VECTORS, MATRICES and COMPLEX NUMBERS

with
International Baccalaureate
questions

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CHAPTER FOUR

APPLICATIONS OF VECTORS

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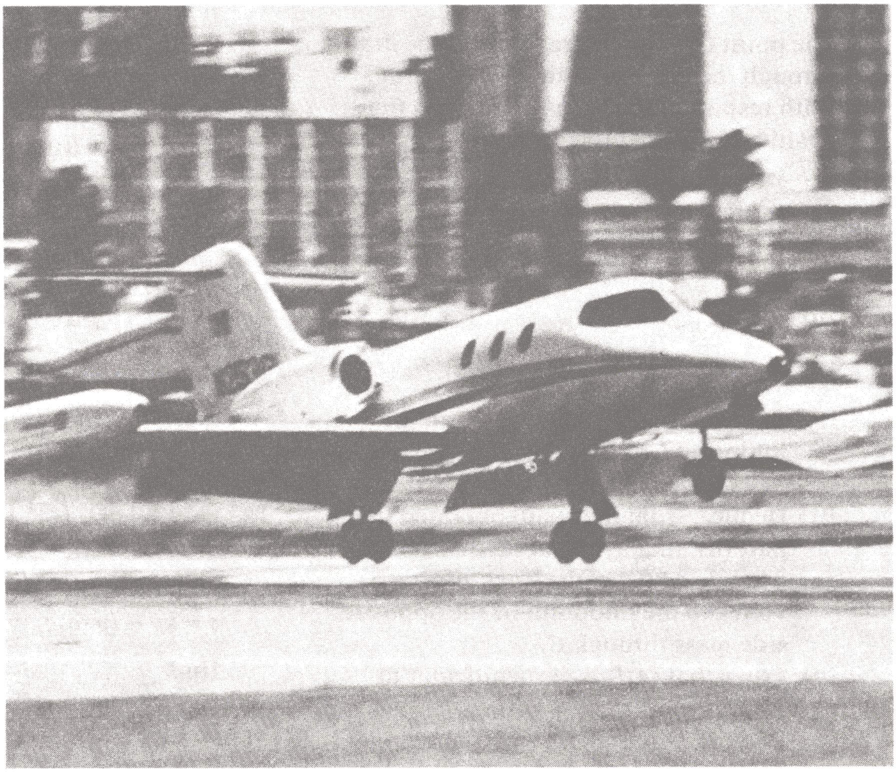
CHAPTER FOUR

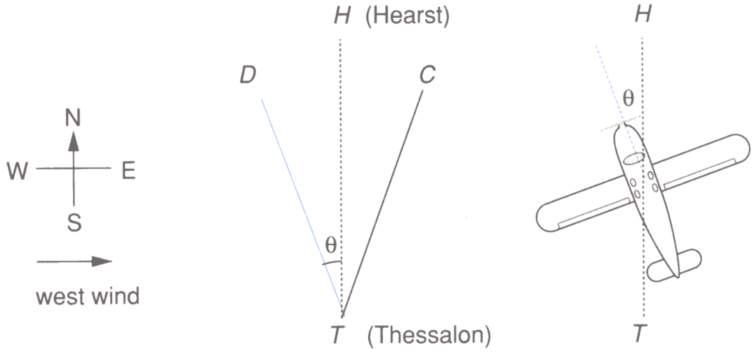
Applications of Vectors



You are in a small airplane in the town of Thessalon, Ontario, ready to fly to Hearst, a distance of 400 km due north. The plane can fly at 200 km/h, and so you would expect the trip to take two hours. However, the 200 km/h is the plane's *airspeed*, in other words, its speed relative to the air around it. This is not necessarily the same as the plane's *groundspeed*.

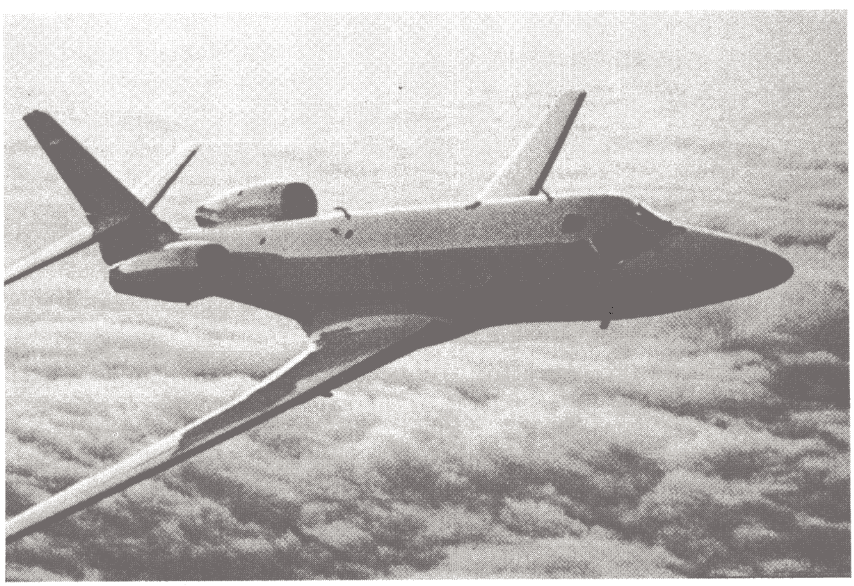
Naturally, if there is a strong wind against you (known as a headwind), you would expect the trip to take longer. Similarly, if there is a strong wind behind you (known as a tailwind), your trip would be shorter.

On this day, a wind is blowing at 50 km/h from the west, and your pilot cannot simply head north. Indeed, if she points the plane due north, along the line *TH*, the wind will blow the airplane along a line such as *TC*. Clearly, in order to reach Hearst, she will need to point the airplane in a direction to the west of line *TH*, parallel to a line such as *TD*. The plane should then continue to fly partly 'sideways', as shown in the diagram, and the wind will keep pushing it so that it remains over the line *TH*.



Her problem will be to determine the angle θ between *TH* and *TD*. The stronger the wind from the west, the greater angle θ must be; the greater the angle θ, the more her groundspeed will be reduced. She will also need to determine this groundspeed, v, to estimate the time of arrival in Hearst.

To find these two numbers, θ and v, she will use a mathematical model that will be able to represent both the speed and the direction of the airplane. Vectors provide an excellent model for this type of situation, and for other problems in physics, as you shall see in this chapter.



4.1 Forces as Vectors

Since a vector has magnitude and direction, and a force also has magnitude and direction, the theory of vectors can be applied to forces.

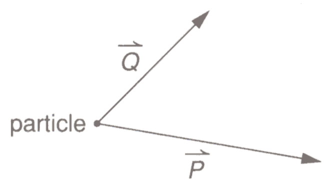
In fact, the theory of vectors grew out of investigations on forces. (See the introduction to chapter 1.)

The branch of physics called mechanics is divided into two main sections, called **statics** and **dynamics**. The latter is a study of how objects change their motion under the action of forces, while statics examines the relationship of forces acting upon stationary (or other non-accelerating) objects.

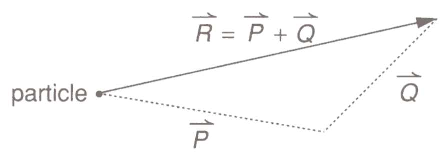
You will be taking an elementary look at **statics of a particle** in this section. (A particle is the name used for an object small enough to be considered as a point.)

Forces are measured in **newtons** (symbol N). In general, the **gravitational force on a mass of *m* kg is *mg* N**, where g is the acceleration due to gravity. On earth, $g \doteq 9.8$ m/s^2. Thus, the force due to gravity on a mass of 1 kg is about 9.8 N. Alternatively, a mass of 1 kg is said to **weigh** 9.8 N.

The fundamental reason which allows you to apply vector theory to forces is that two forces can be combined in the same way that vectors are *added*. In other words, if a particle is being acted upon by two forces $\vec{P}$ and $\vec{Q}$ as shown, these will have the *same effect* on the particle as the force $\vec{R} = \vec{P} + \vec{Q}$.



or



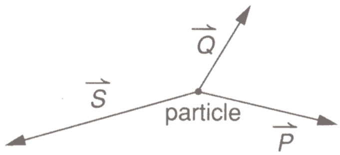
$\vec{R}$ is called the **resultant** of $\vec{P}$ and $\vec{Q}$.

This is also true for any number of forces. That is, the *resultant* of a number of forces is the *vector sum* of those forces.

The effect will be to pull the particle in the direction of $\vec{R}$, and thus accelerate it, or increase its speed.

If the particle does not change its motion, or its state of rest, then the particle remains in a **state of equilibrium**. The particle must then be acted upon simultaneously by a force which counteracts or cancels the effect of $\vec{R}$. Consider the force $\vec{S}$, acting in the opposite direction to that of $\vec{R}$, but with the same magnitude.

You would then have the following situation:



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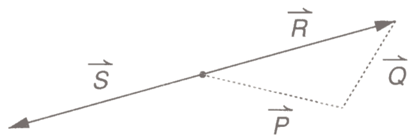


The particle is *in equilibrium* under the effect of $\vec{P}$, $\vec{Q}$, and $\vec{S}$, or under the effect of $\vec{R}$ and $\vec{S}$.

$\vec{S}$ is called the **equilibrant** of $\vec{R}$.
Conversely, $\vec{R}$ is called the equilibrant of $\vec{S}$.

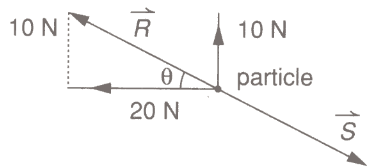
Note: $\vec{S} + \vec{R} = \vec{0}$ or $\vec{S} = -\vec{R}$
$\vec{P} + \vec{Q} + \vec{S} = \vec{0}$

DEFINITION Whenever a particle is in equilibrium, the vector sum of all forces acting upon it is $\vec{0}$.

In the examples of this chapter, unless it is specified otherwise, all magnitudes of forces will be calculated to 3 significant digits, and all angles will be calculated to the nearest degree.

Example 1 A force of 20 N acts west, and a force of 10 N acts north on a particle. Find the direction and magnitude of the resultant and the equilibrant of these two forces.

Solution



Let the resultant be $\vec{R}$. Then by the theorem of Pythagoras,
$|\vec{R}|^2 = 20^2 + 10^2 \Rightarrow |\vec{R}| = \sqrt{500}$ or $|\vec{R}| \doteq 22.4$

The angle θ shown is such that

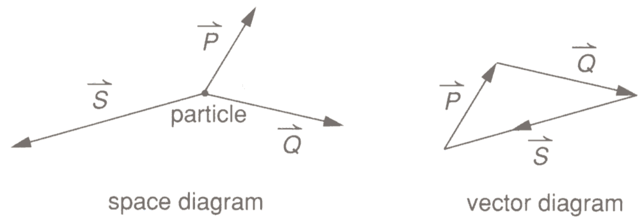
$$\tan\theta = \frac{10}{20} = \frac{1}{2} \Rightarrow \theta \doteq 27^\circ$$

Therefore, the resultant is the force $\vec{R}$ of 22.4 N, acting at bearing $270^\circ + 27^\circ = 297^\circ$ on the particle.

From the diagram, you can see that the equilibrant is thus $\vec{S}$, a force of 22.4 N acting at bearing $90^\circ + 27^\circ = 117^\circ$ on the particle. ■

Vector Diagrams

The diagrams you have seen so far in this section are called **space diagrams** or **position diagrams**, because they try to portray the reality of the particle being tugged at or pushed from different directions. Since vectors can be drawn anywhere, however, you can also represent the forces $\vec{P}$, $\vec{Q}$, $\vec{S}$ by joining the tip of one vector to the tail of another.



This often clarifies relationships between vectors, and facilitates calculations, as you will see in the forthcoming examples. The diagram on the right is known as a **vector diagram**.

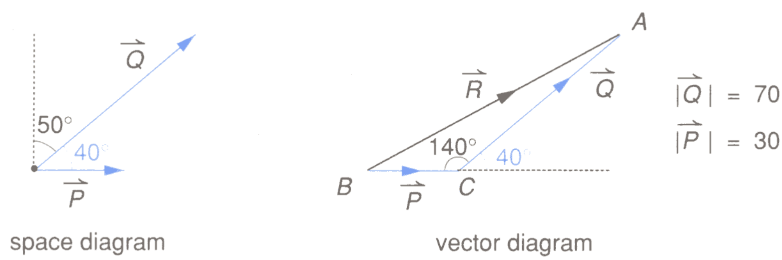
The vector diagram of forces in equilibrium will always be a closed polygon, since the sum of all the vectors is $\vec{0}$.

The particle itself is not represented in the vector diagram.

Force vectors are sometimes described as *fixed* vectors (as opposed to *free* vectors), because they must act on a particular point.

Example 2 Two forces $\vec{P}$ and $\vec{Q}$ act on a particle. $\vec{P}$ points due east and has a magnitude of 30 N. $\vec{Q}$ points on bearing 050° and has a magnitude of 70 N. Find the resultant of $\vec{P}$ and $\vec{Q}$.

Solution



The space diagram indicates how $\vec{P}$ and $\vec{Q}$ act on the particle.
The vector diagram, denoted triangle ABC, shows the resultant $\vec{R}$ and the relationship $\vec{P} + \vec{Q} = \vec{R}$.

Notice that the angle between the forces on the space diagram is $90° - 50° = 40°$.

Notice further that the angle ACB in the vector diagram is $180° - 40° = 140°$.

By the cosine law in the vector diagram

$$|\overrightarrow{AB}|^2 = |\overrightarrow{BC}|^2 + |\overrightarrow{AC}|^2 - 2|\overrightarrow{BC}||\overrightarrow{AC}| \cos (\sphericalangle ACB)$$

$$\text{thus } |\vec{R}|^2 = |\vec{P}|^2 + |\vec{Q}|^2 - 2|\vec{P}||\vec{Q}| \cos 140^\circ$$

$$= 30^2 + 70^2 - (2)(30)(70)(-0.766\ldots)$$

$$= 900 + 4900 + 3217.3\ldots$$

$$= 9017.3\ldots$$

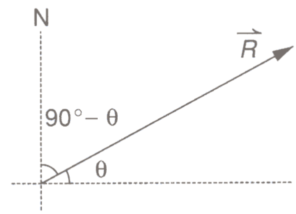
$$\text{thus } \quad |\vec{R}| = \sqrt{9017.3\ldots} = 94.959\ldots \doteq 95.0$$

The direction of $\vec{R}$ can be obtained by using angle B.
By the sine law in the vector diagram

$$\frac{|\overrightarrow{BC}|}{\sin A} = \frac{|\overrightarrow{CA}|}{\sin B} = \frac{|\overrightarrow{AB}|}{\sin C}$$

You need only use the second equation. Let angle $B = \theta^\circ$.

$$\text{Thus } \frac{|\vec{Q}|}{\sin \theta} = \frac{|\vec{R}|}{\sin 140^\circ}$$



$$\sin \theta = \frac{|\vec{Q}| \sin 140^\circ}{|\vec{R}|}$$

$$= \frac{(70)(0.6427\ldots)}{94.959}$$

$$= 0.4738\ldots$$

Thus $\theta \doteq 28^\circ$.

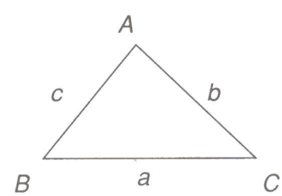
The bearing of $\vec{R}$ is $90^\circ - 28^\circ = 62^\circ$.

The resultant of $\vec{P}$ and $\vec{Q}$ is a force $\vec{R}$ of magnitude 95.0 N, acting on bearing 062°. ■

In this chapter, you will be making extensive use of the theorem of Pythagoras, the sine law and the cosine law, as in the above examples. To avoid cluttering the diagrams with unnecessary letters, the formulas for these laws will not be fully quoted at each instance of their use. You can refer to the formulas for these laws on page 542.

Solution of Triangles

The cosine law
$a^2 = b^2 + c^2 - 2bc \cos A$

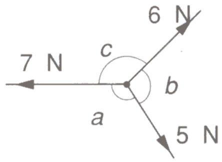


The sine law

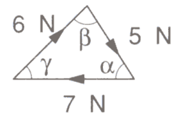
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Example 3 Three forces of 5 N, 6 N, and 7 N act simultaneously on a particle, which remains in a state of equilibrium. What are the angles between the forces?

Solution



space diagram



vector diagram

Since the forces are in equilibrium, the vector diagram is a triangle. You can use this triangle to perform the calculations. The angles α, β, and γ which you will find are the *supplements* of the required angles a, b, and c respectively, between the forces.

Using the cosine law to find α:

$$6^2 = 5^2 + 7^2 - (2)(5)(7)\cos\alpha$$

$$\cos\alpha = \frac{25 + 49 - 36}{70} = \frac{38}{70} = 0.5428\ldots \Rightarrow \alpha \doteq 57.12°$$

Using the cosine law again for β:

$$7^2 = 5^2 + 6^2 - (2)(5)(6)\cos\beta$$

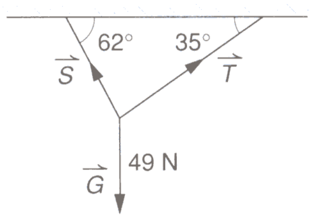
$$\cos\beta = \frac{25 + 36 - 49}{60} = \frac{12}{60} = 0.2 \Rightarrow \beta \doteq 78.46°$$

Thus $\gamma \doteq 180 - (57.12 + 78.46) = 44.42°$

Thus the angles α, β, and γ, to the nearest degree, are respectively 57°, 78°, and 44°, and hence the required angles are:
between the 5 N and 7 N forces, $a = 123°$,
between the 5 N and 6 N forces, $b = 102°$,
between the 6 N and 7 N forces, $c = 136°$. ■

(Observe that, by approximating each angle to nearest degree, in this case $a + b + c \neq 360°$.)

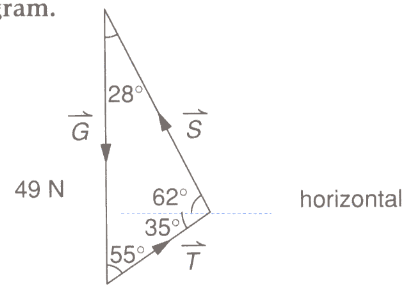
Example 4 A particle of mass 5 kg is suspended from a horizontal ceiling by two strings making angles of 35° and 62° with the ceiling. Calculate the tensions in these strings.



space diagram

Solution Note: The force of gravity on a mass of 5 kg is (5)(9.8) = 49 newtons, acting vertically downward. Call this force $\vec{G}$. Call the tensions in the strings $\vec{S}$ and $\vec{T}$, as shown in the diagram.

Now the particle is in equilibrium, so $\vec{S} + \vec{T} + \vec{G} = \vec{0}$, and this gives the following vector diagram.



By the sine law, $\dfrac{|\vec{T}|}{\sin 28^\circ} = \dfrac{|\vec{S}|}{\sin 55^\circ} = \dfrac{|\vec{G}|}{\sin 97^\circ}$

Also, $|\vec{G}| = 49$,

thus $|\vec{T}| = \dfrac{(49)(\sin 28^\circ)}{\sin 97^\circ} = \dfrac{23.00\ldots}{0.9925\ldots} \doteq 23.2$

and $|\vec{S}| = \dfrac{(49)(\sin 55^\circ)}{\sin 97^\circ} = \dfrac{4.13\ldots}{0.9925} \doteq 40.4$

The tensions in the strings are thus 23.2 N and 40.4 N. ■

Note: An accurately drawn vector diagram would allow you to find the solution by *drawing to scale* instead of carrying out calculations. Indeed, if the length of $\vec{G}$ is drawn to scale as 49 mm, and the angles are correctly drawn, then the lengths of $\vec{S}$ and $\vec{T}$, in mm, will provide the required tensions!

Also the vector diagram seems to indicate that $|\vec{S}| > |\vec{T}|$, whereas the opposite appears to be the case in the space diagram. This is because the space diagram is showing *lengths of strings, not vectors*. The vector diagram is the correct representation of the forces involved, and indeed, the greater tension will occur in the shorter string.

SUMMARY

The *resultant*, $\vec{R}$, of a number of forces is the *vector sum* of those forces. The *equilibrant* of those forces is $-\vec{R}$.

A particle is in *equilibrium* when the vector sum of all forces acting upon it is $\vec{0}$.

The gravitational force on a particle of mass m kg is mg N, where g is the acceleration due to gravity.
(On earth, $g \doteq 9.8$ m/s^2.)

4.1 Exercises

Unless directed otherwise, give all magnitudes in newtons, correct to 3 significant digits, and all angles correct to the nearest degree.
Use $g \doteq 9.8$ m/s^2.

1. Two forces $\vec{P}$ and $\vec{Q}$ act on a particle. Find the resultant and the equilibrant of $\vec{P}$ and $\vec{Q}$ in the following cases.
   a) $|\vec{P}| = 12$ and $\vec{P}$ acts due east
   $|\vec{Q}| = 5$ and $\vec{Q}$ acts due north
   b) $|\vec{P}| = 8$ and $\vec{P}$ acts on bearing 045°
   $|\vec{Q}| = 15$ and $\vec{Q}$ acts on bearing 135°
   c) $|\vec{P}| = 10$ and $\vec{P}$ acts due west
   $|\vec{Q}| = 13$ and $\vec{Q}$ acts on bearing 350°
   d) $|\vec{P}| = 20$ and $\vec{P}$ acts on bearing 058°
   $|\vec{Q}| = 35$ and $\vec{Q}$ acts on bearing 328°
   e) $|\vec{P}| = 510$ and $\vec{P}$ acts on bearing 232°
   $|\vec{Q}| = 425$ and $\vec{Q}$ acts on bearing 105°

2. Two forces $\vec{P}$ and $\vec{Q}$ of magnitudes 30 N and 60 N respectively act on a particle. The angle between $\vec{P}$ and $\vec{Q}$ is 75°. Calculate the magnitude and the direction of the resultant of $\vec{P}$ and $\vec{Q}$.

3. A particle is being pulled by two forces $\vec{F}_1$ and $\vec{F}_2$. $\vec{F}_1$ acts eastward and its magnitude is 250 N. $\vec{F}_2$ acts at a bearing of 10°, and its magnitude is 120 N. Find the magnitude and the direction of the resultant force.

4. Three forces of 10 N, 24 N, and 26 N act simultaneously on a particle, which remains in a state of equilibrium. Calculate the angles between the forces.

5. Repeat question 4 if the forces have magnitudes 15 N, 11 N, and 23 N.

6. The resultant of $\vec{P}$ and $\vec{Q}$ is the force $\vec{F}$, of magnitude 80 N, acting on bearing 070°. $\vec{P}$ has magnitude 25 N and acts due east. Find the magnitude and direction of $\vec{Q}$.

7. Two perpendicular forces of equal magnitude act on a particle. If the resultant has magnitude 100 N, calculate the magnitude of the perpendicular forces.

8. Two forces of equal magnitude acting at an angle of 130° to each other have a resultant of magnitude 42 N. Calculate the magnitude of the two forces.

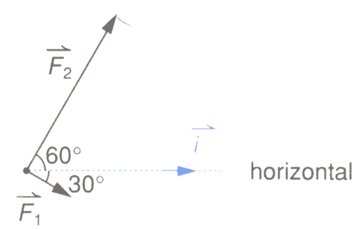
9. A particle of mass 10 kg is suspended from a horizontal ceiling by two strings making angles of 40° and 50° with the ceiling. Calculate the tensions in these strings.

10. Repeat question 9 if the strings make angles of 29° and 41° with the ceiling.

11. A particle of mass 5 kg is suspended by cords from two points A and B on a horizontal ceiling such that $AB = 100$ cm. The lengths of the cords are 80 cm and 70 cm. Calculate the tension in each cord.

12. A force $\vec{P}$ of magnitude 10 N acts along the bearing 060°. Calculate the smallest possible magnitude of a force $\vec{Q}$ such that the resultant of $\vec{P}$ and $\vec{Q}$ acts along the bearing 090°.

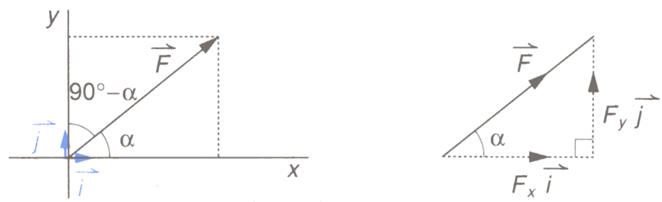
13. Two forces $\vec{F}_1$ and $\vec{F}_2$, of magnitudes 7 N and 24 N, act on a particle. $\vec{F}_1$ acts at 30° to the vector $\vec{i}$, and $\vec{F}_2$ acts at 60° to the vector $\vec{i}$, as shown in the figure. Another force $\vec{F}_3$ of magnitude 30 N acts simultaneously on the particle. Calculate the direction of $\vec{F}_3$ so that the resultant of the three forces is in the direction of $\vec{i}$. (Note: The three forces would then cause the particle to move in the direction of $\vec{i}$.)



4.2 Resolution of Forces

In the last section you learned how to use triangles to find the resultant of two forces. In this section you will learn a second method to solve such problems. This method is the **resolution of forces**. One of its advantages is that it allows you to find the resultant of more than two forces without extra difficulty.

Recall from chapter 3 that a vector $\vec{F}$, making an angle α with $\vec{i}$, can be resolved into two vectors $F_x\vec{i}$ and $F_y\vec{j}$ acting parallel to the x-axis and parallel to the y-axis respectively.



$$\vec{F} = F_x\vec{i} + F_y\vec{j}$$

$F_x\vec{i}$ and $F_y\vec{j}$ are called the *projections* of $\vec{F}$ onto $\vec{i}$ and $\vec{j}$ respectively.

Since $\{\vec{i},\vec{j}\}$ is an orthonormal basis, the scalar multiples F_x and F_y are called the *components* of $\vec{F}$ onto $\vec{i}$ and $\vec{j}$. Recall further that components can be obtained by using the dot product.

$F_x = \vec{F}\cdot\vec{i} = |\vec{F}||\vec{i}|\cos\alpha = |\vec{F}|\cos\alpha$, since $|\vec{i}| = 1$.

$F_y = \vec{F}\cdot\vec{j} = |\vec{F}||\vec{j}|\cos(90^\circ - \alpha) = |\vec{F}|\sin\alpha$, since $|\vec{j}| = 1$ and $\cos(90^\circ - \alpha) = \sin\alpha$.

Alternatively, you can see from the right triangle that

$$\cos\alpha = \frac{F_x}{|\vec{F}|} \text{ and } \sin\alpha = \frac{F_y}{|\vec{F}|},$$

giving the same results for F_x and F_y.

FORMULA

A vector $\vec{F}$ making an angle of α with $\vec{i}$ is resolved on $\vec{i}$ and $\vec{j}$ as follows.

$$\vec{F} = |\vec{F}|\cos\alpha\,\vec{i} + |\vec{F}|\sin\alpha\,\vec{j}$$

The following similar result is true in 3-space, and you will have an opportunity to prove it in the exercises.

FORMULA

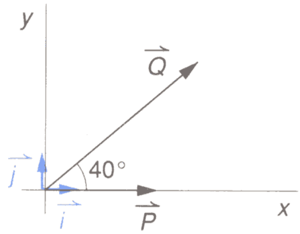
A vector $\vec{F}$ making angles of α, β and γ with $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively is resolved on $\vec{i}$, $\vec{j}$, and $\vec{k}$ as follows.

$$\vec{F} = |\vec{F}|\cos\alpha\,\vec{i} + |\vec{F}|\cos\beta\,\vec{j} + |\vec{F}|\cos\gamma\,\vec{k}$$

The following example matches Example 2 of the previous section, although it is worded differently. Thus you will be able to compare methods.

Example 1 Two forces $\vec{P}$ and $\vec{Q}$ act on a particle and make an angle of 40° with each other. If $|\vec{P}| = 30$ N and $|\vec{Q}| = 70$ N, find the resultant force $\vec{R}$.

Solution Vectors may be drawn anywhere. Draw $\vec{P}$ with its tail at the origin of a two-dimensional coordinate system so that $\vec{P}$ points in the direction of the positive x-axis (that is, in the direction of $\vec{i}$). This is shown in the diagram.



To find the resultant $\vec{R} = \vec{P} + \vec{Q}$, first resolve $\vec{P}$ and $\vec{Q}$ on $\vec{i}$ and $\vec{j}$.

$\vec{P}$: Since $|\vec{P}| = 30$, $\vec{P} = 30\vec{i}$

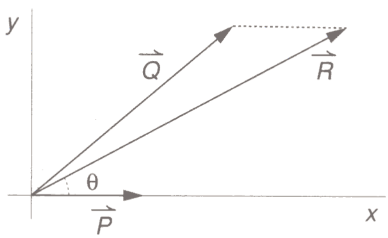
$\vec{Q}$:
$$\begin{aligned}\vec{Q} &= |\vec{Q}| \cos 40^\circ\, \vec{i} + |\vec{Q}| \sin 40^\circ \vec{j}\\ &= (70)(0.7660\ldots)\vec{i} + (70)(0.6427\ldots)\,\vec{j}\\ &\doteq 53.623\,\vec{i} + 44.995\,\vec{j}\end{aligned}$$

Now
$$\begin{aligned}\vec{R} &= \vec{P} + \vec{Q}\\ &= (30\vec{i}) + (53.623\vec{i} + 44.995\vec{j})\\ &= 83.623\vec{i} + 44.995\vec{j}\end{aligned}$$

Magnitude of $\vec{R}$

$$|\vec{R}| = \sqrt{83.623^2 + 44.995^2} = \sqrt{9017.3\ldots} \doteq 95.0$$

Direction of $\vec{R}$



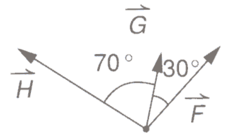
Let θ be the angle between $\vec{R}$ and $\vec{i}$.

Then $\tan\theta = \dfrac{44.995}{83.623} = 0.5380\ldots$, therefore $\theta \doteq 28^\circ$

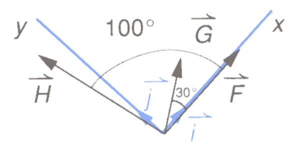
Thus the resultant of $\vec{P}$ and $\vec{Q}$ has a magnitude of 95.0 N and makes an angle of 28° with the force $\vec{P}$
(or an angle of 40° −28° = 22° with force $\vec{Q}$).

The same results are obtained as for Example 2 of the previous section. ■

Example 2 Three coplanar forces $\vec{F}$, $\vec{G}$, and $\vec{H}$ of magnitudes 15 N, 10 N, and 20 N respectively, act on a particle as shown in the diagram. Find the resultant of the three forces.



Solution Vectors can be drawn anywhere. Draw $\vec{F}$ with its tail at the origin of a two-dimensional coordinate system so that $\vec{F}$ points in the direction of $\vec{i}$. This is shown in the diagram. Note that the angle between $\vec{G}$ and $\vec{i}$ is 30°, and the angle between $\vec{H}$ and $\vec{i}$ is 100°.



Then the resultant of the three forces is $\vec{R} = \vec{F} + \vec{G} + \vec{H}$.
Now resolve $\vec{F}$, $\vec{G}$, and $\vec{H}$ on $\vec{i}$ and $\vec{j}$.

$\vec{F}$: Since $|\vec{F}| = 15, \vec{F} = 15\vec{i}$

$\vec{G}$:
$$\begin{aligned}\vec{G} &= |\vec{G}|\cos 30°\ \vec{i} + |\vec{G}|\sin 30°\ \vec{j}\\ &= (10)(0.8660\ldots)\ \vec{i} + (10)(0.5)\ \vec{j}\\ &\doteq 8.660\vec{i} + 5\vec{j}\end{aligned}$$

$\vec{H}$:
$$\begin{aligned}\vec{H} &= |\vec{H}|\cos 100°\ \vec{i} + |\vec{H}|\sin 100°\ \vec{j}\\ &= (20)(-0.1736\ldots)\ \vec{i} + (20)(0.9848\ldots)\ \vec{j}\\ &\doteq -3.472\vec{i} + 19.696\vec{j}\end{aligned}$$

So
$$\begin{aligned}\vec{R} = \vec{F} + \vec{G} + \vec{H} &= (15\vec{i}) + (8.66\vec{i} + 5\vec{j}) + (-3.472\vec{i} + 19.696\vec{j})\\ &= 20.188\vec{i} + 24.696\vec{j}\end{aligned}$$

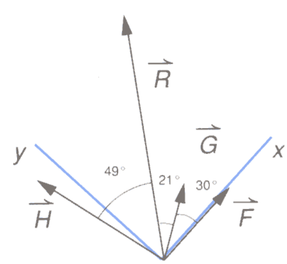
Magnitude of $\vec{R}$

$$|\vec{R}| = \sqrt{20.188^2 + 24.696^2} = \sqrt{1017.44\ldots} \doteq 31.9$$

Direction of $\vec{R}$

Let θ be the angle between $\vec{R}$ and $\vec{i}$.

Then $\tan\theta = \dfrac{24.696}{20.188} = 1.2233\ldots$, thus $\theta \doteq 51°$

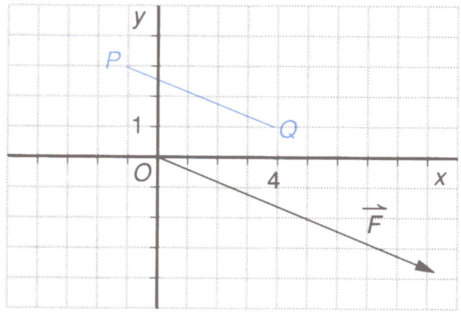


Therefore, the resultant of forces $\vec{F}$, $\vec{G}$ and $\vec{H}$ has magnitude 31.9 N and makes an angle of 51° with the force $\vec{F}$ (or an angle of 51° − 30° = 21° with the force $\vec{G}$, or an angle of 100° − 51° = 49° with the force $\vec{H}$). ■

Some vector problems on forces are formulated in component form, and it is convenient to treat them by using components, as in the next example.

Example 3 Given points $P(-1,3)$ and $Q(4,1)$, a force $\vec{F}$ of 10 N acts in the direction of $\overrightarrow{PQ}$. Resolve $\vec{F}$ parallel to the coordinate axes.

Solution



The force $\vec{F}$ points in the same direction as

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \overrightarrow{(4,1)} - \overrightarrow{(-1,3)} = \overrightarrow{(5,-2)}$,

so $\vec{F} = k\overrightarrow{(5,-2)}$ where k is a positive number.

Now $|\vec{F}| = |k\overrightarrow{(5,-2)}| = k|\overrightarrow{(5,-2)}| = k\sqrt{5^2 + 2^2} = k\sqrt{29}$

but you know that $|\vec{F}| = 10$,

so $$k\sqrt{29} = 10 \text{ or } k = \frac{10}{\sqrt{29}}$$

Thus the force $$\vec{F} = \frac{10}{\sqrt{29}}\overrightarrow{(5,-2)}$$

in other words, the force can be resolved as

$$\vec{F} = \frac{50}{\sqrt{29}}\vec{i} - \frac{20}{\sqrt{29}}\vec{j}.$$

The components of $\vec{F}$ in the x and y directions are respectively

$\frac{50}{\sqrt{29}}$ and $-\frac{20}{\sqrt{29}}$ ■

Friction

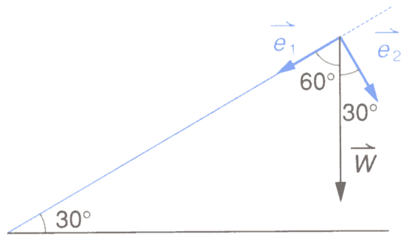
If you place a marble and a book on your desk, then gently incline the surface of your desk, you will notice that the marble rolls off almost immediately, whereas the book will remain motionless. This is because the marble encounters virtually no resistance, and so accelerates down the incline. The book, however, will not slide unless the incline is made a lot steeper. The book is held in equilibrium by a frictional force acting parallel to the surface of your desk.

These two situations will be investigated in Examples 4 and 5.

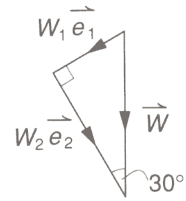
Example 4 A marble of mass 50 g is on a plane inclined at an angle 30° to the horizontal.

Resolve the force of gravity on the marble into two rectangular components, one of which causes the marble to accelerate.

Solution



space diagram



vector diagram

Note: 50 g is the same as $\frac{50}{1000} = 0.05$ kg. The force of gravity on the marble is its weight, $\vec{W}$, whose magnitude is $(0.05)(9.8) = 0.49$ N. The weight acts vertically downwards.

You are asked to resolve $\vec{W}$ in the directions of the unit vectors
$\vec{e}_1$ (parallel to the plane) and
$\vec{e}_2$ (perpendicular to the plane).

Note: The angle between $\vec{W}$ and the horizontal is 90°. Thus the angle θ between $\vec{W}$ and $\vec{e}_1$ is $(180 - 90 - 30)° = 60°$. It follows that the angle between $\vec{W}$ and $\vec{e}_2$ is $(90 - 60)° = 30°$. This is the angle used in the vector diagram.

The marble is prevented by the plane from moving in the direction of $\vec{W}$. It can only move parallel to the plane, downhill, along the direction of $\vec{e}_1$.
In this direction, the component of $\vec{W}$ is
$W_1 = |\vec{W}| \sin 30° = (0.49)(1)(0.5\ldots) = 0.245$
In the direction of $\vec{e}_2$, the component of $\vec{W}$ is
$W_2 = |\vec{W}| \cos 30° = (0.49)(1)(0.866\ldots) = 0.424$

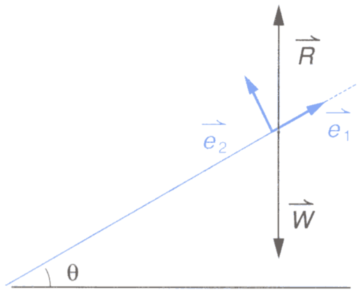
The components of the gravitational force on the marble are 0.245 N down the plane, causing the marble to accelerate, and 0.424 N perpendicular to the plane. ■

Example 5

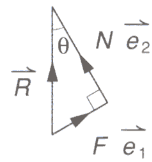
A particle of mass m kg is in equilibrium on a rough plane inclined at an angle θ to the horizontal. The equilibrant of the weight is a force called a **reaction**, acting vertically upwards.

Resolve this reaction into a force perpendicular to the plane (called a **normal reaction**), and a force along the plane (called a **frictional force**).

Solution



space diagram

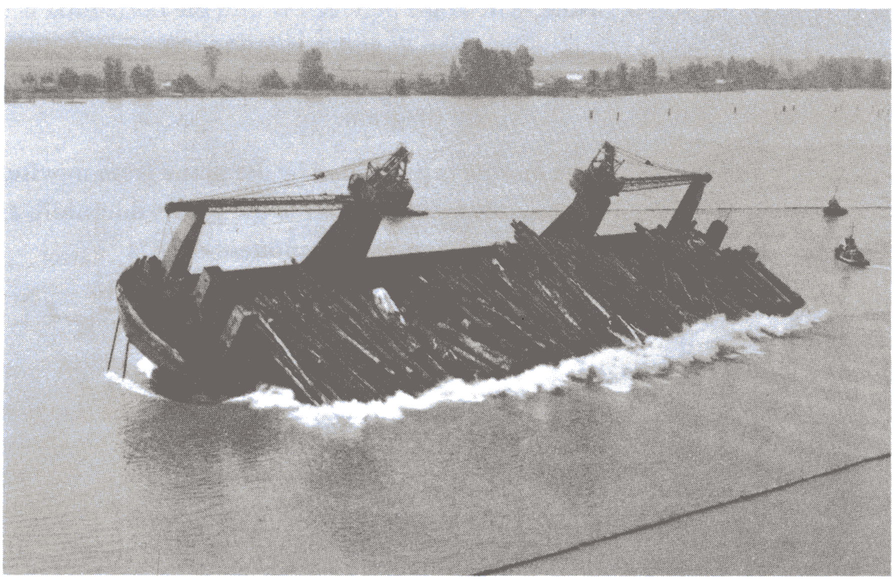


vector diagram

Let the weight be $\vec{W}$, the equilibrant be $\vec{R}$, to be resolved into the normal reaction $N\vec{e}_2$, and the force of friction $F\vec{e}_1$.

The magnitude of the equilibrant, equal to the magnitude of the weight, is mg.

Thus the normal component (the normal reaction) is $N = |\vec{R}|\cos\theta = mg\cos\theta$ and the parallel component (the frictional force) is $F = |\vec{R}|\sin\theta = mg\sin\theta$. ■



The Direction of a Force in 3-space

The following example will show you how to treat a three-dimensional problem on forces.

Example 6 The forces $\vec{P}$ and $\vec{Q}$ are such that $\vec{P} = \overrightarrow{(3,0,2)}$ and $\vec{Q} = \overrightarrow{(-4,6,7)}$. Calculate the magnitude and direction of their resultant $\vec{R}$.

Solution $\vec{R} = \vec{P} + \vec{Q} = \overrightarrow{(3,0,2)} + \overrightarrow{(-4,6,7)} = \overrightarrow{(-1,6,9)}$.

Thus the magnitude of $\vec{R}$, $|\vec{R}| = \sqrt{1^2 + 6^2 + 9^2} = \sqrt{118} \doteq 10.9$.

Now the direction of $\vec{R}$ cannot be specified merely by a 'slope' or 'bearing', since these concepts are meaningful only in 2-space.

A direction in 3-space is specified by the angles α, β and γ that it makes with $\vec{i}$, $\vec{j}$ and $\vec{k}$ respectively.

Now $\vec{R} \cdot \vec{i} = |\vec{R}||\vec{i}| \cos\alpha = |\vec{R}| \cos\alpha$, since $|\vec{i}| = 1$.

Thus $\cos\alpha = \dfrac{\vec{R} \cdot \vec{i}}{|\vec{R}|}$, and similarly $\cos\beta = \dfrac{\vec{R} \cdot \vec{j}}{|\vec{R}|}$ and $\cos\gamma = \dfrac{\vec{R} \cdot \vec{k}}{|\vec{R}|}$

Thus $\cos\alpha = -\dfrac{1}{\sqrt{118}}$, $\cos\beta = \dfrac{6}{\sqrt{118}}$, and $\cos\gamma = \dfrac{9}{\sqrt{118}}$

giving $\alpha \doteq 95°$, $\beta \doteq 56°$, and $\gamma \doteq 34°$. ■

SUMMARY

A vector $\vec{F}$ making an angle of α with $\vec{i}$ is resolved on $\vec{i}$ and $\vec{j}$ as follows.

$$\vec{F} = |\vec{F}| \cos\alpha\ \vec{i} + |\vec{F}| \sin\alpha\ \vec{j}$$

A vector $\vec{F}$ making angles of α, β, and γ with $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively is resolved on $\vec{i}$, $\vec{j}$, and $\vec{k}$ as follows.

$$\vec{F} = |\vec{F}| \cos\alpha\ \vec{i} + |\vec{F}| \cos\beta\ \vec{j} + |\vec{F}| \cos\gamma\ \vec{k}$$

The direction of a vector $\vec{v}$ in $\mathbb{V}_3$ is specified by the angles α, β, and γ that it makes with $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively.

$$\cos\alpha = \frac{\vec{v} \cdot \vec{i}}{|\vec{v}|}, \cos\beta = \frac{\vec{v} \cdot \vec{j}}{|\vec{v}|}, \text{ and } \cos\gamma = \frac{\vec{v} \cdot \vec{k}}{|\vec{v}|}$$

4.2 Exercises

(Unless directed otherwise, give all magnitudes in this exercise in newtons, correct to 3 significant digits, and all angles correct to the nearest degree.)
Use $g \doteq 9.8$ m/s^2.

1. Two forces $\vec{P}$ and $\vec{Q}$ act on a particle. Calculate the magnitude and the direction of their resultant in the following cases.
   a) $\vec{P} = \overrightarrow{(2,7)}$ and $\vec{Q} = \overrightarrow{(4,1)}$
   b) $\vec{P} = \overrightarrow{(-50,30)}$ and $\vec{Q} = \overrightarrow{(35,60)}$

2. Show that a vector $\vec{F}$ making angles of α, β and γ with $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively is resolved on $\vec{i}$, $\vec{j}$, and $\vec{k}$ as follows.
   $\vec{F} = |\vec{F}| \cos \alpha \, \vec{i} + |\vec{F}| \cos \beta \, \vec{j} + |\vec{F}| \cos \gamma \, \vec{k}$

3. Force $\vec{P} = \overrightarrow{(2,8,-1)}$ and force $\vec{Q} = \overrightarrow{(3,1,5)}$.
   a) Calculate the magnitude of the resultant of $\vec{P}$ and $\vec{Q}$.
   b) Specify the direction of the resultant by finding the angles it makes with $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively.

4. Resolve the force $\vec{F}$ in two orthogonal directions so that one component makes an angle $\theta°$ with $\vec{F}$ in the following cases.
   a) $|\vec{F}| = 20$, $\theta = 30°$
   b) $|\vec{F}| = 100$, $\theta = 10°$
   c) $|\vec{F}| = 100$, $\theta = 80°$
   d) $|\vec{F}| = 50$, $\theta = 130°$

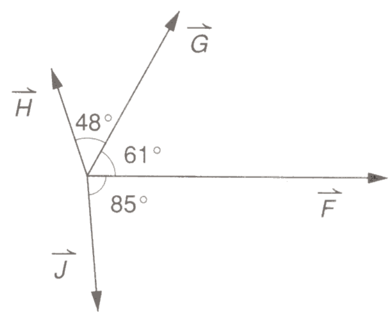
The next two questions are the same as 2 and 6 of 4.1 Exercises. Here, solve them by resolving the forces on $\vec{i}$ and $\vec{j}$.

5. Two forces $\vec{P}$ and $\vec{Q}$ of magnitudes 30 N and 60 N respectively act on a particle. The angle between $\vec{P}$ and $\vec{Q}$ is 75°. Calculate the magnitude and the direction of the resultant of $\vec{P}$ and $\vec{Q}$.

6. The resultant of $\vec{P}$ and $\vec{Q}$ is the force $\vec{F}$, of magnitude 80 N, acting on bearing 070°. $\vec{P}$ has magnitude 25 N and acts due east. Calculate the magnitude and direction of force $\vec{Q}$.

7. Two forces $\vec{P}$ and $\vec{Q}$ act on a particle and make an angle of 115° with each other. If $|\vec{P}| = 16$ N and $|\vec{Q}| = 21$ N, find the magnitude and direction of the resultant force.

8. Four coplanar forces $\vec{F}$, $\vec{G}$, $\vec{H}$, and $\vec{J}$, of magnitudes 40 N, 25 N, 15 N, and 18 N respectively, act on a particle as shown in the figure. Find the magnitude and the direction of the resultant of the four forces.

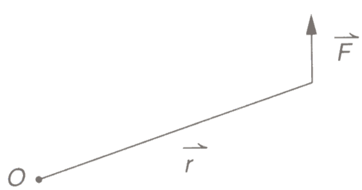


9. Given points $P(4,8)$ and $Q(2,-3)$, a force $\vec{F}$ of 50 N acts in the direction of $\overrightarrow{PQ}$. Resolve $\vec{F}$ on $\vec{i}$ and $\vec{j}$.

10. Given points $P(2,8,-5)$ and $Q(3,-1,2)$, a force $\vec{F}$ of 300 N acts in the direction of $\overrightarrow{PQ}$.
    a) Resolve $\vec{F}$ on $\vec{i}$, $\vec{j}$, and $\vec{k}$.
    b) Find the angles between $\vec{F}$ and each of $\vec{i}$, $\vec{j}$, $\vec{k}$.

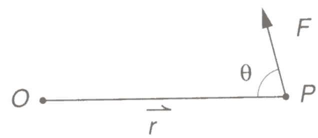
11. A Startrak Space Craft taking off from Earth is being acted upon by three forces. Its main engine is pushing it east with 200 000 N, its vertical jet is pushing it up with 150 000 N, and the north wind is pushing it southward with 30 000 N. By considering a unit vector $\vec{i}$ pointing due south, a unit vector $\vec{j}$ point due east, and a unit vector $\vec{k}$ pointing upward, find the magnitude and direction of the resultant of these three forces on the craft.

12. The moment $\vec{M}$ with respect to the origin O of a force $\vec{F}$ is the cross product $\vec{M} = \vec{r} \times \vec{F}$, where $\vec{r}$ is the position vector of any point on the line along which the force acts. Calculate the following moments.
    a) $\vec{r} = \overrightarrow{(0,3,0)}$; $\vec{F} = \overrightarrow{(2,5,6)}$
    b) $\vec{r} = \overrightarrow{(-2,1,-1)}$; $\vec{F} = \overrightarrow{(4,-3,0)}$



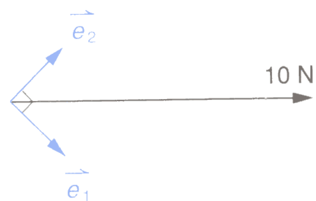
13. The moment of a force about a point is the effectiveness of that force to produce rotation about that point.

    The force $\vec{F}$ acts at the point P, as shown.



    a) What is the angle between $\vec{r}$ and $\vec{F}$?
    b) Write an expression for the moment of force $\vec{F}$ about O, in terms of $|\vec{r}|$, $|\vec{F}|$, and θ. (See question 12.)
    c) For which angle θ is the magnitude of the moment a maximum? a minimum?

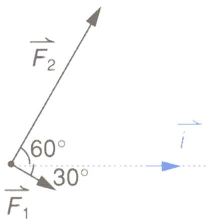
14. Resolve a force of 10 N into two equal rectangular components.



15. Resolve a force of 60 N along two orthogonal directions such that the components are in the ratio 2 : 3. Calculate the angle between the 60 N force and the larger projection.

16. A 150 kg bobsleigh is about to be released on an icy slope inclined at 35° to the horizontal. Calculate the force that must be applied parallel to the slope to keep the bobsleigh stationary.

17. Repeat question 13 of 4.1 Exercises by resolving the forces on $\vec{i}$ and $\vec{j}$.
Two forces $\vec{F_1}$ and $\vec{F_2}$, of magnitudes 7 N and 24 N, act on a particle. $\vec{F_1}$ acts at 30° to the vector $\vec{i}$, and $\vec{F_2}$ acts at 60° to the vector $\vec{i}$, as shown in the figure. Another force $\vec{F_3}$ of magnitude 30 N acts simultaneously on the particle. Calculate the direction of $\vec{F_3}$ so that the resultant of the three forces is in the direction of $\vec{i}$. (Note: The three forces would then cause the particle to move in the direction of $\vec{i}$.)



18. A force $\vec{F} = \overrightarrow{(40,60)}$ is pulling a particle up an inclined plane parallel to the vector $\overrightarrow{(9,2)}$. Calculate the component of the force that is acting parallel to the plane.

19. A particle of mass 20 kg is in equilibrium on a rough plane inclined at 26° to the horizontal.

a) Resolve the equilibrant into a normal reaction and a frictional force.

b) Calculate the number μ, where $|\vec{F}| = \mu\,|\vec{N}|$. (If the particle is about to slide, μ is called the **coefficient of static friction**.)

c) If the plane were smooth, calculate the *horizontal* force required to stop the particle from sliding down the plane.

20. Repeat question 19 for a mass of *m* kg on a plane inclined at θ° to the horizontal.

21. In the following diagrams, the suspended mass is 10 kg.

a) Calculate the tensions in the strings for figure a).

b) Calculate the tension in the string and the pushing force (called a **thrust**) in the strut for figure b).

figure 1

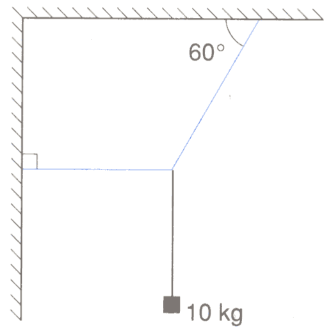
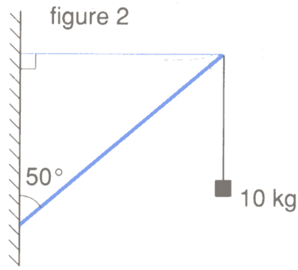
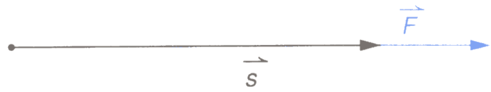


figure 2

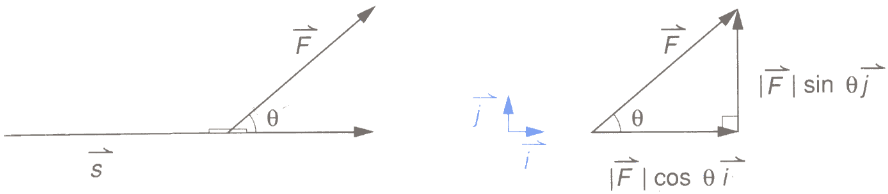


4.3 Work

When a force $\vec{F}$ pulls a particle in its direction along a displacement $\vec{s}$, the 'work' thus done by the force $\vec{F}$ is defined as $|\vec{F}||\vec{s}|$



However, the displacement s is not necessarily in the same direction as the force. For example, if a sled is tugged along a horizontal path, you might have the situation described in this diagram.



In this general case, work is defined as follows.

work done = (magnitude of force *in direction of motion*) (distance moved)

or $\quad W = (|\vec{F}|\cos\theta)(|\vec{s}|)$

or $\quad W = \vec{F} \cdot \vec{s}$

Notice, however, that the sled will only move in the desired direction along the path if θ is acute, that is, if $\theta < 90°$. The cosine of an acute angle is positive, hence *work is a positive scalar*.

If force is measured in newtons, and distance is measured in metres, then work is measured in newton metres (N · m), or joules (J).
1 newton metre equals 1 joule.

Task	Work done (approximate)
Pick up a pencil from the floor	2 J
Push a loaded shopping cart up a short ramp	500 J
Push a car out of a (level) driveway	8000 J

In the above example, you can resolve the force parallel and perpendicular to the path by writing

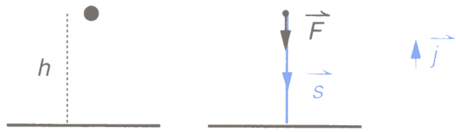
$$\vec{F} = |\vec{F}|\cos\theta\,\vec{i} + |\vec{F}|\sin\theta\,\vec{j}$$

Since the movement is horizontal here, you can say that the parallel force, $|\vec{F}|\cos\theta\,\vec{i}$, does all the work, while the vertical force, $|\vec{F}|\sin\theta\,\vec{j}$, does no work.

In the examples that follow, work will be calculated to the nearest J.

Example 1 A stone of mass m is dropped to the ground from a height h. Calculate the work done by gravity.

Solution



Let the gravitational force be $\vec{F}$ and the displacement be $\vec{s}$.

The magnitude of the gravitational force on the stone is mg.
The stone will travel in the direction of this force, thus the work done
$W = (mg)(h)$
$= mgh.$

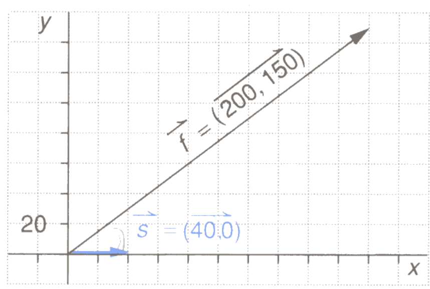
Alternatively, using vectors to describe the situation, using $\vec{j}$ as a unit vector pointing directly upwards:
$\vec{F} = -mg\vec{j}$
$\vec{s} = -h\vec{j}$
Work done $= \vec{F} \cdot \vec{s}$

$$\begin{aligned} W &= (-mg\vec{j}) \cdot (-h\vec{j}) \\ &= (-mg)(-h)\vec{j} \cdot \vec{j} \\ &= (mgh)(1) \\ &= mgh. \end{aligned}$$

■

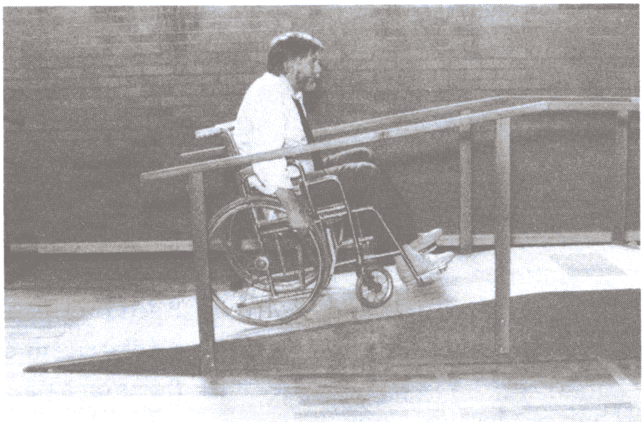
Example 2 A force $\vec{f} = \overrightarrow{(200,150)}$, whose magnitude is measured in newtons, is required to pull a heavy cart along the displacement $\vec{s} = \overrightarrow{(40,0)}$. The magnitude of $\vec{s}$ is measured in metres.
Calculate the magnitude of $\vec{f}$, the magnitude of $\vec{s}$, and the work done by $\vec{f}$.

Solution



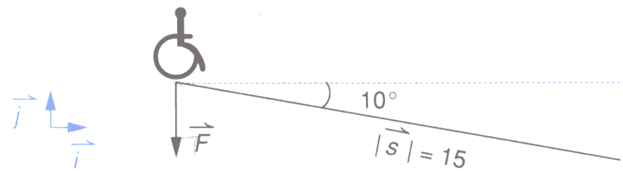
The magnitude of the force, $|\vec{f}| = \sqrt{200^2 + 150^2} = 250$.
The distance, $|\vec{s}| = 40$.
The work done, $W = \vec{f} \cdot \vec{s} = \overrightarrow{(200,150)} \cdot \overrightarrow{(40,0)}$
$= (200)(40) + 0 = 8000.$

Thus the magnitude of the force is 250 N, the distance moved is 40 m, and the work done is 8000 J. ■



Example 3 A man in a wheelchair moves 15 m down a ramp inclined at 10° to the horizontal. The mass of the man and the wheelchair together is 80 kg. Calculate the work done.

Solution Let $\vec{F}$ be the gravitational force on the man and wheelchair. The magnitude of $\vec{F}$ in newtons is $|\vec{F}| = mg = (80)(9.8) = 784$.



Using the horizontal unit vector $\vec{i}$ and the vertical unit vector $\vec{j}$ shown in the diagram, the displacement vector $\vec{s}$ can be written in component form as follows.

$$\vec{s} = \overrightarrow{(15\cos 10^\circ, -15\sin 10^\circ)}$$

and $\vec{F}$ can be written $\vec{F} = \overrightarrow{(0, -784)}$

Thus the work done is
$$\begin{aligned} \vec{F}\cdot\vec{s} &= 0 + (-784)(-15\sin 10^\circ) \\ &= (784)(15)(0.1736\ldots) \\ &= 2042.1\ldots \end{aligned}$$

The work done is about 2042 J. ■

Note: The work is said to be done *by gravity* when the man moves *down* the ramp. In order to move the same distance *up* the ramp, the man must produce an equal amount of work.

SUMMARY

Work done = (magnitude of force *in direction of motion*)(distance moved)

or $W = |\vec{F}|\cos\theta|\vec{s}|$

or $W = \vec{F}\cdot\vec{s}$

4.3 Exercises

In the following, give answers correct to 3 significant digits where appropriate.

1. Find the work done by $\vec{F}$ as it acts on a particle through a displacement $\vec{s}$ in the following cases.
   a) $\vec{F} = \overrightarrow{(20,40)}$, $\vec{s} = \overrightarrow{(1,3)}$
   b) $\vec{F} = \overrightarrow{(30,-5)}$, $\vec{s} = \overrightarrow{(1,3)}$
   c) $\vec{F} = \overrightarrow{(15,20,-22)}$, $\vec{s} = \overrightarrow{(-4,10,-7)}$

2. A force $\vec{F}$ of 1000 N moves a particle along a path that is at an angle of 42° with the force.
   a) Resolve $\vec{F}$ parallel to and perpendicular to the path.
   b) Calculate the work done if $\vec{F}$ moves the particle 30 m.

3. A tractor is pulling a disabled barge along a canal. The tension in the towrope is 2500 N, and the towrope makes an angle of 25° with the direction of the canal. Calculate the work done in moving the barge 600 m along the canal.

4. A shopper pushes a loaded supermarket cart of mass 20 kg up a ramp inclined at 15° to the horizontal. Calculate the work done if the ramp is 10 m long.

5. A force $\vec{F} = \overrightarrow{(10,10,20)}$ attempts to pull a particle along a displacement $\vec{s} = \overrightarrow{(3,-7,2)}$. Calculate the work done. Explain your answer.

6. A force $\vec{F} = \overrightarrow{(-5,-8,-10)}$ pulls a particle from point $P(25,14,20)$ to point $Q(0,-3,4)$. Calculate the work done.

7. A force $\vec{p}$ of magnitude 50 N acts in the same direction as $\overrightarrow{AB}$ where A has coordinates (5,−6,7) and B has coordinates (8,1,8). Calculate the work done in the following cases.
   a) $\vec{P}$ acts on a particle along the displacement $\overrightarrow{AB}$.
   b) $\vec{P}$ acts on a particle along the displacement $\overrightarrow{MN}$, where M has coordinates (−2,−4,−10) and N has coordinates (4,6,9).

8. The work done by gravity as a book of mass M kg drops to the floor from a height h m is equal to the **kinetic energy** of the book just before it hits the ground. The kinetic energy is given by the formula $\frac{1}{2}Mv^2$, where v is the speed of the book in m/s just before it hits the ground.
   a) It is given that $v = 7$. Calculate h.
   b) It is given that $h = 2$. Calculate v.

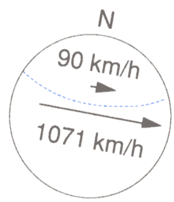
9. a) A pen of mass 100 g is dropped to the floor from a height of 1.5 m. Calculate the work done by gravity.
   b) As the pen falls to the ground, its speed v increases. Just before it hits the ground, the pen has a kinetic energy of $(0.05)v^2$ J, equal to the work done by gravity during the fall. Calculate the speed of the pen at this moment.

4.4 Velocities as Vectors

The word *speed* describes the rate at which *distance* is covered. (Speed is a scalar quantity.) The word *velocity* describes the rate at which *displacement* changes; displacement, unlike distance, includes direction as part of its definition, thus velocity also depends on direction. Like forces, velocities have both magnitude and direction, and furthermore, they can be combined following the laws of vector addition. Hence you can use vectors to represent velocities.

Relative Velocity

It is important to understand that velocities are always 'relative', never 'absolute'. For example, when you say that a truck approaching Winnipeg is travelling due east at 90 km/h, you should realize that this is its velocity *relative to the surface of the earth*. If, for instance, you want to describe its velocity relative to the centre of the earth, you must take into account the velocity of the earth's surface near Winnipeg. That is, at latitude 50° N, the earth's surface moves eastward at 1071 km/h. Thus the velocity of the truck relative to the centre of the earth is 1071 km/h faster than its surface velocity: 1071 + 90 = 1161 km/h, eastward.



The following notation is useful.

$\vec{v}_{TS}$ = velocity of the *Truck* relative to the earth's *Surface*
$\vec{v}_{SC}$ = velocity of the earth's *Surface* (Winnipeg) relative to the earth's *Centre*
$\vec{v}_{TC}$ = velocity of the *Truck* relative to the earth's *Centre*

In this case, the speeds, that is, the magnitudes of the velocities, are such that $|\vec{v}_{TC}| = |\vec{v}_{TS}| + |\vec{v}_{SC}|$, because the vectors involved have the same direction. However, if the vectors are *not* collinear, then the velocity of the truck is obtained by the following *vector addition* of relative velocities.

FORMULA

$$\vec{v}_{TC} = \vec{v}_{TS} + \vec{v}_{SC}$$

In other words, for points T, S, and C moving relative to each other, the velocity of T relative to C is the vector sum of the velocity of T relative to S, and the velocity of S relative to C.

Notice that this formula can be expressed also as a subtraction as follows.

FORMULA

$$\vec{v}_{SC} = \vec{v}_{TC} - \vec{v}_{TS}$$

In other words, the velocity of S relative to C equals the velocity of T relative to some point C, minus the velocity of T relative to S.

Although the verbal descriptions of the relative velocities above may seem difficult to learn, you might appreciate that *the order of letters involved in the notation match exactly those you are familiar with from the triangle law of vector addition, and of vector subtraction*. That is, for example, $\overrightarrow{TC} = \overrightarrow{TS} + \overrightarrow{SC}$.

Similarly, the following observation matches the definition of a negative vector.

Saying "the truck is moving eastward at 90 km/h relative to the surface" is equivalent to saying "the surface is moving westward at 90 km/h relative to the truck". You are motionless inside the truck; the scenery is travelling 'backward'. Thus, if $\vec{v}_{ST}$ is the vector representing this latter velocity, then

$$\vec{v}_{ST} = -\vec{v}_{TS}.$$

Thus the subtraction formula could be written

$$\vec{v}_{SC} = \vec{v}_{CT} - (-\vec{v}_{ST})$$

or

$$\vec{v}_{SC} = \vec{v}_{ST} - \vec{v}_{CT}$$

If a fly is crawling along the truck's dashboard, and you want to find the velocity of the fly relative to the centre of the earth, $\vec{v}_{FC}$, you will require three velocity vectors:

$\vec{v}_{FT}$ = velocity of the fly (relative to the truck)
$\vec{v}_{TS}$ = velocity of the truck (relative to the surface)
$\vec{v}_{SC}$ = velocity of the earth's surface (relative to the centre)

The resultant velocity, $\vec{v}_{FC}$, is obtained by a vector addition of these velocities, that is,

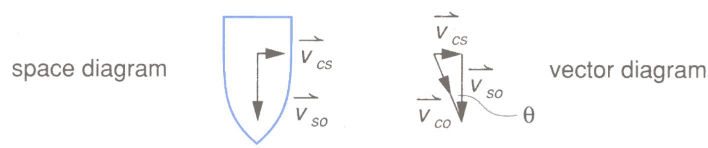
$$\vec{v}_{FC} = \vec{v}_{FT} + \vec{v}_{TS} + \vec{v}_{SC}$$

Example 1

An oil tanker is sailing due south at 60 km/h. The captain, during his daily workout routine, is running perpendicularly across the ship, from starboard (right side) to port (left side), at 25 km/h. What is the captain's velocity relative to the ocean?

Solution

$\vec{v}_{SO}$ represents the velocity of the ship relative to the ocean,
$\vec{v}_{CS}$ represents the captain's velocity relative to the ship, and
$\vec{v}_{CO}$ represents the captain's velocity relative to the ocean, then
$\vec{v}_{CO} = \vec{v}_{CS} + \vec{v}_{SO}$, as indicated in the diagram.



You can draw your vectors where you want them. The diagram that attempts to portray the position of the ship is called the space diagram. The diagram designed purely for vector algebra is the vector diagram.

By the theorem of Pythagoras,

$$|\vec{v}_{CO}|^2 = 25^2 + 60^2 = 4225$$

so $|\vec{v}_{CO}| = \sqrt{4225} = 65$,

and $\tan\theta = \dfrac{25}{60} = 0.4166\ldots$, so $\theta \doteq 23°$.

Thus the bearing of $\vec{v}_{CO}$ is $180° - 23° = 157°$.

The captain's velocity relative to the ocean is 65 km/h at a bearing 157°. ■

In the next two examples, alternate solutions that use more traditional notation will also be presented. If you now look back at the introduction to this chapter, you will find that these examples should enable you to solve the airplane pilot's problem.

Note that wind direction always indicates the direction that the wind is blowing *from*, not blowing to.

Example 2 A pilot is heading her plane due north at an airspeed of 160 km/h, while a wind is blowing from the east at 75 km/h. At what speed is she actually travelling, and in what direction, relative to the ground?

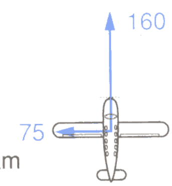
Solution You want

the velocity of the plane relative to the ground, $\vec{v}_{PG}$ (or $\vec{r}$)

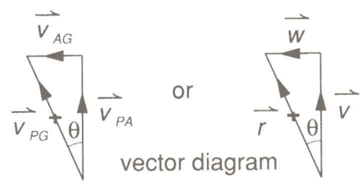
given the plane's velocity relative to the air, $\vec{v}_{PA}$ (or $\vec{v}$)

and the wind velocity, that is,

the velocity of the air relative to the ground, $\vec{v}_{AG}$ (or $\vec{w}$)



space diagram



vector diagram

Observe the diagrams carefully. The plane's groundspeed will be the magnitude of the vector $\vec{v}_{PG}$ (or $\vec{r}$), and the plane will actually fly in the direction θ west of north, *although its nose will always be pointing north.*

The vector sum is thus

$\vec{v}_{PG} = \vec{v}_{PA} + \vec{v}_{AG}$

From Pythagoras,

$|\vec{v}_{PG}|^2 = 75^2 + 160^2 = 31225$

$|\vec{v}_{PG}| = 176.70\ldots \doteq 177$

or using traditional notation you can say

$\vec{r} = \vec{v} + \vec{w}$

From Pythagoras,

$|\vec{r}|^2 = 75^2 + 160^2 = 31225$

$|\vec{r}| = 176.70\ldots \doteq 177$

Also $\tan\theta = \dfrac{75}{160} = 0.46875 \Rightarrow \theta \doteq 25°$.

Thus her plane is actually travelling at 177 km/h, at a bearing 335°, relative to the ground.

(177 km/h is known as the groundspeed of the plane.) ■

Example 3

A jet is to fly from Toronto to Ottawa. The bearing of Ottawa from Toronto is 055°, and the distance is 370 km. The airspeed of the jet is 500 km/h, and there is a prevailing wind blowing from bearing 350° at 160 km/h. Calculate the following.

a) The direction in which the jet should head, to the nearest degree.

b) The time taken, to the nearest minute, to fly from Toronto to Ottawa.

Solution

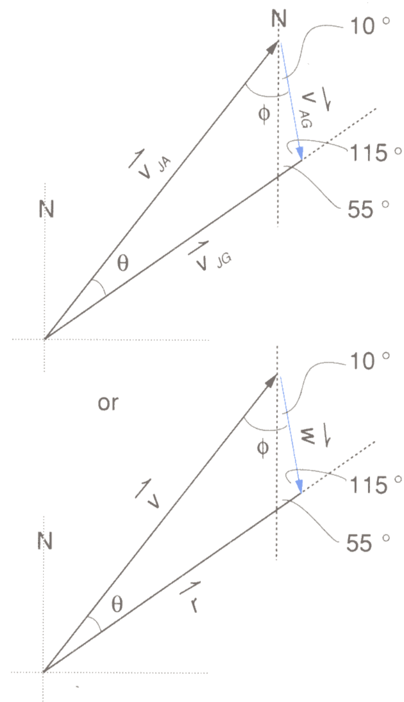
Let the jet's air velocity be $\vec{v}_{JA}$ (or $\vec{v}$)
the jet's ground velocity be $\vec{v}_{JG}$ (or $\vec{r}$)
the wind velocity be $\vec{v}_{AG}$ (or $\vec{w}$)

Then $\vec{v}_{JG} = \vec{v}_{JA} + \vec{v}_{AG}$ or $\vec{r} = \vec{v} + \vec{w}$.

You know that $|\vec{v}_{JA}| = |\vec{v}| = 500$, $|\vec{v}_{AG}| = |\vec{w}| = 160$.

Note carefully the angles marked in each diagram. If the wind is from bearing 350°, that means the angle between the wind and the south is 10°. This has been shown in the diagrams. From the space diagram, you can calculate that the angle between the Toronto-Ottawa line and the wind is $(180 - 10 - 55)^\circ = 115^\circ$. This angle is marked in the vector diagrams, where it will be used.

space diagram



vector diagram

a) By the sine law in the vector diagram

$$\frac{160}{\sin\theta} = \frac{500}{\sin 115^\circ}$$

$$\sin\theta = \frac{160 \sin 115^\circ}{500} = 0.2900\ldots \quad \text{so } \theta = 16.85\ldots^\circ$$

The jet must head about 17° left of the bearing 055°, and thus follow the heading $(055 - 17)^\circ = 038^\circ$.

b) In order to calculate the flight time from Toronto to Ottawa, you need to find the jet's groundspeed, that is, $|\vec{v}_{JB}|$ or $|\vec{r}|$. You must use the sine law again in the vector diagram. First you need the angle ϕ.

$$\phi = 180^\circ - (115^\circ + 16.85\ldots^\circ) = 48.14\ldots^\circ$$

Using the sine law again,

$$\frac{|\vec{v}_{JG}|}{\sin 48.14\ldots^\circ} = \frac{500}{\sin 115^\circ}$$

$$|\vec{v}_{JG}| = \frac{500 \sin 48.14\ldots^\circ}{\sin 115^\circ} = 410.8\ldots$$

or

$$\frac{|\vec{r}|}{\sin 48.14\ldots^\circ} = \frac{500}{\sin 115^\circ}$$

$$|\vec{r}| = \frac{500 \sin 48.14\ldots^\circ}{\sin 115^\circ} = 410.8\ldots$$

Now $\text{speed} = \dfrac{\text{distance}}{\text{time}}$, or $\text{time} = \dfrac{\text{distance}}{\text{speed}}$

Since the distance to be covered is 370 km, the time taken is

$$\frac{370}{410.8\ldots} = 0.9004\ldots, \text{ measured in hours.}$$

The time in minutes is thus $(0.9004)(60) = 54.02\ldots$

The flight time from Toronto to Ottawa is about 54 minutes. ■

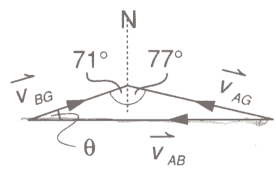
In the final example, the subscript notation should help you to avoid fundamental errors, such as adding the wrong vectors.

Example 4 A small airplane A is flying on bearing 283° at 200 km/h, and another, B, is flying on bearing 071° at 140 km/h. What is the velocity of A relative to B?

Solution

space diagrams

$\vec{v}_{AG}$ represents the velocity of A relative to the ground,
$\vec{v}_{BG}$ represents the velocity of B relative to the ground, and
$\vec{v}_{AB}$ represents the velocity of A relative to B,
then $|\vec{v}_{AG}| = 200$, $|\vec{v}_{BG}| = 140$, and $|\vec{v}_{AB}|$ is unknown.



You can find $\vec{v}_{AB}$ from the vector sum
$\vec{v}_{AB} + \vec{v}_{BG} = \vec{v}_{AG}$,
according to the vector diagram.
As you transcribe the space diagram into the vector diagram, note that the angle between $\vec{v}_{BG}$ and $\vec{v}_{AG}$ is 71° + 77° = 148°.
The cosine law gives

$$|\vec{v}_{AB}|^2 = 140^2 + 200^2 - (2)(140)(200)\cos 148° = 107090.69\ldots$$

so that $|\vec{v}_{AB}| = 327.24\ldots \doteq 327$.
Using the sine law to find the angle θ between $\vec{v}_{AB}$ and $\vec{v}_{BG}$:

$$\frac{200}{\sin\theta} = \frac{327.24\ldots}{\sin 148°}$$

$$\sin\theta = \frac{200\sin 148°}{327.24\ldots} = 0.3238\ldots \Rightarrow \theta \doteq 19°.$$

Thus the angle ϕ between $\vec{v}_{AG}$ and $\vec{v}_{AB}$ is 180° − (148° + 19°) = 13°.
Hence, according to B, A appears to be moving at 327 km/h on a bearing 360° − 77° − 13° = 270°. In other words, B perceives that A is moving away towards the west. ■

SUMMARY

(In the following, given points P and Q moving relative to each other, the notation $\vec{v}_{PQ}$ represents the velocity of P relative to Q.)

Velocity is a vector whose magnitude is called speed.

For any points P and Q moving relative to each other, $\vec{v}_{PQ} = -\vec{v}_{QP}$

Given any points A, B, C moving relative to each other,
$\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$ or $\vec{v}_{BC} = \vec{v}_{AC} - \vec{v}_{AB}$ or $\vec{v}_{BC} = \vec{v}_{BA} - \vec{v}_{CA}$

4.4 Exercises

(Unless directed otherwise, give all speeds in this exercise correct to the nearest km/h, all times correct to the nearest minute, and all angles correct to the nearest degree.)

1. In the following, given points P and Q moving relative to each other, the notation $\vec{v}_{PQ}$ represents the velocity of P relative to Q.

   Find the magnitude and direction of $\vec{v}_{PQ}$ in the following cases. (Speeds are in km/h.)
   a) $|\vec{v}_{PA}| = 40$ pointing due east
      $|\vec{v}_{AQ}| = 25$ pointing due north
   b) $|\vec{v}_{PA}| = 100$ on bearing 280°
      $|\vec{v}_{AQ}| = 35$ on bearing 190°
   c) $|\vec{v}_{PA}| = 550$ on bearing 072°
      $|\vec{v}_{AQ}| = 420$ on bearing 120°
   d) $|\vec{v}_{PA}| = 18$ on bearing 072°
      $|\vec{v}_{AQ}| = 12$ on bearing 300°

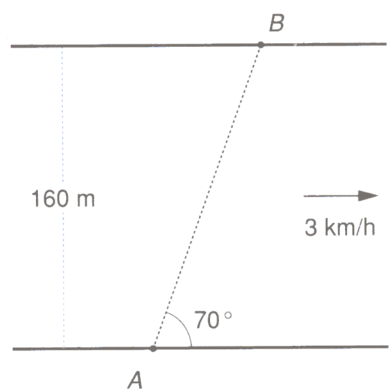
2. The Transcanadian train is travelling westward at 100 km/h. An RCMP officer is on the train. Calculate the magnitude and direction of the RCMP officer's velocity, relative to the ground, in the following cases.
   a) The RCMP officer is walking towards the front of the train at 5 km/h.
   b) The RCMP officer is running towards the back of the train at 15 km/h.
   c) The RCMP officer is running directly across the train, from the north side to the south side, at 12 km/h.

3. A plane is heading due east at an airspeed of 200 km/h, while a wind is blowing from the north at 50 km/h. Find the groundspeed and actual direction of the plane.

4. A is 1000 km due west of B. A plane whose airspeed is 500 km/h flies from A to B, then immediately back to A, on a day when the wind is blowing from the west at 100 km/h. Calculate the total time taken.

5. The Prime Minister asks her pilot to fly her immediately from Ottawa to Toronto on a day when the wind is blowing from the west at 120 km/h. Toronto is 370 km from Ottawa on a bearing 235°. The jet has an airspeed of 900 km/h. Find the following.
   a) the heading the pilot should follow
   b) the flight time

6. A swimmer, whose speed in still water is 3 km/h, wishes to cross a 160 m wide river from A to B as shown in the figure. The river is flowing at 3 km/h.
   a) In what direction should the swimmer head?
   b) How long will it take him to get to B?



7. Repeat question 6 with the following data. The width of the river is w, the speed of the flow is v, the speed of the swimmer in still water is v, and the angle between AB and the direction of the flow is θ.

8. A ship S is sailing on a bearing of 160° at 30 km/h. Another ship T is sailing on a bearing of 250° at 40 km/h.
   a) Calculate the velocity of S relative to T.
   b) Calculate the velocity of T relative to S.

9. A ship is travelling due west at 50 km/h. The smoke emanating from the ship's funnel makes an angle of 25° with the ship's wake. Calculate the speed of the wind if it is known that the wind is blowing from the north.

Mathematics and Aircraft Navigation

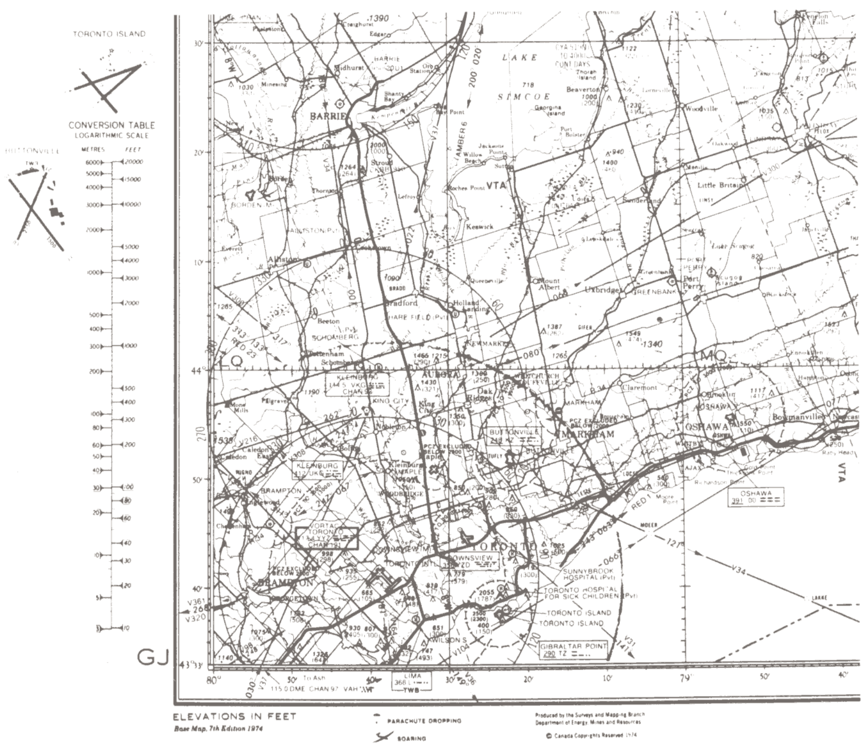
One of the requirements of a private pilot's licence is that you execute at least one long-distance solo flight from your home airport to other airports and return.

Before departing on a cross-country flight, you must prepare a flight plan for review by your instructor and submission to Air Traffic Control. The wind conditions forecast for the flight must be considered. This might involve the following telephone conversation.

"Hello. Toronto Flight Service? Can you give me information for a VFR flight (1) by light aircraft to Barrie this afternoon?"

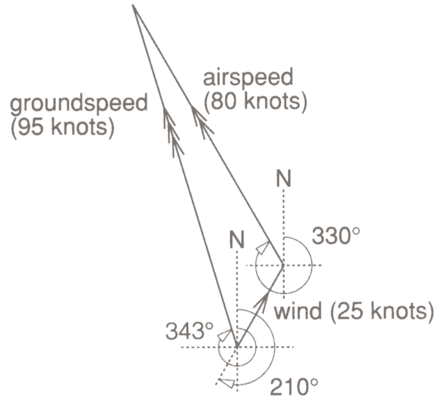
"The weather is clear. The temperature at 4500 ft is 10°C, and the wind is from 210° at 25 knots."

Aircraft altitudes are still measured in feet, and windspeeds are given in knots, or nautical miles per hour. One nautical mile (2) is the distance spanned by one minute of arc, that is, 1/60th of a degree of longitude.



By measurement on the map, you find that the bearing of Barrie airport from Toronto Island is 343°, at a distance of 49 nautical miles (3). You know that your plane has an airspeed of 80 knots.

Your heading and groundspeed are then calculated or measured according to the vector diagram.

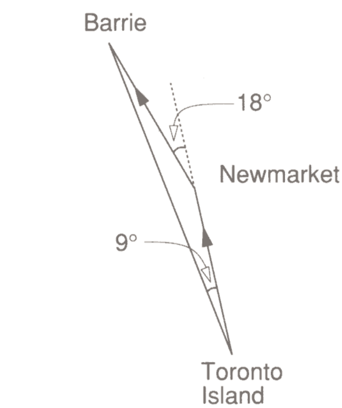


Thus you must fly a "true heading" of 330° to reach Barrie. Your groundspeed will be 95 knots. However, the magnetic north in this part of the world is 10° west of "true north". Thus, you must add 10° to obtain the *magnetic heading*, 330° + 10° = 340°, that you will follow by reference to the magnetic compass in the aircarft.

Note: Your estimated flight time to Barrie is $\frac{\text{distance}}{\text{speed}} = \frac{49}{95} = 0.516$ hours or 31 minutes, to the nearest minute.

HELP!
Due to a wind change, or inaccurate forecast, you find that you are about to fly over Newmarket after about 15 minutes. This is about 9° off course.

A quick sketch shows that you must alter your heading by about 2 × 9° = 18° to the left to reach Barrie. Thus you now turn to magnetic bearing 340° − 18° = 322°, and keep your eyes open to find Barrie airport about 15 minutes later.



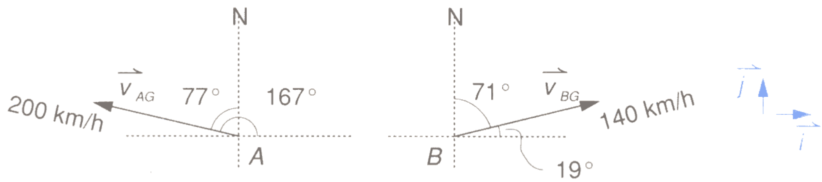
(1) A flight according to Visual Flight Rules.
(2) 1 foot (ft) ≐ 0.3048 m and 1 nautical mile (nm) ≐ 1.852 km.
(3) The usual scale of an air navigation map is 1 : 500 000. You measure 18.2 cm on the map, which represents 91 km, that is 91 × 1.852 ≐ 49 nm. Alternatively, you can measure along a meridian, in minutes of arc, for this distance (degrees and minutes are indicated at the side of the map).

4.5 Velocities Using Components

In this section, you will see that a variety of problems involving velocities can be solved by resolving vectors, or by using components.

The first example is the same as Example 4 of the previous section. This will allow you to compare methods.

Example 1 A small airplane A is flying on bearing 283° at 200 km/h, and another, B, is flying on bearing 071° at 140 km/h. Calculate the direction and magnitude of the velocity of A relative to B.



Solution Resolve the velocities in the eastward and northward directions by using the unit vectors $\vec{i}$ and $\vec{j}$ shown. Note the angles with $\vec{i}$ shown in the diagrams, and recall the resolution formula used in section 4.2, namely: a vector $\vec{v}$ making an angle α with $\vec{i}$ is resolved on $\vec{i}$ and $\vec{j}$ as follows.

$\vec{v} = |\vec{v}| \cos \alpha \, \vec{i} + |\vec{v}| \sin \alpha \, \vec{j}$

Then the velocity of A, $\vec{v}_{AG} = (200) \cos 167^\circ \, \vec{i} + (200) \sin 167^\circ \, \vec{j}$

$= -(194.87\ldots)\vec{i} + (44.99\ldots)\vec{j}$

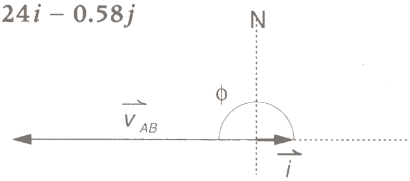
and the velocity of B, $\vec{v}_{BG} = (140) \cos 19^\circ \, \vec{i} + (140) \sin 19^\circ \, \vec{j}$

$= (132.37\ldots)\vec{i} + (45.57\ldots)\vec{j}$

You want the velocity of A relative to B, $\vec{v}_{AB} = \vec{v}_{AG} - \vec{v}_{BG}$

Thus $\vec{v}_{AB} \doteq (-194.87\vec{i} + 44.99\vec{j}) - (132.37\vec{i} + 45.57\vec{j})$

$= -327.24\vec{i} - 0.58\vec{j}$



so $|\vec{v}_{AB}| = \sqrt{327.24^2 + 0.58^2} = 327.24\ldots \doteq 327$

and the angle of $\vec{v}_{AB}$ with $\vec{i}$ is ϕ where

$\vec{v}_{AB} \cdot \vec{i} = |\vec{v}_{AB}||\vec{i}| \cos \phi$

so $\cos \phi = \dfrac{-327.24}{327.24} = -1 \Rightarrow \phi = 180^\circ.$

Thus, as you found before, B perceives that A is moving at 327 km/h towards the west. ■

Some vector problems on velocities are formulated in component form, and it is convenient to treat them by using components, as in the following example.

Example 2 A ship is travelling with velocity $\vec{v}_{SO} = (20\vec{i} + 30\vec{j})$, and a forklift on board is travelling relative to the ship with velocity $\vec{v}_{FS} = (-5\vec{i} + 6\vec{j})$. An ant is crawling on the engine cover of the forklift, with velocity $\vec{v}_{AF} = (\vec{i} - \vec{j} + 3\vec{k})$ relative to the forklift.
(All velocities in this example are given in km/h.)

Calculate the velocity and the speed of the ant relative to the ocean.

Solution The required velocity is

$$\vec{v}_{AO} = \vec{v}_{AF} + \vec{v}_{FS} + \vec{v}_{SO}$$
$$= \overrightarrow{(20,30,0)} + \overrightarrow{(-5,6,0)} + \overrightarrow{(1,-1,3)}$$
$$= \overrightarrow{(16,35,3)}.$$

Thus the speed is

$$|\vec{v}_{AO}| = \sqrt{16^2 + 35^2 + 3^2} = \sqrt{1490} = 38.60\ldots$$

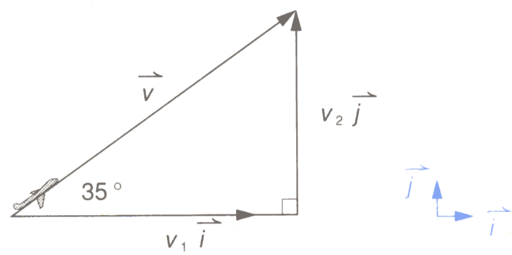
The ant's speed relative to the ocean is about 38.6 km/h. ■

Example 3 The velocity of a jet shortly after takeoff is given by the vector $\vec{v}$ whose magnitude is 500 km/h, and whose direction is 35° from the horizontal, pointing upwards.

a) Calculate the vertical speed of the jet, in m/s.
b) Calculate the speed of the jet's shadow on the ground, if the sun is directly overhead.

Solution The diagram shows that the velocity $\vec{v}$ of the jet can be resolved into a horizontal vector and a vertical vector as follows.

$$\vec{v} = v_1\vec{i} + v_2\vec{j}$$



where v_1 represents the speed of the shadow along the ground, and v_2 represents the vertical speed, or rate of climb, of the jet.

a) $v_2 = |\vec{v}|\sin 35° = (500)(0.5735\ldots) = 286.78\ldots$, in km/h.

To convert to m/s, you must multiply by 1000 and divide by 3600:

vertical speed in m/s is $(286.78\ldots)\left(\frac{1000}{3600}\right) = 79.66$

The rate of climb is about 80 m/s .

b) $v_1 = |\vec{v}|\cos 35° = (500)(0.8191\ldots) = 409.57\ldots$

The horizontal speed of the shadow is about 410 km/h. ■

4.5 Exercises

(Unless directed otherwise, give all magnitudes in this exercise correct to 3 significant digits, and all angles correct to the nearest degree.)
In the following, given points P and Q moving relative to each other, the notation $\vec{v}_{PQ}$ represents the velocity of P relative to Q.

1. Find $\vec{v}_{PQ}$ and $|\vec{v}_{PQ}|$ in the following cases.
   a) $\vec{v}_{PA} = \overrightarrow{(3,1)}$ and $\vec{v}_{AQ} = \overrightarrow{(-2,5)}$
   b) $\vec{v}_{PA} = \overrightarrow{(4,-3,6)}$ and $\vec{v}_{QA} = \overrightarrow{(8,2,-1)}$
   c) $\vec{v}_{AP} = \overrightarrow{(3,2,1)}$ and $\vec{v}_{QA} = \overrightarrow{(9,-6,-5)}$

2. A small airplane P is flying on bearing 165° at 400 km/h, and another airplane, Q, is flying on bearing 220° at 320 km/h. Calculate the magnitude and the direction of the following.
   a) the velocity of P relative to Q
   b) the velocity of Q relative to P

3. A train is travelling with velocity $(36\vec{i} + 48\vec{j})$, and the food trolley is travelling relative to the train with velocity $(6\vec{i} - 4\vec{j})$. A ladybug is crawling on the food trolley with relative velocity $(\vec{i} + \vec{j} - 2\vec{k})$. Calculate the velocity and the speed of the ladybug relative to the ground. (All velocities in this question are given in km/h.)

4. A ship is travelling with velocity $(-25\vec{i} + 50\vec{j})$, and the captain's robot is travelling relative to the ship with velocity $(4\vec{i} - 3\vec{j})$. A mantis is crawling on the robot, with velocity $(-\vec{i} + \vec{j} - 2\vec{k})$ relative to the robot. An ant is crawling on the mantis, with velocity $(2\vec{i} - 2\vec{j} + \vec{k})$ relative to the mantis. (All velocities in this question are given in m/s.)
   Calculate the velocity and the speed relative to the ocean, of
   a) the robot
   b) the mantis
   c) the ant.

5. Repeat question 3 of 4.4 Exercises by using components.

   A plane is heading due east at an airspeed of 200 km/h, while a wind is blowing from the north at 50 km/h.
   Find the groundspeed and actual direction of the plane.

6. Repeat question 5 of 4.4 Exercises by using components.

   The Prime Minister asks her pilot to fly her immediately from Ottawa to Toronto on a day when the wind is blowing from the west at 120 km/h. Toronto is 370 km from Ottawa on a bearing 235°. The jet has an airspeed of 900 km/h.
   Find the following.
   a) the heading the pilot should follow
   b) the flight time

7. A bus is travelling at a steady speed of 20 m/s in the direction of $\vec{i}$ on a rainy day. Raindrops, which are falling vertically (in the direction of $-\vec{j}$), make traces on the side windows of the bus. These traces are inclined at 25° to the horizontal. Calculate the following.

   a) the horizontal component of the drops' velocity with respect to the ground
   b) the horizontal component of the drops' velocity with respect to the bus
   c) the drops' velocity with respect to the ground
   d) the drops' velocity and speed with respect to the bus

8. Repeat question 8 of 4.4 Exercises by using components.

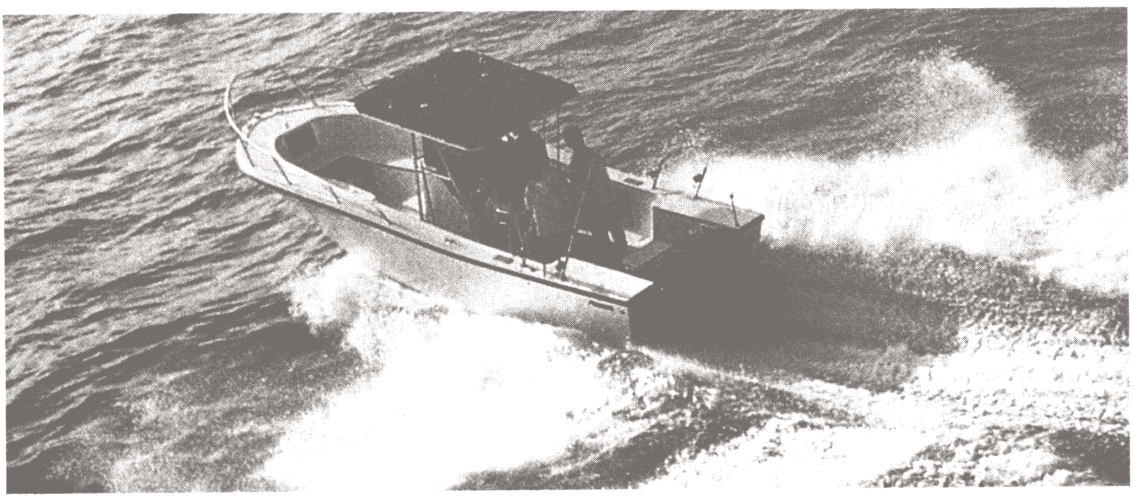
   A ship S is sailing on a bearing of 160° at 30 km/h. Another ship T is sailing on a bearing of 250° at 40 km/h.
   a) Calculate the velocity of S relative to T.
   b) Calculate the velocity of T relative to S.

9. A motor boat is moving at 15 km/h in the direction 048°. The wind is pushing the motorboat 3 km/h westward, and a current is pushing the motorboat 5 km/h northward. Calculate the direction and magnitude of the motorboat due to its own power.

10. A plane P is heading on bearing 060° at 450 km/h, while a plane Q is heading on bearing 340° at 400 km/h, at a time when the air is still.
    a) Calculate the velocity of P relative to Q.
    b) If the wind now starts blowing at 150 km/h from the east, calculate the velocity of P relative to Q.

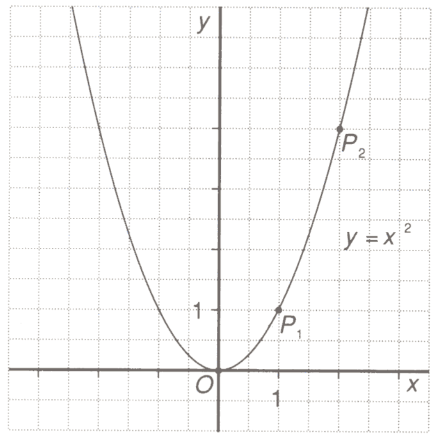
11. A first class passenger on an airliner opens a bottle. The cork pops off and travels at 100 km/h directly across the plane, from the right side to the left side. The plane is heading due east with an airspeed of 500 km/h, and there is a wind blowing from the southwest (that is, bearing 225°) at 120 km/h. Calculate the direction and the magnitude of the velocity of the cork relative to the ground.

12. An airplane pilot heads his plane due east and maintains an airspeed of 170 km/h. After flying for 30 minutes, he finds himself over a village which is 100 km east and 22 km north of his starting point. Find the magnitude and direction of the wind velocity.



In Search of Vector Functions

The points of $\mathbb{R}^2$ whose coordinates (x,y) satisfy the equation $y = x^2$ ① form a parabola.



Notice that if $x = t$ ②

and $y = t^2$ ③

where $t \in \mathbb{R}$ is called a **parameter**, then eliminating t between the equations ② and ③ yields $y = x^2$, the equation ①.

The parabola can be represented
either by the equation ①, called a **Cartesian equation**,
or by the system of equations ② and ③, called **parametric equations**.

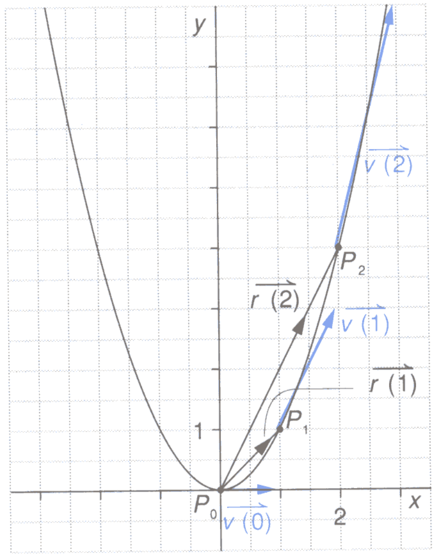
Each value of t gives a specific point on the parabola.
For example,
when $t = 1$, ② gives $x = 1$ and ③ gives $y = 1$.
So $t = 1$ gives the point (1,1).
When $t = 2$, ② gives $x = 2$ and ③ gives $y = 4$.
So $t = 2$ gives the point (2,4).

Now think of $P(x,y)$ as a particle travelling along the curve.
If t is a measure of time, the particle is
at $P_0(0,0)$ at time $t = 0$,
at $P_1(1,1)$ at time $t = 1$,
at $P_2(2,4)$ at time $t = 2$, etc.

Thus the parametric equations can represent the movement of a particle along the parabola, starting at (0,0) and going into the first quadrant.

Now if $\vec{i}$ and $\vec{j}$ are unit vectors along the x and y axes respectively, then the position vector of the particle is $\overrightarrow{OP} = x\vec{i} + y\vec{j}$ or, using ② and ③,

$$\overrightarrow{OP} = t\vec{i} + t^2\vec{j}.$$



$\overrightarrow{OP}$ depends on time, and $\overrightarrow{OP}$ is a vector.
Thus, $\overrightarrow{OP}$ can be called a **vector function** of time,
$\overrightarrow{OP} = \overrightarrow{r(t)} = t\vec{i} + t^2\vec{j}$ ④

The movement of the particle along the parabola can also be described by the vector function ④.

From ④, $\overrightarrow{r(0)} = 0\vec{i} + 0\vec{j} = \vec{0} = \overrightarrow{OP_0}$,
$\overrightarrow{r(1)} = 1\vec{i} + 1^2\vec{j} = \vec{i} + \vec{j} = \overrightarrow{OP_1}$,
$\overrightarrow{r(2)} = 2\vec{i} + 2^2\vec{j} = 2\vec{i} + 4\vec{j} = \overrightarrow{OP_2}$, etc.

The Calculus of Vector Functions

It can be shown that vector functions can be differentiated term by term, just like ordinary functions.

Differentiating ④ yields the velocity vector function
$\overrightarrow{r'(t)} = \overrightarrow{v(t)} = 1\vec{i} + 2t\vec{j} = \vec{i} + 2t\vec{j}$

Observe that $\overrightarrow{v(0)} = \vec{i} + 0\vec{j} = \vec{i}$
$\overrightarrow{v(1)} = \vec{i} + 2\vec{j}$
$\overrightarrow{v(2)} = \vec{i} + 4\vec{j}$

These are the velocity vectors of the particle at times 0, 1, 2 respectively. (These velocity vectors are represented by coloured directed line segments in the figure.)

Differentiating again yields the acceleration vector function.

Example 1 A particle in $\mathbb{R}^2$ moves such that $x = \cos t$ and $y = \sin t$, where t represents time. That is, the position vector of the particle is $\overrightarrow{r(t)} = \cos t\vec{i} + \sin t\vec{j}$.

a) Describe the motion of the particle for $t \geqslant 0$.

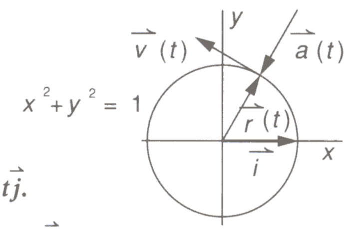
b) Find the velocity of the particle at any time, $\overrightarrow{v(t)}$; compare it to $\overrightarrow{r(t)}$.

c) Find the acceleration of the particle at any time, $\overrightarrow{a(t)}$; compare it to $\overrightarrow{r(t)}$.

Solution a) $x = \cos t$ and $y = \sin t$.

Squaring and adding these two equations eliminates t as follows.
$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, or $x^2 + y^2 = 1$.
(See the trigonometrical identities, page 542.)
Thus the particle moves in a circle, centre O, radius 1.

The position vector $\overrightarrow{r(t)} = \cos t\vec{i} + \sin t\vec{j}$ represents a 'moving radius' of this circle.
At the beginning of the motion, when $t = 0$,
$\overrightarrow{r(0)} = \cos 0\vec{i} + \sin 0\vec{j} = 1\vec{i} + 0\vec{j} = \vec{i}$



b) Differentiating $\overrightarrow{r(t)} = \cos t\vec{i} + \sin t\vec{j}$

yields $\overrightarrow{r'(t)} = \overrightarrow{v(t)} = -\sin t\vec{i} + \cos t\vec{j}$.

Note: $\overrightarrow{r(t)} \cdot \overrightarrow{v(t)} = (\cos t\vec{i} + \sin t\vec{j}) \cdot (-\sin t\vec{i} + \cos t\vec{j})$
$= -\cos t \sin t + \sin t \cos t = 0$

Thus, at any given time t, $\overrightarrow{r(t)}$ and $\overrightarrow{v(t)}$ are perpendicular.
This indicates that the velocity $\overrightarrow{v(t)}$ of the particle is *tangential* to the circle, as expected.

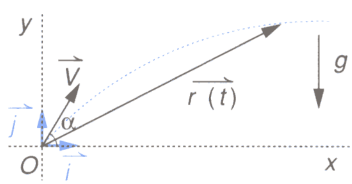
c) Differentiating $\overrightarrow{v(t)} = -\sin t\vec{i} + \cos t\vec{j}$

yields $\overrightarrow{v'(t)} = \overrightarrow{a(t)} = -\cos t\vec{i} - \sin t\vec{j}$. ■

Note: $\overrightarrow{a(t)} = -\overrightarrow{r(t)}$.
Thus, the acceleration of a particle travelling steadily around a circle is *along the radius, pointing towards the centre of the circle*. This indicates that the particle, while travelling in a direction perpendicular to the radius at any instant, keeps 'trying' to turn toward the centre. (This is known as **centripetal** acceleration.)

Example 2 A projectile is fired with initial speed V at an angle α to the horizontal. It is subject only to the acceleration due to gravity, g.
Find a vector function describing the motion of the projectile.

Solution



Let $\overrightarrow{r(t)} = x\vec{i} + y\vec{j}$ be the position vector of the projectile, from the point O from which it is fired, at any time t.

Thus x and y are the components of $\overrightarrow{r(t)}$ on $\vec{i}$ and $\vec{j}$, the standard basis vectors, at any time t.

You know that the acceleration acts vertically downwards, so
$\overrightarrow{r''(t)} = -g\vec{j}$.
Integrating (that is, antidifferentiating) gives
① $\overrightarrow{r'(t)} = -gt\vec{j} + \vec{P}$, where $\vec{P}$ is a constant vector.
But you know $\overrightarrow{r'(0)} = V\cos\alpha\vec{i} + V\sin\alpha\vec{j}$, so
$V\cos\alpha\vec{i} + V\sin\alpha\vec{j} = (-g)(0)\vec{j} + \vec{P}$
hence $\vec{P} = V\cos\alpha\vec{i} + V\sin\alpha\vec{j}$.
Thus ① becomes $\overrightarrow{r'(t)} = -gt\vec{j} + V\cos\alpha\vec{i} + V\sin\alpha\vec{j}$
or $\overrightarrow{r'(t)} = V\cos\alpha\vec{i} + (V\sin\alpha - gt)\,\vec{j}$
Integrating again gives
② $\overrightarrow{r(t)} = Vt\cos\alpha\vec{i} + \left(Vt\sin\alpha - \frac{1}{2}gt^2\right)\vec{j} + \vec{Q}$, where $\vec{Q}$ is a constant vector.

But you know that $\overrightarrow{r(0)} = \vec{0}$, so $\vec{0} = 0\vec{i} + 0\vec{j} + \vec{Q}$, hence $\vec{Q} = \vec{0}$

Thus ② becomes $\overrightarrow{r(t)} = Vt\cos\alpha\vec{i} + \left(Vt\sin\alpha - \frac{1}{2}gt^2\right)\vec{j}$.

The Cartesian equation of this trajectory can be obtained by eliminating t from the two equations

$$\begin{cases} x = Vt\cos\alpha \\ y = Vt\sin\alpha - \frac{1}{2}gt^2 \end{cases}$$

and

This gives

③ $y = \dfrac{-gx^2}{2V^2\cos^2\alpha} + (\tan\alpha)\,x$ which has the form $y = ax^2 + bx + c$. Thus, the path of a projectile is in the shape of a parabola. ■

Activities

1. Verify that the relationship between x and y of Example 2 is the equation ③.
2. Find the range of the projectile by setting $y = 0$.
3. Find the greatest height of the projectile by setting $y' = 0$.
4. A particle in $\mathbb{R}^2$ moves such that $x = a\cos t$ and $y = a\sin t$, where t represents time, and a is a constant.
   a) Describe the motion of the particle for $t \geqslant 0$.
   b) Find the velocity of the particle at any time, $\overrightarrow{v(t)}$, and compare it to $\overrightarrow{r(t)}$.
   c) Find the acceleration of the particle at any time, $\overrightarrow{a(t)}$, and compare it to $\overrightarrow{r(t)}$.
5. Repeat question 4 for $x = a\cos t$ and $y = b\sin t$.

Summary

- The resultant, $\vec{R}$, of a number of forces is the vector sum of those forces. The equilibrant of those forces is $-\vec{R}$.
- A particle is in equilibrium when the vector sum of all forces acting upon it is $\vec{0}$.
- The gravitational force on a particle of mass m kg is mg N, where g is the acceleration due to gravity.
(On earth, $g \doteq 9.8$ m/s^2)
- A vector $\vec{F}$ making an angle of α with $\vec{i}$ is resolved on $\vec{i}$ and $\vec{j}$ as follows.
$\vec{F} = |\vec{F}| \cos \alpha \, \vec{i} + |\vec{F}| \sin \alpha \, \vec{j}$
- A vector $\vec{F}$ making angles of α, β and γ with $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively is resolved on $\vec{i}$, $\vec{j}$, and $\vec{k}$ as follows.
$\vec{F} = |\vec{F}| \cos \alpha \, \vec{i} + |\vec{F}| \cos \beta \, \vec{j} + |\vec{F}| \cos \gamma \, \vec{k}$
- The direction of a vector $\vec{v}$ in $\mathbb{V}_3$ is specified by the angles α, β, and γ that it makes with $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively.
$$\cos \alpha = \frac{\vec{v} \cdot \vec{i}}{|\vec{v}|}, \quad \cos \beta = \frac{\vec{v} \cdot \vec{j}}{|\vec{v}|}, \text{ and } \cos \gamma = \frac{\vec{v} \cdot \vec{k}}{|\vec{v}|}$$
- Work done, W, by a force $\vec{F}$ acting along a displacement $\vec{s}$
= (magnitude of force *in direction of motion*) (distance moved) or
$W = \vec{F} \cdot \vec{s}$

(In the following, given points P and Q moving relative to each other, the notation $\vec{v}_{PQ}$ represents the velocity of P relative to Q.)

- Velocity is a vector whose magnitude is called speed.
- For any points P and Q moving relative to each other,
$\vec{v}_{PQ} = -\vec{v}_{QP}$
- Given any points A, B, C moving relative to each other,
$\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$
or $\vec{v}_{BC} = \vec{v}_{AC} - \vec{v}_{AB}$
or $\vec{v}_{BC} = \vec{v}_{BA} - \vec{v}_{CA}$

Inventory

Complete each of the following statements.

1. A force or a velocity can be represented by a ______.
2. An object small enough to be considered as a point is called a ______.
3. Forces are measured in ______.

4. A mass of 1 kg weighs about ____.

5. Two forces or two velocities can be combined in the same way that vectors are ____.

6. If $\vec{P}$ and $\vec{Q}$ are forces, $\vec{R} = \vec{P} + \vec{Q}$ is called the ____ force.

7. If $\vec{R}$ and $\vec{S}$ are forces, and $\vec{R} + \vec{S} = \vec{0}$, then $\vec{S}$ is called the ____ of $\vec{R}$.

8. When the vector sum of all forces acting on a particle is $\vec{0}$, the particle is said to be in ____.

9. A vector diagram helps you to solve a physical problem. The reality of the physical situation is portrayed in a ____ diagram.

10. A vector $\vec{v}$ making an angle of α with $\vec{i}$ is resolved on $\vec{i}$ and $\vec{j}$ as follows. $\vec{v} =$ ____ $\vec{i} +$ ____ $\vec{j}$.

11. A vector $\vec{v}$ making angles of α, β, and γ with $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively is resolved on $\vec{i}$, $\vec{j}$, and $\vec{k}$ as follows.
$\vec{v} =$ ____ $\vec{i} +$ ____ $\vec{j} +$ ____ $\vec{k}$.

12. If $\vec{F} = 3\vec{e}_1 + 5\vec{e}_2$, then the numbers 3 and 5 are called the ____ of $\vec{F}$ on $\vec{e}_1$ and $\vec{e}_2$.

13. An object on a rough inclined plane is kept in equilibrium by a frictional force. This frictional force acts ____ to the plane.

14. The ____ of a vector $\vec{v}$ in $\mathbb{V}_3$ is specified by the angles α, β, and γ that it makes with ____, ____, and ____ respectively.

15. 'Speed' describes the ____ of velocity. 'Distance' describes the magnitude of ____.

16. The work done by a force is a *vector/scalar*. (Delete the incorrect term.)

17. Work is measured in ____. (These are also the units of energy.)

18. The work done by a force is the ____ of the force vector and the displacement vector.

19. The work done by a force acting perpendicularly to the displacement is ____.

20. The symbol $\vec{v}_{AB}$ describes the velocity of A ____ to B.

21. Complete the following relative velocity formulas.

    a) $\vec{v}_{AC} = \vec{v}_{AB} +$ ____

    b) $\vec{v}_{BC} = \vec{v}_{AC} -$ ____

    c) ____ $= \vec{v}_{BA} - \vec{v}_{CA}$.

Review Exercises

Where appropriate, give all angles correct to the nearest degree, all times correct to the nearest minute, and all other numerical answers correct to 3 significant digits.
Given points P and Q moving relative to each other, the notation $\overrightarrow{v_{PQ}}$ represents the velocity of P relative to Q.
Use $g \doteq 9.8\text{ m/s}^2$.

1. Two forces $\vec{P}$ and $\vec{Q}$ of magnitudes 80 N and 50 N respectively act on a particle. The angle between $\vec{P}$ and $\vec{Q}$ is 62°. Calculate the magnitude and the direction of the resultant of $\vec{P}$ and $\vec{Q}$.

2. A particle is being pulled by two forces $\vec{F_1}$ and $\vec{F_2}$. $\vec{F_1}$ acts vertically upward and its magnitude is 400 N. $\vec{F_2}$ acts at an angle of 20° to the horizontal, and its magnitude is 500 N.
   a) Find the magnitude and the direction of the resultant force.
   b) Find the direction and magnitude of the equilibrant.

3. Three forces of 50 N, 85 N, and 40 N act simultaneously on a particle, which remains in a state of equilibrium. Calculate the angles between the forces.

4. Is it possible for a particle to remain in equilibrium under the action of three forces whose magnitudes are 60 N, 30 N, and 20 N? Explain.

5. a) Two perpendicular forces $\vec{P}$ and $\vec{Q}$ are such that $2|\vec{P}| = 5|\vec{Q}|$. Given that their resultant has magnitude 90 N, calculate $|\vec{P}|$ and $|\vec{Q}|$.
   b) Two perpendicular forces $\vec{P}$ and $\vec{Q}$ are such that $|\vec{P}| = 2|\vec{Q}|$. If their resultant has magnitude 55 N, calculate $|\vec{P}|$ and $|\vec{Q}|$.

6. Two forces of magnitude 15 N acting on a particle have a resultant of magnitude 5 N. Calculate the angle between the two forces.

7. A particle of mass 8 kg is suspended by cords from two points A and B on a horizontal ceiling such that $AB = 2$ m. The lengths of the cords are 1.6 m and 1.1 m. Calculate the tension in each cord.

8. Force $\vec{P} = \overrightarrow{(50,-26,14)}$ and force $\vec{Q} = \overrightarrow{(16,-10,-45)}$.
   a) Calculate the magnitude of the resultant of $\vec{P}$ and $\vec{Q}$.
   b) Specify the direction of the resultant by finding the angles it makes with $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively.

9. Four coplanar forces $\vec{F}$, $\vec{G}$, $\vec{H}$, and $\vec{J}$ are such that their resultant is 2000 N along bearing 330°. $\vec{F}$ has magnitude 800 N and acts along 030°. $\vec{G}$ has magnitude 600 N and acts along 180°. $\vec{H}$ has magnitude 800 N and acts along 233°. Calculate the direction and magnitude of the force $\vec{J}$.

10. Given points $P(4,8,-3)$ and $Q(1,-2,5)$, a force $\vec{F}$ of 100 N acts in the direction of $\overrightarrow{PQ}$. Resolve $\vec{F}$ on $\vec{i}$, $\vec{j}$, and $\vec{k}$.

11. Resolve a force of 100 N into equal components along three mutually orthogonal directions.

12. A 70 kg luge is about to be released on an icy slope inclined at 50° to the horizontal. Calculate the force that must be applied parallel to the slope to keep the luge stationary.

13. A particle of mass 50 kg is in equilibrium on a rough plane inclined at 32° to the horizontal.
    Resolve the equilibrant into a normal reaction and a frictional force.

14. In the following diagrams, the suspended mass is 10 kg.

a) Calculate the tension in the strings for figure a).

b) Calculate the tension in the string, and the pushing force (called a thrust) in the strut for figure b).

figure a

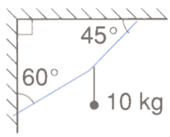
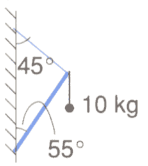


figure b



15. A particle of mass M kg remains stationary on a smooth plane inclined at an angle of $\theta°$ to the horizontal. It is held in equilibrium by a horizontal force. Show that the magnitude of this force in newtons is $Mg \sin \theta \cos \theta$.

16. A force $\vec{P}$ of magnitude 300 N acts in the same direction as $\overrightarrow{AB}$ where A has coordinates $(-1,5,8)$ and B has coordinates $(3,4,9)$. Calculate the work done in the following cases.

a) $\vec{P}$ acts on a particle along the displacement $\overrightarrow{AB}$.

b) $\vec{P}$ acts on a particle along the displacement $\overrightarrow{MN}$, where M has coordinates $(4,4,-7)$ and N has coordinates $(2,-1,3)$.

17. A force $\vec{F}$ of 800 N moves a particle along a path that is at an angle of 82° with the force.

a) Resolve $\vec{F}$ parallel to and perpendicular to the path.

b) Calculate the work done if $\vec{F}$ moves the particle 10 m.

18. A traveller in an airport pushes a loaded luggage cart of mass 45 kg up a ramp inclined at 12° to the horizontal. Calculate the work done if the ramp is 20 m long.

19. A microlight aircraft is heading due west at an airspeed of 90 km/h, while a wind is blowing from the south at 40 km/h. Find the groundspeed and actual direction of the aircraft.

20. An airliner is to take off from Winnipeg to fly to Montréal, 1850 km away on a bearing 094°. The captain prepares her flight plan according to the following data. At the altitude at which the airliner is to fly, the wind is blowing from bearing 330° at 100 km/h. The airspeed of the airliner is 900 km/h.

a) Calculate the heading that the airliner should take.

b) Calculate the flight time.

21. A ship S is sailing on bearing of 005° at 25 km/h. Another ship T is sailing on a bearing of 160° at 32 km/h.

a) Calculate the velocity of S relative to T.

b) Calculate the velocity of T relative to S.

22. A hovercraft is travelling with velocity $(30\vec{i} - 42\vec{j})$, and a trolley is being pushed relative to the hovercraft with velocity $(-5\vec{i} - 6\vec{j})$. A fly is crawling on the trolley with relative velocity

$$\left(-\vec{i} + \frac{1}{2}\vec{j} - \vec{k}\right).$$

Calculate the velocity and the speed of the fly relative to the sea. (All velocities in this question are given in km/h.)