VECTORS, MATRICES
and
COMPLEX NUMBERS

with
International Baccalaureate
questions

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CHAPTER THREE

THE MULTIPLICATION OF VECTORS

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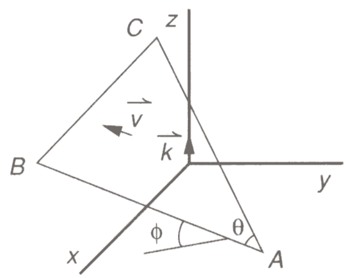
CHAPTER THREE

The Multiplication of Vectors

Problem

a) Given the points $A(1,3,-1)$, $B(2,-5,3)$, and $C(4,-1,6)$, you are asked to find the angle BAC or θ. Note that, although the diagram may help, it does not lend itself to discovering a simple solution through elementary trigonometry.



There is a way to proceed, as follows.

1. Calculate the lengths of AB, AC, and BC, that is, the lengths of the sides of the triangles ABC.
2. Use the cosine law (see page 542) in the triangle ABC to find the angle BAC.

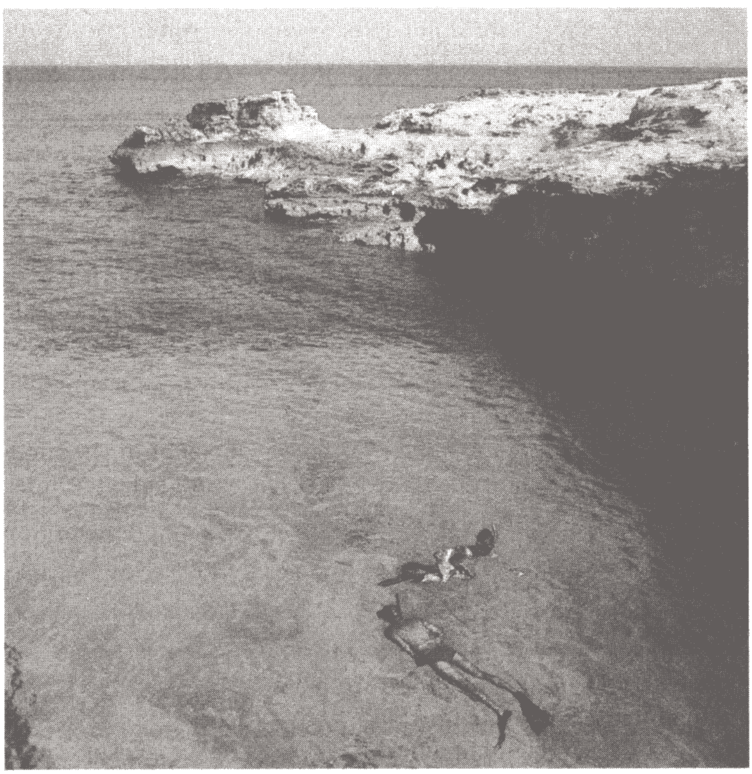
However, this involves a lot of arithmetic. You shall see that by defining an operation called the **dot product** of vectors, a remarkably short and elegant method of calculating this angle can be devised.

b) Now consider the plane defined by the points A, B and C. You are asked to determine a direction, or a vector $\vec{v}$, that is perpendicular to the plane ABC. This means finding a vector that is perpendicular to every line in the plane ABC. It can be shown that it is sufficient to find a vector perpendicular to two linearly independent vectors coplanar with ABC. However, the task is still not simple. You will see that the **cross product** of vectors provides a simple way of determining such a vector $\vec{v}$.

c) You are now asked to calculate the angle between AB and the xy-plane. Recall from chapter 1, the *In Search of* on page 17, that the angle between a line and a plane is the smallest angle that can be defined between the line and a line in the plane. (It turns out to be the angle between the line and its 'perpendicular projection' in the plane.) Let this angle be ϕ. Once again, note that finding ϕ is not obvious, even with a diagram. You will see that the simplest way to determine this angle is by using the dot product between the vector $\overrightarrow{AB}$ and a vector that is perpendicular to the xy-plane (called a **normal** vector to the plane). In the diagram $\vec{k}$ is such a vector.

d) A more challenging task to undertake with ordinary trigonometry would be to calculate the angle between two planes, such as ABC and the xy-plane. This can be accomplished by finding a normal to each plane with the cross product, then calculating the angle α between the normals.

The types of problem described in c) and d) will be investigated further in chapter 6. Once you have learned to multiply vectors, you will appreciate that vector analysis is a very powerful tool that brings 3-space geometric problems to a level hardly more difficult than problems in 2-space. You will be using products of vectors extensively in the rest of your work on vectors in this book.

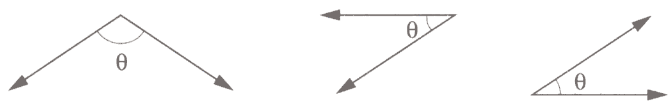


3.1 Projections and Components

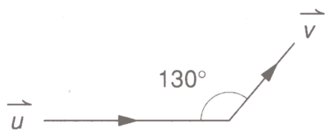
Before investigating the products of vectors, you need to know how a vector can be projected onto another. This projection will depend on the angle between the vectors.

The Angle between Two Vectors

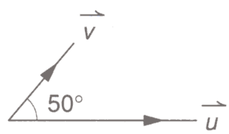
is defined as the angle θ between the vectors when they are drawn with a common tail. Note that $0^\circ \leqslant \theta \leqslant 180^\circ$.



Example 1 What is the angle between the vectors $\vec{u}$ and $\vec{v}$ shown?



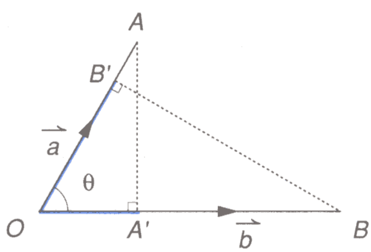
Solution Draw the vectors again so that they have a common tail.



The angle between $\vec{u}$ and $\vec{v}$ is $180^\circ - 130^\circ = 50^\circ$. ■

Projections

Given the vectors $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$ such that θ is the angle between $\vec{a}$ and $\vec{b}$.

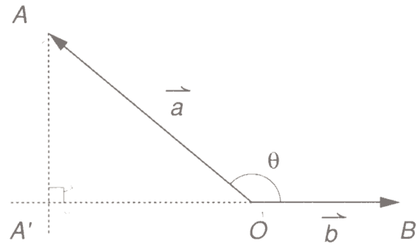


Let A' be the foot of the perpendicular from A to OB,
and B' be the foot of the perpendicular from B to OA.
Then the vector $\overrightarrow{OA'}$ is known as the **orthogonal projection**,
or **projection, of $\vec{a}$ on $\vec{b}$**, and
the vector $\overrightarrow{OB'}$ is known as the orthogonal projection,
or projection, of $\vec{b}$ on $\vec{a}$.

Note: The projection of $\vec{a}$ on $\vec{b}$ is parallel to $\vec{b}$, and the projection of $\vec{b}$ on $\vec{a}$ is parallel to $\vec{a}$.

Now $|\overrightarrow{OA'}| = ||\overrightarrow{OA}| \cos \theta| = ||\vec{a}| \cos \theta|$.

If $\theta < 90^\circ$ it should be clear that the direction of vector $\overrightarrow{OA'}$ is the same as the direction of $\vec{b}$. In the case when $\theta > 90^\circ$, the diagram is as shown.



If you now draw a perpendicular to OB from A, you find that OB needs to be extended beyond O. (This extension is described by saying that "BO is produced".)

Now $\overrightarrow{OA'}$ is still parallel to $\vec{b}$, but is in *the opposite direction*.

The fact that $\cos \theta$ is negative in this case is critical in the definition that follows.

To ensure that the projection of $\vec{a}$ on $\vec{b}$ is a vector, you will need a unit vector in the direction of $\vec{b}$, namely $\vec{e}_b = \frac{1}{|\vec{b}|}\vec{b}$.

DEFINITION

The projection of $\vec{a}$ on $\vec{b}$ is the vector

$$\vec{p} = |\vec{a}| \cos \theta \, \vec{e}_b = \frac{|\vec{a}| \cos \theta \, \vec{b}}{|\vec{b}|}$$

Observe that this definition does give the correct direction for $\overrightarrow{OA'}$.

If $\theta < 90^\circ$, then $\cos \theta > 0$, and $\overrightarrow{OA'}$ has the direction of $\vec{b}$, but if $\theta > 90^\circ$, then $\cos \theta < 0$, and $\overrightarrow{OA'}$ has the direction opposite to $\vec{b}$.

Note: The projection of $\vec{a}$ on $\vec{b}$ does not depend on the length of $\vec{b}$. For example, the projection of $\vec{a}$ on $3\vec{b}$ (which has the same direction as $\vec{b}$), is

$$\frac{|\vec{a}| \cos \theta}{|3\vec{b}|} 3\vec{b} = \frac{|\vec{a}| \cos \theta}{|\vec{b}|} \vec{b} = \overrightarrow{OA'}.$$

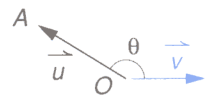
Alternatively, recall from section 1.8 that $\vec{e}_b = \vec{e}_{3b}$.

Example 2 The angle between vectors $\vec{u}$ and $\vec{v}$ is θ, where $\theta > 90°$. Find an expression for

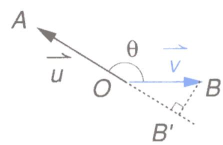
a) the projection of $\vec{v}$ on $\vec{u}$,

b) the projection of $3\vec{v}$ on $\vec{u}$.

Draw each projection.



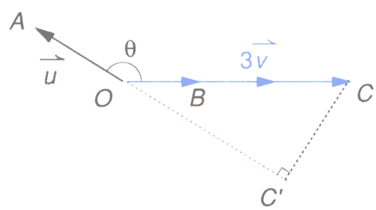
Solution a) Draw the vectors with a common tail so that $\overrightarrow{OA} = \vec{u}$, $\overrightarrow{OB} = \vec{v}$.
Let the projection of $\vec{v}$ on $\vec{u}$ be $\overrightarrow{OB'}$.
B' is the intersection of AO produced and the perpendicular to $\vec{u}$ from B.



The required projection is $\overrightarrow{OB'} = |\vec{v}| \cos\theta \, \vec{e}_u$.

(Notice that since $\theta > 90°$, $\overrightarrow{OB'}$ will have a direction opposite that of $\overrightarrow{OA}$. This is confirmed by the diagram.)

b) Let the projection of $3\vec{v}$ on $\vec{u}$ be $\overrightarrow{OC'}$.



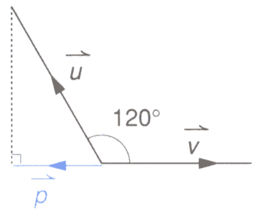
Thus, $\overrightarrow{OC'} = |3\vec{v}| \cos\theta \, \vec{e}_u = 3(|\vec{v}| \cos\theta \, \vec{e}_u) = 3\overrightarrow{OB'}$ ■

Components

In the definition of the orthogonal projection of $\vec{a}$ on $\vec{b}$, the scalar $c = |\vec{a}| \cos\theta$ which multiplies the unit vector $\vec{e}_b$ is called the **component of $\vec{a}$ on $\vec{b}$**, or **component of $\vec{a}$ in the direction of $\vec{b}$.**

Example 3 Vectors $\vec{u}$ and $\vec{v}$ make an angle of 120° with each other. If $|\vec{u}| = 6$ and $|\vec{v}| = 5$, calculate the projection of $\vec{u}$ on $\vec{v}$, and the component of $\vec{u}$ on $\vec{v}$.

Solution

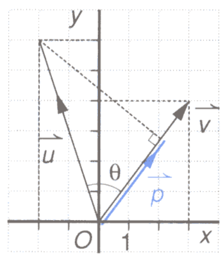


If the unit vector in the direction of $\vec{v}$ is $\vec{e}_v$,
then the projection of $\vec{u}$ on $\vec{v}$ is
$\vec{p} = |\vec{u}| \cos 120° \, \vec{e}_v = 6(-0.5)\, \vec{e}_v = -3\vec{e}_v$.
Thus the component of $\vec{u}$ in the direction of $\vec{v}$ is -3. ■

Note: The length of $\vec{v}$ is *irrelevant*, since there is only one unit vector in the direction of $\vec{v}$.

Example 4 The angle θ between vectors $\vec{u} = \overrightarrow{(-2,6)}$ and $\vec{v} = \overrightarrow{(3,4)}$ is such that $\cos\theta = 0.5692$. Find each of the following, correct to 3 significant digits.

a) the component of $\vec{u}$ on $\vec{v}$
b) the projection of $\vec{u}$ on $\vec{v}$



Solution **a)** Let the component of $\vec{u}$ on $\vec{v}$ be c.
Then $c = |\vec{u}| \cos\theta = \sqrt{(-2)^2 + 6^2}(0.5692)$
$= \sqrt{40}(0.5692) = 3.599\ldots$ or 3.60,
correct to 3 significant digits.

b) Let the projection of $\vec{u}$ on $\vec{v}$ be $\vec{p}$.
Then $\vec{p} = c\vec{e}_v$
where the unit vector in the direction of $\vec{v}$,

$$\vec{e}_v = \frac{1}{|\vec{v}|}\vec{v} = \frac{1}{\sqrt{3^2+4^2}}\overrightarrow{(3,4)} = \frac{1}{5}\overrightarrow{(3,4)} = \overrightarrow{(0.6,0.8)}$$

so $\vec{p} = (3.60)\overrightarrow{(0.6,0.8)} = \overrightarrow{(2.16,2.88)}$. ■

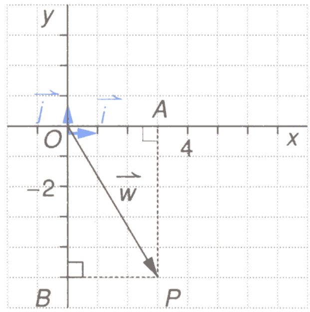
Note: $\vec{p}$ is parallel to $\vec{v}$.

You will see from the following example that the definition of *component* in this section agrees with the previous meaning given to the term.

Example 5

Find the projections and components of the vector $\vec{w} = \overrightarrow{(3,-5)}$ in the direction of $\vec{i}$, and in the direction of $\vec{j}$.

Solution



$\vec{w} = \overrightarrow{OP} = \overrightarrow{(3,-5)}$ is the position vector of the point (3,−5).
Draw a perpendicular from P to A on the x-axis, and
to B on the y-axis.

Now the length of OA is 3, and since $\overrightarrow{OA}$ is in the same direction as $\vec{i}$,
the projection of $\overrightarrow{(3,-5)}$ on $\vec{i}$ is $3\vec{i}$, and
the component of $\overrightarrow{(3,-5)}$ on $\vec{i}$ is 3.

The length of OB is 5, but since $\overrightarrow{OB}$ and $\vec{j}$ are in *opposite* directions,
the projection of $\overrightarrow{(3,-5)}$ on $\vec{j}$ is $-5\vec{j}$, and
the component of $\overrightarrow{(3,-5)}$ on $\vec{j}$ is −5. ■

Alternative Solution

Recall that in section 1.9, you learned that $\overrightarrow{(3,-5)} = 3\vec{i} - 5\vec{j}$.
Thus,
$3\vec{i}$ and $-5\vec{j}$ are the projections of $\overrightarrow{(3,-5)}$ onto $\vec{i}$ and $\vec{j}$ respectively;
3 and −5 are the components of $\overrightarrow{(3,-5)}$ on $\vec{i}$ and $\vec{j}$ respectively.

Resolution of a Vector

The components in Example 5 are sometimes called **rectangular components**, because they refer to mutually orthogonal directions.

Writing a vector in terms of its projections on mutually orthogonal directions is called *resolving* the vector in those directions.

Thus, in Example 5, by writing $\vec{w} = 3\vec{i} - 5\vec{j}$, you are resolving $\vec{w}$ in the directions of $\vec{i}$ and $\vec{j}$.

You can also say that you are *resolving* a vector when you express that vector as a linear combination of an orthonormal basis.

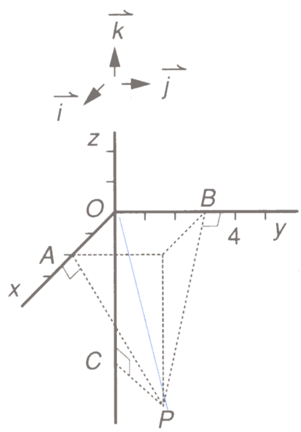
The word 'resolution' is not necessarily confined to cases of mutually orthogonal directions. In this book, however, it shall always refer to mutually orthogonal directions in order to avoid confusion. It is unfortunate that the vocabulary pertaining to the ideas of this section is not standardized in all books on vectors. Some books refer to our projections as 'components', and consider 'projections' to be lengths, that is, positive scalars.
You must beware of this.

Example 6 Resolve the vector $\vec{v} = \overrightarrow{(2,3,-5)}$ in the directions of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

Solution From section 1.9, $\overrightarrow{(2,3,-5)} = 2\vec{i} + 3\vec{j} - 5\vec{k}$. ■

Note: $2\vec{i}$, $3\vec{j}$, $-5\vec{k}$ are the projections of $\vec{v}$, and 2, 3, −5 are the components of $\vec{v}$ in the required directions.

Notice also that, if $\vec{v} = \overrightarrow{OP}$, the projection of $\vec{v}$ in the direction of $\vec{i}$ is $\overrightarrow{OA}$ where PA is perpendicular to the x-axis, as shown in the diagram. Similarly, the projections of $\vec{v}$ in the directions of $\vec{j}$ and $\vec{k}$ are $\overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively.



SUMMARY

The projection of $\vec{a}$ on $\vec{b}$ is the vector

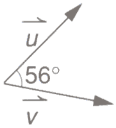
$$\vec{p} = |\vec{a}| \cos\theta \, \vec{e}_b = \frac{|\vec{a}| \cos\theta \, \vec{b}}{|\vec{b}|}$$

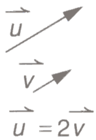
The scalar $c = |\vec{a}| \cos\theta$ that multiplies the unit vector $\vec{e}_b$ is called the component of $\vec{a}$ on $\vec{b}$, or the component of $\vec{a}$ in the direction of $\vec{b}$.

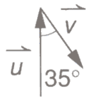
Writing $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$ is called *resolving* $\vec{v}$ in the directions of $\vec{i}$, $\vec{j}$, and $\vec{k}$. (x, y, and z are the components of $\vec{v}$, and $x\vec{i}$, $y\vec{j}$, and $z\vec{k}$ are the projections of $\vec{v}$ in the directions of $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively.)

3.1 Exercises

1. State the angle between the following pairs of vectors.

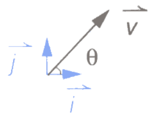
a) 

b) 

c) 

2. Given that the following vectors are in $\mathbb{V}_2$, and that θ is the angle between each vector and $\vec{i}$, find the component of each vector on $\vec{i}$ and $\vec{j}$ (correct to 3 decimal places).

a) $|\vec{a}| = 5$, $\theta = 40°$
b) $|\vec{b}| = 7$, $\theta = 110°$
c) $|\vec{c}| = 13$, $\theta = 90°$.



3. Vectors $\vec{u}$ and $\vec{v}$ make an angle of 68° with each other. If $|\vec{u}| = 4$ and $|\vec{v}| = 3$, find the following.

a) the component of $\vec{u}$ on $\vec{v}$ and the projection of $\vec{u}$ on $\vec{v}$
b) the component of $\vec{v}$ on $\vec{u}$ and the projection of $\vec{v}$ on $\vec{u}$

4. Calculate the component of $\vec{i}$ on $\vec{v}$, given that the angle between $\vec{i}$ and $\vec{v}$ is 110°.

5. State the projections and the components of the following in the directions of $\vec{i}$ and $\vec{j}$.

a) $\overrightarrow{(2,-3)}$ b) $\overrightarrow{(1,0)}$ c) $3\overrightarrow{(-5,1)}$

6. State the projections and the components of the following vectors in the directions of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

a) $\overrightarrow{(1,-4,1)}$ b) $\overrightarrow{(2,0,3)}$ c) $-2\overrightarrow{(1,1,0)}$

7. Resolve the following vectors on $\vec{i}$, $\vec{j}$, $\vec{k}$.

$\vec{u} = \overrightarrow{(4,5,0)}$ $\vec{v} = \overrightarrow{(-2,-3,1)}$

8. Given $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$, state the following.

a) the component of $\vec{v}$ on $\vec{i}$
b) the projection of $\vec{v}$ on $\vec{j}$
c) the projection of $\vec{v}$ on $-\vec{k}$

9. Calculate the component of the following on $\vec{v} = \overrightarrow{(1,1)}$.

a) $\vec{i}$ b) $\vec{j}$

10. If the angle between $\vec{u}$ and $\vec{v}$ is θ, show that $|\vec{u}|$ (the component of $\vec{v}$ on $\vec{u}$) $= |\vec{v}|$ (the component of $\vec{u}$ on $\vec{v}$).

11. Find the following.

a) the component of $\overrightarrow{(2,-5)}$ on $\vec{i}$
b) the component of $\vec{i}$ on $\overrightarrow{(2,-5)}$
c) the projection of $\overrightarrow{(-3,4)}$ on $\vec{j}$
d) the projection of $\vec{j}$ on $\overrightarrow{(-3,4)}$

12. The component of $\vec{u}$ in the direction of $\vec{v}$ is zero, where $\vec{v} \neq 0$. What can you deduce about $\vec{u}$ and/or $\vec{v}$?

13. Given two non-zero vectors $\vec{u}$ and $\vec{v}$, what can you deduce about $\vec{u}$ and $\vec{v}$ where

a) the component of $\vec{u}$ on $\vec{v}$ is equal to the component of $\vec{v}$ on $\vec{u}$?
b) the projection of $\vec{u}$ on $\vec{v}$ is equal to the projection of $\vec{v}$ on $\vec{u}$?

14. The vector $\sqrt{2}\,\vec{i}$ is resolved into two equal rectangular components. What are they?

15. The vector $\overrightarrow{(4,5)}$ has components a and $2a$ when resolved along two perpendicular lines. Calculate the value of a.

16. The vectors $\vec{u} = \overrightarrow{(1,\sqrt{3})}$ and $\vec{v} = \overrightarrow{(-2\sqrt{3},6)}$ make an angle of 60° with each other. Find each of the following.

a) the component of $\vec{u}$ on $\vec{v}$
b) the projection of $\vec{u}$ on $\vec{v}$

17. The projection of $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ on a plane Π is defined to be the vector $\vec{p}$, where $\vec{p}$ is parallel to Π and $\vec{v} - \vec{p}$ is perpendicular to Π.
If $\vec{v} = 5\vec{i} - 12\vec{j} + 2\vec{k}$, calculate

a) the component of $\vec{v}$ on the xy-plane
b) the projection of $\vec{v}$ on the yz-plane.

3.2 The Dot Product

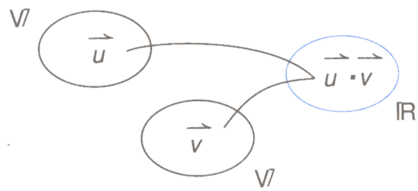
So far you have learned to add and subtract vectors. You have seen that it is possible to multiply a vector $\vec{u}$ by a real number (scalar) to obtain a vector parallel to vector $\vec{u}$. All of these operations produce another vector.

You are now ready to find out how to multiply vectors. There are actually two kinds of vector products, symbolized by $\vec{u} \cdot \vec{v}$ and $\vec{u} \times \vec{v}$. The first product, $\vec{u} \cdot \vec{v}$, is a scalar, while the second, $\vec{u} \times \vec{v}$, is a vector.

The product $\vec{u} \cdot \vec{v}$ is called the 'dot' product, or 'scalar' product of two vectors. The **dot product** takes two *vectors*, $\vec{u}$ and $\vec{v} \in \mathbb{V}$, and returns a *real number*, as follows, where θ is the angle between $\vec{u}$ and $\vec{v}$.

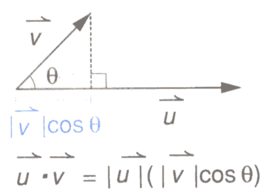
DEFINITION

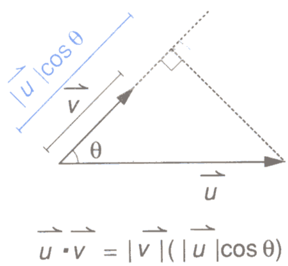
$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta, 0° \leqslant \theta \leqslant 180°$$



Note: You *must* use a dot (·) when writing out this product.

Geometrically, you can see that the dot product is the product of the *length of one vector* with the *component of the other vector* along the first. The result $\vec{u} \cdot \vec{v}$ is indeed a scalar. (Be careful about this!)





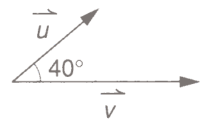
If the component of one of the vectors is directed *opposite* the other vector, the dot product will be negative. If you look back at the definition, you will note that this occurs when the angle θ is greater than 90°; indeed, in that case, the cosine of θ is negative.

The dot product has some very interesting and powerful applications, particularly in trigonometry, which you will be discovering in the next sections.

Example 1 Find $\vec{u}\cdot\vec{v}$ if $|\vec{u}| = 3$, $|\vec{v}| = 5$, $\theta = 40°$, and draw a diagram showing $\vec{u}$ and $\vec{v}$.

Solution

$$\begin{aligned}\vec{u}\cdot\vec{v} &= |\vec{u}||\vec{v}|\cos\theta\\ &= (3)(5)\cos 40°\\ &= (15)(0.7660\ldots)\\ &\doteq 11.5\end{aligned}$$

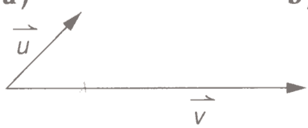


■

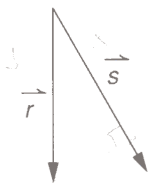
(Notice that you do *not* know the precise direction of the vectors here; you merely know that they are at 40° to each other. However, you still have enough information to find the dot product.)

Example 2 Find the dot product of the following vectors *by measuring* the component of the first vector along the second vector, and *by measuring* the second vector. Use the centimetre as unit.

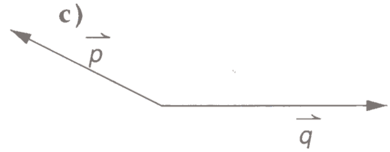
a)



b)



c)

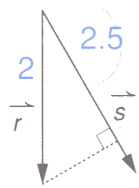


$\vec{u}\cdot\vec{v}$ $\qquad$ $\vec{r}\cdot\vec{s}$ $\qquad$ $\vec{p}\cdot\vec{q}$

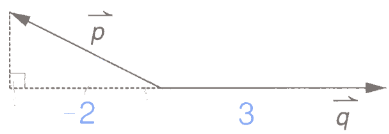
Solution

a)

b)



c)



$\vec{u}\cdot\vec{v} = 1 \times 4 = 4$ $\qquad$ $\vec{r}\cdot\vec{s} = 2 \times 2.5 = 5$ $\qquad$ $\vec{p}\cdot\vec{q} = -2 \times 3 = -6$ ■

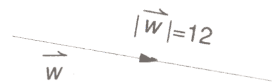
Note: You are measuring $(|\vec{u}|\cos\theta)$ and $|\vec{v}|$, etc.
But $|\vec{u}|\cos\theta|\vec{v}| = |\vec{u}||\vec{v}|\cos\theta$.

The following example should assist you in discovering some useful properties of the dot product.

Example 3 Calculate the following dot products

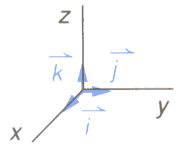
a) $\vec{w}\cdot\vec{w}$, where $|\vec{w}| = 12$.
b) $\vec{i}\cdot\vec{k}$

Solution a) $\vec{w} \cdot \vec{w} = 12 \times 12 \times \cos 0^\circ = 12^2 = 144.$



PROPERTY The dot product of a vector with itself is the square of its length.

b) $\vec{i} \cdot \vec{k} = 1 \times 1 \times \cos 90^\circ = 0.$



PROPERTY The dot product of two perpendicular vectors is zero. ■

In the exercises, you will be proving these and other properties.
The next example should also lead you to the discovery of a property.

Example 4

Given $\vec{u} = \overrightarrow{(1,0)}$, $\vec{v} = \overrightarrow{(-1,\sqrt{3})}$, calculate the following dot products.

a) $\vec{u} \cdot \vec{v}$ b) $(6\vec{u}) \cdot (2\vec{v})$

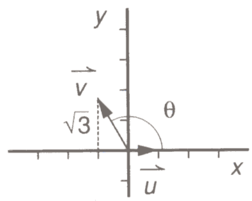
Solution a) $|\vec{u}| = \sqrt{1^2 + 0^2} = 1$

$|\vec{v}| = \sqrt{(-1)^2 + \sqrt{3}^2} = 2.$

If θ is the angle between $\vec{u}$ and $\vec{v}$,

then $\tan \theta = \dfrac{\sqrt{3}}{-1} = -\sqrt{3}$, giving $\theta = 120^\circ$.

Thus $\vec{u} \cdot \vec{v} = (1)(2) \cos 120^\circ = (2)(-0.5) = -1.$

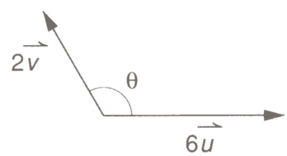


b) $|6\vec{u}| = |6||\vec{u}| = (6)(1) = 6$

$|2\vec{v}| = |2||\vec{v}| = (2)(2) = 4$

The angle between $\vec{u}$ and $\vec{v}$ is the same as the angle between $6\vec{u}$ and $2\vec{v}$, namely 120°.

Thus $\overrightarrow{(6u)} \cdot \overrightarrow{(2v)} = (6)(4) \cos 120^\circ = -12.$ ■



Notice that the results obtained in this example indicate that $(6\vec{u}) \cdot (2\vec{v}) = (6 \times 2)(\vec{u} \cdot \vec{v})$, which illustrates the following property.

PROPERTY For any vectors $\vec{u}$, $\vec{v}$, and scalars m, n

$(m\,\vec{u}) \cdot (n\,\vec{v}) = (mn)(\vec{u} \cdot \vec{v})$

In the next section, you will also discover an alternative method of finding the dot product of two vectors expressed in component form; it will allow you to do a question like Example 4 above more quickly.

SUMMARY The dot product of vectors $\vec{u}$ and $\vec{v}$, having an angle θ between them when drawn with a common tail, is the scalar

$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta \quad (0^\circ \leqslant \theta \leqslant 180^\circ)$

The dot product equals:
(length of one vector) (component of the other vector along the first)

3.2 Exercises

1. Calculate the dot product of the following pairs of vectors correct to 3 significant digits, given that θ is the angle between them.

a) $|\vec{u}| = 4, |\vec{v}| = 6, \theta = 60°$

b) $|\vec{w}| = 7, |\vec{t}| = 3, \theta = 27°$

c) $|\vec{a}| = 6, |\vec{b}| = 7.5, \theta = 90°$

d) $|\vec{c}| = 8, |\vec{d}| = 4, \theta = 0°$

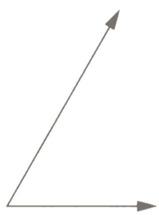
e) $|\vec{e}| = 12, |\vec{f}| = 15, \theta = 91°$

2. Calculate (approximately) the dot products of the following vectors by measurement of lengths and projections, using the centimetre as a unit.

a)

b)

c)



3. Calculate the following dot products.

a) $\vec{j} \cdot \vec{i}$ b) $\vec{i} \cdot \vec{i}$ c) $\vec{k} \cdot (-\vec{k})$

4. Given that $\vec{u}, \vec{v}, \vec{w}$ are vectors, and $m, n \in \mathbb{R}$, state which of the following are vectors, and which are scalars.

a) $2\vec{v}$

b) $|2\vec{v}|$

c) $\vec{v}(m - n)$

d) $m(\vec{u} + \vec{v})$

e) $\vec{u} \cdot \vec{v}$

f) $\vec{u} \cdot (\vec{v} + \vec{w})$

g) $\vec{v} + \vec{u}$

h) $\vec{v} - \vec{u}$

i) $-m\,\vec{u}$

5. Prove that the dot product of two unit vectors is equal to the cosine of the angle between them.

6. Prove that the dot product of a vector with itself is equal to the square of its length.

7. Prove that the dot product of two perpendicular vectors is zero.

8. Given vectors $\vec{u}, \vec{v}$ at an angle θ to each other, and scalar m, prove that $(m\,\vec{u}) \cdot \vec{v} = m(\vec{u} \cdot \vec{v})$ in the following cases.

a) m is positive and θ is acute

b) m is negative and θ is acute

c) m is positive and θ is obtuse

d) m is negative and θ is obtuse

9. Using the results of question 8, prove that, if n is another scalar, $(m\,\vec{u}) \cdot (n\,\vec{v}) = (mn)\,\vec{u} \cdot \vec{v}$.

10. If $\vec{u} \cdot \vec{v} = 0$, is the angle between $\vec{u}$ and $\vec{v}$ necessarily 90°?

11. ABC is an equilateral triangle whose sides have length 10 units. Calculate the following.

a) $\overrightarrow{AB} \cdot \overrightarrow{AC}$ b) $\overrightarrow{AB} \cdot \overrightarrow{BC}$

12. Given that $|\vec{p}| = 10$ and $|\vec{q}| = 3$, calculate the angle θ between the vectors $\vec{p}$ and $\vec{q}$ in the following cases. Give your answers to the nearest degree.

a) $\vec{p} \cdot \vec{q} = 30$ b) $\vec{p} \cdot \vec{q} = -5$ c) $\vec{p} \cdot \vec{q} = 0$

13. Given any three vectors $\vec{u}, \vec{v}, \vec{w}$, which of the following expressions are meaningful? Justify your answers.

a) $\vec{u} + (\vec{v} \cdot \vec{w})$

b) $(\vec{u} + \vec{v}) \cdot \vec{w}$

c) $\vec{u} \cdot \vec{v} \cdot \vec{w}$

d) $\vec{u} \cdot (\vec{v} \cdot \vec{w})$

e) $(\vec{u} \cdot \vec{v})\vec{w}$

f) $\vec{u}(\vec{v} \cdot \vec{w})$

14. a) Given that vectors $\vec{a}$ and $\vec{b}$ of $\mathbb{V}_2$ make angles of 45° and 60° respectively with $\vec{i}$, where $|\vec{a}| = 4\sqrt{2}$, and $|\vec{b}| = 8$, find the exact value of $\vec{a} \cdot \vec{i}$ and $\vec{b} \cdot \vec{i}$.

b) Use your result to part a) to comment on the following.

If $\vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w}$, is it necessarily true that $\vec{u} = \vec{v}$?

15. Prove that for any vectors $\vec{u}$ and $\vec{v}$,

$|\vec{u} \cdot \vec{v}| \leqslant |\vec{u}|\,|\vec{v}|$

When does the equality hold?

3.3 Properties of the Dot Product

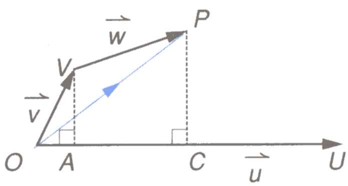
Commutativity

By definition, $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta$.
Thus, $\vec{v} \cdot \vec{u} = |\vec{v}||\vec{u}| \cos \theta$, which is the same real number as $\vec{u} \cdot \vec{v}$.
Hence, for any vectors $\vec{u}$ and $\vec{v}$,
$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$, *the dot product is commutative.*

Distributivity over Addition

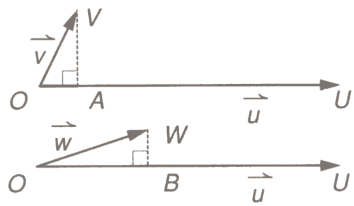
Look at the diagrams to help you see the results of the following expressions. The vectors are drawn so that
$\vec{u} = \overrightarrow{OU}$, $\vec{v} = \overrightarrow{OV}$, $\vec{w} = \overrightarrow{OW}$, and $\vec{v} + \vec{w} = \overrightarrow{OP}$.

$x = \vec{u} \cdot (\vec{v} + \vec{w})$



$$x = \overrightarrow{OU} \cdot \overrightarrow{OP}$$
$$= |\overrightarrow{OU}| \text{ (component of } \overrightarrow{OP} \text{ on } \overrightarrow{OU})$$
$$= (OU)(OC)$$

$y = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$



$$y = \overrightarrow{OU} \cdot \overrightarrow{OV} + \overrightarrow{OU} \cdot \overrightarrow{OW}$$
$$= |\overrightarrow{OU}| (OA) + |\overrightarrow{OU}| (OB)$$
$$= (OU)(OA + OB)$$
$$= (OU)(OA + AC) = (OU)(OC)$$

This indicates the following property.

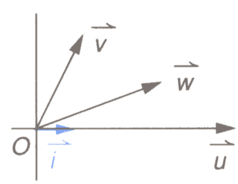
For any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

That is, *the dot product is distributive over vector addition.*

Algebraic Proof of the Distributivity Property

Draw $\vec{u}$ along the positive x-axis, that is, in the direction of $\vec{i}$, with its tail at (0,0).

Let $\vec{v} = \overrightarrow{(v_1, v_2)}$ and $\vec{w} = \overrightarrow{(w_1, w_2)}$, thus $\vec{v} + \vec{w} = \overrightarrow{(v_1 + w_1, v_2 + w_2)}$.



Now $\vec{u} \cdot \vec{v} = |\vec{u}|$ (the component of $\vec{v}$ on $\vec{u}$)
$= |\vec{u}|$ (the component of $\vec{v}$ on $\vec{i}$) $= |\vec{u}| (v_1)$
and $\vec{u} \cdot \vec{w} = |\vec{u}|$ (the component of $\vec{w}$ on $\vec{u}$)
$= |\vec{u}|$ (the component of $\vec{w}$ on $\vec{i}$) $= |\vec{u}| (w_1)$

therefore $\vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} = |\vec{u}|(v_1) + |\vec{u}|(w_1) = |\vec{u}|(v_1 + w_1)$
but $\vec{u} \cdot (\vec{v} + \vec{w}) = |\vec{u}|$ (the component of $[\vec{v} + \vec{w}]$ on $\vec{u}$)
$= |\vec{u}|$ (the component of $[\vec{v} + \vec{w}]$ on $\vec{i}$) $= |\vec{u}| (v_1 + w_1)$
Therefore, $\vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} = \vec{u} \cdot (\vec{v} + \vec{w})$ as required.

There follows a summary of the major properties of the dot product. The first two properties have been proved above. You have demonstrated properties 3, 4 and 5 in the examples and problems of the last section.

PROPERTIES

1. $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$ — commutative
2. $\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$ — distributive over +
3. $(k\vec{u})\cdot(m\vec{v})=(km)(\vec{u}\cdot\vec{v})$
4. $\vec{u}\cdot\vec{u}=|\vec{u}|^2$
5. If $\vec{u}$ is perpendicular to $\vec{v}$, then $\vec{u}\cdot\vec{v}=0$.
6. $\vec{i}\cdot\vec{i}=1,\ \vec{j}\cdot\vec{j}=1,\ \vec{k}\cdot\vec{k}=1$
7. $\vec{i}\cdot\vec{j}=\vec{j}\cdot\vec{i}=\vec{j}\cdot\vec{k}=\vec{k}\cdot\vec{j}=\vec{k}\cdot\vec{i}=\vec{i}\cdot\vec{k}=0$

Property 6 follows from property 4 (recall that $\vec{i}$, $\vec{j}$, and $\vec{k}$ are unit vectors), and property 7 follows from property 5.

NOTATION It is also possible to write the dot product $\vec{u}\cdot\vec{u}$ as $\vec{u}^2$.

Now you are ready to use these properties to find a formula for calculating the dot product of vectors in component form. Indeed, if $\vec{u}=u_1\vec{i}+u_2\vec{j}$ and $\vec{v}=v_1\vec{i}+v_2\vec{j}$,

$$\begin{aligned}\vec{u}\cdot\vec{v}&=(u_1\vec{i}+u_2\vec{j})\cdot(v_1\vec{i}+v_2\vec{j})\\&=(u_1\vec{i})\cdot(v_1\vec{i})+(u_1\vec{i})\cdot(v_2\vec{j})+(u_2\vec{j})\cdot(v_1\vec{i})+(u_2\vec{j})\cdot(v_2\vec{j}) &&\text{property 2}\\&=(u_1v_1)(\vec{i}\cdot\vec{i})+(u_1v_2)(\vec{i}\cdot\vec{j})+(u_2v_1)(\vec{j}\cdot\vec{i})+(u_2v_2)(\vec{j}\cdot\vec{j}) &&\text{property 3}\\&=(u_1v_1)(1)+(u_1v_2)(0)+(u_2v_1)(0)+(u_2v_2)(1) &&\text{properties 6 and 7}\\&=u_1v_1+u_2v_2\end{aligned}$$

FORMULA So $\vec{u}=\overrightarrow{(u_1,u_2)}$ and $\vec{v}=\overrightarrow{(v_1,v_2)}\Rightarrow\vec{u}\cdot\vec{v}=u_1v_1+u_2v_2$

In the exercises, you will be demonstrating in a similar manner that in $\mathbb{V}_3$

FORMULA $\vec{u}=\overrightarrow{(u_1,u_2,u_3)}$ and $\vec{v}=\overrightarrow{(v_1,v_2,v_3)}\Rightarrow\vec{u}\cdot\vec{v}=u_1v_1+u_2v_2+u_3v_3$

Note: The result in each case is indeed a real number (or scalar) as expected.

Now you can use these new formulas to redo Example 4 of section 3.2.

Example 1 Find the dot product in the following cases.

a) $\vec{u}=\overrightarrow{(1,0)},\ \vec{v}=\overrightarrow{(-1,\sqrt{3})}$ b) $6\vec{u}=\overrightarrow{(6,0)},\ 2\vec{v}=\overrightarrow{(-2,2\sqrt{3})}$

Solution a) $\vec{u}\cdot\vec{v}=(1)(-1)+(0)(\sqrt{3})=-1$ b) $\vec{u}\cdot\vec{v}=(6)(-2)+(0)(2\sqrt{3})=-12$ ■

Notice that the method is a lot quicker, if the vectors' components are known. The fact that you now have two methods of calculating dot products will help you to make more discoveries.

Example 2 Find the angle θ between $\vec{u} = \overrightarrow{(1,2,5)}$ and $\vec{v} = \overrightarrow{(-1,-3,4)}$.

Solution The definition of the dot product states $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta$.
This is an opportunity to use both methods of calculating the dot product.

$$\overrightarrow{(1,2,5)} \cdot \overrightarrow{(-1,-3,4)} = |\overrightarrow{(1,2,5)}||\overrightarrow{(-1,-3,4)}| \cos \theta$$

$$(1)(-1) + (2)(-3) + (5)(4) = \sqrt{1^2 + 2^2 + 5^2}\sqrt{(-1)^2 + (-3)^2 + 4^2} \cos \theta$$

$$-1 - 6 + 20 = \sqrt{30}\sqrt{26} \cos \theta$$

Thus $\cos \theta = \dfrac{13}{\sqrt{(30 \times 26)}} = 0.46547\ldots$ so that $\theta \doteq 62°$. ■

(Recall that this is the angle between the vectors *when they are drawn with a common tail*.)

Example 3 Find the value of x if $\overrightarrow{(-2,0,-6)} \cdot \overrightarrow{(1,3,x)} = 10$.

Solution The dot product is $(-2)(1) + (0)(3) + (-6)(x) = 10$
hence $-2 - 6x = 10$ or $x = -2$. ■

Example 4 If $\vec{u} = \overrightarrow{(3,-1)}$, find a two-dimensional vector perpendicular to $\vec{u}$.

Solution Let $\vec{v} = \overrightarrow{(x,y)}$ be perpendicular to $\vec{u}$.

Then $\overrightarrow{(3,-1)} \cdot \overrightarrow{(x,y)} = 0$

$3x + (-y) = 0$ or $y = 3x$.

If x is any real number, say $x = k$, then $y = 3k$.

So $\vec{v} = \overrightarrow{(k,3k)}$ is perpendicular to $\vec{u}$, no matter what the value of k. For instance, if $k = 2$, then $\vec{v} = \overrightarrow{(2,6)}$, which is perpendicular to $\vec{u}$. ■

The formulas on page 126 calculate the dot product of vectors expressed in components with the basis vectors $\vec{i},\vec{j}$ or $\vec{i},\vec{j},\vec{k}$. Would similar results hold for *any basis*? In the demonstration on page 126, you used facts such as $\vec{i} \cdot \vec{i} = 1$ and $\vec{i} \cdot \vec{j} = 0$ (properties 6 and 7). If you were using *any* basis, this would not necessarily be true, so the answer is NO.

The formulas stated are true because:

1. $\vec{i}, \vec{j}, \vec{k}$ are *unit vectors*, and
2. $\vec{i}, \vec{j}, \vec{k}$ are *mutually perpendicular* or orthogonal.

A basis which has these two qualities is called an **orthonormal basis**. Similar formulas for the calculation of the dot product would hold true only in an orthonormal basis.

SUMMARY

A basis of a vector space is orthonormal if

1. the basis vectors are all unit vectors
2. the basis vectors are all mutually perpendicular

In an orthonormal basis of $\mathbb{V}_2$,
$\vec{u} = \overrightarrow{(u_1,u_2)}$ and $\vec{v} = \overrightarrow{(v_1,v_2)}$
$\Rightarrow \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2$

In an orthonormal basis of $\mathbb{V}_3$,
$\vec{u} = \overrightarrow{(u_1,u_2u_3)}$ and $\vec{v} = \overrightarrow{(v_1,v_2v_3)}$
$\Rightarrow \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$

3.3 Exercises

1. Find the dot product $\vec{u} \cdot \vec{v}$ if
   a) $\vec{u} = \overrightarrow{(1,2,4)}$ $\vec{v} = \overrightarrow{(2,0,3)}$
   b) $\vec{u} = \overrightarrow{(4,5)}$ $\vec{v} = \overrightarrow{(2,-4)}$
   c) $\vec{u} = \overrightarrow{(4,5)}$ $\vec{v} = \overrightarrow{(-5,4)}$
   d) $\vec{u} = \overrightarrow{(1,0,0)}$ $\vec{v} = \overrightarrow{(6,17,-32)}$
   e) $\vec{u} = \overrightarrow{(-1,3,4)}$ $\vec{v} = \overrightarrow{(-2,2,-2)}$

   What can you say about the vectors in parts c) and e)?

2. If $\vec{e}$ is a unit vector, what is $\vec{e} \cdot \vec{e}$?

3. Use the component formula to calculate the following in $\mathbb{V}_3$.
   a) $\vec{i} \cdot \vec{k}$ b) $\vec{j} \cdot \vec{j}$

4. Find the value of t if the vectors $\vec{u} = \overrightarrow{(3,t,-2)}$ and $\vec{v} = \overrightarrow{(4,-1,5)}$ are perpendicular.

5. Find the angle between the following pairs of vectors. Give your answers to the nearest degree.
   a) $\vec{a} = \overrightarrow{(3,7)}$ $\vec{b} = \overrightarrow{(2,1)}$
   b) $\vec{a} = \overrightarrow{(-5,0)}$ $\vec{b} = \overrightarrow{(0,1)}$
   c) $\vec{a} = \overrightarrow{(-5,1)}$ $\vec{b} = \overrightarrow{(1,2)}$
   d) $\vec{a} = \overrightarrow{(2,-3,1)}$ $\vec{b} = \overrightarrow{(5,0,-6)}$

6. Calculate k given that
   a) $\overrightarrow{(3,9)} \cdot \overrightarrow{(k,-1)} = 0$
   b) $\overrightarrow{(3,k,2)} \cdot \overrightarrow{(0,5,4,)} = 6$

7. a) Calculate the two values of k if $\vec{u} \cdot \vec{v} = 3$ where $\vec{u} = \overrightarrow{(k,1,4)}$ and $\vec{v} = \overrightarrow{(k,2k,-3)}$.
   b) For each of the values of k found in a), calculate the angle between $\vec{u}$ and $\vec{v}$.

8. Calculate $\vec{v} \cdot \vec{v}$ in the following cases.
   a) $\vec{v} = \overrightarrow{(3,-2)}$ c) $\vec{v} = \overrightarrow{(x,y,z)}$
   b) $\vec{v} = \overrightarrow{(-1,4,3)}$

9. a) Find a vector $\vec{p}$ perpendicular to the vector $\vec{q} = \overrightarrow{(4,-5)}$.
   b) Normalize $\vec{p}$.

10. Given the vectors $\vec{u} = \overrightarrow{(1,-3,2)}$, $\vec{v} = \overrightarrow{(-4,1,1)}$, and $\vec{w} = \overrightarrow{(2,0,5)}$, calculate the following.
    a) $2\vec{u} \cdot \vec{v}$
    b) $(\vec{u} + \vec{v}) \cdot \vec{w}$
    c) $-4(\vec{v} \cdot \vec{w})$
    d) $\vec{u} \cdot \vec{v} - \vec{w} \cdot \vec{v}$
    e) $(2\vec{u} - \vec{v}) \cdot (2\vec{u} + \vec{v})$

11. If $\vec{v} \neq \vec{0}$, prove that the angle between $\vec{v}$ and $-\vec{v}$ is 180°.

12. For any vector $\vec{v}$ of $\mathbb{V}_3$, prove that $(\vec{v} \cdot \vec{i})\vec{i} + (\vec{v} \cdot \vec{j})\vec{j} + (\vec{v} \cdot \vec{k})\vec{k} = \vec{v}$

13. $\vec{p}$ and $\vec{q}$ are unit vectors at an angle of 60° with each other.
    a) Calculate $(\vec{p} - 3\vec{q}) \cdot (\vec{p} - 3\vec{q})$.
    b) Hence find the unit vector in the direction of $\vec{p} - 3\vec{q}$.

14. $\vec{u}$, $\vec{v}$, and $\vec{w}$ are three distinct non-zero vectors. $\vec{v} \perp$ to both $\vec{u}$ and $\vec{w}$.
    a) If $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{v} \cdot (\vec{u} - \vec{w})$, prove that $\vec{w}$ is perpendicular to $(\vec{u} + \vec{v})$.
    b) If $(\vec{u} \cdot \vec{v})\vec{w} = (\vec{v} \cdot \vec{w})\vec{u}$, prove that $\vec{u} \| \vec{w}$.

15. The angle between the vectors $\vec{a}$ and $\vec{b}$ is θ where $\cos\theta = \frac{3}{7}$. If $\vec{a} = \overrightarrow{(2,3,-1)}$ and $\vec{b} = \overrightarrow{(-1,k,1)}$, find the possible values of k, correct to 3 decimal places.

16. A triangle is such that its three sides represent the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$. By expressing $\vec{c}$ in terms of $\vec{a}$ and $\vec{b}$, prove the cosine law. That is, prove that $|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos C$, where C is the angle between $\vec{a}$ and $\vec{b}$.

17. The position vectors of points A, B, C relative to the origin O are $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively. AB is perpendicular to OC and BC is perpendicular to OA. Prove that OB and AC are perpendicular.

3.4 Applications: The Dot Product and Trigonometry

In this section you will use the dot product to prove some trigonometric results. These proofs are very much neater than the proofs you have seen before, due to the power of vector algebra.

Finding Components

You can use the dot product to create another formula to obtain the component of a vector in the direction of another vector. Indeed, if the angle between $\vec{u}$ and $\vec{v}$ is θ, then the component of $\vec{u}$ on $\vec{v}$ is $|\vec{u}| \cos \theta$.

But $\quad |\vec{u}||\vec{v}| \cos \theta = \vec{u} \cdot \vec{v}$

Dividing each side by the scalar $|\vec{v}|$ gives

$$|\vec{u}| \cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$$

Thus the **component of $\vec{u}$ on $\vec{v}$**, $c = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \vec{u} \cdot \vec{e}_v$,

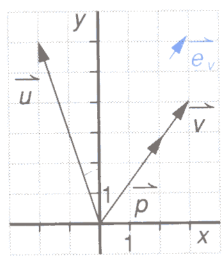
where $\vec{e}_v$ is the unit vector in the direction of $\vec{v}$.

Thus, the **projection of $\vec{u}$ on $\vec{v}$**, $\vec{p} = (\vec{u} \cdot \vec{e}_v)\, \vec{e}_v$

The following example matches Example 4 of section 3.1.

Example Given $\vec{u} = \overrightarrow{(-2,6)}$ and $\vec{v} = \overrightarrow{(3,4)}$, find the following.

a) the component of $\vec{u}$ on $\vec{v}$ **b)** the projection of $\vec{u}$ on $\vec{v}$



Solution **a)** The unit vector $\vec{e}_v = \dfrac{1}{|\vec{v}|}\vec{v} = \dfrac{1}{\sqrt{3^2 + 4^2}} \overrightarrow{(3,4)} = \dfrac{1}{5} \overrightarrow{(3,4)} = \overrightarrow{(0.6,0.8)}$

Thus the component of $\vec{u}$ on $\vec{v}$ is $c = \overrightarrow{(-2,6)} \cdot \overrightarrow{(0.6,0.8)} = 3.6$

b) The projection of $\vec{u}$ on $\vec{v}$ is thus

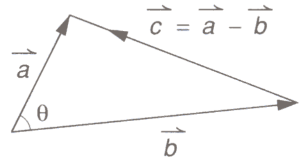
$\vec{p} = (3.6)\vec{e}_v = 3.6\overrightarrow{(0.6,0.8)} = \overrightarrow{(2.16,2.88)}$ ■

Note: It is not necessary to know the angle between $\vec{u}$ and $\vec{v}$, as in 3.1.

You will find other applications of the dot product in chapter 4, but some of these are interesting and spectacular enough to be introduced now.

A Proof of the Cosine Law using the Dot Product

In the figure, any vectors $\vec{a}$ and $\vec{b}$ are sketched with their tails in common.
The angle between $\vec{a}$ and $\vec{b}$ is θ.
The vector $\vec{c}$ which 'closes the triangle' as shown is equal to $\vec{a} - \vec{b}$. Now



$\vec{c}\cdot\vec{c} = (\vec{a} - \vec{b})\cdot(\vec{a} - \vec{b})$

$= \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$ — using property 2

$|\vec{c}|^2 = |\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{b}|^2$ — using properties 1 and 4

$|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$ — using the definition

which is already the cosine law!

From this fast proof you might appreciate the power of the dot product, and the vastness of the possible applications of vectors to mathematics. If you look back at a traditional proof of the cosine law, you will see how much more concise this one is.

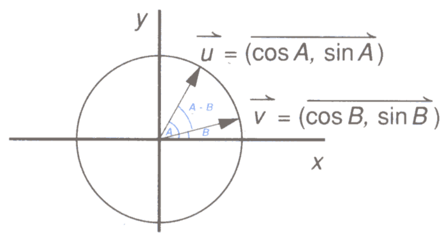
Finding Compound Angle Identities

Another fundamental result in trigonometry is the set of compound angle identities, that is, the formulas for the sine, cosine, and tangent of the sum and difference of two angles (see page 542).

Finding an expression for, say, $\cos(A - B)$ in terms of trigonometrical ratios of A or B only, by applying basic trigonometry, is a much longer process.

Once again, the dot product of vectors will allow you to arrive at a result remarkably quickly.

Consider two unit vectors $\vec{u}$ and $\vec{v}$, making angles A and B with the horizontal respectively, as shown.



In component form, the vectors are $\vec{u} = \overrightarrow{(\cos A, \sin A)}$ and $\vec{v} = \overrightarrow{(\cos B, \sin B)}$.

$|\vec{u}| = \sqrt{\cos^2 A + \sin^2 A} = 1$, as expected, and $|\vec{v}| = 1$. Now

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(A - B) = (1)(1)\cos(A - B) = \cos(A - B)$ ①

but by using components

$\vec{u}\cdot\vec{v} = \overrightarrow{(\cos A, \sin A)}\cdot\overrightarrow{(\cos B, \sin B)} = \cos A\cos B + \sin A\sin B.$ ②

From ① and ②, $\cos(A - B) = \cos A\cos B + \sin A\sin B$! !

The speed and conciseness of this proof, compared to traditional proofs, is even more striking than the comparison in the case of the cosine law.

3.4 Exercises

1. a) Calculate, to the nearest degree, the angle between the vector $\vec{v} = \overrightarrow{(2,3,-5)}$ and the three coordinate axes.
   b) What is the component of $\vec{v}$ on each of $\vec{i}$, $\vec{j}$, and $\vec{k}$?

2. Calculate, to the nearest degree, the angle between $\overrightarrow{PQ}$ and $\vec{i}$, if P and Q are the following points.
   a) $P(2,1)$, $Q(3,7)$
   b) $P(-2,0)$, $Q(4,5)$
   c) $P(-2,-2,1)$, $Q(4,3,2)$
   d) $P(5,3,1)$, $Q(1,-1,1)$

3. Using the points of question 2, state the component of $\overrightarrow{PQ}$ on $\vec{i}$ in each case.

4. Calculate the component of $\vec{u}$ on $\vec{v}$, and the component of $\vec{v}$ on $\vec{u}$, in the following cases.
   a) $\vec{u} = \overrightarrow{(-3,5)}$, $\vec{v} = \overrightarrow{(1,2)}$
   b) $\vec{u} = \overrightarrow{(-1,1,1)}$, $\vec{v} = \overrightarrow{(2,4,-5)}$

5. Show that the triangle ABC with vertices $A(2,2,2)$, $B(-2,-4,0)$, $C(0,-12,2)$ has an obtuse angle at B.

6. Determine the angles of the triangle PQR in the following cases, to the nearest degree.
   a) $P(3,-1)$, $Q(4,4)$, $R(-2,-3)$
   b) $P(2,0,1)$, $Q(5,1,-3)$, $R(-4,2,7)$

The next three questions match questions 7, 9, and 13 of 3.1 Exercises. Use the dot product to find the solutions here.

7. Resolve the following vectors on $\vec{i}$, $\vec{j}$, and $\vec{k}$.
   $\vec{u} = \overrightarrow{(4,5,0)}$ $\qquad$ $\vec{v} = \overrightarrow{(-2,-3,1)}$

8. Calculate the projection of the following on $\vec{v} = \overrightarrow{(1,1)}$.
   a) $\vec{i}$ $\qquad$ b) $\vec{j}$

9. Given two non-zero vectors $\vec{u}$ and $\vec{v}$, what can you deduce about $\vec{u}$ and $\vec{v}$ where
   a) the component of $\vec{u}$ on $\vec{v}$ is equal to the component of $\vec{v}$ on $\vec{u}$?
   b) the projection of $\vec{u}$ on $\vec{v}$ is equal to the projection of $\vec{v}$ on $\vec{u}$?

10. Given the vectors $\vec{u} = \overrightarrow{(2,10)}$ and $\vec{v} = \overrightarrow{(-3,-2)}$, find each of the following, correct to 3 significant digits.
   a) the component of $\vec{u}$ on $\vec{v}$
   b) the projection of $\vec{u}$ on $\vec{v}$

11. Resolve the vector $\vec{v} = \overrightarrow{(-2,3)}$ onto the vectors $\vec{a} = \overrightarrow{(1,1)}$ and $\vec{b} = \overrightarrow{(-1,1)}$.

12. Use the dot product to determine whether or not the following points determine a right-angled triangle.
   a) $A(2,1)$, $B(6,5)$, $C(3,0)$
   b) $A(2,1)$, $B(3,-1)$, $C(6,5)$
   c) $A(1,-1,5)$, $B(2,3,-4)$, $C(3,5,-3)$

13. A circle of centre O has a diameter PR. C is any other point on the circle.
   a) If $\overrightarrow{OR} = \vec{r}$, state the vector $\overrightarrow{OP}$ in terms of $\vec{r}$.
   b) If $\overrightarrow{OC} = \vec{c}$, express the vectors $\overrightarrow{RC}$ and $\overrightarrow{PC}$ in terms of $\vec{r}$ and $\vec{c}$.
   c) Calculate the dot product $\overrightarrow{RC} \cdot \overrightarrow{PC}$.
   d) Hence deduce the value of an angle inscribed in a semi-circle.

14. A fact about a circle is that any angle inscribed in a given segment of a circle is constant. A converse of this theorem can be described as follows. If A, B, C, and D are four points such that $\measuredangle ABD = \measuredangle ACD$, then A, B, C, and D lie on a circle. Use this fact to prove that the following four points lie on a circle. (Such points are called cyclic or **concyclic**).

   $A(-2,-2)$, $B(-1,5)$, $C(6,4)$, $D(7,1)$.

3.5 The Cross Product

The second product of two vectors $\vec{u}$ and $\vec{v}$ is written $\vec{u}\times\vec{v}$, and is called the **cross product** or **vector product** of $\vec{u}$ and $\vec{v}$.

The cross product is a regular binary operation, unlike the dot product which gives you a scalar result. In other words, the result of the cross product of two vectors is also a vector.

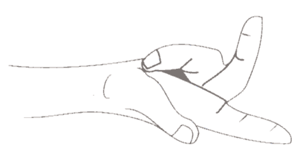
When writing this product, you *must* use a 'cross' ($\times$) as shown:

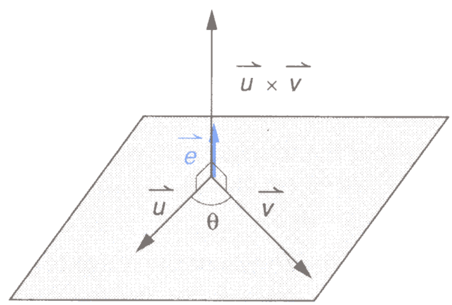
DEFINITION

$$\vec{u}\times\vec{v} = (|\vec{u}||\vec{v}|\sin\theta)\vec{e}$$

where θ is the angle between the two vectors when they are drawn with a common tail, and $\vec{e}$ is a unit vector *perpendicular* to both $\vec{u}$ and $\vec{v}$ chosen in such a way that the triple $[\vec{u}, \vec{v}, \vec{e}]$ forms a right-handed system.

(See chapter 1, section 2.
Using your right hand, the directions of $\vec{u}$, $\vec{v}$, and $\vec{e}$ are represented by your thumb, your first finger and your second finger respectively.)





The diagram shows the direction of $\vec{e}$ and hence of $\vec{u}\times\vec{v}$.

PROPERTY

If $\vec{u}$ and $\vec{v}$ were interchanged, then $\vec{e}$ would be pointing in precisely the opposite direction. (Again, check this with your right hand.) This indicates that $\vec{v}\times\vec{u} = -\vec{u}\times\vec{v}$. This will be confirmed in the following example.

Note: Since a three-dimensional system is required for its definition, the cross product *is meaningless in* $\mathbb{V}_2$.

Example 1 In each of the following, find the cross products, if possible.

a) If $|\vec{u}| = 4$, $|\vec{v}| = 5$, $\theta = 30°$, find $\vec{u}\times\vec{v}$.

b) If $\vec{u} = \overrightarrow{(2,5)}$, $\vec{v} = \overrightarrow{(-1,1)}$, find $\vec{u}\times\vec{v}$.

c) If $\vec{u} = \overrightarrow{(2,5,0)}$, $\vec{v} = \overrightarrow{(-1,1,0)}$, find both $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$.

Solution a) $\vec{u}\times\vec{v} = (|\vec{u}||\vec{v}|\sin\theta)\vec{e} = (4\times5\times\sin 30°)\vec{e} = (20)(0.5)\vec{e} = 10\vec{e}$

Note: *The direction of $\vec{e}$, and hence of $\vec{u}\times\vec{v}$, is not known,* unless the exact directions of $\vec{u}$ and of $\vec{v}$ are known. In this example, you do *not* have that information.

b) $\vec{u}\times\vec{v}$ is not defined in $\mathbb{V}_2$, so this vector product is *impossible*.

c) $|\vec{u}| = \sqrt{2^2+5^2+0^2} = \sqrt{29}$, and $|\vec{v}| = \sqrt{(-1)^2+1^2+0^2} = \sqrt{2}$.

You can obtain the angle θ between $\vec{u}$ and $\vec{v}$ by using the dot product.

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$, so $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$

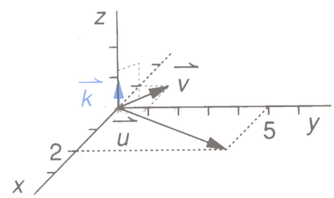
$$\cos\theta = \frac{\overrightarrow{(2,5,0)}\cdot\overrightarrow{(-1,1,0)}}{\sqrt{29}\sqrt{2}}$$

$$= \frac{(2)(-1)+(5)(1)+(0)(0)}{\sqrt{58}} = \frac{3}{\sqrt{58}} = 0.3939\ldots \Rightarrow \theta \doteq 67°.$$

Thus $\vec{u}\times\vec{v} = |\vec{u}||\vec{v}|\sin\theta\vec{e}$

$= \sqrt{29}\sqrt{2}\sin 67°\ \vec{e}$

$= (\sqrt{58})(0.9191\ldots) = 7\vec{e}$.



But vectors $\vec{u}$ and $\vec{v}$ are in the xy-plane, as the diagram shows. The triple $[\vec{u}, \vec{v}, \vec{k}]$ forms a right-handed system. Thus, $\vec{e} = \vec{k}$, therefore $\vec{u}\times\vec{v} = 7\vec{k}$, or $\overrightarrow{(0,0,7)}$.

If $\vec{u}$ and $\vec{v}$ are interchanged, the second finger of your right hand points in the *opposite direction* to the direction of $\vec{k}$, namely in the direction of $-\vec{k}$.

Thus $\vec{v}\times\vec{u} = |\vec{v}||\vec{u}|\sin\theta\ (-\vec{k})$

$= \sqrt{2}\sqrt{29}\sin 113°\ (-\vec{k})$

$= -7\vec{k}$. ■

Geometrical interpretation of the Cross Product

Consider a parallelogram $ABCD$ where $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{AD} = \vec{v}$. Let H be the foot of the perpendicular from D to AB.

If θ is the angle between $\vec{u}$ and $\vec{v}$, then $DH = |\vec{v}|\sin\theta$.

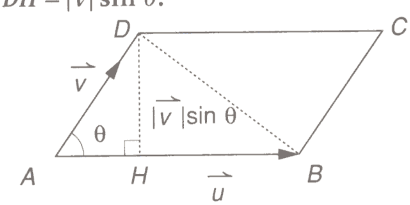
The area of this parallelogram is

(base)(height)

$= (AB)(DH)$

$= |\vec{u}|(|\vec{v}|\sin\theta)$

$= |\vec{u}\times\vec{v}|$.



In other words, the magnitude of $\vec{u}\times\vec{v}$ is the area of the parallelogram whose sides are represented by the vectors $\vec{u}$ and $\vec{v}$.

Example 2 A parallelogram $ABCD$ is such that $|\overrightarrow{AB}| = 6$, $|\overrightarrow{AD}| = 5$, and $\measuredangle BAD$ is 28°. Calculate the area of the parallelogram, correct to 1 decimal place.

Solution The area of $ABCD$ is
$|\overrightarrow{AB} \times \overrightarrow{AD}| = |\overrightarrow{AB}||\overrightarrow{AD}| \sin 28^\circ = (6)(5)(0.4694\ldots) \doteq 14.1.$ ■

The following examples should help you to discover further properties of the cross product.

Example 3 Calculate a) $\vec{i} \times \vec{i}$, b) $\vec{i} \times \vec{j}$ c) $\vec{k} \times \vec{j}$

Solution a) $\vec{i} \times \vec{i} = (1)(1)(\sin 0^\circ)\vec{e} = 0\vec{e} = \vec{0}$

Note: You cannot find a *unique* direction for $\vec{e}$, because there are many vectors perpendicular to $\vec{i}$ in $\mathbb{V}_3$. However, this is not important, since the coefficient of this vector is 0. Thus the result is the zero vector, $\vec{0}$, which is considered to have *any* direction.

PROPERTY The cross product of a vector with itself is the zero vector.

b) $\vec{i} \times \vec{j} = (1)(1)(\sin 90^\circ)\vec{k} = \vec{k}$
c) $\vec{k} \times \vec{j} = (1)(1)(\sin 90^\circ)(-\vec{i}) = -\vec{i}$ ■

right-handed system

PROPERTY The cross product of any two distinct standard basis vectors of $\mathbb{V}_3$ gives ±(the third standard basis vector).

Example 4 Calculate $(3\vec{i}) \times (4\vec{j})$.

Solution $(3\vec{i}) \times (4\vec{j}) = (3)(4)(\sin 90^\circ)\vec{k} = 12\vec{k}$. ■

Compare the result of this example to that of Example 3b).

PROPERTY For any vectors $\vec{u}$, $\vec{v}$, and scalars m, n, $(m\,\vec{u}) \times (n\,\vec{v}) = (mn)\vec{u} \times \vec{v}$.

You might note that the vectors in the examples of this section were in one of the major planes of $\mathbb{V}_3$. If they had not been, you would have had difficulty in determining the direction of $\vec{e}$. The method of the next section will allow you to find the cross product of *any* two vectors of $\mathbb{V}_3$.

SUMMARY The cross product of vectors $\vec{u}$ and $\vec{v}$ is the vector $\vec{u} \times \vec{v} = |\vec{u}||\vec{v}| \sin \theta\ \vec{e}$, where $\vec{e}$ is a unit vector perpendicular to $\vec{u}$ and $\vec{v}$ such that the triple $[\vec{u}, \vec{v}, \vec{e}]$ forms a right-handed system.

The magnitude of $\vec{u} \times \vec{v}$ is the area of the parallelogram whose sides are represented by the vectors $\vec{u}$ and $\vec{v}$.

$\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$.

3.5 Exercises

1. Calculate $\vec{u}\times\vec{v}$ for each of the following, where $\vec{u}$ and $\vec{v}$ are vectors of $\mathbb{V}_3$. Specify the direction as precisely as you can in each case.
   a) $|\vec{u}| = 5, |\vec{v}| = 2$, and the angle between $\vec{u}$ and $\vec{v}$ is $30°$
   b) $|\vec{u}| = 12, |\vec{v}| = 7$, and the angle between $\vec{u}$ and $\vec{v}$ is $150°$
   c) $|\vec{u}| = 3, |\vec{v}| = 6$, and $\vec{u}\cdot\vec{v} = 8$
   d) $|\vec{u}| = 4, |\vec{v}| = 5$, and $\vec{u}\cdot\vec{v} = -10$
   e) $\vec{u} = \overrightarrow{(0,6,3)}$ and $\vec{v} = \overrightarrow{(0,-1,5)}$
   f) $\vec{u} = \overrightarrow{(0,-1,5)}$ and $\vec{v} = \overrightarrow{(0,6,3)}$

2. Calculate
   a) $\vec{j}\times\vec{j}$
   b) $\vec{j}\times\vec{k}$
   c) $\vec{j}\times\vec{i}$.

3. A parallelogram $ABCD$ is such that $|\overrightarrow{AB}| = 10, |\overrightarrow{AD}| = 4$, and the angle BAD is $42°$. Calculate the area of the parallelogram, correct to 1 decimal place.

4. A triangle ABC is such that $|\overrightarrow{AB}| = 15$, $|\overrightarrow{AC}| = 12$, and $\overrightarrow{AB}\cdot\overrightarrow{AC} = -100$. Calculate the area of the triangle, correct to 1 decimal place.

5. Given any three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ of $\mathbb{V}_3$, determine which of the following expressions are meaningful.
   a) $\vec{u}\times(\vec{v}\cdot\vec{w})$
   b) $\vec{u}\times(\vec{v}+\vec{w})$
   c) $\vec{u}\times(\vec{v}-\vec{w})$
   d) $(\vec{u}\times\vec{v})\vec{w}$
   e) $(\vec{u}\times\vec{v})\times\vec{w}$
   f) $(\vec{u}\cdot\vec{v}) + (\vec{u}\times\vec{v})$

6. Prove that the cross product of a vector with itself is the zero vector.

7. Prove that, for any vectors $\vec{u}$ and $\vec{v}$ of $\mathbb{V}_3$, $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$.

8. Suppose m is any scalar, and $[\vec{u},\vec{v},\vec{e}]$ is any right-handed system of vectors, where $\vec{e}$ is a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$. Prove that $(m\,\vec{u})\times\vec{v} = m(\vec{u}\times\vec{v})$ in the following cases.
   a) m is positive
   b) m is negative

9. Using the results of question 8, prove that, if n is another scalar, $(m\,\vec{u})\times(n\,\vec{v}) = (mn)(\vec{u}\times\vec{v})$

10. Prove that $\vec{u}\times\vec{v} = \vec{0}$ if and only if $\vec{u}$ and $\vec{v}$ are linearly dependent.

11. Consider the standard basis vectors $\vec{i}$, $\vec{k}$, and the vector $\vec{u} = \overrightarrow{(1,1,0)}$ of $\mathbb{V}_3$.
    a) Explain why $(\vec{i}\times\vec{u})\times\vec{k} = \vec{0}$.
    b) Does $\vec{i}\times(\vec{u}\times\vec{k}) = \vec{0}$?
    c) State whether or not the cross product is associative.

12. Prove that, for any vectors $\vec{u}$ and $\vec{v}$ of $\mathbb{V}_3$, $\vec{u}\cdot(\vec{u}\times\vec{v}) = 0$.

13. The standard basis vectors of $\mathbb{V}_3$ are $\vec{i}$, $\vec{j}$, and $\vec{k}$. Prove that the cross product of any two distinct standard basis vectors in $\mathbb{V}_3$ gives ±(the third standard basis vector). (You may use a general argument, or treat this case by case.)

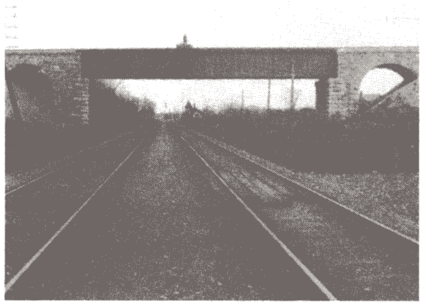
14. Given that the vectors $\vec{u}$ and $\vec{v}$ of $\mathbb{V}_3$ make an angle θ with each other, prove the following.
    a) $(\vec{u}\cdot\vec{u})(\vec{v}\cdot\vec{v}) - (\vec{u}\cdot\vec{v})^2 = |\vec{u}|^2|\vec{v}|^2\sin^2\theta$
    b) $|\vec{u}\times\vec{v}| = \sqrt{(\vec{u}\cdot\vec{u})(\vec{v}\cdot\vec{v}) - (\vec{u}\cdot\vec{v})^2}$

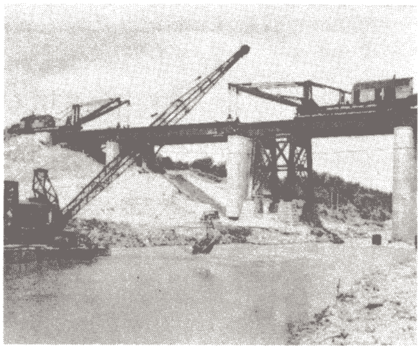
15. a) If $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$, does it necessarily follow that $\vec{b} = \vec{c}$?
    b) If $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$ *and* $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$, does it necessarily follow that $\vec{b} = \vec{c}$?

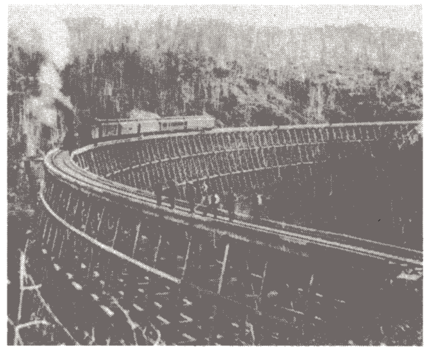
16. Verify that $(\vec{i}+\vec{j})\times\vec{k} = \vec{i}\times\vec{k} + \vec{j}\times\vec{k}$. (This illustrates that the cross product is distributive over vector addition.)

M A K I N G

Bridges







Bridges have been in existence for a long time. The first people on earth probably used a fallen tree to go from one side of a stream to the other.

In the 18th, 19th and the early 20th centuries much of the work done in designing bridges was based on the success of previously constructed bridges. If an earlier construction had no problems, bigger, larger, and more elaborate versions of the old structures were erected. When a bridge did fail, then engineers would locate the source of the problem and add more safety features in future structures.

With the advent of the railroad in Canada and the United States in the 19th century much work was done developing strong truss structures that would enable the construction of longer bridges. One problem that had to be overcome was caused by the vibrations set off in a bridge by a train passing over the bridge.

Designers were not always successful. In 1877 a bridge in Ashtabula, Ohio, failed as a train passed over it. 90 people were killed. Two years later, a train consisting of a locomotive and six passenger coaches fell from a collapsing bridge into the Firth of Tay in Scotland, killing over 100 passengers. During the period from 1870 to 1890, truss bridges in the United States failed at the rate of 25 per year. Something needed to be done.

In 1934, C.E. Inglis completed a long study called *A Mathematical Treatise on Vibrations in Railway Bridges*. In this treatise he wrote, "Mathematical analysis is required to indicate the lines along which experiments should proceed, and experiment, in its turn, is necessary to check the validity of theoretical predictions and to prevent mathematics running off the scent and barking, so to speak, up the wrong tree."

Nowadays one expects engineers to make use of mathematics in their designs of bridges. Indeed, calculations and simulations based on appropriate mathematical models form the basis of structural engineering. The use of theory, rather than experience, to design bridges, has been in part motivated by the desire for safety, and to use new materials that will cut costs and speed construction.

Theory has allowed designers of bridges to come closer to the limits of safety while still maintaining a large margin of safety. But nevertheless, designers still run the risk of structural failure, especially if they do not take into account all physical situations.

One notable failure was 'Galloping Gertie'. Galloping Gertie was the nickname given to a bridge in the United States built to cross the Puget Sound in Tacoma, Washington.

From the time it opened on July 1, 1940, the bridge achieved a certain popularity and notoriety due to its tendency to sway in the wind. People would drive over the bridge just to get a roller coaster feeling. But all was not well. Just a few months later, on November 7, 1940, the bridge collapsed into the Puget Sound.

After the collapse of this bridge, new studies were made to try to prevent similar disasters. In the case of Gertie, the static analysis of the bridge had been done correctly but proper attention had not been paid to aerodynamical considerations.

Now mathematicians, computer scientists, and engineers act together with architects to design bridges. Computer simulations of the design features and workings of a bridge can be produced graphically on a computer screen. These new bridges should be beautiful, functional bridges that do not collapse.

3.6 Properties of the Cross Product

In the examples and exercises of the last section, you proved the first four of the following properties of the cross product.

For any vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ and any scalar m,

PROPERTIES

1. $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$ [not commutative]
2. $\vec{u}\times\vec{u} = \vec{0}$
3. $\vec{i}\times\vec{j} = \vec{k}$, $\vec{j}\times\vec{k} = \vec{i}$, $\vec{k}\times\vec{i} = \vec{j}$
4. $(m\,\vec{u})\times\vec{v} = m(\vec{u}\times\vec{v})$
5. $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v}+\vec{u}\times\vec{w}$ [distributive over +]

This last property will be proved later in this section. Before this can be done, you need to investigate the following product.

Triple Scalar Product

The expression $(\vec{a}\times\vec{b})\cdot\vec{c}$ is known as a **triple scalar product.**

Note 1 $(\vec{a}\times\vec{b})\cdot\vec{c}$ is meaningful, since $\vec{a}\times\vec{b}$ and $\vec{c}$ are both vectors. Thus the dot product *can* be obtained, giving a *scalar* as a final result.

2 "$\vec{a}\times(\vec{b}\cdot\vec{c})$" is meaningless. You cannot perform a cross product with a vector and a scalar.

Consider a parallelepiped whose sides emanating from O are represented by vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, where $\vec{a}$, $\vec{b}$, $\vec{c}$ form a right-handed system. Let the height of the parallelepiped be represented by the vector $\overrightarrow{HC} = \vec{h}$.

The volume V of the parallelepiped can be calculated as follows.

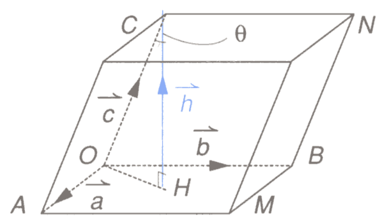
V = (base area) (height)

$= (OAMB)(\overrightarrow{HC})$

$= |\vec{a}\times\vec{b}||\vec{h}|$

$= |\vec{a}\times\vec{b}||\vec{c}|\cos\theta|$

$= |(\vec{a}\times\vec{b})\cdot\vec{c}|$ ①



Since the vector $\vec{a}\times\vec{b}$ has the same direction as $\vec{h}$ (which is perpendicular to the base), the angle between $\vec{a}\times\vec{b}$ and $\vec{c}$ is the same as the angle between $\vec{h}$ and $\vec{c}$, namely θ. Thus the dot product $(\vec{a}\times\vec{b})\cdot\vec{c}$ is equal, by definition, to $|\vec{a}\times\vec{b}||\vec{c}|\cos\theta|$. Furthermore, since $[\vec{a},\vec{b},\vec{c}]$ is a right-handed system, the angle θ is acute ($0^\circ < \theta \leqslant 90^\circ$), so $\cos\theta$ is positive. Thus, the dot product is positive, and the absolute value signs of line ① are not required.

Therefore $V = (\vec{a}\times\vec{b})\cdot\vec{c}$.

Similarly, by using parallelogram *OBNC* instead of parallelogram *OAMB* as a base, you can calculate the volume of the parallelepiped as

$V = (\vec{b} \times \vec{c}) \cdot \vec{a}$,

and since the dot product is commutative,

$V = \vec{a} \cdot (\vec{b} \times \vec{c})$.

(Notice that $[\vec{b},\vec{c},\vec{a}]$ or $[\vec{a},\vec{b},\vec{c}]$ are both right-handed systems.)

Thus, $V = (\vec{a} \times \vec{b}) \cdot \vec{c}$ and $V = \vec{a} \cdot (\vec{b} \times \vec{c})$.

From this argument, the following property of triple scalar products can be deduced.

PROPERTY

$\vec{a} \times \vec{b} \cdot \vec{c} = \vec{a} \cdot \vec{b} \times \vec{c}$

In other words, *as long as the order of the vectors remains the same, the dot and the cross of a triple scalar product can be interchanged*!

Notice that brackets are not essential, since either expression is meaningful only if the cross product is performed *first*.

Proof of the Distributivity of × over + (property 5)

Consider $\vec{r} = \vec{u} \times (\vec{v} + \vec{w}) - \vec{u} \times \vec{v} - \vec{u} \times \vec{w}$, where $\vec{u}$, $\vec{v}$, and $\vec{w}$ are any vectors. Then

$$\begin{aligned}
\vec{r} \cdot \vec{r} &= \vec{r} \cdot [\vec{u} \times (\vec{v} + \vec{w}) - \vec{u} \times \vec{v} - \vec{u} \times \vec{w}] \\
&= \underline{\vec{r} \cdot \vec{u} \times} (\underline{\vec{v}} + \underline{\vec{w}}) - \underline{\vec{r} \cdot \vec{u} \times \vec{v}} - \underline{\vec{r} \cdot \vec{u} \times \vec{w}} && \text{dot product is distributive} \\
&= \vec{r} \times \vec{u} \cdot (\vec{v} + \vec{w}) - \vec{r} \times \vec{u} \cdot \vec{v} - \vec{r} \times \vec{u} \cdot \vec{w} && \text{triple scalar product property} \\
&= \vec{r} \times \vec{u} \cdot \vec{v} + \vec{r} \times \vec{u} \cdot \vec{w} - \vec{r} \times \vec{u} \cdot \vec{v} - \vec{r} \times \vec{u} \cdot \vec{w} && \text{dot product is distributive} \\
&= 0.
\end{aligned}$$

Thus $\vec{r} = \vec{0}$, that is,

$$\vec{u} \times (\vec{v} + \vec{w}) - \vec{u} \times \vec{v} - \vec{u} \times \vec{w} = \vec{0}, \text{ thus}$$

$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$

which completes the proof.

Now by using the properties of the cross product listed at the beginning of this section, you can find a formula for calculating the cross product of vectors expressed in component form in the orthonormal basis $\vec{i}$, $\vec{j}$, $\vec{k}$.

If $\vec{u} = \overrightarrow{(u_1,u_2,u_3)}$ and $\vec{v} = \overrightarrow{(v_1,v_2,v_3)}$,

$$\begin{aligned}
\vec{u} \times \vec{v} &= (u_1\vec{i} + u_2\vec{j} + u_3\vec{k}) \times (v_1\vec{i} + v_2\vec{j} + v_3\vec{k}) \\
&= u_1v_1(\vec{i} \times \vec{i}) + u_1v_2(\vec{i} \times \vec{j}) + u_1v_3(\vec{i} \times \vec{k}) \\
&\quad + u_2v_1(\vec{j} \times \vec{i}) + u_2v_2(\vec{j} \times \vec{j}) + u_2v_3(\vec{j} \times \vec{k}) && \text{properties} \\
&\quad + u_3v_1(\vec{k} \times \vec{i}) + u_3v_2(\vec{k} \times \vec{j}) + u_3v_3(\vec{k} \times \vec{k}) && \text{4 and 5} \\
&= u_1v_2\vec{k} - u_1v_3\vec{j} - u_2v_1\vec{k} + u_2v_3\vec{i} + u_3v_1\vec{j} - u_3v_2\vec{i} && \text{properties 2 and 3}
\end{aligned}$$

FORMULA

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}$$

This is a particularly difficult formula to remember. It is easier to recall it in the form of a 3×3 determinant.

The 3×3 determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

where a 2×2 determinant, such as $\begin{vmatrix} e & f \\ g & h \end{vmatrix} = eh - fg$.

You will be learning more about determinants in chapter 7.

If $\vec{u} = \overrightarrow{(u_1,u_2,u_3)}$ and $\vec{v} = \overrightarrow{(v_1,v_2,v_3)}$

FORMULA

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Note: The result is indeed a vector.

Because $\vec{u}\times\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$, the cross product can be used to find a vector perpendicular to two given vectors.

Example 1

In each of the following cases, use the cross product to find a vector perpendicular to both $\vec{u}$ and $\vec{v}$.

a) $\vec{u} = \overrightarrow{(2,5,0)}$ and $\vec{v} = \overrightarrow{(-1,1,0)}$

b) $\vec{u} = \overrightarrow{(1,2,3)}$ and $\vec{v} = \overrightarrow{(-2,5,6)}$

Solution

a) $$\vec{u}\times\vec{v} = \overrightarrow{(2,5,0)}\times\overrightarrow{(-1,1,0)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 5 & 0 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= \vec{i}[(5)(0) - (0)(1)] - \vec{j}[(2)(0) - (0)(-1)] + \vec{k}[(2)(1) - (5)(-1)]$$

$$= \vec{i}[0 - 0] - \vec{j}[0 - 0] + \vec{k}[2 - (-5)]$$

$$= 7\vec{k}$$

Note: The result is indeed perpendicular to both $\vec{u}$ and $\vec{v}$, which are in the xy-plane.

(Compare this to the solution of Example 1 c) in the previous section.)

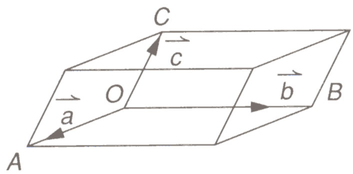
b) $$\vec{u}\times\vec{v} = \overrightarrow{(1,2,3)}\times\overrightarrow{(-2,5,6)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ -2 & 5 & 6 \end{vmatrix}$$

$$= \vec{i}[(2)(6) - (3)(5)] - \vec{j}[(1)(6) - (3)(-2)] + \vec{k}[(1)(5) - (2)(-2)]$$

$$= \vec{i}[12 - 15] - \vec{j}[6 - (-6)] + \vec{k}[5 - (-4)]$$

$$= \overrightarrow{(-3,-12,9)}$$ ■

Example 2 Given $\vec{a} = \overrightarrow{(2,1,1)}$, $\vec{b} = \overrightarrow{(0,-1,1)}$, and $\vec{c} = \overrightarrow{(-1,3,0)}$, use both versions of the triple scalar product to find the volume of the parallelepiped shown.



Solution A volume must be positive. To avoid checking whether or not $[\vec{a},\vec{b},\vec{c}]$ forms a right-handed system, use the absolute value of the triple scalar product.

The volume is either $|(\vec{a}\times\vec{b})\cdot\vec{c}|$ or $|\vec{a}\cdot(\vec{b}\times\vec{c})|$.

$$\text{Now } \vec{a}\times\vec{b} = \overrightarrow{(2,1,1)}\times\overrightarrow{(0,-1,1)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= \vec{i}(1+1) - \vec{j}(2-0) + \vec{k}(-2-0)$$

$$= 2\vec{i} - 2\vec{j} - 2\vec{k}$$

and so $|(\vec{a}\times\vec{b})\cdot\vec{c}| = |\overrightarrow{(2,-2,-2)}\cdot\overrightarrow{(-1,3,0)}| = |-2+(-6)+0| = 8.$

Alternatively,

$$\vec{b}\times\vec{c} = \overrightarrow{(0,-1,1)}\times\overrightarrow{(-1,3,0)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 1 \\ -1 & 3 & 0 \end{vmatrix}$$

$$= \vec{i}(0-3) - \vec{j}(0+1) + \vec{k}(0-1)$$

$$= -3\vec{i} - \vec{j} - \vec{k}$$

and so $|\vec{a}\cdot(\vec{b}\times\vec{c})| = |\overrightarrow{(2,1,1)}\cdot\overrightarrow{(-3,-1,-1)}| = |-6-1-1| = 8.$

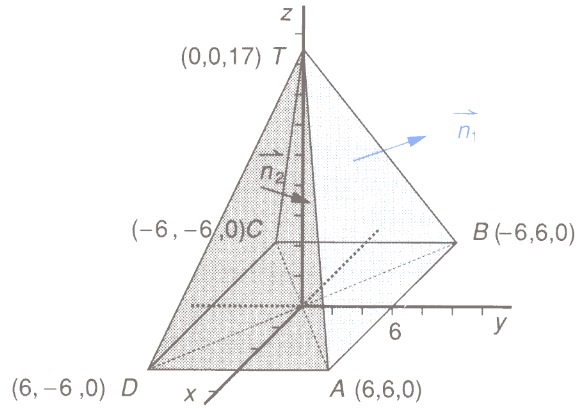
The same volume is obtained in both cases. ■

Example 3 Prove that $\vec{a}\times\vec{b}\cdot\vec{c} = 0$ if and only if $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent.

Solution

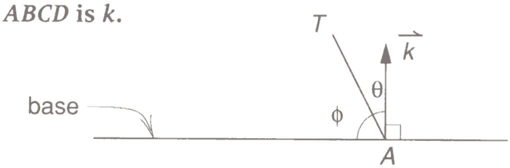
1. Given $\vec{a}\times\vec{b}\cdot\vec{c} = 0$, prove that $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly dependent.
   If $\vec{a}\times\vec{b}\cdot\vec{c} = 0$ then $\vec{a}\times\vec{b}$ is perpendicular to $\vec{c}$.
   But $\vec{a}\times\vec{b}$ is also perpendicular to both $\vec{a}$ and $\vec{b}$,
   so $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar.
2. Given $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly dependent, prove that $\vec{a}\times\vec{b}\cdot\vec{c} = 0$.
   If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar,
   $\vec{a}\times\vec{b}$ is perpendicular to the plane of $\vec{a}$ and $\vec{b}$,
   and since $\vec{c}$ is in the plane of $\vec{a}$ and $\vec{b}$,
   $\vec{c}$ is also perpendicular to $\vec{a}\times\vec{b}$.
   Hence $\vec{a}\times\vec{b}\cdot\vec{c} = 0$. ■

Example 4 A right square pyramid $ABCDT$ whose base has side 12 units and whose height is 17 units is positioned in 3-space with coordinates $A(6,6,0)$, $B(-6,6,0)$, $C(-6,-6,0)$, $D(6,-6,0)$, and $T(0,0,17)$. Calculate the following, giving your answers to the nearest degree.



a) the angle between the edge AT and the base $ABCD$
b) the angle between the planes ABT and ADT

Solution a) A normal vector to the base $ABCD$ is $\vec{k}$.



The diagram shows that you must find ϕ, the complement of the angle θ between $\overrightarrow{AT}$ and $\vec{k}$.
First calculate the angle θ.
$\overrightarrow{AT} = \overrightarrow{OT} - \overrightarrow{OA} = \overrightarrow{(0,0,17)} - \overrightarrow{(6,6,0)} = \overrightarrow{(-6,-6,17)}$
By the dot product

$$\overrightarrow{AT} \cdot \vec{k} = |\overrightarrow{AT}||\vec{k}| \cos\theta$$
$$\overrightarrow{(-6,-6,17)} \cdot \overrightarrow{(0,0,1)} = \sqrt{(-6)^2 + (-6)^2 + 17^2}\,(1)(\cos\theta)$$
$$17 = \sqrt{361}\cos\theta, \text{ or}$$
$$\cos\theta = \frac{17}{19} = 0.8947\ldots \Rightarrow \theta \doteq 27^\circ$$

Thus the required angle is $\phi = 90^\circ - \theta \doteq 63^\circ$

b) In order to find the angle between the planes, first find the angle between normal vectors to each plane.
A normal vector for plane ABT is $\vec{n}_1 = \overrightarrow{TA} \times \overrightarrow{TB}$
A normal vector for plane ADT is $\vec{n}_2 = \overrightarrow{TA} \times \overrightarrow{TD}$
(Note that, by the right-handed rule of the cross product, $\vec{n}_1$ points outside the pyramid, but $\vec{n}_2$ points inside.)

Now from a), $\overrightarrow{TA} = -\overrightarrow{AT} = \overrightarrow{(6,6,-17)}$.
Also, $\overrightarrow{TB} = \overrightarrow{OB} - \overrightarrow{OT} = \overrightarrow{(-6,6,-17)}$ and $\overrightarrow{TD} = \overrightarrow{OD} - \overrightarrow{OT} = \overrightarrow{(6,-6,-17)}$

$$\text{Thus, } \vec{n}_1 = \overrightarrow{(6,6,-17)} \times \overrightarrow{(-6,6,-17)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 6 & -17 \\ -6 & 6 & -17 \end{vmatrix}$$
$$= \vec{i}(0) - \vec{j}(-204) + \vec{k}(72)$$
$$= \overrightarrow{(0,204,72)} = 12\overrightarrow{(0,17,6)}$$

$$\text{Also } \vec{n}_2 = \overrightarrow{(6,6,-17)} \times \overrightarrow{(6,-6,-17)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 6 & -17 \\ 6 & -6 & -17 \end{vmatrix}$$
$$= \vec{i}(-204) - \vec{j}(0) + \vec{k}(-72)$$
$$= \overrightarrow{(-204,0,-72)} = -12\overrightarrow{(17,0,6)}$$

Let the angle between $\vec{n}_1$ and $\vec{n}_2$ be α.

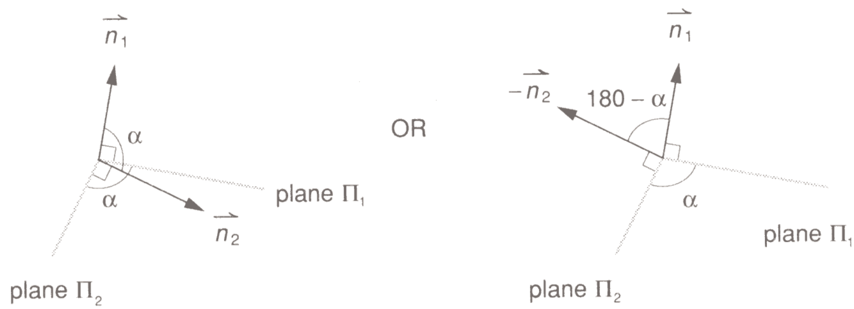
$$\text{Then } \vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2| \cos\alpha$$

$$12\overrightarrow{(0,17,6)} \cdot [-12\overrightarrow{(17,0,6)}] = 12\sqrt{17^2 + 6^2}\ 12\sqrt{17^2 + 6^2} \cos\alpha$$

$$-144(36) = 144(\sqrt{325})^2 \cos\alpha$$

$$\cos\alpha = -\frac{36}{325} = -0.1107\ldots \Rightarrow \alpha \doteq 96°.$$

This is the required angle between the planes. ■



Observe from the diagrams that the angle α between two planes could be either *equal* to the angle between two normals, or equal to the *supplement* of this angle.

If the planes Π_1 and Π_2 are infinite mathematical planes, there are actually two possible angles between them, α, and $180° - \alpha$.

However, if the two planes refer to real physical objects such as the pyramid of Example 4, you must decide which of the two angles is appropriate to describe the physical situation.

The Angle between Two Vectors

If you are given two non-zero vectors $\vec{u}$ and $\vec{v}$ in component form, recall (Example 2, section 3.3) that you can calculate the angle between them by using the dot product.

The cross product, $\vec{u}\times\vec{v} = |\vec{u}||\vec{v}|\sin\theta\, \vec{e}$, where θ is the required angle and $\vec{e}$ is a unit vector so that $\vec{u}$, $\vec{v}$, $\vec{e}$ form a right-handed system, can also be used to calculate θ.

To find $\sin\theta$, you can equate the *lengths* of the above vectors.
Thus, $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$, since $\sin\theta$ is never negative (because $0^\circ \leqslant \theta \leqslant 180^\circ$).

Hence, $\sin\theta = \dfrac{|\vec{u}\times\vec{v}|}{|\vec{u}||\vec{v}|}$

However, this will produce two solutions for θ, in the range $0^\circ \leqslant \theta \leqslant 180^\circ$. One solution is the angle between the vectors, and the other is the supplement of this angle. You must select the correct angle.

The dot product is more useful than the cross product to determine the angle between two vectors, because it produces only one solution (the correct solution) in the range $0^\circ \leqslant \theta \leqslant 180^\circ$.

SUMMARY

The triple scalar product property
$\vec{a}\times\vec{b}\cdot\vec{c} = \vec{a}\cdot\vec{b}\times\vec{c}$

If $\vec{u} = \overrightarrow{(u_1,u_2,u_3)}$ and $\vec{v} = \overrightarrow{(v_1,v_2,v_3)}$, then

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}$$

3.6 Exercises

1. Calculate $\vec{u} \times \vec{v}$ in the following cases.
   a) $\vec{u} = \vec{i} = \overrightarrow{(1,0,0)}$ and $\vec{v} = \vec{j} = \overrightarrow{(0,1,0)}$
   b) $\vec{u} = \overrightarrow{(2,3,5)}$ and $\vec{v} = \overrightarrow{(1,0,4)}$
   c) $\vec{u} = \overrightarrow{(-1,8,-3)}$ and $\vec{v} = \overrightarrow{(2,-2,-5)}$
   d) $\vec{u} = \overrightarrow{(1,-3,2)}$ and $\vec{v} = \overrightarrow{(2,-6,4)}$

2. Find a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$ for the vectors given in question 1.

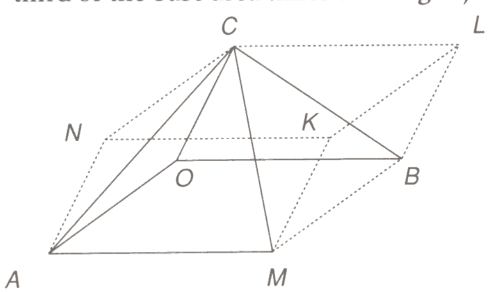
3. Simplify the following.
   a) $\vec{p} \times (\vec{p} + \vec{q})$
   b) $(\vec{p} + \vec{q}) \times (\vec{p} + \vec{q})$
   c) $\vec{p} \cdot \vec{q} \times \vec{p}$
   d) $\vec{p} \times (\vec{q} + \vec{r}) \cdot \vec{q}$

4. Find the area of the parallelogram $ABCD$ if $\overrightarrow{AB} = \overrightarrow{(1,9,2)}$, and $\overrightarrow{AD} = \overrightarrow{(-2,1,7)}$.

5. Find the area of the triangle ABC in the following cases.
   a) $\overrightarrow{AB} = \overrightarrow{(1,2,-3)}$, $\overrightarrow{AC} = \overrightarrow{(4,4,1)}$
   b) the vertices are $A(12,5,7)$, $B(4,10,13)$, and $C(8,-3,-1)$.

6. Three sides of a parallelepiped represent the vectors $\vec{u} = \overrightarrow{(3,2,1,)}$, $\vec{v} = \overrightarrow{(-4,0,2)}$, and $\vec{w} = \overrightarrow{(5,1,-1)}$. Calculate the volume of parallelepiped.

7. The parallelepiped $OAMBLCNK$ shown is such that $\overrightarrow{OA} = \overrightarrow{(1,4,4)}$, $\overrightarrow{OB} = \overrightarrow{(-1,2,1)}$, and $\overrightarrow{OC} = \overrightarrow{(6,-2,-3)}$. Calculate the volume of the pyramid $OAMBC$.
   (Formula: the volume of a pyramid is one third of the base area times the height.)



8. In the pyramid $OAMBC$ of question 7, show that $\overrightarrow{OM} = \overrightarrow{(0,6,5)}$. Then calculate the following, giving your answers correct to the nearest degree.
   a) the angle between BC and the base $OAMB$
   b) the angle between MC and the base $OAMB$
   c) the angle between the planes MBC and $OAMB$
   d) the angle between the planes MBC and MAC

9. $|\vec{u}| = 5, |\vec{v}| = 6, |\vec{u} \times \vec{v}| = 21$. Determine whether or not this information is sufficient to find the angle θ between $\vec{u}$ and $\vec{v}$.

10. Given $\vec{u} = \overrightarrow{(4,-6,0)}$, $\vec{v} = \overrightarrow{(2,1,1)}$, and $\vec{w} = \overrightarrow{(1,3,-5)}$.
    a) Calculate $(\vec{u} \times \vec{v}) \times \vec{w}$ and $\vec{u} \times (\vec{v} \times \vec{w})$. (These are known as **triple vector products**).
    b) Draw a conclusion about the associativity of the cross product.

11. Confirm the result of 3.5 Exercises, question 11, as follows. Given $\vec{i} = \overrightarrow{(1,0,0)}$, $\vec{u} = \overrightarrow{(1,1,0)}$, and $\vec{k} = \overrightarrow{(0,0,1)}$, show that $(\vec{i} \times \vec{u}) \times \vec{k} = \vec{0}$, but $\vec{i} \times (\vec{u} \times \vec{k}) \neq \vec{0}$.

12. Prove that the three $\mathbb{V}_3$ vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are non-coplanar if and only if $\vec{u} \cdot \vec{v} \times \vec{w} \neq 0$.

13. Use the cross product to show that the vectors $\vec{a} = \overrightarrow{(2,-1,3)}$, $\vec{b} = \overrightarrow{(1,2,0)}$, and $\vec{c} = \overrightarrow{(1,-13,9)}$ are coplanar (hence linearly dependent).

14. Show that $\frac{1}{2}(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \vec{b} \times \vec{a}$.

15. $A(2,-1,4)$, $B(3,1,-5)$, and $C(1,1,1)$ are three points in a plane Π. Use the cross product to find a vector perpendicular to the plane Π.

Summary

- The angle θ between two vectors $\vec{a}$ and $\vec{b}$ is the angle formed when the vectors are drawn with a common tail.
- The dot product of $\vec{a}$ and $\vec{b}$ is the scalar
  $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$
- The orthogonal projection of $\vec{a}$ on $\vec{b}$ is the vector

  $$\vec{p} = |\vec{a}| \cos \theta \frac{\vec{b}}{|\vec{b}|} = |\vec{a}| \cos \theta \, \vec{e}_b = (\vec{a} \cdot \vec{e}_b) \, \vec{e}_b$$

where $\vec{e}_b$ is the unit vector in the direction of $\vec{b}$.

- The component of $\vec{a}$ on $\vec{b}$ is the scalar coefficient of the unit vector in the projection,

  $$c = |\vec{a}| \cos \theta = \vec{a} \cdot \vec{e}_b = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

- *Properties of the dot product*
  1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
  2. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
  3. $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b})$
  4. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
  5. If $\vec{a}$ is perpendicular to $\vec{b}$, then $\vec{a} \cdot \vec{b} = 0$.
  6. $\vec{i} \cdot \vec{i} = 1, \vec{j} \cdot \vec{j} = 1, \vec{k} \cdot \vec{k} = 1$
  7. $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{i} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{j} = \vec{k} \cdot \vec{i} = \vec{i} \cdot \vec{k} = 0$
- In $\mathbb{V}_2$, $\vec{a} = \overrightarrow{(a_1,a_2)}$ and $\vec{b} = \overrightarrow{(b_1,b_2)}$
  $\Rightarrow \vec{a} \cdot \vec{b} = a_1b_2 + a_2b_2$
- In $\mathbb{V}_3$, $\vec{a} = \overrightarrow{(a_1,a_2,a_3)}$ and $\vec{b} = \overrightarrow{(b_1,b_2,b_3)}$
  $\Rightarrow \vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
- Vectors that are all unit, and mutually perpendicular, are said to form an orthonormal basis of the vector space.
- The cross product of vectors $\vec{a}$ and $\vec{b}$ is the vector
  $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \vec{e}$,
  where $\vec{e}$ is a unit vector perpendicular to $\vec{a}$ and $\vec{b}$, such that the triple $[\vec{a},\vec{b},\vec{e}]$ forms a right-handed system.
- *Properties of the cross product*
  1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
  2. $\vec{a} \times \vec{a} = \vec{0}$
  3. $\vec{i} \times \vec{j} = \vec{k}, \vec{j} \times \vec{k} = \vec{i}, \vec{k} \times \vec{i} = \vec{j}$
  4. $(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b})$
  5. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

- The triple scalar product property
  $\vec{a}\times\vec{b}\cdot\vec{c} = \vec{a}\cdot\vec{b}\times\vec{c}$
- If $\vec{a} = \overrightarrow{(a_1,a_2,a_3)}$ and $\vec{b} = \overrightarrow{(b_1,b_2,b_3)}$, then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \text{ where } \begin{vmatrix} c & f \\ e & d \end{vmatrix} = cd - fe.$$

Inventory

1. The angle between two vectors is defined as the angle between them when they are drawn with a common ______.
2. The projection of the vector $\overrightarrow{(2,-3)}$ on the x-axis is ______.
3. The component of the vector $\overrightarrow{(2,-3)}$ on the y-axis is ______.
4. Writing the vectors $\overrightarrow{(2,-3)}$ as $2\vec{i} - 3\vec{j}$ is called ______ the vector in the directions of $\vec{i}$ and $\vec{j}$.
5. The dot product combines two vectors to produce a ______.
6. The cross product combines two vectors to produce a ______.
7. The ______ product is not defined in $\mathbb{V}_2$.
8. The dot product of $\vec{u}$ and $\vec{v}$ equals (length of $\vec{u}$) (component of ______ on ______).
9. The magnitude of the cross product of $\vec{u}$ and $\vec{v}$ equals the area of the ______ whose adjacent sides represent the vectors ______ and ______.
10. The ______ product is not commutative.
11. The dot product of a unit vector with itself equals ______.
12. The cross product of a unit vector with itself equals ______.
13. The dot product of two perpendicular vector equals ______.
14. If the angle between two vectors is obtuse, then the dot product of those two vectors is ______, and vice-versa.
15. The expression $\vec{a}\times\vec{b}\cdot\vec{c}$ is known as a ______ ______ ______.
16. The expression $\vec{a}\times\vec{b}\cdot\vec{c}$ is equal to ______.
17. The product $\vec{u}\times\vec{v}$ yields a vector that is ______ to both $\vec{u}$ and $\vec{v}$.

Review Exercises

1. State the projections and the components of the following vectors in the directions of $\vec{i}$, $\vec{j}$ and $\vec{k}$.
   a) $\overrightarrow{(6,-5,-3)}$
   b) $\overrightarrow{(-\sqrt{2},1,\sqrt{3})}$
   c) $-3\overrightarrow{(1,-1,2)}$

2. Given that the following vectors are in $\mathbb{V}_2$, and that θ is the angle between each vector and $\vec{i}$, find the component of each vector on $\vec{i}$ and $\vec{j}$. Express your answers correct to 3 decimal places.
   a) $|\vec{a}| = 2$, $\theta = 55°$
   b) $|\vec{b}| = 4$, $\theta = 161°$
   c) $|\vec{c}| = 5$, $\theta = 180°$.

3. What can you deduce about $\vec{u}$ and/or $\vec{v}$ in the following cases?
   a) The component of $\vec{u}$ in the direction of $\vec{v}$ is equal to $|\vec{u}|$.
   b) The component of $\vec{u}$ in the direction of $\vec{v}$ is equal to $-|\vec{u}|$.

4. Calculate the dot product of the following pairs of vectors, given that θ is the angle between them.
   a) $|\vec{u}| = 7$, $|\vec{v}| = 1$, $\theta = 35°$
   b) $|\vec{w}| = 2$, $|\vec{t}| = 5$, $\theta = 120°$
   c) $|\vec{a}| = 10$, $|\vec{b}| = 10$, $\theta = 157°$

5. In question 4, calculate the component of the first vector on the second in each case.

6. *ABCDEF* is a regular hexagon whose sides have length 2 units. Calculate the following.
   a) $\overrightarrow{AB} \cdot \overrightarrow{BC}$
   b) $\overrightarrow{AF} \cdot \overrightarrow{FE}$
   c) $\overrightarrow{AF} \cdot \overrightarrow{BC}$
   d) $\overrightarrow{AB} \cdot \overrightarrow{ED}$
   e) $\overrightarrow{AB} \cdot \overrightarrow{BD}$
   f) $\overrightarrow{AF} \cdot \overrightarrow{BE}$

7. Find the dot product $\vec{u} \cdot \vec{v}$ if
   a) $\vec{u} = \overrightarrow{(2,4)}$ $\quad \vec{v} = \overrightarrow{(0,3)}$
   b) $\vec{u} = \overrightarrow{(-5,0,12)}$ $\quad \vec{v} = \overrightarrow{(3,-4,-2)}$
   c) $\vec{u} = \overrightarrow{(2,7)}$ $\quad \vec{v} = \overrightarrow{(21,-6)}$
   d) $\vec{u} = \overrightarrow{(1,2,5)}$ $\quad \vec{v} = \overrightarrow{(3,1,-1)}$

8. Find the value of k if the vectors $\vec{u} = \overrightarrow{(-8,6,7)}$ and $\vec{v} = \overrightarrow{(k,-1,2)}$ are perpendicular.

9. Prove that, for any vectors $\vec{u}$ and $\vec{v}$,
   a) $(\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$
   $= |\vec{u}|^2 + 2\vec{u} \cdot \vec{v} + |\vec{v}|^2$
   b) $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = |\vec{u}|^2 - |\vec{v}|^2$

10. Apply the result of question 9 a) to two perpendicular vectors $\vec{u}$ and $\vec{v}$ to prove the theorem of Pythagoras.

11. Two adjacent sides of a rhombus represent the vectors $\vec{u}$ and $\vec{v}$. Using the result of question 9 b), prove that the diagonals of a rhombus are perpendicular.

12. $\vec{a}$ and $\vec{b}$ are unit vectors at an angle of 135° to each other. Use the dot product to find an exact expression for the unit vector in the direction of $2\vec{a} - \vec{b}$. (See the table of trigonometric ratios on page 543.)

13. Given the vectors $\vec{u} = \overrightarrow{(2,-1)}$ and $\vec{v} = \overrightarrow{(-2,-3)}$, find each of the following.
    a) the angle between $\vec{u}$ and $\vec{v}$
    b) the component of $\vec{u}$ on $\vec{v}$
    c) the projection of $\vec{u}$ on $\vec{v}$
    d) the component of $\vec{v}$ on $\vec{u}$
    e) the projection of $\vec{v}$ on $\vec{u}$

14. The angle between the vectors $\vec{u}$ and $\vec{v}$ is θ where $\cos\theta = \frac{1}{\sqrt{6}}$. If $\vec{u} = \overrightarrow{(2,-1,1)}$ and $\vec{v} = \overrightarrow{(a,3,4)}$, find the possible values of a.

15. a) Prove that $\vec{u} = \vec{v}$ if and only if $\vec{u} \cdot \vec{p} = \vec{v} \cdot \vec{p}$ for *every* vector $\vec{p}$.
 b) Hence prove that in $\mathbb{V}_2$ it is sufficient to verify this relationship for *two* linearly independent vectors $\vec{p}_1$ and $\vec{p}_2$.
 c) How many vectors would be needed to verify the relationship in $\mathbb{V}_3$?

16. Determine the angles of the triangle PQR in the following cases.
 a) $P(-2,4)$, $Q(7,-9)$, $R(0,3)$
 b) $P(5,4,1)$, $Q(8,-1,-3)$, $R(9,4,4)$

17. OAB is a triangle with $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$. M is the midpoint of OA, and N is the midpoint of OB.
 a) Express $\overrightarrow{AN}$ and $\overrightarrow{BM}$ in terms of $\vec{a}$ and $\vec{b}$.
 b) If $|\overrightarrow{AN}| = |\overrightarrow{BM}|$, prove that $|\vec{a}| = |\vec{b}|$.

 (This proves that if two medians of a triangle are equal in length, then the triangle is isosceles.)

18. $OABC$ is a parallelogram with $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OC} = \vec{c}$. Evaluate $\overrightarrow{AC} \cdot \overrightarrow{AC} + \overrightarrow{OB} \cdot \overrightarrow{OB}$ in terms of $\vec{a}$ and $\vec{c}$ to prove the following theorem.
 The sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the sides.

19. Calculate $\vec{u} \times \vec{v}$ for each of the following, where $\vec{u}$ and $\vec{v}$ are vectors of $\mathbb{V}_3$.
 a) $|\vec{u}| = 3$, $|\vec{v}| = 2$, and the angle between $\vec{u}$ and $\vec{v}$ is 25°.
 b) $|\vec{u}| = 4$, $|\vec{v}| = 1$, and the angle between $\vec{u}$ and $\vec{v}$ is 110°.
 c) $|\vec{u}| = 5$, $|\vec{v}| = 6$, and $\vec{u} \cdot \vec{v} = -5$.
 d) $\vec{u} = \overrightarrow{(9,-2,4)}$ and $\vec{v} = \overrightarrow{(3,-1,0)}$.

20. Find the two unit vectors that are perpendicular to $\vec{a} = \overrightarrow{(1,6,8)}$ and $\vec{b} = \overrightarrow{(4,-2,-5)}$.

21. Calculate the area of the triangle whose vertices are $P(10,-3,9)$, $Q(-1,4,2)$, and $R(0,5,-6)$.

22. If $\vec{a} = \overrightarrow{(-1,3,6)}$ and $\vec{b} = \overrightarrow{(-2,-2,5)}$, determine whether or not it is possible to find the angle θ between $\vec{a}$ and $\vec{b}$ by using the cross product exclusively.

23. In each of the following, use the triple scalar product $(\vec{a} \times \vec{b}) \cdot \vec{c}$ to determine whether or not the three vectors are linearly dependent.
 a) $\overrightarrow{(0,1,3)}$, $\overrightarrow{(-3,5,2)}$, and $\overrightarrow{(-6,11,7)}$
 b) $\overrightarrow{(1,2,3)}$, $\overrightarrow{(-3,0,4)}$, and $\overrightarrow{(-1,4,6)}$
 c) $\overrightarrow{(4,1,8)}$, $\overrightarrow{(-2,1,0)}$, and $\overrightarrow{(0,3,16)}$
 d) $\overrightarrow{(1,2,4)}$, $\overrightarrow{(2,-3,-1)}$, and $\overrightarrow{(-1,-9,-13)}$
 e) $\overrightarrow{(3,5,1)}$, $\overrightarrow{(2,-2,-1)}$, and $\overrightarrow{(-4,-4,0)}$

24. Choose specific vectors in $\mathbb{V}_3$ to show that the cross product is not associative. That is, show by counterexample that $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$.

25. a) Find a vector $\vec{p}$ perpendicular to $\vec{q} = \overrightarrow{(4,-1,-2)}$ and perpendicular to $\vec{r} = \overrightarrow{(0,3,1)}$.
 b) Normalize $\vec{p}$.

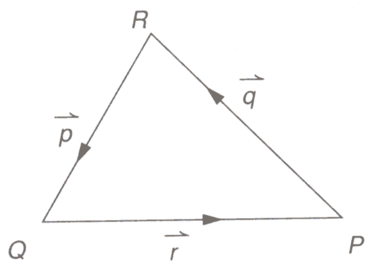
26. Given a vector $\vec{v} \neq \vec{0}$ and a scalar k, use the dot product to calculate the angle between $\vec{v}$ and $k\vec{v}$ in the following cases.
 a) $k > 0$ b) $k < 0$

27. In question 26, discuss the case where $k = 0$.

28. Given any three vectors $\vec{p}$, $\vec{q}$, $\vec{r}$ of $\mathbb{V}_3$, prove the following.
 a) $\vec{p} \cdot \vec{q} \times \vec{r} = \vec{q} \cdot \vec{r} \times \vec{p} = \vec{r} \cdot \vec{p} \times \vec{q}$ (This is known as the **cyclic property** of triple scalar products.)
 b) $\vec{p} \cdot \vec{q} \times \vec{r} = -\vec{q} \cdot \vec{p} \times \vec{r}$.

29. a) Show that $(\vec{u}\times\vec{v})\times\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$. That is, show that $(\vec{u}\times\vec{v})\times\vec{w}=k\vec{u}+s\vec{v}$, where k and s are scalars.
b) Show similarly that $\vec{u}\times(\vec{v}\times\vec{w})=m\vec{v}+n\vec{w}$, where m and n are scalars.
c) Using the results of parts a) and b), describe the triples $\vec{u}, \vec{v}, \vec{w}$ for which the cross product is associative.

30. Given the triangle PQR, whose sides represent the vectors $\vec{p}, \vec{q}$, and $\vec{r}$ as shown on the diagram.



a) Prove that $\vec{p}+\vec{q}+\vec{r}=\vec{0}$.
b) Write and then simplify the relation obtained by carrying out the cross product of $\vec{p}$ with each side of the relation in a).
c) Repeat step b) by using $\vec{q}$, then $\vec{r}$.
d) From your results, prove the sine law in triangle PQR (see page 542).

31. The vertices of the triangle ABC have position vectors $\vec{a}, \vec{b}$, and $\vec{c}$ respectively from an origin O. Prove that the area of ABC is $\frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$.

32. A plane contains the points A, B, and C whose position vectors are $\vec{a}, \vec{b}$, and $\vec{c}$ respectively. Prove that the vector $\vec{n}=\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$ is normal to the plane.

33. Let $\vec{i}, \vec{j}, \vec{k}$ be unit vectors along the coordinate axes of a three-dimensional rectangular Cartesian coordinate system, and let $\vec{a}, \vec{b}, \vec{c}$ be defined by the equations
$\vec{a}=-\vec{i}+\vec{j}+\vec{k}$, $\vec{b}=\vec{i}-\vec{j}+\vec{k}$,
$\vec{c}=\vec{i}+\vec{j}-\vec{k}$.
a) Find the angle between the vectors $\vec{a}$ and $\vec{b}$, giving your answer in degrees, correct to 1 decimal place.
b) Given that O is the origin and $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the points A, B, C respectively, find
i) the lengths of the sides of the triangle OAB, leaving your answers in surd form, and
ii) the magnitudes of its angles.
c) What are the lengths of the sides of the triangle ABC and the magnitudes of its angles?
d) Find the position vector of the mid-point D of $[BC]$, and write down the position vectors of the mid-points E, F of $[CA]$ and $[AB]$ respectively. Hence find the volume of the tetrahedron $ODEF$.
e) By considering the ratio of the areas of the triangles ABC and DEF, or otherwise, prove that the volume of the tetrahedron $OABC$ is $\frac{2}{3}$ units³.

(84 S)

34. In a rectangular Cartesian coordinate system the points O, A, B and C have coordinates (0,0), (4,8), (4,–2) and (–8,–6) respectively.
a) Show the points O, A, B and C in a sketch, taking $\frac{1}{2}$ cm as a unit.
b) Prove that $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are perpendicular.
c) Given $\overrightarrow{OD}=\overrightarrow{OB}+\overrightarrow{OC}$, find the coordinates of the point D.

d) The point C' lies on the line (AB) so that $\overrightarrow{OC'} = \alpha\overrightarrow{OC}$. Calculate the value of α.

e) i) Given that the point E lies on the y-axis and $\overrightarrow{OA} \cdot \overrightarrow{AE} = 0$, calculate the coordinates of the point E.

ii) It is further given that $\overrightarrow{OF} = \overrightarrow{OE} - \overrightarrow{OA}$. Calculate the coordinates of the point F.

f) Calculate

i) the area of the rectangle $OAEF$, and

ii) the area of the parallelogram $OBDC$.

(85 SMS)

35. The point O is the centre of the circle drawn through the vertices of the triangle ABC. With respect to the point O as origin the position vectors of the points A, B and C are $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively, so that $|\vec{a}| = |\vec{b}| = |\vec{c}|$. The points G and H are such that $\overrightarrow{OG} = \frac{1}{3}(\vec{a} + \vec{b} + \vec{c})$ and

$\overrightarrow{OH} = \vec{a} + \vec{b} + \vec{c}$

i) Prove that $\overrightarrow{AG} = \frac{1}{3}(\vec{b} + \vec{c} - 2\vec{a})$

ii) Prove that (AG) passes through the midpoint L of $[BC]$ and find the ratio $AG : GL$.

iii) Deduce, from the result obtained in part (ii), that the three medians of triangle ABC, i.e. the lines joining a vertex to the midpoint of the opposite side, pass through G.

iv) Prove that (AH) is perpendicular to (BC).

v) Deduce, from the result obtained in part (iv), that the three perpendiculars, from the vertices of triangle ABC to the opposite sides, pass through H.

(85 H)

36. In a rectangular Cartesian coordinate system the points A, B and C have coordinates $(-6,1)$, $(-2,4)$ and $(1,0)$ respectively.

i) a) Given the column vector $\overrightarrow{AB} = \begin{pmatrix} p \\ q \end{pmatrix}$, find the values of p and q.

b) Given $\overrightarrow{AB} = \overrightarrow{DC}$, find the coordinates of the point D.

c) Show that I, the midpoint of $[AC]$, is also the midpoint of $[BD]$.

d) Show that $ABCD$ is a square.

ii) Calculate the coordinates of the points E and F such that

$\overrightarrow{BE} = \frac{1}{4}\overrightarrow{BI}$ and $\overrightarrow{CF} = 2\overrightarrow{CE}$.

iii) a) By finding the column vectors for each of $\overrightarrow{AF}$ and $\overrightarrow{DB}$ show that

$\overrightarrow{AF} = \frac{3}{4}\overrightarrow{DB}$.

b) Continue the argument

$\overrightarrow{AF} = \overrightarrow{AC} + \overrightarrow{CF} = 2(\overrightarrow{IC} + \overrightarrow{CE})$

to confirm that $\overrightarrow{AF} = \frac{3}{4}\overrightarrow{DB}$.

(84 SMS)

37. Given that

$\vec{p} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, $\vec{q} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

and that

$\vec{s} = k\vec{p} - (\vec{p} \cdot \vec{q})\vec{q}$, where $k \in \mathbb{R}$,

find

a) the value of $p \cdot \vec{q}$, and

b) the value of the constant k such that the directions of $\vec{s}$ and $\vec{q}$ are at right angles.

(87 S)