VECTORS, MATRICES and COMPLEX NUMBERS

with
International Baccalaureate
questions

John EGSGARD
and
Jean-Paul GINESTIER

CHAPTER TWO

LINEAR DEPENDENCE

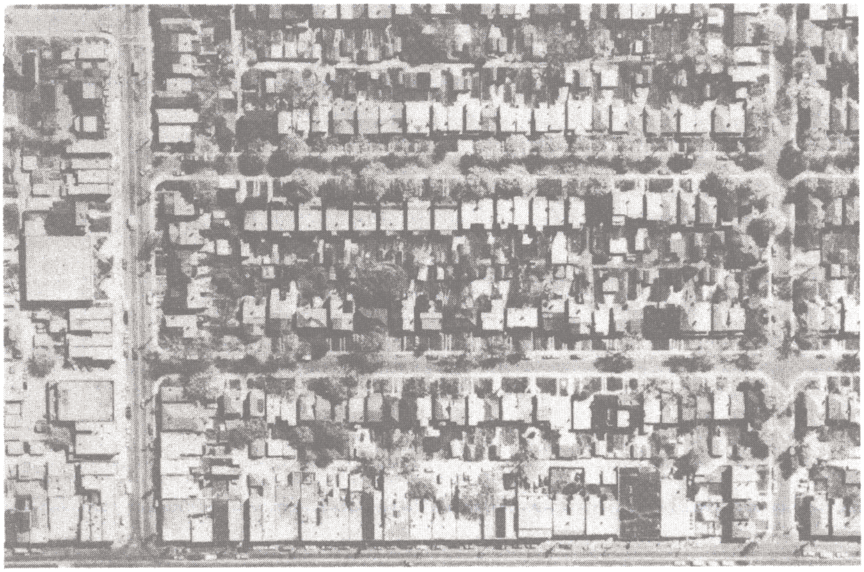
e-mails johnegs@rogers.com
& jean-paul.ginestier@uwc.net

ISBN of Gage Edition 0-7715-3503-1

For orders contact:
Jean-Paul Ginestier
Red Cross Nordic UWC
N-6968 FLEKKE
Norway
Tel +47 5773 7000
Fax +47 5773 7001
e-mail jean-paul.ginestier@uwc.net

CHAPTER TWO

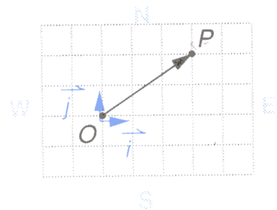
Linear Dependence



The photographs show two cities A and B. The streets of city A run east to west and south to north. The streets of city B go east to west and southwest to northeast.

If you wish to locate an intersection P in city A you can use origin O and position vector $\overrightarrow{OP}$. The diagram shows two perpendicular unit vectors $\vec{i}$ and $\vec{j}$ where $|\vec{i}|$ equals one city block east to west, and $|\vec{j}|$ equals one city block south to north.

City A



Since intersection P is three blocks east and two blocks north of O, $\overrightarrow{OP} = 3\vec{i} + 2\vec{j}$.

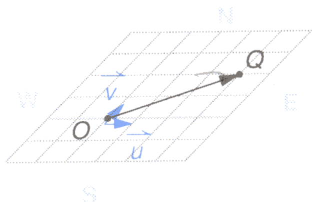
Every intersection point in city A can be expressed as a *combination* of a scalar times $\vec{i}$, plus a scalar times $\vec{j}$. Such a combination is called a **linear combination** of vectors $\vec{i}$ and $\vec{j}$.

Observe that the vector $3\vec{i}$ depends on vector $\vec{i}$. Vector $3\vec{i}$ is described as being **linearly dependent** with vector $\vec{i}$.

Recall that $3x + 2y$ is a *linear* expression, and that $3x + 2y = k$ defines a *linear* relation.

To locate an intersection Q in city B you will need origin O and two unit vectors $\vec{u}$ and $\vec{v}$. These vectors are *not* perpendicular to each other. Observe that $|\vec{u}|$ is one city block east to west while $|\vec{v}|$ is one city block southwest to northeast.

City B



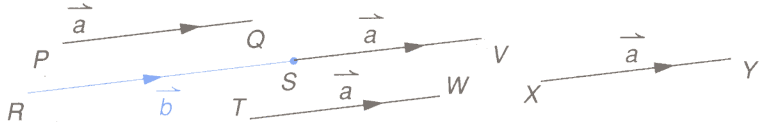
Since the point Q is three blocks east and two blocks northeast of point O, $\overrightarrow{OQ} = 3\vec{u} + 2\vec{v}$. Again vector $\overrightarrow{OQ}$ is called a linear combination of vectors $\vec{u}$ and $\vec{v}$, and the vector $3\vec{u}$ is described as being linearly dependent with vector $\vec{u}$.

In this chapter you will learn more about the linear combinations of vectors and about the dependence of vectors on each other.

2.1 Linear Dependence of Two Vectors

Vectors are not directed line segments but vectors can be represented by directed line segments. A vector in $\mathbb{V}_2$ can also be represented by an ordered pair; a vector in $\mathbb{V}_3$ by an ordered triple. There is a relationship that is derived from the directed line segment representation of a vector. This relationship is called *linear dependence*.

Suppose $\vec{a}$ and $\vec{b}$ can be represented by the parallel directed line segments $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ respectively.



In this situation you can write $\vec{a} \| \vec{b}$. But every directed line segment which is congruent to, parallel to, and in the same direction as $\overrightarrow{PQ}$ represents the vector $\vec{a}$. In particular, the directed line segment $\overrightarrow{SV} = \vec{a}$. Because the three points R, S, and V are collinear, $\vec{a}$ and $\vec{b}$ are called *collinear vectors*. Note that $\vec{a}$ and $\vec{b}$ can also be described as being parallel. Thus, for vectors *there is no distinction made between vectors that are parallel and vectors that are collinear*. The symbol $\|$ when used with vectors can be read either "parallel to" or "collinear with".

Two vectors that are collinear or parallel are *linearly dependent vectors*. *The zero vector $\vec{0}$ is parallel and linearly dependent with every other vector.*

DEFINITION Two vectors $\vec{a}$ and $\vec{b}$ are linearly dependent if and only if $\vec{a} \| \vec{b}$.

What is the algebraic condition for two vectors to be linearly dependent? In section 1.5 you learned that every scalar multiple of $\vec{a}$ is parallel to $\vec{a}$. Hence $\vec{a}$ and $s\vec{a}$, $s \in \mathbb{R}$, are linearly dependent. Also, any two vectors that are linearly dependent are parallel, and so one vector must be a scalar multiple of the other vector.

An Algebraic Condition for Two Vectors to be Linearly Dependent

Two non-zero vectors $\vec{a}$ and $\vec{b}$ are linearly dependent if and only if $\vec{b} = s\vec{a}$, for some $s \in \mathbb{R}$.

Example 1 Given $\vec{c} = 3\vec{a}$ and $\vec{d} = 2\vec{a}$, prove that $\vec{c}$ and $\vec{d}$ are linearly dependent.

Solution *Geometric Proof*

You must prove that $\vec{c} \| \vec{d}$.
Since $\vec{c}$ and $\vec{d}$ are each scalar multiples of $\vec{a}$, each vector is parallel to $\vec{a}$.
But $\vec{c} \| \vec{a}$ and $\vec{d} \| \vec{a}$ implies that $\vec{c} \| \vec{d}$.
Thus, $\vec{c}$ and $\vec{d}$ are linearly dependent.

Algebraic Proof

You must prove that $\vec{c}$ is a scalar multiple of $\vec{d}$ or that $\vec{d}$ is a scalar multiple of $\vec{c}$.

Since $\vec{c} = 3\vec{a}$, therefore $\vec{a} = \frac{1}{3}\vec{c}$.

Thus, $\vec{d} = 2\vec{a} = 2\left(\frac{1}{3}\vec{c}\right) = \frac{2}{3}\vec{c}$.

Therefore, $\vec{c}$ and $\vec{d}$ are linearly dependent. ■

In Example 1, you proved that

$\vec{d} = \frac{2}{3}\vec{c}$ or

$3\vec{d} = 2\vec{c}$ or

$3\vec{d} + (-2)\vec{c} = \vec{0}$.

Thus, real numbers $m = 3$, and $k = -2$ exist such that $m\vec{d} + k\vec{c} = \vec{0}$. This leads to the following alternate form of the first algebraic condition.

Another Algebraic Condition for Two Vectors to be Linearly Dependent.

Two vectors $\vec{a}$ and $\vec{b}$ are linearly dependent if and only if $m, k \in \mathbb{R}$ exist, not both equal to 0, such that $m\vec{a} + k\vec{b} = \vec{0}$.

Proof of this algebraic condition

Part 1 Given $\vec{a}$ and $\vec{b}$ are linearly dependent, prove $m, k \in \mathbb{R}$ exist, not both equal to 0, such that $m\vec{a} + k\vec{b} = \vec{0}$.

Proof: $\vec{a}$ and $\vec{b}$ linearly dependent $\Rightarrow \vec{b} = p\vec{a}$, for some $p \in \mathbb{R}$

$$\Rightarrow \vec{b} - p\vec{a} = \vec{0}$$

$$\Rightarrow (-p)\vec{a} + (1)\vec{b} = \vec{0}$$

Therefore, $m = -p$, $k = 1$ exist, not both 0, such that $m\vec{a} + k\vec{b} = \vec{0}$

Part 2 Given $m, k \in \mathbb{R}$ exist, not both equal to 0, such that $m\vec{a} + k\vec{b} = \vec{0}$, prove $\vec{a}$ and $\vec{b}$ are linearly dependent.

Proof: At least one of k and m cannot be 0.

Suppose $k \neq 0$, then $m\vec{a} + k\vec{b} = \vec{0}$ can be written

$k\vec{b} = -m\vec{a}$ or

$\vec{b} = -\frac{m}{k}\vec{a}$

Thus, $\vec{a}$ and $\vec{b}$ are linearly dependent.

Note: If m and k are both zero, the statement $m\vec{a} + k\vec{b} = \vec{0}$ would imply that $0\vec{a} + 0\vec{b} = \vec{0}$. This statement is true for all vectors $\vec{a}$ and $\vec{b}$, whether or not they are linearly dependent.

Example 2 Which of the following vectors are linearly dependent with $\vec{a} = \overrightarrow{(1,2,-3)}$?

$\vec{b} = \overrightarrow{(4,8,-12)}$, $\vec{c} = \overrightarrow{\left(-\frac{1}{2},-1,\frac{3}{2}\right)}$, $\vec{d} = \overrightarrow{(2,4,-5)}$

Solution You must check to see which vectors are scalar multiples of vector $\vec{a}$.

$\vec{b} = \overrightarrow{(4,8,-12)} = 4\overrightarrow{(1,2,-3)} = 4\vec{a}$

Therefore, $\vec{b}$ and $\vec{a}$ are linearly dependent.

$\vec{c} = \overrightarrow{\left(-\frac{1}{2},-1,\frac{3}{2}\right)} = -\frac{1}{2}\overrightarrow{(1,2,-3)} = -\frac{1}{2}\vec{a}$

Therefore, $\vec{c}$ and $\vec{a}$ are linearly dependent.

If $\vec{d} = k\vec{a}$, $k \in \mathbb{R}$

then $\overrightarrow{(2,4,-5)} = k\overrightarrow{(1,2,-3)} = \overrightarrow{(k,2k,-3k)}$

and $2 = k$, $4 = 2k$, $-5 = -3k$

This forces $k = 2$ and $k = \frac{5}{3}$ at the same time, which is impossible.

Therefore, $\vec{d}$ and $\vec{a}$ are not linearly dependent. ■

DEFINITION Two vectors that are not linearly dependent are *linearly independent*.

Important Facts about Two Linearly Independent Vectors

Geometric

If $\vec{a}$ and $\vec{b}$ are linearly independent, then $\vec{a} \nparallel \vec{b}$.

Algebraic

1. No scalar p exists such that $\vec{a} = p\vec{b}$.
2. If $\vec{a}$ and $\vec{b}$ are linearly independent *and* $m\vec{a} + k\vec{b} = \vec{0}$ then $m = k = 0$.

Intuitively, this last statement says that the only way that you can add two non-parallel vectors to obtain the vector $\vec{0}$ is to multiply each vector by the scalar 0.

Example 3 The two vectors $\vec{a}$ and $\vec{b}$ are linearly independent. If $x\vec{a} + (y - 3)\vec{b} = \vec{0}$, then find the values of x and y.

Solution Since $\vec{a}$ and $\vec{b}$ are linearly independent, and
$x\vec{a} + (y - 3)\vec{b} = \vec{0}$, then
$x = 0$, and $y - 3 = 0$.
Thus $x = 0$, and $y = 3$. ■

2.1 Exercises

1. Vectors $\vec{a}$ and $\vec{b}$ are linearly dependent.
   a) What is the geometric relationship between $\vec{a}$ and $\vec{b}$?
   b) State two algebraic equations that are true relating $\vec{a}$ and $\vec{b}$.
   c) What conditions, if any, are imposed on the scalars in the equations in b)?

2. You are given two vectors $\vec{x}$ and $\vec{y}$ such that $\vec{x} \| \vec{y}$. What is the algebraic relationship between $\vec{x}$ and $\vec{y}$?

3. Use the fact that a vector as a directed line segment can be drawn anywhere to explain how two collinear vectors can be represented by two line segments that are not collinear.

4. If $\vec{z} = w\vec{d}$ where $w \in \mathbb{R}$, then how are vectors $\vec{z}$ and $\vec{d}$ related?

5. Scalars s and t exist, not both 0, such that $s\vec{m} + t\vec{k} = \vec{0}$. How are vectors $\vec{m}$ and $\vec{k}$ related geometrically?

6. Points P and Q are such that $\overrightarrow{PQ} = \vec{a} \neq \vec{0}$.
   a) If R is any point on line PQ explain why $\vec{a}$ and $\overrightarrow{PR}$ are linearly dependent.
   b) If T is any point not on line PQ explain why $\vec{a}$ and $\overrightarrow{PT}$ are linearly independent.

7. Given that $\vec{a} \| \vec{c}$, $\vec{c} \nparallel \vec{b}$, and $\vec{d} = 5\vec{c}$, which of the following vectors are linearly dependent with $\vec{a}$?
   a) $\vec{u} = 3\vec{a}$
   b) $\vec{v} = -2\vec{b}$
   c) $\vec{w} = 7\vec{c}$
   d) $\vec{r} = \pi\vec{c}$
   e) $\vec{t} = -6\vec{d}$?

8. List all sets of parallel vectors from among the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{u}$, $\vec{v}$, $\vec{w}$, $\vec{r}$, and $\vec{t}$ of question 7.

9. Given $\vec{a} = \overrightarrow{(2,3)}$,
   a) write three vectors collinear with $\vec{a}$
   b) write three vectors linearly dependent with $\vec{a}$
   c) write one vector linearly independent with $\vec{a}$.

10. Given $\vec{b} = \overrightarrow{(4,1,3)}$,
    a) write three vectors collinear with $\vec{b}$
    b) write three vectors linearly dependent with $\vec{b}$
    c) write one vector linearly independent with $\vec{b}$.

11. $\vec{a}$ and $\vec{b}$ are linearly independent. Use the following equations to name vectors that are linearly dependent with $\vec{a}$.
    $7\vec{c} + 4\vec{a} = \vec{0}$
    $5\vec{b} + 2\vec{d} = \vec{0}$
    $m\vec{a} - k\vec{e} = \vec{0}$, $m, k \in \mathbb{R}, m \neq 0$.

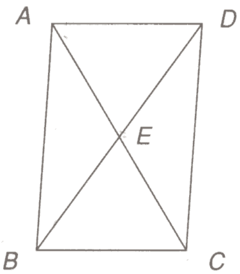
12. a) $\vec{a}$ and $\vec{b}$ are linearly independent. What is the geometric relationship between $\vec{a}$ and $\vec{b}$?
    b) $\vec{x} \nparallel \vec{y}$. Are $\vec{x}$ and $\vec{y}$ necessarily linearly independent? Explain.

13. a) $\vec{p}$ and $\vec{q}$ are linearly independent vectors and $r\vec{p} + v\vec{q} = \vec{0}$. What conclusion can you draw about the scalars r and v?
    b) Scalars m and k exist, both equal to 0, such that $m\vec{a} + k\vec{b} = \vec{0}$. Are $\vec{a}$ and $\vec{b}$ necessarily linearly independent?

14. Which of the following pairs of vectors are linearly dependent? Justify your answer.
    a) $\overrightarrow{(1,2)}$, $\overrightarrow{(4,8)}$
    b) $\overrightarrow{(-3,2)}$, $\overrightarrow{(-6,3)}$
    c) $\overrightarrow{(8,-2)}$, $\overrightarrow{\left(-1,\frac{1}{4}\right)}$
    d) $\overrightarrow{(3,6,2)}$, $\overrightarrow{(6,12,4)}$
    e) $\overrightarrow{(-1,-1,6)}$, $\overrightarrow{(6,6,-36)}$
    f) $\overrightarrow{(4,0,1)}$, $\overrightarrow{(4,1,1)}$

15. *ABCD* is a parallelogram. Name vectors linearly dependent with each of the following.

a) $\overrightarrow{AB}$ c) $\overrightarrow{AE}$
b) $\overrightarrow{AC}$ d) $\overrightarrow{BD}$



16. Explain why each of the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{BD}$ from question 15 is linearly independent with each of the other vectors.

17. Given the points $P(-2,4)$, $Q(-3,7)$, and $R(-4,10)$,

a) write $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ in component form
b) prove that $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are linearly dependent
c) use part b) to draw a geometric conclusion about the points *P*, *Q*, and *R*.

18. Given the points $A(1,3,-2)$, $B(5,5,4)$, and $C(-1,2,-5)$,

a) write $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in component form
b) prove that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly dependent
c) use part b) to draw a geometric conclusion about the points *A*, *B*, and *C*.

19. a) $\vec{a} \nparallel \vec{b}$. For what real numbers *m* and *k* is $m\vec{a} + k\vec{b} = \vec{0}$? Are $\vec{a}$ and $\vec{b}$ linearly dependent?
b) $\vec{a} \parallel \vec{b}$, and $|\vec{a}| = |\vec{b}|$. For what real numbers *m* and *k* is $m\vec{a} + k\vec{b} = \vec{0}$? Are $\vec{a}$ and $\vec{b}$ linearly dependent?

20. Given that $\vec{a}$ and $\vec{b}$ are linearly independent, find the values of the scalars in each of the following.

a) $s\vec{a} + t\vec{b} = \vec{0}$
b) $r\vec{a} + (3 - m)\,\vec{b} = \vec{0}$
c) $(x - 1)\vec{a} + (y + 2)\,\vec{b} = \vec{0}$
d) $(2z - 6)\,\vec{a} + (7 + 3k)\,\vec{b} = \vec{0}$

21. $\vec{a}$ and $\vec{b}$ are linearly independent non-zero vectors such that $3\vec{a} + k\vec{b} = m\vec{a} - 5\vec{b}$. Find the values of the scalars *k* and *m*.

22. $\vec{p}$ and $\vec{q}$ are linearly independent non-zero vectors where
$5c\vec{p} + d\vec{r} - 6\vec{q} = \vec{0}$ and
$3\vec{q} + 2\vec{p} + \vec{r} = \vec{0}$.
Find the values of the real numbers *c* and *d*.

23. Given that
$\vec{a} = 3\vec{b} - 2\vec{c} + 4\vec{d}$,
$\vec{e} = 2\vec{b} + 6\vec{c} - 2\vec{d}$,
$\vec{f} = 4\vec{b} - 10\vec{c} + 10\vec{d} + \vec{e}$,
prove that $\vec{a}$ and $\vec{f}$ are linearly dependent.

24. Given $\vec{a} = m\vec{c} \neq \vec{0}$, and $\vec{b} = k\vec{d} \neq \vec{0}$, where $\vec{c}$ and $\vec{d}$ are linearly independent,

a) use a geometric argument to prove that $\vec{a}$ and $\vec{b}$ are linearly independent
b) use an algebraic argument to prove that $\vec{a}$ and $\vec{b}$ are linearly independent.

25. $\vec{a}$ and $\vec{b}$ are linearly dependent and *P*, *Q*, *R* are points such that $\overrightarrow{PQ} = \vec{a}$ and $\overrightarrow{PR} = \vec{b}$. *D* is any point on the line containing points *P* and *Q*. Prove that scalars *m* and *k* exist such that $\overrightarrow{QD} = m\vec{a}$, and $\overrightarrow{QD} = k\vec{b}$.

26. $\vec{a}$ and $\vec{b}$ are linearly independent. *O*, *A*, *B*, and *C* are points such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, and $\overrightarrow{OC} = 4\vec{a} - 3\vec{b}$. Prove that points *A*, *B*, and *C* are collinear.

In Search of a Solution for a System with Three Variables: Elimination

Problem

Solve the linear system

$3x - y + 2z = -1$	①
$5x + 3y + z = 17$	②
$x - 2y - 3z = 11$	③

Solution

One method you can employ to find the solution of a system of 3 equations in 3 variables is *elimination*. Another method using matrices will be found on page 95.

The first step in elimination is to *obtain two equations in two variables by eliminating one variable* from each of two different pairs of the original equations.

Eliminate z from ① and ②.

2 × ②	$10x + 6y + 2z = 34$	④
	$3x - y + 2z = -1$	①
④ − ①	$7x + 7y = 35$	⑤

Eliminate z from ② and ③.

3 × ②	$15x + 9y + 3z = 51$	⑥
	$x - 2y - 3z = 11$	③
⑥ + ③	$16x + 7y = 62$	⑦

Now eliminate one of the variables, say y, from the two equations in two variables, ⑤ and ⑦.

⑤ − ⑦	$-9x = -27$
	$x = 3$

By back-substitution you can find $y = 2$ and $z = -4$, giving the solution $(x,y,z) = (3,2,-4)$

2.2 Linear Dependence of Three Vectors

In the last section you studied the linear dependence of *two* vectors. What could it mean to say that *three* vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent?

If two vectors $\vec{a}$ and $\vec{b}$ are linearly dependent, recall that the algebraic condition between the vectors can be expressed in two equivalent ways.

1. m and k exist, not both 0, such that $m\vec{a} + k\vec{b} = \vec{0}$.
2. Some $s \in \mathbb{R}$ exists such that $\vec{b} = s\vec{a}$

The definition chosen for the meaning of the linear dependence of three vectors will be an extension of the first algebraic condition for two vectors to be linearly dependent.

DEFINITION

Three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent if and only if m, k, and p exist, not all equal to 0, such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0} \quad m, k, p \in \mathbb{R}$.

Example 1 will demonstrate the geometric significance of this equation.

Example 1 Suppose $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent vectors in $\mathbb{V}_3$ such that $3\vec{a} + 2\vec{b} - 4\vec{c} = \vec{0}$. Find a geometric relationship among $\vec{a}$, $\vec{b}$, and $\vec{c}$.

Solution Three vectors in 3-space are not usually coplanar. You will show that the above algebraic condition forces the three vectors to be parallel to the same plane.

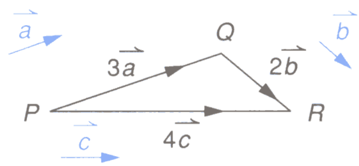
Points P, Q, and R can be selected such that $\overrightarrow{PQ} = 3\vec{a}$, and $\overrightarrow{QR} = 2\vec{b}$.

Then $\overrightarrow{PQ} + \overrightarrow{QR} - 4\vec{c} = \vec{0}$.

Thus, $4\vec{c} = \overrightarrow{PQ} + \overrightarrow{QR}$.

But, $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

thus, $4\vec{c} = \overrightarrow{PR}$.



Now any three points lie in one plane, that is, are coplanar. Thus, points P, Q, and R are coplanar, and segments PQ, QR, and PR lie in this same plane. So, $3\vec{a}$, $2\vec{b}$, and $4\vec{c}$ all lie in this plane. Since $\vec{a} \| 3\vec{a}$ and $\vec{b} \| 2\vec{b}$ and $\vec{c} \| -4\vec{c}$, then $\vec{a}$, $\vec{b}$, and $\vec{c}$ are parallel to this plane.

Hence, $\vec{a}$, $\vec{b}$, and $\vec{c}$ are parallel to the same plane. ■

Note: *Two* vectors are *collinear* if directed line segments that represent them can be translated so that each segment lies along the same line.

DEFINITION

Three vectors are said to be *coplanar* if directed line segments that represent them can be translated so that each segment lies in the same plane Π. These vectors can be coplanar in any one of a family of planes parallel to plane Π.

You should visualize this definition using the top of your desk and three pencils as $\vec{a}$, $\vec{b}$, and $\vec{c}$ to realize that three vectors being coplanar is the exception rather than the rule.

Two non-collinear vectors always lie in the same plane. (Place two pencils, $\vec{a}$ and $\vec{b}$, on your desk. If you introduce any third vector (a third pencil, $\vec{c}$), it does not have to lie in this plane. (The pencil can make a non-zero angle with the desk.)

PROPERTY

Three vectors are linearly dependent if and only if the three vectors are coplanar.

Proof of coplanar property

Part 1 Given three linearly dependent vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ in $\mathbb{V}_3$, prove $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar.

Proof:

You will need to show that three line segments representing $\vec{a}$, $\vec{b}$, and $\vec{c}$, can be drawn in such a way that the line segments form a triangle. The sides of a triangle must lie in the same plane.

Since $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly dependent, then $m, k, p \in \mathbb{R}$ exist such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$, where not all of m, k, and p are equal to 0.

Suppose $m \neq 0$. Then distinct points P and Q exist such that $\overrightarrow{PQ} = m\vec{a}$. Let R be the point such that $\overrightarrow{QR} = k\vec{b}$.

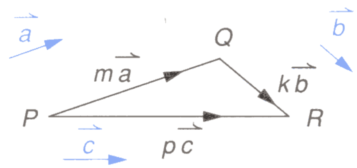
Then $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$ becomes

$\overrightarrow{PQ} + \overrightarrow{QR} + p\vec{c} = \vec{0}$.

Thus, $p\vec{c} = -(\overrightarrow{PQ} + \overrightarrow{QR})$.

But $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

thus, $p\vec{c} = -\overrightarrow{PR}$.



Now any three points lie in one plane, that is, are coplanar. Thus, points P, Q, and R are coplanar, and segments PQ, QR, and PR lie in this same plane. So vectors $m\vec{a}$, $k\vec{b}$ and $p\vec{c}$ all lie in this plane. Since $\vec{a} \parallel m\vec{a}$, $\vec{b} \parallel k\vec{b}$ and $\vec{c} \parallel p\vec{c}$, then $\vec{a}$, $\vec{b}$, $\vec{c}$ are parallel to this plane.

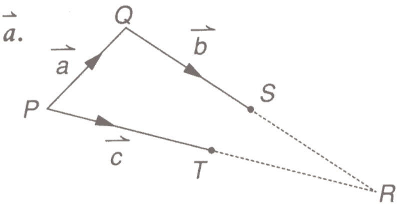
Hence, $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar.

Part 2 Given $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar, prove $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent, that is, prove that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$, where not all of m, k, and p are equal to 0, $m, k, p \in \mathbb{R}$.

Proof:

Case 1: No two vectors are linearly dependent.

Select points P and Q such that $\overrightarrow{PQ} = \vec{a}$.



Since $\vec{b} \nparallel \vec{a}$, a point S, not on line PQ, exists such that $\overrightarrow{QS} = \vec{b}$.

Since $\vec{c} \nparallel \vec{a}$, and $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar, then a point T exists such that $\overrightarrow{PT} = \vec{c}$, where T is in the plane PQS, but T is not in line PQ.

Since $\vec{b} \nparallel \vec{c}$, the lines containing segments PT and QS must intersect at some point, say R.

Thus, for some real numbers k and t,
$\overrightarrow{QR} = k\vec{b}$, and $\overrightarrow{PR} = t\vec{c}$

but $\quad \overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP} = \vec{0}$

thus $\quad \vec{a} + k\vec{b} - t\vec{c} = \vec{0}$

thus $\quad (1)\vec{a} + k\vec{b} + (-t)\vec{c} = \vec{0}$.

Therefore, $m = 1$, $k = k$, and $p = -t$ exist, not all 0, such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$.

Therefore, $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent.

Case 2: Two vectors are linearly dependent.

Suppose $\vec{a}$ and $\vec{b}$ are linearly dependent. Then $w \in \mathbb{R}$ exists such that $\vec{b} = w\vec{a}$.

Thus, $w\vec{a} - \vec{b} = \vec{0}$.

Thus, $w\vec{a} + (-1)\vec{b} + (0)\vec{c} = \vec{0}$.

Therefore, $m = w$, $k = -1$, and $p = 0$ exist, not all 0, such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$.

Therefore, $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent.

A relationship similar to the second algebraic condition for two dependent vectors is true for three linearly dependent vectors.

You know that $\vec{a}$, $\vec{b}$, and $\vec{c}$ being linearly dependent implies that m, k, and p exist, not all 0, such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$.

Suppose $m \neq 0$. Then $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$ can be written

$$m\vec{a} = -k\vec{b} - p\vec{c}, \text{ or}$$

$$\vec{a} = -\frac{k}{m}\vec{b} + -\frac{p}{m}\vec{c}$$

This relationship is described by saying that vector $\vec{a}$ is a *linear combination* of vectors $\vec{b}$ and $\vec{c}$.

Thus the previous property can now be written in terms of linear combinations of vectors.

PROPERTY

$\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent if and only if *at least one* vector can be expressed as a linear combination of the other two vectors.

Example 2

a) Prove the vectors $\vec{d} = \overrightarrow{(2,3)}$, $\vec{e} = \overrightarrow{(6,1)}$, $\vec{f} = \overrightarrow{(4,2)}$, are linearly dependent.

b) Express one of the vectors as a linear combination of the other two vectors.

Solution

a) A geometric proof is simpler here. You must show that $\vec{d}$, $\vec{e}$, and $\vec{f}$ are coplanar.

The three vectors $\vec{d}$, $\vec{e}$, and $\vec{f}$ are 2-space vectors (in $\mathbb{V}_2$), so they are coplanar in the plane of this 2-space. But three coplanar vectors are linearly dependent. Thus, the three vectors $\vec{d}$, $\vec{e}$, and $\vec{f}$ are linearly dependent.

b) If $\vec{d}$ is a linear combination of $\vec{e}$ and $\vec{f}$, then

$$\vec{d} = s\vec{e} + t\vec{f},$$

or $\overrightarrow{(2,3)} = s\overrightarrow{(6,1)} + t\overrightarrow{(4,2)}$

or $\overrightarrow{(2,3)} = \overrightarrow{(6s + 4t, s + 2t)}$

Equating components gives

$$2 = 6s + 4t$$
$$3 = s + 2t$$

Solving these equations gives $s = -1$ and $t = 2$.

Thus, $\vec{d}$ can be expressed as a linear combination of $\vec{e}$ and $\vec{f}$, namely $\vec{d} = -\vec{e} + 2\vec{f}$. ■

Note: Part b) of Example 2 provides an algebraic proof for part a) but the algebraic argument is more complex than the geometric one.

Example 3 Prove the following vectors are linearly dependent. Express one of the vectors as a linear combination of the other two vectors.
$\vec{a} = \overrightarrow{(2,1,3)}, \vec{b} = \overrightarrow{(3,-5,4)}, \vec{c} = \overrightarrow{(12,-7,17)}$

Solution You can use either of the two equivalent conditions for linear dependence to prove that $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent. You will see a proof using both conditions.

When employing the first condition as in method 1, additional algebra must be performed to obtain a linear combination. If the second condition is used as in method 2, the linear combination appears as part of the proof.

Method 1 Prove m, k, and p exist, not all 0, such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$.

You must find m, k, and p such that

$$m\overrightarrow{(2,1,3)} + k\overrightarrow{(3,-5,4)} + p\overrightarrow{(12,-7,17)} = \overrightarrow{(0,0,0)}$$

or, $$\overrightarrow{(2m,m,3m)} + \overrightarrow{(3k,-5k,4k)} + \overrightarrow{(12p,-7p,17p)} = \overrightarrow{(0,0,0)}$$

or, $$\overrightarrow{(2m + 3k + 12p, m - 5k - 7p, 3m + 4k + 17p)} = \overrightarrow{(0,0,0)}$$

Hence, $2m + 3k + 12p = 0$ ①
$m - 5k - 7p = 0$ ②
$3m + 4k + 17p = 0$ ③

You can solve this system by the method of elimination or by the use of matrices as described on page 73 and page 95 respectively. Elimination will be used here.

Eliminate k from ① and ②

5 × ① + 3 × ② $13m + 39p = 0$ ④

Eliminate k from ② and ③

4 × ② + 5 × ③ $19m + 57p = 0$ ⑤

Eliminate m from ④ and ⑤

④ ÷ 13 − ⑤ ÷ 19 $0 + 0p = 0$, which is true for all values of p.

Select $p = 1$. From ④, $m = -3$. Substituting into ① gives $k = -2$.

Hence, real numbers $m = -3$, $k = -2$, $p = 1$ exist, not all zero, such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$.

Hence the three vectors are linearly dependent.

Using these values, $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$ gives $-3\vec{a} - 2\vec{b} + \vec{c} = \vec{0}$

Solving for $\vec{a}$, $-3\vec{a} = 2\vec{b} - \vec{c}$ or $\vec{a} = -\frac{2}{3}\vec{b} + \frac{1}{3}\vec{c}$.

This expresses $\vec{a}$ as a linear combination of $\vec{b}$ and $\vec{c}$.

Method 2 Prove that at least one of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is a linear combination of the other two vectors.

Suppose $\vec{a} = m\vec{b} + k\vec{c}$. Then,

$$\overrightarrow{(2,1,3)} = m\overrightarrow{(3,-5,4)} + k\overrightarrow{(12,-7,17)}$$

or $\overrightarrow{(2,1,3)} = \overrightarrow{(3m,-5m,4m)} + \overrightarrow{(12k,-7k,17k)}$

or $\overrightarrow{(2,1,3)} = \overrightarrow{(3m + 12k,-5m - 7k,4m + 17k)}$

Thus, $2 = 3m + 12k$ ①
$1 = -5m - 7k$ ②
$3 = 4m + 17k$ ③

Eliminating m from ① and ②

$5 \times$ ① $+ 3 \times$ ② gives $13 = 39k$, or $k = \frac{1}{3}$.

Substituting in ② gives $1 = -5m - 7\left(\frac{1}{3}\right)$, or $m = -\frac{2}{3}$.

Because m and k must satisfy all three equations you *must* check these values in equation ③.

Substituting in ③, $L.S. = 3$, $R.S. = 4\left(-\frac{2}{3}\right) + 17\left(\frac{1}{3}\right) = \frac{9}{3} = 3 = L.S.$

Since $\vec{a} = -\frac{2}{3}\vec{b} + \frac{1}{3}\vec{c}$ is a linear combination of $\vec{b}$ and $\vec{c}$, then $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent. ■

Note: Method 2 will not work if $\vec{a}$ and $\vec{b}$ are multiples of each other. In this case, it will be necessary to show either $\vec{a}$ or $\vec{b}$ is a linear combination of the other two vectors. Method 1 will always work.

SUMMARY

Linear Dependence of Two or Three Vectors

	two vectors: $\vec{a}$, $\vec{b}$	three vectors: $\vec{a}$, $\vec{b}$, $\vec{c}$
geometric condition	$\vec{a} \parallel \vec{b}$	$\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar
algebraic conditions	1. m, k exist, not both 0, such that $m\vec{a} + k\vec{b} = \vec{0}$, or 2. $\vec{b} = s\vec{a}$, for some $s \in \mathbb{R}$	1. m, k, p exist, not all 0, such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$, or 2. at least one is a linear combination of the other two; for example $\vec{c} = s\vec{a} + t\vec{b}$ $s, t \in \mathbb{R}$

2.2 Exercises

1. Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent.
   a) What is the geometric relationship among $\vec{a}$, $\vec{b}$, and $\vec{c}$?
   b) State two algebraic equations that are true relating $\vec{a}$, $\vec{b}$, and $\vec{c}$.
   c) What conditions, if any, are imposed on the scalars in the equations in b)?

2. You are given three coplanar vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$. State two equations showing the algebraic relationship among $\vec{x}$, $\vec{y}$, and $\vec{z}$.

3. Explain why three coplanar vectors can be represented by three directed line segments that are not coplanar.

4. a) The vector $\vec{k}$ is a linear combination of the vectors $\vec{d}$ and $\vec{e}$. What is the geometric relationship among $\vec{k}$, $\vec{d}$, and $\vec{e}$?
   b) Scalars a, b, and c exist for the vectors in a) such that $a\vec{k} + b\vec{d} + c\vec{e} = \vec{0}$. What, if anything, must be true about the scalars a, b, and c?

5. Vectors $\vec{a}$ and $\vec{b}$ are linearly dependent. Vectors $\vec{c}$ and $\vec{d}$ are such that $\vec{c} = 3\vec{a} + 2\vec{b}$ and $\vec{d} = 4\vec{a} + 2\vec{c}$.
   a) Use a geometric argument to show that $\vec{c}$ and $\vec{d}$ lie in the plane of $\vec{a}$ and $\vec{b}$.
   b) Use an algebraic argument to show that $\vec{c}$ and $\vec{d}$ lie in the plane of $\vec{a}$ and $\vec{b}$.

6. a) Prove that $\vec{0} = \overrightarrow{(0,0)}$ is linearly dependent with every vector in $\mathbb{V}_2$.
   b) Prove that $\vec{0} = \overrightarrow{(0,0,0)}$ is linearly dependent with every pair of vectors in $\mathbb{V}_3$.

7. Prove that the vectors $\vec{a} = \overrightarrow{(4,1,2)}$, $\vec{b} = \overrightarrow{(-1,0,3)}$, and $\vec{c} = \overrightarrow{(2,1,8)}$ are linearly dependent.

8. Prove the vectors in each of the following are linearly dependent. In each case express one of the vectors as a linear combination of the other two vectors.
   a) $\vec{a} = \overrightarrow{(2,3,-4)}$, $\vec{b} = \overrightarrow{(-1,6,2)}$, $\vec{c} = \overrightarrow{(8,-3,-16)}$
   b) $\vec{d} = \overrightarrow{(-5,-6)}$, $\vec{e} = \overrightarrow{(2,3)}$, $\vec{f} = \overrightarrow{(-1,-3)}$
   c) $\vec{g} = \overrightarrow{(4,0,1)}$, $\vec{h} = \overrightarrow{(-8,15,-12)}$, $\vec{n} = \overrightarrow{(0,3,-2)}$
   d) $\vec{p} = \overrightarrow{(6,-4)}$, $\vec{q} = \overrightarrow{(-3,9)}$, $\vec{r} = \overrightarrow{(-3,2)}$

9. The vectors $\vec{a} = \overrightarrow{(0,-1,-4)}$, $\vec{b} = \overrightarrow{(1,5,-1)}$, and $\vec{c} = \overrightarrow{(3,k,5)}$ are linearly dependent. Find the value of k.

10. The vectors $\vec{x} = \overrightarrow{(8,-1,3)}$, $\vec{y} = \overrightarrow{(-4,2,m)}$, and $\vec{z} = \overrightarrow{(4,5,7)}$ are coplanar. Find the value of m.

11. Given the vectors $\vec{u} = \overrightarrow{(1,2)}$, $\vec{v} = \overrightarrow{(3,2)}$, and $\vec{w} = \overrightarrow{(-1,3)}$. If possible, write $\vec{a} = \overrightarrow{(3,6)}$ as a linear combination of each of the following.
    a) $\vec{u}$ only
    b) $\vec{v}$ only
    c) $\vec{u}$ and $\vec{v}$ only
    d) $\vec{u}$, $\vec{v}$, $\vec{w}$

12. Repeat question 11 for $\vec{a} = \overrightarrow{(-1,2)}$.

13. Given the vectors $\vec{u} = \overrightarrow{(2,0,0)}$, $\vec{v} = \overrightarrow{(0,-1,2)}$, $\vec{w} = \overrightarrow{(1,0,3)}$, and $\vec{t} = \overrightarrow{(0,0,1)}$. If possible, write $\vec{a} = \overrightarrow{(0,-3,6)}$ as a linear combination of each of the following.
    a) $\vec{u}$ and $\vec{v}$ only
    b) $\vec{u}$, $\vec{v}$, and $\vec{w}$ only
    c) $\vec{u}$, $\vec{v}$, $\vec{w}$, and $\vec{t}$

14. Repeat question 13 for $\vec{a} = \overrightarrow{(5,-1,3)}$.

15. If any two of three vectors are linearly dependent, then prove that the three vectors are linearly dependent.

16. Given $\vec{a} = \overrightarrow{(2,1,-3)}$, $\vec{b} = \overrightarrow{(0,2,5)}$ and $\vec{c} = \overrightarrow{(-4,-2,6)}$, prove that constants m, k, and p, not all zero, exist such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$

2.3 Linearly Independent Vectors and Basis Vectors

Two vectors can be linearly dependent or linearly independent but not both. You know that three vectors can be linearly dependent. You are now ready to define linear independence for three vectors. You will do this in such a way that three vectors (in general, *any* number of vectors) can be linearly dependent or linearly independent, but not both.

DEFINITION

Three vectors that are not linearly dependent are *linearly independent.*

Geometrically this definition suggests that three linearly independent vectors cannot be coplanar.

Also, three vectors being linearly dependent means that m, k, and p exist, not all equal to 0, such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$. This fact implies that if the only values that can be found for m, k, and p such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$ are $m = k = p = 0$, then $\vec{a}$, $\vec{b}$ and $\vec{c}$ are not linearly dependent. Hence, the three vectors are linearly independent.

SUMMARY

Linear Independence of Two or Three Vectors

	two vectors: $\vec{a}, \vec{b}$	three vectors: $\vec{a}, \vec{b}, \vec{c}$
geometric condition	$\vec{a} \nparallel \vec{b}$	$\vec{a}, \vec{b}, \vec{c}$ are not coplanar
algebraic conditions	1. If $m\vec{a} + k\vec{b} = \vec{0}$ then $m = k = 0$, or 2. no s exists such that $\vec{b} = s\vec{a}$, $s \in \mathbb{R}$	If $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$ then $m = k = p = 0$

Two linearly independent vectors you have met in $\mathbb{V}_2$ are the unit vectors $\vec{i}$ and $\vec{j}$. In chapter 1 you learned that any vector in $\mathbb{V}_2$ can be written $m\vec{i} + k\vec{j}$, where m and k are scalars. Thus every vector in the plane of $\vec{i}$ and $\vec{j}$ can be expressed as a linear combination of $\vec{i}$ and $\vec{j}$.

In Example 1 you will find that a similar fact is true for every pair $\vec{a}, \vec{b}$ of linearly independent vectors, that is, any vector in the plane of $\vec{a}$ and $\vec{b}$ can be expressed as a linear combination of $\vec{a}$ and $\vec{b}$.

Example 1

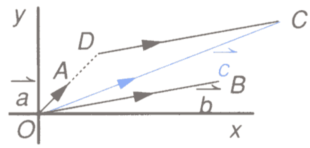
If $\vec{a}$ and $\vec{b}$ are two linearly independent vectors, and $\vec{c}$ is any other vector coplanar with $\vec{a}$ and $\vec{b}$, prove that $\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$.

Solution

You will need to show that scalars s and t exist such that $\vec{c} = s\vec{a} + t\vec{b}$.

Since $\vec{a}$ and $\vec{b}$ are linearly independent, they are not parallel. But they are coplanar. Now the vectors $\vec{a}$ and $\vec{b}$ can be translated in their common plane so that each becomes a position vector with its tail at the origin of a 2-space coordinate system. Let $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$

Case 1: $\vec{c}$ is not parallel to either $\vec{a}$ or $\vec{b}$.



Let $\vec{c} = \overrightarrow{OC}$. From C draw a line parallel to OB intersecting OA or OA extended at point D. Then $\overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{DC}$.
Since D is in OA, a real number s exists such that $\overrightarrow{OD} = s\vec{a}$.
Since CD is parallel to OB, a real number t exists such that $\overrightarrow{DC} = t\vec{b}$.

Thus, $\overrightarrow{OC} = s\vec{a} + t\vec{b}$ or $\vec{c} = s\vec{a} + t\vec{b}$. Hence, $\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$.

Case 2: $\vec{c}$ is parallel to either $\vec{a}$ or $\vec{b}$.

Suppose $\vec{c}$ is parallel to $\vec{a}$. Then a real number p exists such that $\vec{c} = p\vec{a}$.
Hence, $\vec{c} = p\vec{a} + 0\vec{b}$. Hence, $\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$. ■

The results of Example 1 are true for any two linearly independent vectors in $\mathbb{V}_2$. Two linearly independent vectors form a **basis** in the 2-space plane in which they lie.

DEFINITION

Any two linearly independent vectors $\vec{a}$ and $\vec{b}$ form a basis for $\mathbb{V}_2$.

If $\vec{c} = s\vec{a} + t\vec{b}$, then s and t are called the **components** of $\vec{c}$ in the basis $\{\vec{a},\vec{b}\}$.

Example 2

Prove that the components of a vector in the $\mathbb{V}_2$ basis $\{\vec{a},\vec{b}\}$ are unique.

Solution

Suppose that a vector $\vec{c}$ has two pairs of components (s,t) and (p,q) in basis $\{\vec{a},\vec{b}\}$.

You will need to show that (s,t) and (p,q) are the same ordered pair.

Because both ordered pairs are components, you have two equations for $\vec{c}$, namely $\vec{c} = s\vec{a} + t\vec{b}$ and $\vec{c} = p\vec{a} + q\vec{b}$.

Thus, $s\vec{a} + t\vec{b} = p\vec{a} + q\vec{b}$ or, $(s - p)\vec{a} + (t - q)\vec{b} = \vec{0}$.
Since $\{\vec{a},\vec{b}\}$ is a basis in $\mathbb{V}_2$, $\vec{a}$ and $\vec{b}$ are linearly independent.

Hence, $s - p = 0$, and $t - q = 0$, or $s = p$ and $t = q$. Thus, the components of $\vec{c}$ in $\{\vec{a},\vec{b}\}$ are unique. ■

The most important basis in $\mathbb{V}_2$ consists of the unit vectors $\vec{i} = \overrightarrow{(1,0)}$ and $\vec{j} = \overrightarrow{(0,1)}$. You will prove $\vec{i}$ and $\vec{j}$ form a basis for $\mathbb{V}_2$ in Example 4.

You have now seen that if two vectors in 2-space or in 3-space are linearly independent then any third vector in the same plane can be written as a linear combination of the two vectors.

As you will see in the following example, every vector in 3-space can be written as a linear combination of three linearly independent vectors.

In other words, it is impossible for four 3-space vectors to be linearly independent.

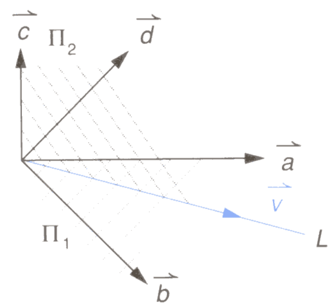
Example 3

If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are three linearly independent vectors, and $\vec{d}$ is any other vector, then prove that $m, k, p \in \mathbb{R}$ exist such that $\vec{d} = m\vec{a} + k\vec{b} + p\vec{c}$.

Solution You will make use of the fact that the plane containing vectors $\vec{a}$ and $\vec{b}$ intersects with the plane containing $\vec{c}$ and $\vec{d}$.

$\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ can be translated so that each becomes a position vector with its tail at the origin of the 3-space coordinate system. Since the three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly independent, the three vectors are not coplanar.

Let Π_1 be the plane containing $\vec{a}$ and $\vec{b}$.
Let Π_2 be the plane containing $\vec{c}$ and $\vec{d}$.
Since Π_1 and Π_2 are distinct non-parallel planes, they intersect in a line L.
Let $\vec{v}$ be any vector along line L. Then $\vec{v}$ lies in the plane Π_1. Hence $\vec{v}$ is a linear combination of $\vec{a}$ and $\vec{b}$.



Thus, $\vec{v} = s\vec{a} + t\vec{b}$ ①

Also $\vec{v}$ lies in Π_2, so $\vec{v} = w\vec{d} + r\vec{c}$ ②

From ① and ②

$s\vec{a} + t\vec{b} = w\vec{d} + r\vec{c}$

Thus, $w\vec{d} = s\vec{a} + t\vec{b} - r\vec{c}$

or, $\vec{d} = \frac{s}{w}\vec{a} + \frac{t}{w}\vec{b} - \frac{r}{w}\vec{c}$ ($w \neq 0$, see note).

Thus, $\vec{d} = m\vec{a} + k\vec{b} + p\vec{c}$,

where $m = \frac{s}{w}$, $k = \frac{t}{w}$, $p = -\frac{r}{w}$. ■

Note: The scalar w can not be 0, otherwise $\vec{v}$ and $\vec{c}$ would be scalar multiples and hence collinear. Thus, $\vec{c}$ would be coplanar with $\vec{b}$ and $\vec{a}$, which is not true.

If three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly independent then the vectors can not be coplanar but must exist in 3-space. Example 3 shows that any other vector in 3-space can be expressed as a linear combination of these vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

The vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are said to form a basis for $\mathbb{V}_3$.

DEFINITION Any three linearly independent vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ form a basis for $\mathbb{V}_3$.

If $\vec{d} = m\vec{a} + k\vec{b} + p\vec{c}$, then the scalars m, k, and p are called the *components* of $\vec{d}$ in the basis $\{\vec{a},\vec{b},\vec{c}\}$.

The most important 3-space basis consists of the unit vectors $\vec{i} = \overrightarrow{(1,0,0)}$, $\vec{j} = \overrightarrow{(0,1,0)}$, and $\vec{k} = \overrightarrow{(0,0,1)}$ that you met in chapter 1. You will prove $\vec{i}$, $\vec{j}$ and $\vec{k}$ form a basis for $\mathbb{V}_3$ in Example 4.

Example 4

a) Prove that the vectors $\vec{i} = \overrightarrow{(1,0)}$ and $\vec{j} = \overrightarrow{(0,1)}$ form a basis for $\mathbb{V}_2$.

b) Prove that the vectors $\vec{i} = \overrightarrow{(1,0,0)}$, $\vec{j} = \overrightarrow{(0,1,0)}$, and $\vec{k} = \overrightarrow{(0,0,1)}$ form a basis for $\mathbb{V}_3$.

Solution

a) Vectors $\vec{i}$ and $\vec{j}$ form a basis for $\mathbb{V}_2$ if they are linearly independent. Hence, you must prove that $m\vec{i} + t\vec{j} = \vec{0}$ implies that $m = t = 0$.

Suppose $m\vec{i} + t\vec{j} = \vec{0}$

Thus $m\overrightarrow{(1,0)} + t\overrightarrow{(0,1)} = \overrightarrow{(0,0)}$

or $\overrightarrow{(m,0)} + \overrightarrow{(0,t)} = \overrightarrow{(0,0)}$

or $\overrightarrow{(m + 0,0 + t)} = \overrightarrow{(0,0)}$

Equating components you obtain $m = 0$ and $t = 0$. Hence, $\vec{i}$ and $\vec{j}$ form a basis for $\mathbb{V}_2$.

b) Vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ form a basis for $\mathbb{V}_3$ if they are linearly independent. Hence, you must prove that $m\vec{i} + t\vec{j} + p\vec{k} = \vec{0}$ implies that $m = t = p = 0$. Suppose $m\vec{i} + t\vec{j} + p\vec{k} = \vec{0}$

Thus $m\overrightarrow{(1,0,0)} + t\overrightarrow{(0,1,0)} + p\overrightarrow{(0,0,1)} = \overrightarrow{(0,0,0)}$

or $\overrightarrow{(m,0,0)} + \overrightarrow{(0,t,0)} + \overrightarrow{(0,0,p)} = \overrightarrow{(0,0,0)}$

or $\overrightarrow{(m + 0 + 0,0 + t + 0,0 + 0 + p)} = \overrightarrow{(0,0,0)}$

or $\overrightarrow{(m,t,p)} = \overrightarrow{(0,0,0)}$

Equating components you obtain $m = 0$, $t = 0$, and $p = 0$. Hence, $\vec{i}$, $\vec{j}$, and $\vec{k}$ form a basis for $\mathbb{V}_3$. ■

In Search of Vectors in Spaces with Dimension Higher than Three

Vectors in 2-space can be represented geometrically by directed line segments or algebraically by ordered pairs. The set of such vectors is called $\mathbb{V}_2$. Vectors in 3-space can be represented geometrically by directed line segments or algebraically by ordered 3-tuples. The set of such vectors is called $\mathbb{V}_3$.

Are there also sets of vectors $\mathbb{V}_4$, $\mathbb{V}_5$, $\mathbb{V}_6$, and so on?

In that part of *Finite Mathematics* called matrices you will find that ordered 4-tuples, ordered 5-tuples etc. are used to represent such things as the inventory of a factory or the wins of various teams.

City	Monitors	Printers	Disk Drives	Keyboards
Weston	10	12	25	15
Guelph	20	24	44	25
Kingston	12	15	28	10

City of Team	Games Played	Games Won	Games Lost	Games Tied	Points For	Points Against
Toronto	1	1	0	0	21	20
Ottawa	1	1	0	0	20	11
Winnipeg	1	0	1	0	11	20
Hamilton	1	0	1	0	20	21

Each set of ordered n-tuples, $n \in \mathbb{N}$, makes up the set of vectors $\mathbb{V}_n$ *provided that* an addition rule can be defined so that addition has all of the properties held by $\mathbb{V}_2$ and by $\mathbb{V}_3$. When a set of ordered n-tuples has these properties $\mathbb{V}_n$ is called an *n-dimensional vector space*. There is no readily available geometric model for vectors of more than three dimensions. (The corner of a room provides a model for three mutually perpendicular axes. Can you imagine *four* mutually perpendicular axes?) Nevertheless, ordered n-tuples can exhibit all the other properties of vectors—all you lose is the geometric model.

Properties of an n-dimensional vector space $\mathbb{V}_n$

Definition of addition

If $\vec{a} = \overrightarrow{(a_1, a_2, a_3, \ldots, a_n)}$ and $\vec{b} = \overrightarrow{(b_1, b_2, b_3, \ldots, b_n)}$ then
$\vec{a} + \vec{b} = \overrightarrow{(a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots, a_n + b_n)}$

Properties of addition

$\mathbb{V}_n$ is closed: $\vec{a}, \vec{b} \in \mathbb{V}_n$ implies $(\vec{a} + \vec{b}) \in \mathbb{V}_n$
Addition is associative: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
A neutral element $\vec{0} \in \mathbb{V}_n$ exists such that $\vec{a} + \vec{0} = \vec{0} + \vec{a}$ for all $\vec{a} \in \mathbb{V}_n$
Each $\vec{a} \in \mathbb{V}_n$ has an inverse $-\vec{a}$ such that $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$

Definition of scalar multiplication

If k is a scalar and $\vec{a} = \overrightarrow{(a_1, a_2, a_3, \ldots, a_n)}$ then $k\vec{a} = \overrightarrow{(ka_1, ka_2, ka_3, \ldots, ka_n)}$

Properties of scalar multiplication

$k\vec{a} \in \mathbb{V}_n$
$(km)\vec{a} = k(m\vec{a})$ [m is a scalar.]
$k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$
$(k + m)\vec{a} = k\vec{a} + m\vec{a}$

The linear dependence and linear independence of vectors in $\mathbb{V}_n$ can be defined in an manner similar to that for $\mathbb{V}_2$ and $\mathbb{V}_3$.

k vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots, \vec{a}_k$ are linearly dependent if and only if k real numbers exist, $m_1, m_2, m_3, \ldots, m_k$, not all zero, such that
$\sum_{i=1}^{k} m_i\vec{a}_i = \vec{0}$

k vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots, \vec{a}_k$ are linearly independent if they are not linearly dependent.

If the vectors are linearly independent, then $\sum_{i=1}^{k} m_i\vec{a}_i = \vec{0}$ implies

$m_1 = m_2 = m_3 = \ldots = m_k = 0.$

Any k linearly independent vectors in a space of k dimensions form a basis for $\mathbb{V}_k$.

Activity

Find applications of vector spaces of dimension higher than three.

2.3 Exercises

1. Vectors $\vec{s}$, $\vec{t}$, and $\vec{r}$ are not coplanar, and $w\vec{s} + x\vec{t} + z\vec{r} = \vec{0}$. What, if anything, is true about the scalars w, x, and z?

2. $\vec{x} \nparallel \vec{y}$, and $\vec{z}$ does not lie in the plane of $\vec{x}$ and $\vec{y}$. Which of $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ are linearly dependent with $\vec{x}$ and $\vec{y}$?
   $\vec{a} = 3\vec{x} + 5\vec{y}$ $\quad \vec{b} = 4\vec{a} + 3\vec{x}$
   $\vec{c} = 3\vec{x} + 5\vec{z}$ $\quad \vec{d} = \vec{a} - \vec{b}$

3. Vectors $\vec{a}$ and $\vec{b}$ are linearly independent. Scalars x, y, m, and k are such that $\vec{c} = x\vec{a} + y\vec{b}$ and $\vec{d} = m\vec{a} + k\vec{c}$. Prove that $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ are coplanar.

4. If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly independent, then prove that $\vec{u} = \vec{a} + 2\vec{b} + \vec{c}$, $\vec{v} = \vec{a} + 3\vec{b} - 2\vec{c}$, and $\vec{w} = \vec{a} + \vec{b} + 4\vec{c}$ are linearly dependent.

5. Prove that $\vec{a} = \overrightarrow{(4,1,2)}$, $\vec{b} = \overrightarrow{(-1,0,3)}$, and $\vec{d} = \overrightarrow{(3,1,4)}$ are linearly independent.

6. Determine whether or not the three vectors in each of the following are linearly dependent. In each case state the geometric significance of the result.
   a) $\overrightarrow{(0,1,3)}$, $\overrightarrow{(-3,5,2)}$, and $\overrightarrow{(-6,11,7)}$
   b) $\overrightarrow{(1,2,3)}$, $\overrightarrow{(-3,0,4)}$, and $\overrightarrow{(-1,4,6)}$
   c) $\overrightarrow{(4,1,8)}$, $\overrightarrow{(-2,1,0)}$, and $\overrightarrow{(0,3,16)}$
   d) $\overrightarrow{(1,2,4)}$, $\overrightarrow{(2,-3,-1)}$, and $\overrightarrow{(-1,-9,-13)}$
   e) $\overrightarrow{(3,5,1)}$, $\overrightarrow{(2,-2,-2)}$, and $\overrightarrow{(-4,-4,0)}$

7. Given the vectors $\vec{a} = \overrightarrow{(2,-5)}$ and $\vec{b} = \overrightarrow{(3,1)}$.
   a) Prove $\vec{a}$ and $\vec{b}$ form a basis for $\mathbb{V}_2$.
   b) Express each of the following vectors as a linear combination of $\vec{a}$ and $\vec{b}$.
   $\vec{c} = \overrightarrow{(7,-9)}$ $\vec{d} = \overrightarrow{(-2,-29)}$ $\vec{e} = \overrightarrow{(6,-15)}$

8. Which of the following pairs of vectors form a basis for $\mathbb{V}_2$?
   a) $\overrightarrow{(3,4)}$, $\overrightarrow{(2,3)}$
   b) $\overrightarrow{(1,-3)}$, $\overrightarrow{(4, 2)}$
   c) $\overrightarrow{(4,-6)}$, $\overrightarrow{(6,-9)}$

9. Given the vectors $\vec{a} = \overrightarrow{(4,1,0)}$, $\vec{b} = \overrightarrow{(2,-3,4)}$, and $\vec{c} = \overrightarrow{(6,1,4)}$.
   a) Prove the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ form a basis for $\mathbb{V}_3$.
   b) Express each of the following vectors as a linear combination of $\vec{a}$, $\vec{b}$, and $\vec{c}$.
   $\vec{d} = \overrightarrow{(-8,-4,-8)}$ $\vec{e} = \overrightarrow{(18,13,4)}$
   $\vec{f} = \overrightarrow{(14,6,4)}$.

10. If $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$, then either $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar or $m = k = p = 0$. Prove.

11. Vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly independent. Vectors $\vec{d}$, $\vec{b}$ and $\vec{c}$ are not coplanar.
    $\vec{d} = m\vec{a} + k\vec{b} + p\vec{c} \neq \vec{0}$.
    $\vec{e} = r\vec{b} + t\vec{c} \neq \vec{0}$
    Prove that $\vec{d}$ and $\vec{e}$ are linearly independent
    a) using a geometric argument
    b) using an algebraic argument.

12. Given vectors $\vec{a} = \overrightarrow{(w + 1,3w + 1)}$, and $\vec{b} = \overrightarrow{(2,w + 2)}$.
    a) If $\{\vec{a},\vec{b}\}$ is *not* a basis for $\mathbb{V}_2$ find the value(s) for the scalar w.
    b) If $\{\vec{a},\vec{b}\}$ is a basis for $\mathbb{V}_2$ find the value(s) for the scalar w.

13. Given vectors $\vec{a} = \overrightarrow{(m,0,0)}$, $\vec{b} = \overrightarrow{(0,m,1)}$, and $\vec{c} = \overrightarrow{(0,1,m)}$.
    a) If $\{\vec{a},\vec{b},\vec{c}\}$ is *not* a basis for $\mathbb{V}_3$ find the value(s) for the scalar m.
    b) If $\{\vec{a},\vec{b},\vec{c}\}$ is a basis for $\mathbb{V}_3$ find the value(s) for the scalar m.

14. a) State an algebraic condition for four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ to be linearly dependent.
    b) The three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ form a basis for $\mathbb{V}_3$, and $\vec{d}$ is any other 3-space vector. Prove that $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ are linearly dependent.

The Prisoners' Dilemma—a Game

The game known as the *Prisoners' Dilemma* was introduced in 1950 by the Canadian-born mathematician Albert W. Tucker. The game involves the scenario of two suspects in a crime who are prevented from communicating with each other. Each is given one of two choices.

C: co-operation: maintain that both are innocent
N: nonco-operation: accuse the other of having committed the crime alone

It is usually in the individual's self-interest to accuse the other. Yet when both accuse, they reach a bad outcome. What is good for the prisoners *as a pair*, is to maintain that both are innocent.

This simple model can be used for a crucial international problem—the arms race between the USA and the USSR. Each of these two superpowers can independently select one of two policies.

C: co-operation: disarm, or at least agree to a partial ban on armaments
N: nonco-operation: heavily arm in preparation for any war contingency

Here, as in the original game of *Prisoners' Dilemma*, there are four possible outcomes, as indicated by the four ordered pairs (USA's choice, USSR's choice) that follow.

(C,C) Both the USA and the USSR co-operate by choosing to disarm. Most people would see this as the most preferred outcome, even though there are certain risks.

(N,N) Both the USA and the USSR refuse to co-operate by deciding to arm. From a global standpoint, most people would agree that this is the worst possible outcome.

(N,C) The USA decides not to co-operate and arm while the USSR elects to co-operate and disarm. This unilateral disarmament by the USSR would be the one preferred most of all by the USA and the one least preferred by the USSR.

(C,N) The USSR decides to arm while the USA elects to disarm. This would be the 'worst possible' outcome for the USA and the 'best' for the USSR.

The following matrix models this game.

To consider this game mathematically it is customary to assign a payoff from 0 to 5 for each event. Here (0,5) signifies a payoff of 0 for the USA and a payoff of 5 for the USSR.

		USSR	
		C	N
USA	C	both disarm (4,4)	favours USSR (0,5)
	N	favours USA (5,0)	arms race (2,2)

Should the USA select strategy C or strategy N? The USA can see what happens if the USSR selects C. The USA will receive a payoff of 4 for co-operating by disarming but a payoff of 5 for arming. Thus, the USA will get a better payoff by arming.

Now if the USSR selects strategy N, that is, the USSR decides to arm, the USA will receive 0 for disarming, and 2 for arming. Again the USA has the better payoff if it chooses to arm.

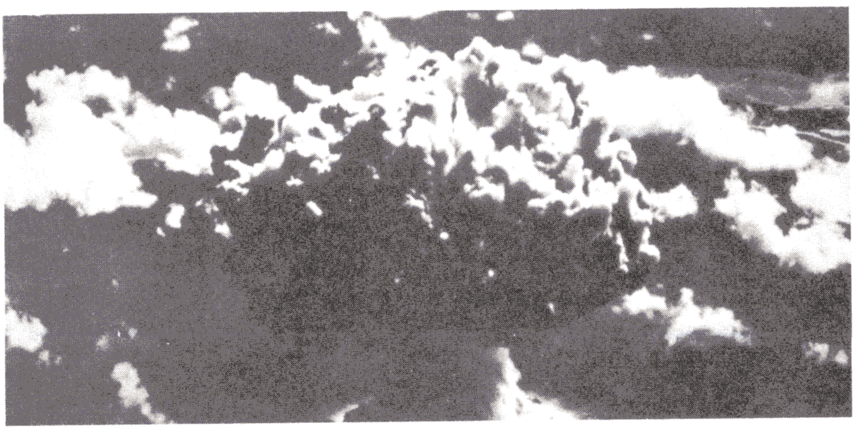
In either case, the USA gets a better payoff by arming than by disarming.

The same argument will lead the USSR to decide upon N, that is, no co-operation by arming. Thus, when each nation attempts to maximize its own payoff independently, the pair is driven into the outcome (N,N) with the payoff (2,2). The best payoff, (4,4), appears unattainable when the game is played in an atmosphere of nonco-operation.

Even if the USA and the USSR agree in advance to pursue together the globally optimal solution (C,C), this outcomes is fraught with problems. If either the USA or the USSR alone reneges on the agreement and secretly arms, it will benefit. Each country would be tempted to go back on its word and select N.

In real life, people, and sometimes nations, do manage to avoid the nonco-operative outcome in the game of *Prisoner's Dilemma*. The game is usually played within a larger context, where other incentives have their part to play. Also, the game is usually played on a repeating basis, so that elements such as reputation and trust also play a role. Players realize the mutual advantages in co-operation. Nations can also resort to other helpful measures, such as better communication, and more reliable inspection procedures.

The game of *Prisoners' Dilemma* pinpoints the dynamics behind a frequently occurring paradox. The nonco-operative outcome is not as satisfactory as a co-operative solution in which one is ready to allow one's own selfish interests to take second place.

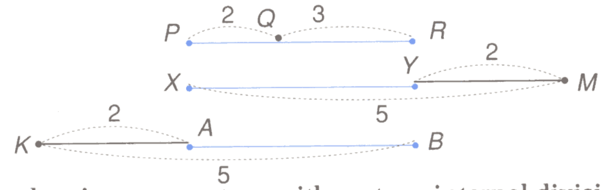


2.4 Points of Division of a Line Segment

When two directed line segments $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are linearly dependent, then the points P, Q, and R lie in a straight line. In this situation any one point divides the line segment joining the other two points into some ratio. Vectors can be used to find a special relationship among these points.

First recall the following about points of division of a line segment.

In the diagram, point Q divides PR *internally* in the ratio $2:3$.
Point M divides XY *externally* in the ratio $5:2$.
Point K divides AB *externally* in the ratio $2:5$.



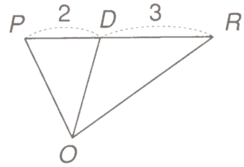
If direction is taken into account, as with vectors, internal division can be distinguished from external division by using negative signs.
Thus, if the direction from point X to point Y is taken to be positive, the point M divides XY externally in the ratio $5:(-2)$.
Similarily, if the direction from point A to point B is taken to be positive, the point K divides AB externally in the ratio $(-2):5$.

By convention the first term of the ratio is associated with the first point mentioned, that is, the division is from the first point mentioned to the dividing point, then from the dividing point to the second point mentioned. Observe that the point of division is closer to the point associated with the smaller term of the ratio.

Example 1 If point D divides the line segment PR internally in the ratio $2:3$, and O is any fixed point, then express $\overrightarrow{OD}$ in terms of $\overrightarrow{OP}$ and $\overrightarrow{OR}$.

Solution 1 The points are related as in the figure.

The key to solving this problem is to write an equation relating any two vectors along line PR and then to replace those vectors with position vectors *by subtraction*.



From the diagram $\overrightarrow{PD} = \frac{2}{5}\overrightarrow{PR}$ ①

But, $\overrightarrow{PD} = \overrightarrow{OD} - \overrightarrow{OP}$, and $\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP}$
Substituting in ① gives

$$\overrightarrow{OD} - \overrightarrow{OP} = \frac{2}{5}(\overrightarrow{OR} - \overrightarrow{OP})$$

$$\overrightarrow{OD} = \frac{2}{5}\overrightarrow{OR} - \frac{2}{5}\overrightarrow{OP} + \overrightarrow{OP}, \text{ hence, } \overrightarrow{OD} = \frac{2}{5}\overrightarrow{OR} + \frac{3}{5}\overrightarrow{OP}.$$ ■

Solution 2 From the figure, replacing $\overrightarrow{OD}$ by the longer route $\overrightarrow{OP} + \overrightarrow{PD}$,

$\overrightarrow{OD} = \overrightarrow{OP} + \overrightarrow{PD}$

But $\overrightarrow{PD} = \frac{2}{5}\overrightarrow{PR}$, and $\overrightarrow{PR} = -\overrightarrow{OP} + \overrightarrow{OR}$

Thus $\overrightarrow{OD} = \overrightarrow{OP} + \overrightarrow{PD}$

$$= \overrightarrow{OP} + \frac{2}{5}\overrightarrow{PR}$$

$$= \overrightarrow{OP} + \frac{2}{5}(-\overrightarrow{OP} + \overrightarrow{OR})$$

$$= \overrightarrow{OP} - \frac{2}{5}\overrightarrow{OP} + \frac{2}{5}\overrightarrow{OR}$$

Thus $\overrightarrow{OD} = \frac{3}{5}\overrightarrow{OP} + \frac{2}{5}\overrightarrow{OR}$, as before. ■

Observe in Example 1,

1. the number 2 in the ratio 2 : 3 is connected to the point P in the diagram but multiplies $\overrightarrow{OR}$ in the equation
2. the number 3 in the ratio 2 : 3 is connected to the point R in the diagram but multiplies $\overrightarrow{OP}$ in the equation
3. the sum of the multipliers of $\overrightarrow{OP}$ and $\overrightarrow{OR}$, namely, $\frac{2}{5} + \frac{3}{5} = 1$
4. the denominator $5 = 2 + 3$.

Do a few more examples like Example 1 to see if these patterns continue.

Example 2 If point D divides the line segment PR externally in the ratio 5 : 3, and O is any fixed point, then express $\overrightarrow{OD}$ in terms of $\overrightarrow{OP}$ and $\overrightarrow{OR}$.

Solution The points are related as in the figure. Again, the key is to write an equation relating any two vectors along PR. From the figure,

$$\overrightarrow{DR} = \frac{3}{5}\overrightarrow{DP}$$

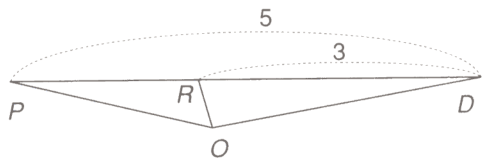
Using subtraction of position vectors for $\overrightarrow{DR}$ and $\overrightarrow{DP}$ gives

$$\overrightarrow{OR} - \overrightarrow{OD} = \frac{3}{5}(\overrightarrow{OP} - \overrightarrow{OD})$$

$$\overrightarrow{OR} - \overrightarrow{OD} = \frac{3}{5}\overrightarrow{OP} - \frac{3}{5}\overrightarrow{OD}$$

$$\overrightarrow{OR} - \frac{3}{5}\overrightarrow{OP} = \frac{2}{5}\overrightarrow{OD}$$

Hence, $\overrightarrow{OD} = \frac{5}{2}\overrightarrow{OR} - \frac{3}{2}\overrightarrow{OP}$. ■



Example 2 shows the same patterns as Example 1, if the ratio is considered as $5:(-3)$,

1. the number 5 in the ratio $5:(-3)$ is connected to the point P in the diagram but multiplies $\overrightarrow{OR}$ in the equation.
2. the number 3 in the ratio $5:(-3)$ is connected to the point R in the diagram but multiplies $\overrightarrow{OP}$ in the equation.
3. the sum of the multipliers of $\overrightarrow{OP}$ and $\overrightarrow{OR}$, namely, $-\frac{3}{2}+\frac{5}{2}=1$.
4. the denominator $2 = 5 + (-3)$.

It appears that the point D that divides segment PR in the ratio $m:k$ satisfies the following relationship. $\overrightarrow{OD}=\frac{k}{m+k}\overrightarrow{OP}+\frac{m}{m+k}\overrightarrow{OR}$

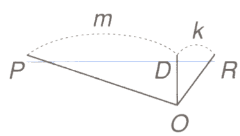
You will prove this in the following argument.

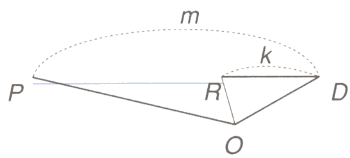
Proof of Internal/External Division Property

Given point D divides the line segment PR in the ratio $m:k$. For internal division m and k are positive. For external division the smaller of m and k is negative. If O is any fixed point, prove that

$$\overrightarrow{OD}=\frac{k}{(m+k)}\overrightarrow{OP}+\frac{m}{(m+k)}\overrightarrow{OR}$$

Proof:





The diagrams for internal and external division indicate that

$$\overrightarrow{PD}=\frac{m}{m+k}\overrightarrow{PR}$$

$$\overrightarrow{OD}-\overrightarrow{OP}=\frac{m}{m+k}(\overrightarrow{OR}-\overrightarrow{OP})$$

$$\overrightarrow{OD}-\overrightarrow{OP}=\frac{m}{m+k}\overrightarrow{OR}-\frac{m}{m+k}\overrightarrow{OP}$$

$$\overrightarrow{OD}=\frac{m+k-m}{m+k}\overrightarrow{OP}+\frac{m}{m+k}\overrightarrow{OR}$$

Thus, $\overrightarrow{OD}=\frac{k}{m+k}\overrightarrow{OP}+\frac{m}{m+k}\overrightarrow{OR}$

Note: Using the notation $\overrightarrow{OD}=\vec{d}$, $\overrightarrow{OP}=\vec{p}$, and $\overrightarrow{OR}=\vec{r}$, the property may be written

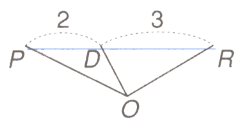
$$\vec{d}=\frac{k}{m+k}\vec{p}+\frac{m}{m+k}\vec{r}.$$

You can now use these results as a short cut for doing Examples 1 and 2.

Example 3 If point D divides the line segment PR as indicated, and O is any fixed point, then express $\overrightarrow{OD}$ in terms of $\overrightarrow{OP}$ and $\overrightarrow{OR}$.
a) internally in the ratio 2 : 3 b) externally in the ratio 5 : 3

Solution

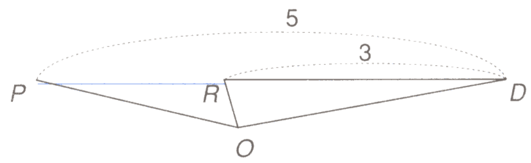
a) D divides PR internally, 2 : 3



Here $m = 2$, $k = 3$, $m + k = 5$

Thus, $\overrightarrow{OD} = \frac{3}{5}\overrightarrow{OP} + \frac{2}{5}\overrightarrow{OR}$

b) D divides PR externally, 5 : (−3)



Here $m = 5$, $k = -3$, $m + k = 2$

Thus $\overrightarrow{OD} = -\frac{3}{2}\overrightarrow{OP} + \frac{5}{2}\overrightarrow{OR}$ ■

Example 4 Find the coordinates of the point D that divides the line segment joining points $P(1,2,3)$ and $R(2,-4,3)$
a) internally in the ratio 5 : 7 b) externally in the ratio 3 : 2

Solution Let the fixed point O be $(0,0,0)$, and D have coordinates (x,y,z).

Thus, $\overrightarrow{OP} = \overrightarrow{(1,2,3)}$, $\overrightarrow{OR} = \overrightarrow{(2,-4,3)}$, and $\overrightarrow{OD} = \overrightarrow{(x,y,z)}$.

a) Here $m = 5$, $k = 7$, and $m + k = 12$.

Thus $\overrightarrow{OD} = \frac{7}{12}\overrightarrow{OP} + \frac{5}{12}\overrightarrow{OR}$, and

$$\overrightarrow{(x,y,z)} = \frac{7}{12}\overrightarrow{(1,2,3)} + \frac{5}{12}\overrightarrow{(2,-4,3)}$$

$$\overrightarrow{(x,y,z)} = \overrightarrow{\left(\frac{7}{12},\frac{14}{12},\frac{21}{12}\right)} + \overrightarrow{\left(\frac{10}{12},-\frac{20}{12},\frac{15}{12}\right)}$$

$$\overrightarrow{(x,y,z)} = \overrightarrow{\left(\frac{17}{12},-\frac{6}{12},\frac{36}{12}\right)}$$

Thus, the point of division D has coordinates $\left(\frac{17}{12},-\frac{1}{2},3\right)$.

b) Here $m = 3$, $k = -2$, and $m + k = 1$.

Thus $\overrightarrow{OD} = -2\overrightarrow{OP} + 3\overrightarrow{OR}$

$$\overrightarrow{(x,y,z)} = -2\overrightarrow{(1,2,3)} + 3\overrightarrow{(2,-4,3)}$$
$$= \overrightarrow{(-2,-4,-6)} + \overrightarrow{(6,-12,9)}$$
$$= \overrightarrow{(4,-16,3)}$$

The point dividing PR externally in the ratio 3 : 2 is $(4,-16,3)$. ■

2.4 Exercises

1. Draw a diagram showing a point D dividing the line segment PR as follows.
   a) internally in the ratio 2 : 5
   b) externally in the ratio 2 : 5

2. Point D divides the line segment PR internally in the ratio 4 : 3, and O is any fixed point. Express $\overrightarrow{OD}$ in terms of $\overrightarrow{OP}$ and $\overrightarrow{OR}$ as follows.
   a) using either method of Example 1
   b) using the point of division formula

3. In each of the following, find the vector $\overrightarrow{OD}$ such that point D divides the line segment PR internally in the indicated ratio, where O is any fixed point.
   a) 3 : 5 b) 4 : 1 c) 7 : 2 d) 6 : 11 e) 1 : 1

4. Point D divides the line segment PR *externally* in the ratio 4 : 3, and O is any fixed point. Express $\overrightarrow{OD}$ in terms of $\overrightarrow{OP}$ and $\overrightarrow{OR}$ as follows.
   a) using the method of Example 2
   b) using the point of division formula

5. In each of the following, find the vector $\overrightarrow{OD}$ such that point D divides the line segment PR externally in the indicated ratio, where O is any fixed point.
   a) 3 : 5 b) 4 : 1 c) 7 : 2 d) 6 : 11 e) 1 : 2

6. The point A divides the segment PQ internally in the ratio 2 : 3. The point B divides AQ externally in the ratio 3 : 4. If O is any fixed point, then express $\overrightarrow{OB}$ in terms of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.

7. Point D divides segment PR in the ratio $m : k$. Describe the position of the point D with respect to points P and R in each of the following cases.
   a) $m \geqslant 0, k \geqslant 0, m \geqslant k$
   b) $m \geqslant 0, k \leqslant 0, |m| \geqslant |k|$
   c) $k = 0$

8. Find the coordinates of the point D that divides the line segment joining $P(7,8)$ and $R(-4,5)$ internally in the ratio 3 : 7.

9. Repeat question 8 for the following ratios.
   a) 2 : 5, internally
   b) 7 : 6, internally
   c) 1 : 1
   d) 11 : 3, externally
   e) 3 : 11, externally
   f) 1 : 2, externally

10. Find the coordinates of the point D that divides the line segment joining $P(3,2,1)$ and $R(5,6,3)$ internally in the ratio 5 : 7.

11. Repeat question 10 for the following ratios.
    a) 5 : 2, internally
    b) 9 : 7, internally
    c) 1 : 1
    d) 3 : 13, externally
    e) 3 : 2, externally
    f) 2 : 1, externally

12. Point Q lies on the line PR. O is any point such that $\overrightarrow{OQ} = s\overrightarrow{OP} + \frac{5}{11}\overrightarrow{OR}$.
    a) Find the value of s.
    b) Into what ratio does the point Q divide the segment PR?

13. The point Q divides the segment PR internally in the ratio 2 : 1. The point A divides the segment PQ externally in the ratio 6 : 5. The point T divides PA internally in the ratio 2 : 3. If O is any point, then express $\overrightarrow{OT}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OR}$.

14. Given the triangle ABC with vertices $A(3,8)$, $B(-1,-6)$, and $C(7,4)$. D, E and F are the midpoints of sides BC, AC, and AB respectively.
    a) Express $\overrightarrow{OD}$, $\overrightarrow{OE}$, and $\overrightarrow{OF}$ as ordered pairs.
    b) Find the coordinates of the point K dividing the median AD internally in the ratio 2 : 1.
    c) Find the coordinates of the point M dividing the median BE internally in the ratio 2 : 1.
    d) Find the coordinates of the point N dividing the median CF internally in the ratio 2 : 1.
    e) Use your results of b), c), and d) to draw conclusions about the intersection of the medians AD, BE, and CF.

In Search of a Solution for a System with Three Variables: Matrices

In an *In Search of* on page 73 you learned the elimination method of solving a linear system. Here you will learn to solve the same system using equivalent matrices.

Problem

Solve the linear system

$$\begin{aligned} 3x - y + 2z &= -1 \quad ① \\ 5x + 3y + z &= 17 \quad ② \\ x - 2y - 3z &= 11 \quad ③ \end{aligned}$$

Solution

The method of matrices reduces the amount of writing that you must do by concentrating only on the coefficients in the three equations. Since the values of the variables do not change under the operations used in the method of elimination, only the coefficients are recorded in an array called a *matrix*. The position of each coefficient in the matrix corresponds to its position in the linear system.

Thus the above linear system is written in matrix form as

$$\begin{bmatrix} 3 & -1 & 2 & -1 \\ 5 & 3 & 1 & 17 \\ 1 & -2 & -3 & 11 \end{bmatrix}$$

This matrix is called the *augmented matrix* of the system. The name indicates that it *includes* the coefficient matrix $\begin{bmatrix} 3 & -1 & 2 \\ 5 & 3 & 1 \\ 1 & -2 & -3 \end{bmatrix}$

A matrix can be replaced by an equivalent matrix with zeros in certain positions by the multiplication of rows by numbers, to make elements equal, and then adding or subtracting rows.

First get 0's in the first position in row ② and in row ③.

$$\begin{matrix} \\ 5 \times \text{row ①} - 3 \times \text{row ②} \\ \text{row ①} - 3 \times \text{row ③} \end{matrix} \begin{bmatrix} 3 & -1 & 2 & -1 \\ 0 & -14 & 7 & -56 \\ 0 & 5 & 11 & -34 \end{bmatrix}$$

Now get a 0 in the second position of row ③.

$$\begin{matrix} \\ \\ 5 \times \text{row ②} + 14 \times \text{row ③} \end{matrix} \begin{bmatrix} 3 & -1 & 2 & -1 \\ 0 & -14 & 7 & -56 \\ 0 & 0 & 189 & -756 \end{bmatrix}$$

From row ③: $189z = -756$ thus $z = -4$.
From row ②: $-14y + 7z = -56$; using $z = -4$, $y = 2$.
From row ①: $3x - y + 2z = -1$; using $z = -4$ and $y = 2$, $x = 3$.

Therefore, the solution is $(x,y,z) = (3,2,-4)$.

Note 1 These matrices are known as *equivalent matrices* because the solution of each is the same as the solution of the original system.

2 The final matrix

$$\begin{bmatrix} 3 & -1 & 2 & -1 \\ 0 & -14 & 7 & -56 \\ 0 & 0 & 189 & -756 \end{bmatrix}$$

is called the *reduced matrix* for the system; that is, when a matrix corresponding to a linear system of three equations in three variables has the form

$$\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{bmatrix}$$

with the triangle of zeros on the left, the matrix is in *row-reduced form*. The values of a, b, c, d, e, f, g, h, and i are not unique because the row reduced form can be reached by different row operations. Nevertheless the solutions are the same.

Two special cases can occur.

Case 1 The last row is
$0\ 0\ 0\ i$
where $i = 0$.
Then $0z = 0$ has an *infinity of solutions*. Hence an infinite number of (x,y,z) exist solving the system.

Here is an example of such a matrix in reduced form.

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Case 2 The last row is
$0\ 0\ 0\ i$
where $i \neq 0$.
Then $0z = i$ has *no solution*. Hence no (x,y,z) exists solving the system.

Here is an example of such a matrix in reduced form.

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Activities

Solve the following linear systems using matrices.

a) $x + 2y - z = 2$
$2x - 3y + z = -1$
$4x + y + 2z = 12$

b) $2x - 3y + 4z = -8$
$3x + 4y + 2z = 13$
$5x + 2y - 3z = 25$

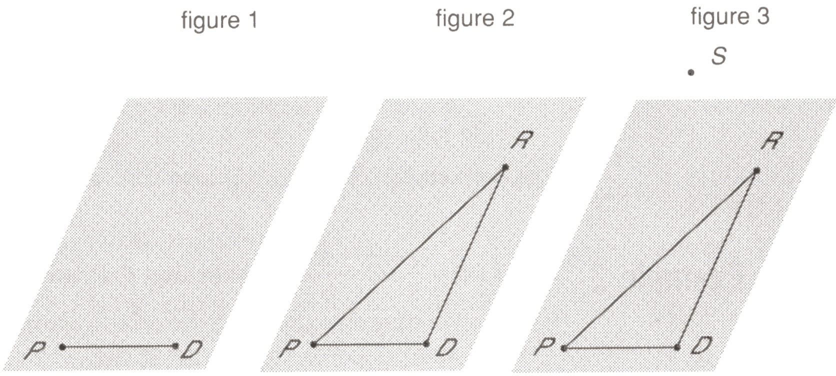
c) $2x + 3y - z = 12$
$3x - 2y + 3z = 1$
$x + 8y - 5z = 23$

d) $2x + 3y - z = 12$
$3x - 2y + 3z = 1$
$x + 8y - 5z = 1$

2.5 Collinear Points and Coplanar Points

In 3-space

1. two distinct points P and D are always collinear, (figure 1)
2. three distinct points P, D, and R are usually not collinear but will always lie in the same plane, (figure 2)
3. four distinct points P, D, R, and S will usually not be coplanar (figure 3).



In this section you will make use of linear dependence of vectors to determine whether or not three points are collinear and whether or not four points are coplanar.

Collinear Points

In the diagram, points P, D, and R lie in the same straight line. Thus, vectors $\overrightarrow{PD}$ and $\overrightarrow{DR}$ are collinear. Hence, $\overrightarrow{PD}$ and $\overrightarrow{DR}$ are linearly dependent.

Similarily, $\overrightarrow{PD}$ and $\overrightarrow{PR}$ are linearly dependent, and also $\overrightarrow{DR}$ and $\overrightarrow{PR}$ are linearly dependent.

Intuitively you should understand that line segment PD parallel to line segment DR with common point D implies that points P, D, and R are collinear points. This leads to the following property.

PROPERTY

If any two of the vectors, $\overrightarrow{PD}$, $\overrightarrow{PR}$, and $\overrightarrow{DR}$ are linearly dependent, then points P, D, and R are collinear.

Example 1 Prove that the points $P(3,2,-1)$, $D(5,4,1)$, and $R(-3,-4,-7)$ are collinear.

Solution $\overrightarrow{PD} = \overrightarrow{OD} - \overrightarrow{OP} = \overrightarrow{(5,4,1)} - \overrightarrow{(3,2,-1)} = \overrightarrow{(2,2,2)}$.

$\overrightarrow{DR} = \overrightarrow{OR} - \overrightarrow{OD} = \overrightarrow{(-3,-4,-7)} - \overrightarrow{(5,4,1)} = \overrightarrow{(-8,-8,-8)}$.

But $\overrightarrow{DR} = -4\overrightarrow{(2,2,2)} = -4\overrightarrow{PD}$.

Therefore, $\overrightarrow{PD}$ and $\overrightarrow{DR}$ are linearly dependent.

Thus, points P, D, and R are collinear. ■

In section 2.4 you learned another fact about three collinear points P, D, and R. If the point D divides the segment PR in the ratio $m:k$ then

$$\overrightarrow{OD} = \frac{k}{m+k}\overrightarrow{OP} + \frac{m}{m+k}\overrightarrow{OR}$$

Note: The sum of the coefficients of $\overrightarrow{OP}$ and $\overrightarrow{OR}$, that is,

$$\frac{k}{m+k} + \frac{m}{m+k} = \frac{m+k}{m+k} = 1$$

The converse of this result is also true, as the following example shows.

Example 2 a) If $\overrightarrow{OD} = \frac{2}{5}\overrightarrow{OP} + \frac{3}{5}\overrightarrow{OR}$, prove that P, D, and R are collinear points.

b) Draw a diagram showing the relationship among the points P, D, and R.

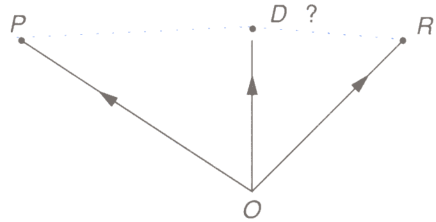
Solution a) You will need to show that two of the vectors represented by the directed line segments $\overrightarrow{PD}$, $\overrightarrow{DR}$, or $\overrightarrow{PR}$ are linearly dependent.

Now $\overrightarrow{PD} = \overrightarrow{OD} - \overrightarrow{OP}$

$$= \frac{2}{5}\overrightarrow{OP} + \frac{3}{5}\overrightarrow{OR} - \overrightarrow{OP}$$

$$= \frac{3}{5}\overrightarrow{OR} - \frac{3}{5}\overrightarrow{OP}$$

$$= \frac{3}{5}(\overrightarrow{OR} - \overrightarrow{OP})$$



Thus, $\overrightarrow{PD} = \frac{3}{5}\overrightarrow{PR}$

Therefore, points P, D, and R are collinear.

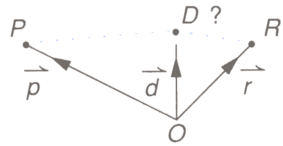
b) Since $\overrightarrow{PD} = \frac{3}{5}\overrightarrow{PR}$, the point D is positioned $\frac{3}{5}$ the distance from P to R as shown in the diagram.

■

The following example proves the result of Example 2 is true for any three collinear points.

Example 3 Given that points P, D, and R have position vectors, with respect to the origin O, of $\vec{p}$, $\vec{d}$, and $\vec{r}$ respectively, such that $\vec{d} = s\vec{p} + t\vec{r}$ where $s + t = 1$, prove that P, D, and R are collinear points.

Solution You will need to show that two of the directed line segments $\overrightarrow{PD}$, $\overrightarrow{DR}$, or $\overrightarrow{PR}$ are linearly dependent.



Now $\overrightarrow{PD} = \vec{d} - \vec{p}$

$= s\vec{p} + t\vec{r} - \vec{p}$

$= t\vec{r} + (s - 1)\vec{p}$ since $s + t = 1$, $s - 1 = -t$

$= t\vec{r} - t\vec{p}$

$= t(\vec{r} - \vec{p})$

Thus, $\overrightarrow{PD} = t\overrightarrow{PR}$

Therefore, points P, D, and R are collinear. ■

PROPERTY The points P, D, and R are collinear if scalars s and t exist such that $\overrightarrow{OD} = s\overrightarrow{OP} + t\overrightarrow{OR}$, where $s + t = 1$.

This property can be described as the *condition for three vectors with tails at a fixed point O to have their tips in a line.*

Example 4
a) A, B, and C are collinear points such that $\overrightarrow{OB} = p\overrightarrow{OA} + (3 - 2p)\overrightarrow{OC}$. Find the value of p.
b) State the ratio into which B divides AC.

Solution
a) Since A, B, and C are collinear points and $\overrightarrow{OB} = p\overrightarrow{OA} + (3 - 2p)\overrightarrow{OC}$ then $p + (3 - 2p) = 1$
Therefore, $p = 2$.
b) Thus $\overrightarrow{OB} = 2\overrightarrow{OA} - 1\overrightarrow{OC}$
Hence B divides AC externally in the ratio $1 : 2$ ■

Coplanar Points

Suppose you have four points P, Q, R, and S such that $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ are linearly dependent. What can you say about the four points? Since $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ are linearly dependent, these three vectors must be coplanar. Hence the points P, Q, R, and S must lie in the same plane. Because the vectors $\overrightarrow{QR}$, $\overrightarrow{QS}$, and $\overrightarrow{RS}$ will also be in this plane, you can use the following property to prove that four points are coplanar.

PROPERTY For the four points P, Q, R, S to be coplanar, three vectors (chosen with a common origin) must be linearly dependent.

Example 5 Prove that the four points $P(1,0,2)$, $D(3,2,0)$, $R(4,1,2)$, and $S(1,4,-4)$ are coplanar.

Solution You must select three of the vectors and show that they are linearly dependent. Suppose you choose the vectors $\overrightarrow{PD}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$, where

$$\overrightarrow{PD} = \overrightarrow{OD} - \overrightarrow{OP} = \overrightarrow{(2,2,-2)}$$
$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \overrightarrow{(3,1,0)}$$
$$\overrightarrow{PS} = \overrightarrow{OS} - \overrightarrow{OP} = \overrightarrow{(0,4,-6)}.$$

You need to find m, k, and p, not all 0, such that

$$m\overrightarrow{PD} + k\overrightarrow{PR} + p\overrightarrow{PS} = \vec{0}$$
$$m\overrightarrow{(2,2,-2)} + k\overrightarrow{(3,1,0)} + p\overrightarrow{(0,4,-6)} = \overrightarrow{(0,0,0)}$$
$$\overrightarrow{(2m,2m,-2m)} + \overrightarrow{(3k,k,0)} + \overrightarrow{(0,4p,-6p)} = \overrightarrow{(0,0,0)}$$
$$\overrightarrow{(2m+3k,2m+k+4p,-2m-6p)} = \overrightarrow{(0,0,0)}$$

Thus
$$\begin{aligned} 2m + 3k \phantom{{}+4p} &= 0 \quad ① \\ 2m + k + 4p &= 0 \quad ② \\ -2m \phantom{{}+k} - 6p &= 0 \quad ③ \end{aligned}$$

You can solve this system by the method of elimination or by the use of matrices as described on page 73 and page 95 respectively. Matrices will be used here.

The augmented matrix for this system of three equations is

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 2 & 1 & 4 & 0 \\ -2 & 0 & -6 & 0 \end{bmatrix}$$

row ① − row ②
row ① + row ③
$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 3 & -6 & 0 \end{bmatrix}$$

3 × row ② − 2 × row ③
$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From row ③: $0p = 0$
Hence, p can be any real number.
From row ②: $2k - 4p = 0$, or
$$k = 2p$$
From row ①: $2m + 3k = 0$, or
$$2m + 3(2p) = 0, \text{ or}$$
$$m = -3p$$

Thus, m, k, and p exist, for example, $p = 1$, $m = -3$, and $k = 2$ such that $m\overrightarrow{PD} + k\overrightarrow{PR} + p\overrightarrow{PS} = \vec{0}$. Thus, vectors $\overrightarrow{PD}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ are linearly dependent.

Hence, points P, D, R, and S are coplanar. ■

2.5 Exercises

1. a) State a vector condition for three points P, Q, and R to be collinear.
   b) State a vector condition for three points A, B, and C to be collinear.

2. Prove that the points $A(2,3)$, $B(-6,5)$, and $C(6,2)$ are collinear.

3. Prove that the points $A(2,3)$, $B(-6,5)$, and $C(-4,6)$ are not collinear.

4. Prove that the points $P(8,2,-4)$, $Q(7,0,-7)$, and $R(10,6,2)$ are collinear.

5. Prove that the points $P(8,2,-4)$, $Q(7,0,-7)$, and $R(10,6,0)$ are not collinear.

6. In each of the following decide whether or not the three points are collinear.
   a) $P(2,0,3)$, $Q(4,1,6)$, $R(6,2,9)$
   b) $X(4,5,6)$, $Y(12,1,-2)$, $Z(0,14,20)$
   c) $A(8,6)$, $B(-1,-7)$, $C(0,-25)$
   d) $D(7,11)$, $E(-3,8)$, $F(-23,2)$

7. For each of the following explain why the points P, D, and R are collinear, where $\overrightarrow{OP} = \vec{p}$, $\overrightarrow{OD} = \vec{d}$, and $\overrightarrow{OR} = \vec{r}$. In each case state the ratio into which point D divides segment PR. O is any point.
   a) $\vec{d} = \frac{1}{3}\vec{p} + \frac{2}{3}\vec{r}$
   b) $\vec{d} = \frac{4}{7}\vec{p} + \frac{3}{7}\vec{r}$
   c) $\vec{d} = -\frac{1}{3}\vec{p} + \frac{4}{3}\vec{r}$
   d) $\vec{d} = 8\vec{p} - 7\vec{r}$

8. Points P, D, and R are collinear and O is any point such that $\overrightarrow{OD} = \frac{2}{3}\overrightarrow{OP} + k\overrightarrow{OR}$. Find the value of k.

9. In each of the following, points P, D, and R are collinear. The position vectors of P, D, and R are $\vec{p}$, $\vec{d}$, and $\vec{r}$ respectively. Find the values of the scalars.
   a) $\vec{d} = m\vec{p} - 4\vec{r}$
   b) $\vec{d} = \frac{7}{9}\vec{p} + n\vec{r}$
   c) $\vec{d} = \frac{5}{3}\vec{p} + s\vec{r}$

10. a) State a vector condition for four points P, D, R, and S to be coplanar.
    b) State a vector condition for four points A, B, C, and D to be coplanar.

11. Prove that the points $P(4,0,3)$, $D(6,3,2)$, $R(3,2,7)$, and $S(5,12,13)$ are coplanar.

12. Prove that the points $P(4,0,3)$, $D(6,3,2)$, $R(3,2,7)$, and $S(5,7,14)$ are not coplanar.

13. In each of the following decide whether or not the four points are coplanar.
    a) $P(2,0,3)$, $Q(4,1,-6)$, $R(14,3,-3)$, $S(-16,-3,-12)$
    b) $W(3,1,2)$, $X(3,2,-1)$, $Y(0,6,4)$, $Z(-3,12,3)$
    c) $A(5,1,3)$, $B(4,3,0)$, $C(7,1,8)$, $D(5,2,6)$
    d) $K(1,6,3)$, $L(-2,-4,-1)$, $M(3,9,4)$, $N(-3,0,1)$

14. A, B, and C are points such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$. Scalars m, k, and p exist where $m \neq 0$ such that $m + k + p = 0$ and $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$. Prove that A, B, and C are collinear points.

15. O, A, B, C, and Z are five points in 3-space such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$, and $\overrightarrow{OZ} = -5\vec{a} + 2\vec{b} + 3\vec{c}$.
    a) Express $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AZ}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$.
    b) Prove that points A, B, C, and Z are coplanar.

16. You are given points O, P, D, R, and A such that $\overrightarrow{OP} = 2\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OR}$ and $\overrightarrow{OA} = k\overrightarrow{OD} + m\overrightarrow{OR}$. If points P, D, and R are collinear, then prove that $4k + 4m = 1$.

17. You are given points O, P, D, and R such that $m\overrightarrow{OD} - k\overrightarrow{OP} + (k - m)\,\overrightarrow{OR} = \vec{0}$. Prove that the points P, D, and R are collinear.

18. A, B, C, D, and E are five points in 3-space such that $\overrightarrow{AD} = \overrightarrow{AB} + \frac{2}{5}(\overrightarrow{EC} - \overrightarrow{EB})$. Prove that the three points B, C, and D are collinear.

2.6 Geometric Proofs Using Linear Independence of Vectors

In your study of geometry you have proved geometric facts using the theorems of Euclidean geometry such as those on congruent triangles, angles in triangles, parallel lines and so on. In this section you will use the linear independence of vectors to prove some geometric facts involved with the division of segments internally or externally. The property of linear independence that you will use is the following.

If $\vec{u}$ and $\vec{v}$ are linearly independent, and $m\vec{u} + k\vec{v} = \vec{0}$, then $m = k = 0$.

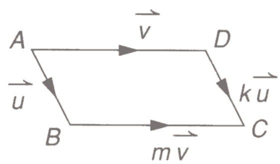
In chapter 5 you will have the opportunity of doing the problems of this section by using vector equations of lines.

Example 1

Prove that the opposite sides of a parallelogram are congruent.

Solution

Given: $AB \| DC$ and $AD \| BC$
Prove: $AB \cong DC$ and $AD \cong BC$



You must first translate the 'Given' and 'Prove' into vector information.

If $\overrightarrow{AB} = \vec{u}$, and $\overrightarrow{AD} = \vec{v}$, then the 'Given: $AB \| DC$ and $AD \| BC$' implies that k, m exist such that $\overrightarrow{DC} = k\vec{u}$, and $\overrightarrow{BC} = m\vec{v}$.

The 'Prove: $AB \cong DC$ and $AD \cong BC$' implies that you must show that $k = m = 1$.

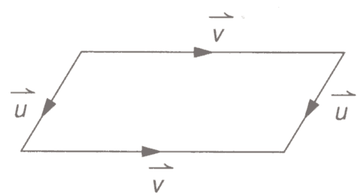
Starting at point A and moving around the parallelogram gives

$$\vec{u} + m\vec{v} - k\vec{u} - \vec{v} = \vec{0}$$

or $(1 - k)\,\vec{u} + (m - 1)\,\vec{v} = \vec{0}$

But $\vec{u}$ and $\vec{v}$ are not parallel, and so are linearly independent.
Thus, $1 - k = 0$, and $m - 1 = 0$.
Hence, $k = m = 1$, as required. ■

Example 1 implies that you may use the following or equivalent figure for problems involving a parallelogram.



2.5 Exercises

1. a) State a vector condition for three points P, Q, and R to be collinear.
   b) State a vector condition for three points A, B, and C to be collinear.

2. Prove that the points $A(2,3)$, $B(-6,5)$, and $C(6,2)$ are collinear.

3. Prove that the points $A(2,3)$, $B(-6,5)$, and $C(-4,6)$ are not collinear.

4. Prove that the points $P(8,2,-4)$, $Q(7,0,-7)$, and $R(10,6,2)$ are collinear.

5. Prove that the points $P(8,2,-4)$, $Q(7,0,-7)$, and $R(10,6,0)$ are not collinear.

6. In each of the following decide whether or not the three points are collinear.
   a) $P(2,0,3)$, $Q(4,1,6)$, $R(6,2,9)$
   b) $X(4,5,6)$, $Y(12,1,-2)$, $Z(0,14,20)$
   c) $A(8,6)$, $B(-1,-7)$, $C(0,-25)$
   d) $D(7,11)$, $E(-3,8)$, $F(-23,2)$

7. For each of the following explain why the points P, D, and R are collinear, where $\overrightarrow{OP} = \vec{p}$, $\overrightarrow{OD} = \vec{d}$, and $\overrightarrow{OR} = \vec{r}$. In each case state the ratio into which point D divides segment PR. O is any point.
   a) $\vec{d} = \frac{1}{3}\vec{p} + \frac{2}{3}\vec{r}$
   b) $\vec{d} = \frac{4}{7}\vec{p} + \frac{3}{7}\vec{r}$
   c) $\vec{d} = -\frac{1}{3}\vec{p} + \frac{4}{3}\vec{r}$
   d) $\vec{d} = 8\vec{p} - 7\vec{r}$

8. Points P, D, and R are collinear and O is any point such that $\overrightarrow{OD} = \frac{2}{3}\overrightarrow{OP} + k\overrightarrow{OR}$. Find the value of k.

9. In each of the following, points P, D, and R are collinear. The position vectors of P, D, and R are $\vec{p}$, $\vec{d}$, and $\vec{r}$ respectively. Find the values of the scalars.
   a) $\vec{d} = m\vec{p} - 4\vec{r}$
   b) $\vec{d} = \frac{7}{9}\vec{p} + n\vec{r}$
   c) $\vec{d} = \frac{5}{3}\vec{p} + s\vec{r}$

10. a) State a vector condition for four points P, D, R, and S to be coplanar.
    b) State a vector condition for four points A, B, C, and D to be coplanar.

11. Prove that the points $P(4,0,3)$, $D(6,3,2)$, $R(3,2,7)$, and $S(5,12,13)$ are coplanar.

12. Prove that the points $P(4,0,3)$, $D(6,3,2)$, $R(3,2,7)$, and $S(5,7,14)$ are not coplanar.

13. In each of the following decide whether or not the four points are coplanar.
    a) $P(2,0,3)$, $Q(4,1,-6)$, $R(14,3,-3)$, $S(-16,-3,-12)$
    b) $W(3,1,2)$, $X(3,2,-1)$, $Y(0,6,4)$, $Z(-3,12,3)$
    c) $A(5,1,3)$, $B(4,3,0)$, $C(7,1,8)$, $D(5,2,6)$
    d) $K(1,6,3)$, $L(-2,-4,-1)$, $M(3,9,4)$, $N(-3,0,1)$

14. A, B, and C are points such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$. Scalars m, k, and p exist where $m \neq 0$ such that $m + k + p = 0$ and $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$. Prove that A, B, and C are collinear points.

15. O, A, B, C, and Z are five points in 3-space such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$, and $\overrightarrow{OZ} = -5\vec{a} + 2\vec{b} + 3\vec{c}$.
    a) Express $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AZ}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$.
    b) Prove that points A, B, C, and Z are coplanar.

16. You are given points O, P, D, R, and A such that $\overrightarrow{OP} = 2\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OR}$ and $\overrightarrow{OA} = k\overrightarrow{OD} + m\overrightarrow{OR}$. If points P, D, and R are collinear, then prove that $4k + 4m = 1$.

17. You are given points O, P, D, and R such that $m\overrightarrow{OD} - k\overrightarrow{OP} + (k - m)\overrightarrow{OR} = \vec{0}$. Prove that the points P, D, and R are collinear.

18. A, B, C, D, and E are five points in 3-space such that $\overrightarrow{AD} = \overrightarrow{AB} + \frac{2}{5}(\overrightarrow{EC} - \overrightarrow{EB})$. Prove that the three points B, C, and D are collinear.

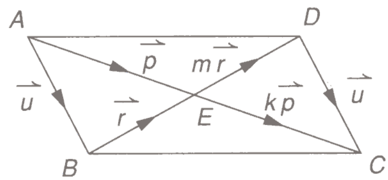
Example 2 Prove that the diagonals of a parallelogram bisect each other.

Solution Given: $ABCD$ is a parallelogram with diagonals intersecting at E.
Prove: $AE \cong EC$, and $BE \cong ED$

Proof: As in the diagram, let $\overrightarrow{AE} = \vec{p}$, and $BE = \vec{r}$, then $\overrightarrow{EC} = k\vec{p}$, and $\overrightarrow{ED} = m\vec{r}$.

(These statements are not needed if the facts are clearly shown on the diagram.)

You must show that $m = k = 1$.
In $\triangle AEB$, $\vec{u} = \vec{p} - \vec{r}$
In $\triangle CED$, $\vec{u} = -m\vec{r} + k\vec{p}$
Thus, $\vec{p} - \vec{r} = -m\vec{r} + k\vec{p}$
or $(1 - k)\,\vec{p} + (-1 + m)\,\vec{r} = \vec{0}$
Since $\vec{p}$ and $\vec{r}$ are not parallel
$1 - k = 0$, and $-1 + m = 0$
Thus, $k = m = 1$, as required. ■



Example 1 and Example 2 indicate a method of solving problems involving parallel or collinear segments where the ratio between the lengths of some segments is required to be found.

Step 1 Express parallel or collinear segments as $\vec{u}$ and $k\vec{u}$, $\vec{v}$ and $m\vec{v}$, etc. (If the ratio of the lengths of some segments is known, then use the terms of the ratio as scalar multipliers.)

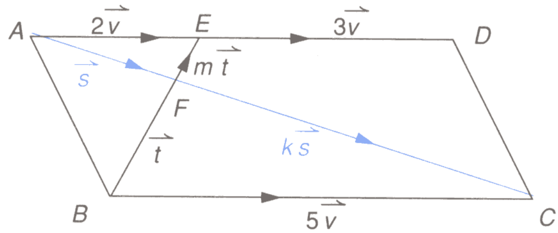
Step 2 a) Write a vector equation involving at least *two* linearly independent vectors and the unknown scalars of step 1.
b) If three linearly independent vectors appear in a), write one more equation involving the three vectors.
c) If four linearly independent vectors appear in a), write two more equations involving the four vectors.

Step 3 Use the vector equations from step 2 to eliminate all but two linearly independent vectors, say $\vec{x}$ and $\vec{y}$.

Step 4 Solve from the equation in step 3 for the required scalars k and m. First, write the equation in the form $A\vec{x} + B\vec{y} = \vec{0}$, then use the fact that $A = B = 0$.

Example 3 In parallelogram $ABCD$, E divides AD in the ratio $2:3$. BE and AC intersect at F. Find the ratio into which F divides AC.

Solution



Step 1

Let $\overrightarrow{AE} = 2\vec{v}$, and $\overrightarrow{ED} = 3\vec{v}$
$\overrightarrow{AF} = \vec{s}$ and $\overrightarrow{FC} = k\vec{s}$
$\overrightarrow{BF} = \vec{t}$ and $\overrightarrow{FE} = m\vec{t}$.
You must solve for the scalars k and m.

Step 2

In $\triangle AFE$: $2\vec{v} = \vec{s} + m\vec{t}$ ①
In $\triangle BFC$: $5\vec{v} = \vec{t} + k\vec{s}$ ②

Step 3

Eliminate vector $\vec{v}$ from ① and ②.
From ①: $10\vec{v} = 5\vec{s} + 5m\vec{t}$
From ②: $10\vec{v} = 2\vec{t} + 2k\vec{s}$
Thus $5\vec{s} + 5m\vec{t} = 2\vec{t} + 2k\vec{s}$
or $(5 - 2k)\vec{s} + (5m - 2)\vec{t} = \vec{0}$

Step 4

Since $\vec{s} \nparallel \vec{t}$
$5 - 2k = 0$ and $5m - 2 = 0$

Hence $k = \frac{5}{2}$, and $m = \frac{2}{5}$.

Using $k = \frac{5}{2}$ gives $\overrightarrow{FC} = \frac{5}{2}\vec{s}$

so $AF:FC = |\vec{s}|:\frac{5}{2}|\vec{s}| = 1:\frac{5}{2}$ or $2:5$

Thus, F divides segment AC in the ratio $2:5$. ■

2.6 Exercises

Use vector methods to solve the following problems.

1. $OBCD$ is a parallelogram. E is the midpoint of side OD. Segments OC and BE intersect at point F. Find the ratio into which OC divides BE.

2. $OBCD$ is a parallelogram. E is the point that divides side OD in the ratio $2:5$. Segments OC and BE intersect at point F. Find the ratio into which OC divides BE.

3. a) In $\triangle OAB$, medians AD and BE intersect at point G. Find the ratios into which G divides AD and BE.
   b) Show the medians of a triangle trisect each other.

4. In $\triangle OBC$, E is the midpoint of side OB. Point F is on side OC such that segment EF is parallel to side BC. Into what ratio does F divide side OC?

5. In $\triangle OBC$, E is the point that divides side OB into the ratio $1:2$. Point F is on side OC such that segment EF is parallel to side BC. Into what ratio does F divide side OC?

6. In $\triangle OBC$, E is the point that divides side OB into the ratio $1:k$, $k \neq 0$. Point F is on side OC such that segment EF is parallel to side BC. Into what ratio does F divide side OC?

7. In $\triangle ABC$, D divides AB in the ratio $1:2$ and E divides AC in the ratio $1:4$. BE and CD intersect at point F. Find the ratios into which F divides each of BE and CD.

8. In parallelogram $PQRS$, A divides PQ in the ratio $2:5$, and B divides SR in the ratio $3:2$. Segments PR and AB intersect at C. Find the ratio into which C divides segment PR.

9. $PQRS$-is a trapezoid with PQ parallel to SR. PR and QS intersect at point A. If A divides segment QS in the ratio $2:3$, then find the ratio into which A divides PR.

10. $ABCD$ is a parallelogram. E is the point that divides side AD in the ratio $1:k$, where $k \geqslant 0$. Segments AC and BE intersect at point F. Find the ratio into which point F divides AC.

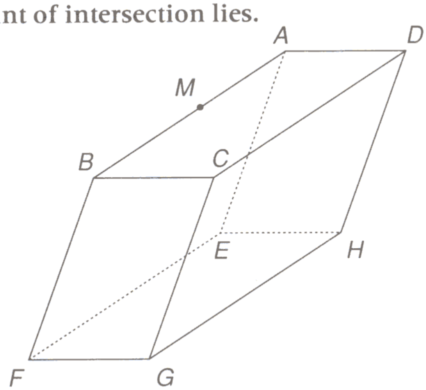
11. Let M be the midpoint of median AD of $\triangle ABC$. BM extended and AC intersect at K. Find the ratio into which K divides AC.

12. $ABCD$ is a trapezoid in which AD is parallel to BC. P and Y divide AB and DC respectively in the same ratio. Q is the point on diagonal AC such that PQ is parallel to BC. Prove that points P, Q, and Y are collinear.

13. In a tetrahedron, prove that the line segments joining a vertex to the centroid of the opposite face intersect at a point that divides the line segments in the ratio $1:3$. (The centroid of a triangle is the point of intersection of the medians. See also question 3.)

14. Show that the point found in question 13 is the same as the point of intersection of the line segments joining the midpoints of opposite edges of a tetrahedron.

15. The box shown, called a parallelepiped, is made up of three pairs of congruent parallelograms. Prove that the diagonals BH and EC intersect, and determine where the point of intersection lies.



16. In the box shown, let M be the midpoint of AB. Prove that MG and FD do not intersect.

Summary

- Two vectors $\vec{a}$ and $\vec{b}$ are *collinear or parallel* if they can be represented by parallel directed line segments. The relationship is written $\vec{a} \| \vec{b}$.
- *Two* non-zero vectors $\vec{a}$ and $\vec{b}$ are *linearly dependent* if and only if $\vec{a} \| \vec{b}$.
- The zero vector $\vec{0}$ is linearly dependent with every vector.
- *Two* vectors that are not linearly dependent are *linearly independent*.
- *Three* vectors are *coplanar* if they can be represented by directed line segments parallel to the same plane.
- *Three* vectors are *linearly dependent* if and only if they are coplanar.
- *Three* vectors that are not linearly dependent are *linearly independent*.

Linear Dependence of Two or Three Vectors

	two vectors: $\vec{a}, \vec{b}$	three vectors: $\vec{a}, \vec{b}, \vec{c}$
geometric condition	$\vec{a} \| \vec{b}$	$\vec{a}, \vec{b}, \vec{c}$ are coplanar
algebraic conditions	1. m, k exist, not both 0, such that $m\vec{a} + k\vec{b} = \vec{0}$, or 2. $\vec{b} = s\vec{a}$, for some $s \in \mathbb{R}$	1. m, k, p exist, not all 0, such that $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$, or 2. *at least one* is a linear combination of the other two; for example $\vec{c} = s\vec{a} + t\vec{b} \quad s, t \in \mathbb{R}$

Linear Independence of Two or Three Vectors

geometric condition	$\vec{a} \nparallel \vec{b}$	$\vec{a}, \vec{b}, \vec{c}$ are not coplanar
algebraic conditions	1. If $m\vec{a} + k\vec{b} = \vec{0}$ then $m = k = 0$, or 2. no s exists such that $\vec{b} = s\vec{a}, \quad s \in \mathbb{R}$	If $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$ then $m = k = p = 0$

Basis Vectors

In $\mathbb{V}_2$ (the set of 2-space vectors), if $\vec{a}$ and $\vec{b}$ are linearly independent, then any other vector in $\mathbb{V}_2$ can be expressed as a *linear combination* of $\vec{a}$ and $\vec{b}$.

The vectors $\vec{a}$ and $\vec{b}$ form a *basis for* $\mathbb{V}_2$. In particular, the unit vectors $\vec{i} = \overrightarrow{(1,0)}$ and $\vec{j} = \overrightarrow{(0,1)}$ form a basis in $\mathbb{V}_2$.

In $\mathbb{V}_3$ (the set of 3-space vectors), if $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly independent, then any other vector in $\mathbb{V}_3$ can be expressed as a *linear combination* of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ form a *basis for* $\mathbb{V}_3$.In particular, the unit vectors $\vec{i} = \overrightarrow{(1,0,0)}$, $\vec{j} = \overrightarrow{(0,1,0)}$, and $\vec{k} = \overrightarrow{(0,0,1)}$ form a basis in $\mathbb{V}_3$.

Collinear Points

1. If any two of the vectors, $\overrightarrow{PD}$, $\overrightarrow{PR}$, and $\overrightarrow{DR}$ are linearly dependent, then points P, D, and R are collinear.
2. If $\overrightarrow{OD} = s\overrightarrow{OP} + t\overrightarrow{OR}$, where $s + t = 1$, then P, D, and R are collinear points.
3. If point D divides the line segment PR in the ratio $m : k$, and O is any fixed point, then

$$\overrightarrow{OD} = \frac{k}{m+k}\overrightarrow{OP} + \frac{m}{m+k}\overrightarrow{OR}$$

For internal division m and k are positive. For external division the smaller of m and k is negative.

Coplanar Points

For the four points P, Q, R, S to be coplanar, three vectors (chosen with a common origin) must be linearly dependent.

Using Vectors in Euclidean Geometry

A method of solving problems involving parallel or collinear segments where the ratio between the lengths of some segments is required can be found on page 103.

Inventory

Answer the following by filling in the blanks.

1. $\vec{a}$ and $\vec{b}$ are linearly dependent. Thus, geometrically, $\vec{a}$ and $\vec{b}$ are ______, and a scalar k exists such that $\vec{b} =$ ______, and scalars s and t exist such that $s\vec{a} +$ ______ $= \vec{0}$.

2. $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent. Thus, geometrically, $\vec{a}$, $\vec{b}$, and $\vec{c}$ are ______, and at least one of $\vec{a}$, $\vec{b}$, and $\vec{c}$ can be written as a linear ______ of the other two; for example, $\vec{c} =$ ______. Also, scalars m, k, and p exist, not all ______, such that ______ $= \vec{0}$.

3. If $\vec{a}$ and $\vec{b}$ are not collinear then $\vec{a}$ and $\vec{b}$ are linearly ______.

4. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are not coplanar, then $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly ______.

5. a) If $\vec{a}$ and $\vec{b}$ are linearly independent and $m\vec{a} + k\vec{b} = \vec{0}$, then $k =$ ______ and $m =$ ______.
   b) If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly independent and $m\vec{a} + k\vec{b} + p\vec{c} = \vec{0}$, then $k =$ ______, $m =$ ______, and $p =$ ______.

6. a) Vectors $\overrightarrow{(3,5)}$ and ______ are linearly dependent.
   b) Vectors $\overrightarrow{(3,5)}$ and ______ are linearly independent.
   c) Vectors $\overrightarrow{(2,3,1)}$ and ______ are linearly dependent.
   d) Vectors $\overrightarrow{(2,3,1)}$ and ______ are linearly independent.

7. If $m\overrightarrow{(1,2)} + k\overrightarrow{(3,4)} = \vec{0}$, then $m + 3k =$ ______, and ______ = ______.

8. If $\vec{a} = k\vec{b}$, then $\vec{a}$ and $\vec{b}$ are ______.

9. If scalars a, b, and c exist, not all 0, such that $a\vec{x} + b\vec{y} + c\vec{z} = \vec{0}$, then $\vec{x}$, $\vec{y}$, and $\vec{z}$ are ______.

10. a) Point Q divides segment PR internally in the ratio 5 : 7. Thus, $\overrightarrow{OQ} =$ ______ $\overrightarrow{OP} +$ ______ $\overrightarrow{OR}$.
    b) Point Q divides segment PR externally in the ratio 5 : 7. Thus, $\overrightarrow{OQ} =$ ______ $\overrightarrow{OP} +$ ______ $\overrightarrow{OR}$.

11. a) $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly dependent. Thus, A, B, and C are ______.
    b) $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ are linearly dependent. Thus, P, Q, R, and S are ______.

12. In 2-space every vector is a linear combination of each pair of ______ vectors.

13. To form a basis in $\mathbb{V}_2$ you need ______ vectors that are linearly ______.

14. To form a basis in $\mathbb{V}_3$ you need ______ vectors that are linearly ______.

Review Exercises

1. $\vec{a}$ and $\vec{b}$ are linearly dependent vectors.
   a) What is the geometric relationship between $\vec{a}$ and $\vec{b}$?
   b) State two algebraic equations that are true relating $\vec{a}$ and $\vec{b}$.
   c) What conditions, if any, are imposed on the scalars in the equations in b)?

2. Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent.
   a) What is the geometric relationship among $\vec{a}$, $\vec{b}$, and $\vec{c}$?
   b) State two algebraic equations that are true relating $\vec{a}$, $\vec{b}$, and $\vec{c}$.
   c) What conditions, if any, are imposed on the scalars in the equations in b)?

3. $\vec{k}$ and $\vec{t}$ are linearly independent vectors and $a\vec{k} + b\vec{t} = \vec{0}$. What conclusion can you draw about the scalars a and b?

4. Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are not coplanar, and $s\vec{a} + t\vec{b} + r\vec{c} = \vec{0}$. What, if anything, is true about the scalars s, t, and r?

5. Given that $\vec{a} \parallel \vec{b}$, and $\vec{b} \nparallel \vec{c}$, and $\vec{d} = 4\vec{a}$, which of the following vectors are linearly dependent with $\vec{a}$?
   a) $2\vec{a}$ b) $-8\vec{b}$ c) $2\vec{c}$ d) $-7\vec{d}$

6. Vectors $\vec{p}$ and $\vec{q}$ are linearly independent.
   $\vec{x} = 3\vec{p}$, $\vec{y} = -2\vec{q}$, $\vec{z} = \frac{1}{2}\vec{q}$, $\vec{w} = 3\vec{y}$
   a) Which of $\vec{x}$, $\vec{y}$, $\vec{z}$, and $\vec{w}$ are not parallel?
   b) Which among the vectors $\vec{x}$, $\vec{p}$, $\vec{q}$, and $\vec{z}$ are linearly dependent with $\vec{w}$?

7. Vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$ are linearly independent. Which of $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ are linearly dependent with $\vec{x}$ and $\vec{y}$?
   $\vec{a} = 4\vec{x} + 5\vec{y}$, $\vec{b} = 4\vec{a} + 3\vec{x}$
   $\vec{c} = 3\vec{x} + 7\vec{z}$, $\vec{d} = 7\vec{a} + 2\vec{b}$

8. Vectors $\vec{a}$ and $\vec{b}$ are not collinear; $\vec{c}$ is not coplanar with $\vec{a}$ and $\vec{b}$. Which of the following vectors are coplanar with $\vec{a}$ and $\vec{b}$?
   $\vec{z} = 5\vec{a} + 3\vec{b}$, $\vec{y} = 2\vec{a} + 4\vec{c}$,
   $\vec{h} = \vec{a} + 2\vec{b}$, $\vec{w} = 3\vec{z} + 4\vec{h}$

9. Given $\vec{a} = \overrightarrow{(3,5)}$.
   a) Write three vectors linearly dependent with $\vec{a}$.
   b) Write one vector linearly independent with $\vec{a}$.

10. Given $\vec{b} = \overrightarrow{(7,1,-2)}$.
    a) Write three vectors collinear with $\vec{b}$.
    b) Write two vectors linearly independent with $\vec{b}$.

11. Vectors $\vec{a}$ and $\vec{b}$ are linearly independent. Vectors $\vec{c}$ and $\vec{d}$ are such that $\vec{c} = 2\vec{a} + 5\vec{b}$ and $\vec{d} = 3\vec{a} - 2\vec{c}$.
    a) Use a geometric argument to show that $\vec{c}$ and $\vec{d}$ lie in the plane of $\vec{a}$ and $\vec{b}$.
    b) Use an algebraic argument to show that $\vec{c}$ and $\vec{d}$ lie in the plane of $\vec{a}$ and $\vec{b}$.

12. Which of the following pairs of vectors are linearly dependent? Justify your answer.
    a) $\overrightarrow{(2,5)}$, $\overrightarrow{(5,2)}$
    b) $\overrightarrow{(-3,2)}$, $\overrightarrow{(1.5,-1)}$
    c) $\overrightarrow{(3,1,2)}$, $\overrightarrow{(9,3,6)}$
    d) $\overrightarrow{(4,2,1)}$, $\overrightarrow{(2,1,1)}$

13. Express the vector $\vec{v} = \overrightarrow{(11,-2)}$ as a linear combination of $\vec{a} = \overrightarrow{(1,-2)}$ and $\vec{b} = \overrightarrow{(2,1)}$.

14. Express the vector $\vec{v} = \overrightarrow{(-5,16,5)}$ as a linear combination of $\vec{a} = \overrightarrow{(1,2,3)}$, $\vec{b} = \overrightarrow{(4,0,1)}$, and $\vec{c} = \overrightarrow{(-1,4,0)}$.

15. Establish whether or not the vectors $\overrightarrow{(2,1,0)}$, $\overrightarrow{(3,1,1,)}$, and $\overrightarrow{(1,0,2)}$ are coplanar.

16. Scalars m and k exist, neither equal to 0, such that $m\vec{a} + k\vec{b} = \vec{0}$. Are $\vec{a}$ and $\vec{b}$ necessarily linearly dependent?

17. a) Given three points A, B, C such that $\overrightarrow{AB} = 6\overrightarrow{BC}$. Explain why you can say that points A, B, and C lie along the same line.

b) Given four points A, B, C, and D such that $\overrightarrow{AB} = 2\overrightarrow{AC} + 4\overrightarrow{BD}$. Explain why you can say the four points are coplanar.

18. a) Vectors $\vec{a}$ and $\vec{b}$ are basis vectors for $\mathbb{V}_2$. Explain what this means.

b) Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are basis vectors for $\mathbb{V}_3$. Explain what this means.

19. Determine whether or not the three vectors in each of the following are linearly dependent. In each case state the geometric significance of the result.

a) $\overrightarrow{(6,0,2)}$, $\overrightarrow{(-3,1,1)}$, and $\overrightarrow{(-1,1,2)}$

b) $\overrightarrow{(1,2,3)}$, $\overrightarrow{(6,11,4)}$, and $\overrightarrow{(0,1,14)}$

c) $\overrightarrow{(4,1,9)}$, $\overrightarrow{(-3,1,1)}$, and $\overrightarrow{(6,3,29)}$

20. Given the vectors $\vec{a} = \overrightarrow{(-2,3)}$ and $\vec{b} = \overrightarrow{(3,5)}$.

a) Prove the two vectors form a basis for $\mathbb{V}_2$.

b) Express the vector $\overrightarrow{(-2,22)}$ as a linear combination of $\vec{a}$ and $\vec{b}$.

21. Given the vectors $\vec{a} = \overrightarrow{(0,1,5)}$, $\vec{b} = \overrightarrow{(2,1,-4)}$ and $\vec{c} = \overrightarrow{(6,4,0)}$

a) Prove the three vectors form a basis for $\mathbb{V}_3$.

b) Express the vector $\overrightarrow{(11,9,-1)}$ as a linear combination of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

22. The vectors $\vec{a} = \overrightarrow{(-4,-1,0)}$, $\vec{b} = \overrightarrow{(-1,5,1)}$, and $\vec{c} = \overrightarrow{(5,17,k)}$ are linearly dependent. Find the value of k.

23. In each of the following, find the vector $\overrightarrow{OQ}$ such that point Q divides the line segment PR internally in the indicated ratio, where O is any fixed point.

a) 11:5

b) 5:7

c) 7:1

d) 6:11

24. In each of the following, find the vector $\overrightarrow{OT}$ such that point T divides the line segment AB externally in the indicated ratio, where O is any fixed point.

a) 1:5 b) 7:3 c) 4:9

25. The point A divides the segment PQ externally in the ratio 4:5. The point B divides AQ internally in the ratio 3:2. If O is any fixed point, then express $\overrightarrow{OB}$ in terms of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.

26. Find the coordinates of the points that divide the line segment joining points $P(2,5,8)$ and $R(-4,1,5)$ in the indicated ratios.

a) 3:1, internally

b) 4:7, externally

27. The point Q divides the line segment PR externally in the ratio 1:2. The point A divides the segment PQ internally in the ratio 4:3. The point T divides PA externally in the ratio 5:6. If O is any point, then express $\overrightarrow{OT}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OR}$.

28. a) State a vector condition for three points P, Q, and R to be collinear.

b) State a vector condition for four points A, B, C, and D to be coplanar.

29. In each of the following decide whether or not the three points are collinear.

a) $A(0,3,2)$, $B(1,5,4)$, $C(3,9,8)$

b) $P(4,1,6)$, $Q(-2,1,-5)$, $R(0,1,2)$

30. In each of the following decide whether or not the four points are coplanar.

a) $A(1,4,-5)$, $B(2,12,-8)$, $C(4,6,-4)$, $D(5,3,-2)$

b) $P(3,2,1)$, $Q(0,2,-1)$, $R(1,0,4)$, $S(0,-2,1)$

31. If O is any point, then explain why the points P, Q, and R are collinear where

$$\overrightarrow{OQ} = \frac{5}{7}\overrightarrow{OP} + \frac{2}{7}\overrightarrow{OR}.$$

State the ratio into which point Q divides segment PR.

32. Points P, Q, and R are collinear and O is any point such that $\overrightarrow{OQ} = 2m\overrightarrow{OP} + k\overrightarrow{OR}$, and $4m + 3k = 5$. Find the values of k and m.

33. $\vec{a}$ and $\vec{b}$ are linearly independent. O, A, B, and C are points such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, and $\overrightarrow{OC} = 6\vec{a} - 5\vec{b}$. Prove that points A, B, and C are collinear.

34. O, A, B, C, and Z are five points in 3-space such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$, and $\overrightarrow{OZ} = 4\vec{a} + 4\vec{b} - 7\vec{c}$.

a) Express $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AZ}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

b) Prove that points A, B, C, and Z are coplanar.

Use vector methods to solve problems 35–39.

35. $ABCD$ is a parallelogram. E is the point that divides side AD in the ratio $4:7$. Segments AC and BE intersect at point F. Find the ratio into which AC divides BE.

36. In a parallelogram $ABCD$, H is the midpoint of AD, and E divides BC in the ratio $3:2$. If BH and AE intersect at M, find the ratio $AM:ME$.

37. In $\triangle ABC$, E is the point that divides side AB into the ratio $3:2$. Point F is on side AC such that segment EF is parallel to side BC. Into what ratio does F divide side AC?

38. In $\triangle ABC$, D divides AB in the ratio $3:2$ and E divides AC in the ratio $5:4$. BE and CD intersect at point F. Find the ratios into which F divides each of BE and CD.

39. In parallelogram $PQRS$, A divides PQ in the ratio $2:1$, and B divides SR in the ratio $3:4$. Segments PR and AB intersect at C. Find the ratio into which C divides segment PR.

40. Vector $\vec{c}$ is a linear combination of the vectors $\vec{b}$ and $\vec{d}$, and $\vec{c} \neq 0$. Vectors $\vec{a}$, $\vec{b}$, and $\vec{d}$ are linearly independent. Prove that $\vec{a}$ and $\vec{d}$ cannot be linearly dependent.

41. Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly independent. O, A, B, C, and Z are points such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$, and $\overrightarrow{OZ} = -4\vec{a} + 2\vec{b} + 3\vec{c}$.

a) Express $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AZ}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

b) Prove that $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AZ}$ are linearly dependent.

c) Draw conclusions about the geometric relationship among the points A, B, C, and Z.

42. The position vectors of A and B are $\vec{i} - 2\vec{j} + \vec{k}$ and $5\vec{i} + 4\vec{j} - 7\vec{k}$ respectively. The point P lies between A and B and is such that $\overrightarrow{AP} = 2\overrightarrow{PB}$. Find the position vector of P.
(83 H)

43. The position vectors of A, B and C are $2\vec{i} - \vec{j} + 3\vec{k}$, $12\vec{i} + 4\vec{j} - 7\vec{k}$ and $6\vec{i} + \vec{j} - \vec{k}$ respectively. Given that $\overrightarrow{AC} = \lambda\overrightarrow{AB}$, find the value of λ.
(87 SMS)

44. R is a point on the line PQ where P has coordinates $(2,7)$ and Q has coordinates $(-2,3)$. If R divides PQ in the ratio $3:-2$ then the coordinates of R are

A. $(-10,-5)$ **B.** $(10,-5)$ **C.** $(-2,-1)$
D. $(2,1)$ **E.** $(10,16)$
(79 S)

45. With respect to the standard basis of $\mathbb{R}^3$ the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are defined by $\boldsymbol{a} = \overrightarrow{(1,2,3)}$, $\boldsymbol{b} = \overrightarrow{(0,1,3)}$, $\boldsymbol{c} = \overrightarrow{(\lambda,0,1)}$ where $\lambda \in \mathbb{R}$. Which of the following statements is true?

A. $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly dependent.
B. $\boldsymbol{a}$ and $\boldsymbol{c}$ are linearly dependent.
C. $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent for all values of λ.
D. $\boldsymbol{a}$ and $\boldsymbol{b}$ form a basis of $\mathbb{R}^3$.
E. $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ form a basis of $\mathbb{R}^3$ if $\lambda = 1$.
(81 H)