions of $\cosh z$ and $\sinh z$ given on page 460, prove the following identities.

a) $\cosh iz = \cos z$ **b)** $\sinh iz = i\sin z$

14. Given $z = x + iy$ and $w = a + ib$, where $x, y, a, b \in \mathbb{R}$, show that one value of z^w is $e^{ax-b\arg z}\, e^{i(bx-a\arg z)}$.

15. If $z = e^w$, then $w = \ln z$, called the *natural logarithm* of z. If $z = re^{iy}$, then $w = \ln z = \ln r + i(y + 2k\pi)$, $k \in \mathbb{Z}$.

Use these definitions to show that the values of $\ln(1 - i)$ are given by

$\ln(1 - i) = \frac{1}{2}\ln 2 + \frac{i\pi(7 + 8k)}{4}$, $k \in \mathbb{Z}$

16. Consider a complex-valued function that can be written in the two forms

$f(\theta) = e^{i\theta}$ ① or $f(\theta) = \cos\theta + i\sin\theta$ ②.

Using the normal rules of differentiation, find $f'(\theta)$ using each of the forms ① and ② and show that these derivatives are equal.

17. Use the ideas on pages 454–455, and your knowledge of the exponential form of a complex number, to graph $z = e^{ix}$, $x \in \mathbb{R}$. (Use a complex z-plane and a real x-axis.)

The Visual Display of Data

René Descartes' (1596–1650) invention of coordinate geometry was the first link established between algebra and geometry. The link is visual, since it gives us a 'picture' of algebraic relations. Today, 'graphing' is the general term used to indicate that any information is displayed visually, rather than by words alone.

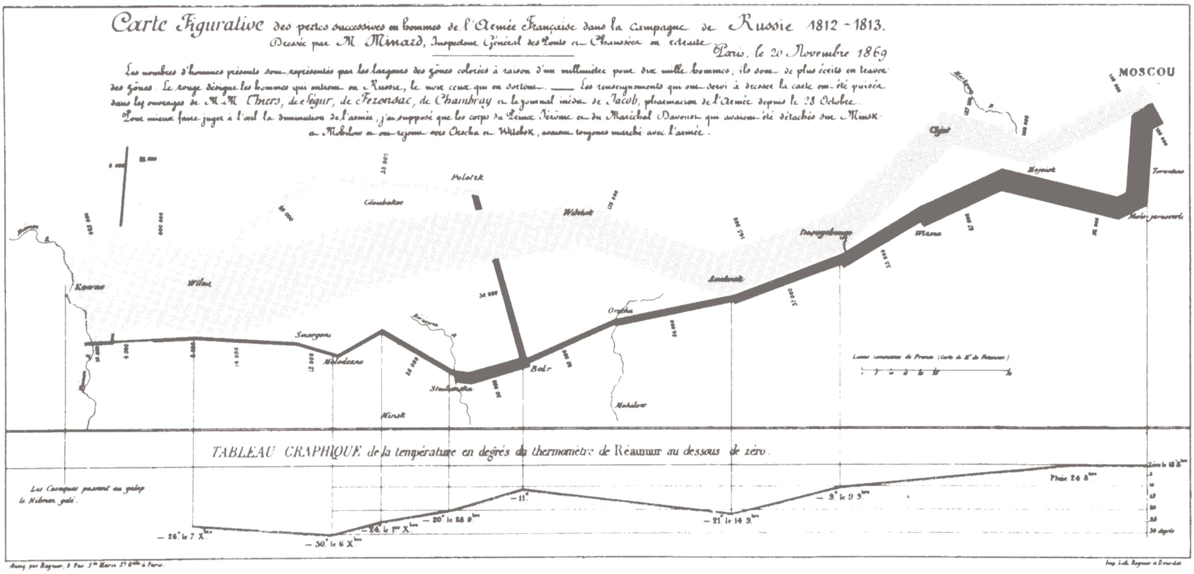
Graphing has made extraordinary advances since the time of Descartes. The recent advent of computers is leading to another great increase in the availability of visual displays of information.

Unfortunately, visual displays are not always good representations of what they try to portray. When used for advertising purposes, only some aspects of the data may be emphasized, while information that is not helpful to the advertiser is either not displayed, or cleverly disguised.

Possibly one of the most eloquent graphics ever drawn is the 'figurative map' drawn in 1869 by the French engineer Charles Joseph Minard (1781–1870) to describe Napoleon's Russian campaign of 1812.

Minard started with a map of the region extending from the Niemen river (the Russian-Polish border at the time) to Moscow. He then indicated not only the route taken by Napoleon's Grande Armée, but also superimposed the size of the army as it progressed towards Moscow. (The size of the army is indicated by the width of the shaded band.) The Russian armies sacked, burned and deserted most cities before Napoleon could reach them, thus cutting off supplies needed by the French. This had a devastating effect on the Grande Armée. Of the 422 000 men who started the campaign at the Niemen river in June 1812, only 100 000 made it to Moscow in September.

The retreat, which started on October 19, also had to contend with an unusually cold winter. Minard shows the retreating army with the darker band, and adds to his graphic a time scale (from October 24 to December 7), and a temperature scale indicating degrees below freezing. Note the immense losses suffered at the Berezina river, swelled by a sudden thaw. The Russians had destroyed the bridge. The Grande Armée made it back to Poland with about 10 000 men.



Observe that six variables are represented on this single diagram: the geographical location of the army (two dimensions), its size, its direction, a time-scale, and a temperature scale for the retreat from Moscow. Few graphics contain so much clearly displayed information.

Notes
1 The "lieue commune" is about 4444 metres.
2 The Réaumur temperature scale is such that water freezes at 0°R, boils at 80°R. Thus, to convert from Réaumur degrees to Celsius degrees, multiply by $\frac{100}{80}$ or 1.25.
3 The abbreviations 8^{bre}, 9^{bre} and X^{bre} refer to October, November and December respectively.

Summary

First Definitions and Properties

- $i^2 = -1$. i is called an *imaginary number*.
- $z = a + bi$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$, is called a *complex number*.
- a is the real part of z, or $a = \text{Re}(z)$.
  b is the imaginary part of z, or $b = \text{Im}(z)$.
  If $b = 0$, z is real. If $b \neq 0$, z is non-real.
- The set of all complex numbers is denoted by $\mathbb{C}$.
- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$
- The *complex plane* is determined by a real axis and an imaginary axis, crossing at 0.
- Complex numbers have all the properties of vectors of $\mathbb{V}_2$.
- There is no order relation in $\mathbb{C}$.

Modulus and Argument—Conjugates

- If $z = x + yi$ is represented by the point P, or the vector $\overrightarrow{OP}$, in the complex plane:
  the *modulus* of z, $|z| = |\overrightarrow{OP}| = \sqrt{x^2 + y^2}$
  the *argument* of z is the angle that $\overrightarrow{OP}$ makes with the positive real axis, that is, $\sin(\arg z) = \dfrac{y}{|z|}$ and $\cos(\arg z) = \dfrac{x}{|z|}$
- The *complex conjugates* $z = x + yi$ and $\bar{z} = x - yi$ are reflections of each other in the real axis.
- $|z - w|$ represents the distance between the points representing z and w in the complex plane.

Properties of $\mathbb{C}$

E. Equality	$a + bi = c + di$ if and only if $a = c$ and $b = d$
S. Sum	$(a + bi) + (c + di) = (a + c) + (b + d)i$
P. Product	$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

Given any numbers z, w and u of $\mathbb{C}$,

1. Closure	$z + w$ and zw belong to $\mathbb{C}$
2. Commutativity	$z + w = w + z$ and $zw = wz$
3. Associativity	$(z + w) + u = z + (w + u)$ and $(zw)u + z(wu)$
4. Distributivity	$z(w + u) = zw + zu$
5. Neutral elements	$z + 0 = 0 + z = z$ and $(z)(1) = (1)(z) = z$
6. Inverse elements	$z + (-z) = (-z) + z = 0$ and $z\left(\dfrac{1}{z}\right) = \left(\dfrac{1}{z}\right)z = 1$, provided that $z \neq 0$

Properties involving Conjugates

Consider two complex numbers z, w, and their conjugates $\overline{z}$, $\overline{w}$.

1. $z + \overline{z} = 2\mathrm{Re}(z)$
2. $z - \overline{z} = 2i\mathrm{Im}(z)$
3. $z\overline{z} = |z|^2$
4. $(\overline{z + w}) = \overline{z} + \overline{w}$
5. $(\overline{zw}) = \overline{z}\,\overline{w}$
6. $(\overline{\overline{z}}) = z$
7. Division: $\dfrac{z}{w} = \dfrac{z\overline{w}}{w\overline{w}} = \dfrac{z\overline{w}}{|w|^2}$

Polar Form

- $z = \underset{\substack{\uparrow \\ \text{Cartesian form}}}{a} + bi = r(\cos \underset{\substack{\uparrow \\ \text{polar form}}}{\theta} + i \sin \theta)$
- $r(\cos \theta + i \sin \theta) = p(\cos \phi + i \sin \phi)$ implies $r = p$ and $\theta = \phi + 2k\pi$ (or $\theta^\circ = \phi^\circ + 360k^\circ$), $k \in \mathbb{Z}$.
- $z = r(\cos \theta + i \sin \theta) \Rightarrow \overline{z} = r[\cos(-\theta) + i \sin(-\theta)]$
  or $\overline{z} = r(\cos \theta - i \sin \theta)$

Multiplication and Division in Polar Form

- $[p(\cos \theta + i \sin \theta)][q(\cos \phi + i \sin \phi)] = pq(\cos[\theta + \phi] + i \sin[\theta + \phi])$
  The modulus of the product is the product of the moduli.
  The argument of the product is the sum of the arguments.
- $\dfrac{p(\cos \theta + i \sin \theta)}{q(\cos \phi + i \sin \phi)} = \dfrac{p}{q}(\cos[\theta - \phi] + i \sin[\theta - \phi])$
  The modulus of the quotient is the quotient of the moduli.
  The argument of the quotient is the difference of the arguments.

De Moivre's Theorem

- $[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta)$, $n \in \mathbb{Q}$.
  If n is not an integer, then $(\cos \theta + i \sin \theta)^n$ is *not unique*.
- Given $z, w \in \mathbb{C}$, and $n \in \mathbb{N}$, the n values of z satisfying $z^n = w$ are called the nth roots of w.

The Fundamental Theorem of Algebra

- A polynomial equation of degree n always has n complex roots.

The Factor Theorem

- If $p(z_k) = 0$, then $(z - z_k)$ is a factor of $p(z)$.

Exponential Form

- $e^z = re^{iy} = r(\cos y + i \sin y)$,
  where $z = x + iy$ and $r = e^x$, $x, y \in \mathbb{R}$. (y in radians)
- $|e^{x+iy}| = e^x$, and $\arg(e^{x+iy}) = y$

Inventory

Complete each of the following statements.

1. $i^2 =$ _____.
2. The numbers i, $4i$, $i\sqrt{2}$ are called _____.
3. Using the real numbers a, b and the number i, a complex number can be written _____.
4. The sets of numbers $\mathbb{R}$ and $\mathbb{C}$ are related such that $\mathbb{R}$ _____ $\mathbb{C}$.
5. The two axes of the complex plane are called _____.
6. A complex number can be real, imaginary, or _____.
7. If the complex numbers $a + bi$ and $c + di$ are equal then _____ and _____.
8. Given $z = a + bi$, $\mathrm{Re}(z) =$ _____, $\mathrm{Im}(z) =$ _____, the complex conjugate $\bar{z} =$ _____, the modulus $|z| =$ _____ the argument arg z is such that tan(_____) = _____.
9. Complex conjugates are _____ of each other in the _____ of the complex plane.
10. The conjugate of the conjugate of z is equal to _____.
11. _____ numbers are added like vectors of $\mathbb{V}_2$.
12. A complex number z whose modulus is r and whose argument is θ can be represented in polar form as $z =$ _____.
13. If two complex numbers are equal, then their moduli are _____ and their arguments differ by _____.
14. When two complex numbers are multiplied, the modulus of the product is the _____ of the moduli.
15. When two complex numbers are divided, the argument of the quotient is the _____ of the arguments.
16. De Moivre's theorem: $[r(\cos\theta + i \sin\theta)]^n = r^n($_____$)$.
17. The fifth roots of unity are the numbers z satisfying the equation _____.
18. A polynomial equation of degree n has _____ complex roots, some of which may be equal.
19. Each equation in $\mathbb{C}$ incorporates _____ equations in $\mathbb{R}$.
20. The distance between the points representing z and w in the complex plane is _____.
21. If $z = x + iy$, then the complex number e^z has modulus _____ and argument _____.
22. $e^{i\pi} =$ _____.

Review Exercises

1. Simplify the following.

- **a)** $(7 + 2i) + (3 - 2i)$
- **b)** $(7 + 2i)(3 - 2i)$
- **c)** $(11 - i)^2$
- **d)** $(1 + i)^3$
- **e)** i^7
- **f)** i^{-4}
- **g)** $\dfrac{1}{-i}$
- **h)** $(2 + i) - (4 - 5i)$
- **i)** $4(-1 + i) - 3(1 + i)$
- **j)** $(1 + 6i)^2 - (1 - 6i)^2$
- **k)** $i(i - 1) - (2 + i)(4 + 3i)$

2. Express in the form $a + ib$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

- **a)** $\dfrac{1 - 3i}{i}$
- **b)** $\dfrac{1 - 3i}{1 + 3i}$
- **c)** $\dfrac{8 + 5i}{-i}$
- **d)** $\dfrac{5 + i}{4 - 2i}$
- **e)** $\dfrac{1}{3 - 4i}$
- **f)** $\dfrac{i}{2 + i}$
- **g)** $\dfrac{1}{6 + i} + \dfrac{1}{6 - 4i}$
- **h)** $\dfrac{1}{(9 - 2i)^2} - \dfrac{1}{9 - 2i}$

3. Simplify the following expressions.

- **a)** $(a + bi)^2 - (a - bi)^2$
- **b)** $\dfrac{1}{a - bi} - \dfrac{1}{a + bi}$
- **c)** $a + bi + \dfrac{1}{a + bi}$

4. Find two numbers whose sum is 10 and whose product is 29.

5. Find the roots of the following equations.

- **a)** $z^2 - 12z + 37 = 0$
- **b)** $z^2 + 4z + 20 = 0$
- **c)** $z^2 = 3z - 5$

6. Find the roots of the equation $z^2 - (4 + i)z + 4i = 0$ by factoring in $\mathbb{C}$.

7. Show that a quadratic equation whose roots are $z = \alpha$ and $z = \beta$ can be written $z^2 - (\alpha + \beta)z + \alpha\beta = 0$.

8. Simplify $(2 + i)^3(5 - 12i)^2(2 - i)^3(5 + 12i)^2$.

9. Find the number k such that

$$\left|\frac{2 - ki}{1 + i}\right| = 5.$$

10. Let $z = 2 + i$,
$w = 3 - 4i$,
$p = -5i$,
$q = -6 - i$,
$u = 4$.

- **a)** Plot the points representing numbers z, w, p, q, u in a complex plane.
- **b)** Find the conjugates $\overline{z}, \overline{w}, \overline{p}, \overline{q}$, and $\overline{u}$, and plot them in the same complex plane.
- **c)** Find the moduli $|z|, |w|, |p|, |q|$, and $|u|$.
- **d)** Find the arguments arg z, arg w, arg p, arg q, arg u.

11.
- **a)** State the complex conjugate $\overline{z}$ of the number $z = a + bi$.
- **b)** Prove that the sum of a complex number and its conjugate is always real.
- **c)** Prove that the product of a complex number and its conjugate is always real.

12.
- **a)** What is the argument of the number -1?
- **b)** Describe the geometric effect of -1 as a multiplier in the complex plane. Does your description apply to real numbers?

13. Given two complex numbers z and w, use a vector analogy to illustrate the following inequalities geometrically.

- **a)** $|z + w| \leqslant |z| + |w|$ (the triangle inequality)
- **b)** $|z - w| \leqslant |z| + |w|$
- **c)** $|z - w| \geqslant ||z| - |w||$

14. Given numbers z and w, use a vector analogy to find an interpretation in the complex plane of

- **a)** $\dfrac{1}{2}z + \dfrac{1}{2}w$
- **b)** $\dfrac{m}{m + n}z + \dfrac{n}{m + n}w$

15. **a)** If $z = a + bi$ and $w = c + di$, prove that $|z - w| = \sqrt{(a - c)^2 + (b - d)^2)}$

b) Use a vector analogy to illustrate geometrically that the distance between the points representing z and w in the complex plane is $|z - w|$.

16. Find quadratic equations in the form $az^2 + bz + c = 0$ with the following roots.

a) $-i$ and $5 - i$ **b)** $a + bi$ and $c - di$

17. **a)** Prove that a polynomial equation of degree n with real coefficients always has at least one real root if n is odd.

b) How many real roots are there if n is even?

18. By solving $z^2 + 4i = 0$, find the two square roots of $-4i$. Locate these roots in the complex plane.

19. Given $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$,

a) calculate z^2

b) plot z and z^2 in a complex plane

c) discuss the statement: "$\sqrt{i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$".

20. The difference of two squares can be factored, but the sum of two squares cannot be factored. Discuss.

21. It is given that $1 + 3i$ is a root of the equation $2z^3 - 9z^2 + 30z - 50 = 0$.

a) Use this information to find all the roots of the equation.

b) Show that the representations of these roots in a complex plane are the vertices of an isosceles triangle.

22. **a)** Verify that $w = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ is a cube root of 1.

b) Calculate w^2 and show that w^2 is also a cube root of 1.

23. Consider the numbers $z = r(\cos\theta + i\sin\theta)$ and $w = p(\cos\phi + i\sin\phi)$, where θ and ϕ are measured in radians. Prove that if $z = w$, then $r = p$ and $\theta = \phi + 2k\pi$, where $k \in \mathbb{Z}$.

24. Consider the equation in z

$z^2 - uz + v = 0$,

where u and v are known to be non-real. Determine whether or not it is possible for this equation to have a real root.

25. **a)** If $z = \cos 45° + i\sin 45°$, calculate z^2.

b) Calculate $(-z)^2$.

c) Use your results to a) and b) to state the two square roots of i in Cartesian form.

d) Calculate z^4.

26. **a)** Verify the identity $9\cos^2\theta - \sin^2\theta - 8 = \cos^2\theta - 9\sin^2\theta$.

b) Use this identity to solve the equation $z^2 - (3\cos\theta + i\sin\theta)z + 2 = 0$.

27. Given $z = r(\cos\theta + i\sin\theta)$, verify that $z\bar{z} = r^2$.

28. Find the modulus and an argument of $z = x + iy$ in the following cases.

a) $x = 0, y > 0$

b) $x < 0, y = 0$

29. Given $z = 3(\cos 67° + i\sin 67°)$ and $w = 2(\cos 123° - i\sin 123°)$, express the following in polar form.

a) $\bar{z}$

b) $\bar{w}$

c) zw

d) $\bar{z}\,\bar{w}$

e) $\frac{z}{w}$

f) $\frac{w}{z}$

30. Given $z = 10 + i$ and $w = 4 - 7i$, express the following in polar form.

a) z

b) w

c) zw

d) $\frac{z}{w}$

e) $\frac{w}{z}$

31. **a)** Calculate the exact modulus and an exact argument of each of the numbers $z = -1 + i\sqrt{3}$ and $w = -1 - i$.

b) Hence state the values of z^3 and w^4.

32. Use the results of question 31 to express the following in polar form.

a) zw **b)** $\frac{z}{w}$ **c)** $\frac{w}{z}$

33. **a)** If $z = \cos\theta + i\sin\theta$, state an argument of z^3.

b) Hence find expressions for $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

34. Calculate in Cartesian form

a) $(\cos 30° + i\sin 30°)^{12}$

b) $(\cos 20° - i\sin 20°)]^{-6}$

c) $(1 + i)^{10}$

d) $(-1 - i\sqrt{3})^{-2}$

e) $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5$

f) $\left(\cos\frac{7\pi}{12} - i\sin\frac{7\pi}{12}\right)^6$

35. Simplify $\dfrac{\cos 3\theta + i\sin 3\theta}{(\cos\theta + i\sin\theta)^2}$

36. Find the modulus and argument of

$$\frac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}$$

37. **a)** Compare the expressions for $(\cos\theta + i\sin\theta)^5$ given by De Moivre's theorem and the binomial expansion to prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

b) By considering the equation $\cos 5\theta = 0$, prove that $\cos\left(\frac{\pi}{10}\right)\cos\left(\frac{3\pi}{10}\right) = \frac{1}{4}\sqrt{5}$.

38. **a)** Verify that each of the following numbers is a sixth root of unity.

$$\alpha = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \beta = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

b) State the six roots of the equation $z^6 - 1 = 0$.

39. Find the fifth roots of -1 in Cartesian form and represent them in a complex plane.

40. **a)** Find a complex number equation for a circle of centre $3 + 4i$ and radius 5.

b) Show that this circle passes through O.

41. Find equations in x and y of the loci described by the following, where $z = x + iy$.

a) $|z - 1 + 3i| = 1$

b) $|2z + i| = 5|z - i|$

42. Describe the locus of a point that moves in the complex plane in such a way that its distance from $-1 + 2i$ is half its distance from the origin.

43. Write the following in exponential form, using exact values.

a) $u = -2 - 2i\sqrt{3}$

b) $v = 5 - 5i$

44. Use the definitions $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ to prove that $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$.

45. **i)** Solve the equation $2z^3 - 3z^2 + 2z + 2 = 0$ given that $z = 1 + i$ is a solution

ii) The complex numbers w and z are related by the equation

$$w = \frac{z - 6i}{z + 8},$$

and the points W and Z in the Argand diagram correspond to w and z respectively.

a) Given that the real part of w is zero, show that Z lies on a circle, and find the centre and radius of this circle.

b) Given that the imaginary part of w is zero, show that Z lies on a straight line, and give the equation of this line.

(88 H)

46. **i)** By first putting $z^2 = w$, or otherwise, find the values of z for which
$z^4 + 2z^2 + 25 = 0, z \in \mathbb{C}$,
giving your answers in the form
$z = x + iy, x, y \in \mathbb{R}$.

ii) It is given that
$z = 2r(\cos\theta + i\sin\theta)$,
$w = z + \dfrac{r^2}{z}, r \in \mathbb{R}^+; w, z \in \mathbb{C}; -\pi < \theta \leqslant \pi$.

a) If $w = u + iv, u, v \in \mathbb{R}$,
show that $\left(\dfrac{2u}{5r}\right)^2 + \left(\dfrac{2v}{3r}\right)^2 = 1$.

b) Find the four values of θ where
$|w| = 2r$,
giving your answers correct to two decimal places.

(88 S)

47. Let

$w = \cos\dfrac{2\pi}{5} + i\sin\dfrac{2\pi}{5}$.

a) Show that $1, w, w^2, w^3$ and w^4 are the 5 roots of the equation
$z^5 = 1, z \in \mathbb{C}$.

b) By factorizing $(z^5 - 1)$, or otherwise, prove that
$1 + w + w^2 + w^3 + w^4 = 0$.

c) Show, by multiplying out and using parts a) and b), that
$(1 - w)(1 - w^2)(1 - w^3)(1 - w^4) = 5$.

d) i) Use the given expression for w to prove that

$(1 - w)(1 - w^4) = 4\sin^2\dfrac{\pi}{5}$.

ii) Work out a similar expression for $(1 - w^2)(1 - w^3)$.

iii) Deduce, from parts c), d)(i) and d)(ii), that

$\sin\dfrac{\pi}{5}\sin\dfrac{2\pi}{5} = \dfrac{1}{4}\sqrt{5}$.

(88 S)

48. **a)** Find the complex roots of the equation
$z^2 - z + 1 = 0$
in the form $p + iq, p - iq$, where $p, q \in \mathbb{R}$.

b) Express the two roots obtained in part a) in the form $r(\cos\theta + i\sin\theta)$ and $r(\cos\theta - i\sin\theta)$, where r and θ are to be determined, $r \in \mathbb{R}^+, 0 \leqslant \theta \leqslant \pi$.

c) Show that
$(z + 1)(z^2 - z + 1) = (z^3 + 1)$.

d) i) Use the results already obtained to write down the modulus and argument of each of the three roots of the equation
$z^3 + 1 = 0$.

ii) Hence plot these roots on a carefully labelled Argand diagram.

iii) Prove that the three plotted points lie at the vertices of an equilateral triangle.

e) By expressing
$z^3 - 3z^2 + 3z = (z - 1)^3 + 1$,
or otherwise, prove that the roots of the equation
$z(z^2 - 3z + 3) = 0$,
when plotted on an Argand diagram, also lie at the vertices of an equilateral triangle.

(86 S)

49. **i) a)** Find, in the form $a + bi$, all the solutions of the equation
$z^3 + 6z = 20$.

b) The points in the Argand diagram, representing the three solutions found in part a) are the vertices of a triangle. Find the angles of this triangle.

c) Show that two of the solutions found in part a) have modulus $\sqrt{10}$ and find their arguments.

ii) Given that $0° \leqslant \theta \leqslant 360°$ solve the equation
$\sin 3\theta + \sin\theta = \cos\theta$

iii) Given that $0° \leqslant \theta \leqslant 360°$ solve, correct to the nearest degree, the equation
$3\cos\theta + 4\sin\theta + 2 = 0$.

(87 H)

50. The complex number z is given by $z = x + iy$, $x, y \in \mathbb{R}$.

a) Find z^2 in terms of x and y.

b) Given that $z^2 = 9 + 40i$,

i. find the possible values of x and y, and

ii. hence solve, for z, the equation $z^2 = 9 + 40i$.

c) On a clearly labelled Argand diagram plot the points P and Q which represent the solutions obtained in part b), placing P in the first quadrant. Plot also the point R representing z^2.

d) Find OP and OQ, leaving your answers in surd form.

e) Determine, to the nearest degree, the value of the angle that (OP) makes with the positive direction of the real axis.

f) Determine, to the nearest degree, the value of the angle POR.

(85 S)

51. i) Solve the simultaneous equations

$$\left.\begin{aligned} z + 2w &= 7 \\ iz + w &= 1 \end{aligned}\right\}$$

and show the solutions on an Argand diagram.

ii) Given that $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, where $r_2 \neq 0$, prove that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}.$$

If $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$ find the modulus and argument of

$$\frac{z_1}{z_2} \text{ and } \frac{1}{z_2^6}$$

(83 H)

52. i) a) Solve the equation $z^3 = 4\sqrt{3} - 4i$, giving your answers in modulus-argument form.

b) The equation $z^3 - z^2 + 3z + 5 = 0$ has $z = -1$ as one of its roots. Find the other two roots, giving your answers in the form $z = a + bi$.

ii) The complex number z satisfies each of the inequalities

a. $-\frac{1}{2}\pi \leqslant \arg z \leqslant 0$,

b. $|z - 1| \leqslant 2$,

c. $|z - 3| \leqslant |z - 1|$.

Show, on a clearly labelled Argand diagram, the region containing the set of points satisfying the three inequalities simultaneously.

(84 H)

53. i) Show that the set of complex numbers which satisfy the equation $|z + 1| = 2|z - 1|$ lie on a circle in the Argand diagram. Find the centre and radius of this circle.

ii) Use the fact that $\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$ to prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. Hence, without using a calculator, prove that

$$\cos 18° = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}$$

and find a similar expression for $\cos 54°$.

(86 H)