VECTORS, MATRICES and COMPLEX NUMBERS

with
International Baccalaureate
questions

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CHAPTER FIVE

EQUATIONS OF LINES IN 2- AND 3-SPACE

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CHAPTER FIVE

Equations of Lines in 2- and 3-space

A city has a number of power lines going out underground in straight lines from points *A* and *B* at different depths below the main distribution centre. The map showing these lines has been accidentally destroyed. Your task is to find out which lines coming from point *A* intersect with a line coming from point *B*. If two lines do not intersect, you are required to determine the shortest distance between these lines, and to find the points on the lines where this shortest distance occurs.

You will be able to solve this problem if you can determine the equation of a line in 3-space.

In 3-space, as in 2-space, two lines can intersect or not intersect. In 2-space two lines that do not intersect must be parallel. But, as explained in section 1.2, in 3-space there are lines that are not parallel *and* do not intersect. You can observe this in your classroom by noticing that a line on the front wall parallel to the floor will not intersect a line on the back wall that is perpendicular to the floor, even though the two lines are not parallel.

As you saw in chapter 1, two lines in 3-space that are not parallel and do not intersect are called *skew lines*.

In this chapter you will learn how to find the equations of lines in 3-space. With this and other information from this chapter you will have the mathematics necessary to solve the 'power line' problem given above.

5.1 The Vector Equation of a Line

You are familiar with equations such as $y = -2x + 6$, $4x - 2y = 15$, $x = 5$, and $y = -3$. Each represents a straight line and is called a *linear equation*. What is meant by saying that $y = -2x + 6$ is an equation for a straight line? If you graph every point P whose coordinates (x,y) satisfy this equation, then the points P will all lie in a straight line. For example, the point $(1,4)$ lies on this line because $x = 1$, $y = 4$ substituted in the equation $y = -2x + 6$ gives $4 = -2(1) + 6$, which is true. Because the line is defined in terms of points $P(x,y)$ in a Cartesian coordinate system, this equation is also called the **Cartesian equation of a line in 2-space**.

You will recall that for the line $y = -2x + 6$, the slope of the line is -2 and the y-intercept is 6, that is, the line intersects the y-axis in the point $(0,6)$. Figure 1 shows the line drawn using the axes of a Cartesian coordinate system. The chart gives the coordinates of several points on the line.

x	4	3	2	1	0	-1	-2
y	-2	0	2	4	6	8	10

The line consists of the set of all points $P(x,y)$ such that $y = 2x + 6$. Note that a point whose coordinates does *not* satisfy this equation does *not* lie on the line. The point $O(0,0)$ is clearly not on the line. If $x = 0$, $y = 0$ are substituted in $y = -2x + 6$, the result $0 = -2(0) + 6$ is not true.

The line described by the Cartesian equation $y = -2x + 6$ can also be defined by an equation that contains vectors, called a **vector equation of a line**. Instead of looking at the coordinates (x,y) of a point P on the line, a vector equation describes the position vector $\overrightarrow{OP}$ of a point P on the line. Figure 2 shows position vectors for several points on the line.

figure 1

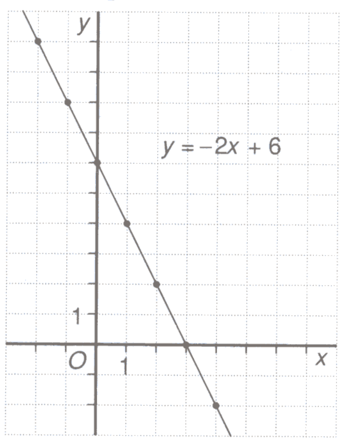
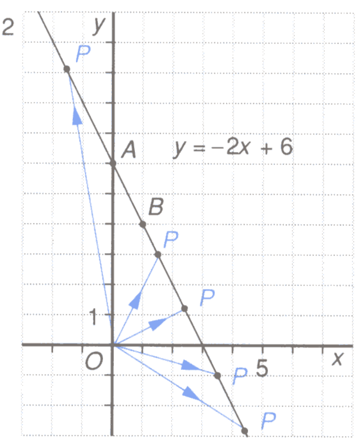
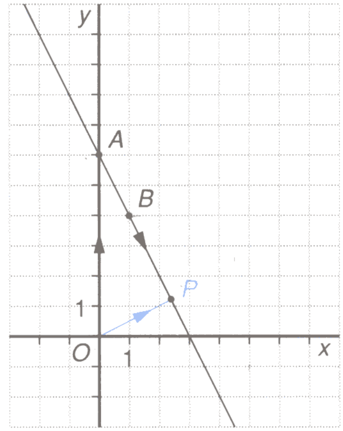


figure 2



Two facts are needed about this line in order to obtain its vector equation. For example, the point $A(0,6)$ and the point $B(1,4)$ are on the line. These points are used to determine a vector parallel to the line, namely $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{(1,4)} - \overrightarrow{(0,6)} = \overrightarrow{(1,-2)}$.

Let P be any point on the line.
From the figure, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$ ①
But $\overrightarrow{OA} = (0,6)$, and $\overrightarrow{AP}$ is a vector collinear with vector $\overrightarrow{AB}$.
Thus, $\overrightarrow{AP} = k\overrightarrow{AB}$ where $k \in \mathbb{R}$.

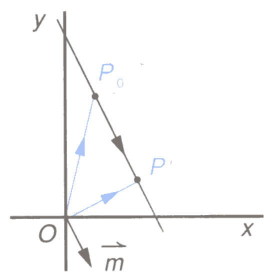


Different values for k will give different positions for the point P on the line.
Substituting in ①
$\overrightarrow{OP} = \overrightarrow{OA} + k\overrightarrow{AB}$
$\overrightarrow{OP} = (0,6) + k\overrightarrow{(1,-2)}$
This is a vector equation for the line through the points A and B.

A vector, such as $\overrightarrow{AB}$, that is collinear with or parallel to a line, is called a **direction vector** of the line.

The above method can be used *to find a vector equation of the line passing through the fixed point P, and having direction vector* $\vec{m}$.

Let P be any point on the line. From the figure,
$$\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}.$$
But $\overrightarrow{P_0P}$ is parallel to $\vec{m}$.
Thus $\overrightarrow{P_0P} = k\vec{m}$, $k \in \mathbb{R}$.
Thus $\overrightarrow{OP} = \overrightarrow{OP_0} + k\vec{m}$.
This is the required vector equation of the line.

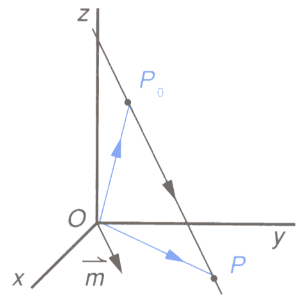


It is customary to use the abbreviations $\vec{r} = \overrightarrow{OP}$ and $\vec{r}_0 = \overrightarrow{OP_0}$

FORMULA

The vector equation of a line is
$\vec{r} = \vec{r}_0 + k\vec{m}$
where
$\vec{r} = \overrightarrow{OP}$, the position vector of *any* point P on the line,
$\vec{r}_0 = \overrightarrow{OP_0}$ the position vector of a *given* point P_0 on the line,
$\vec{m}$ is a vector parallel to the line,
k is any real number and is called a **parameter**.

One of the beautiful mathematical results of vector geometry is that this vector equation of a line in 2-space, namely, $\vec{r} = \vec{r}_0 + k\vec{m}$, is also the vector equation of a line in 3-space. The derivation of this equation for 3-space is identical to its derivation in 2-space. You can easily see that this is true by employing the preceding argument using the diagram below.



However, there is one difference.
In 2-space the vectors $\vec{r}$, $\vec{r}_0$, and $\vec{m}$ have two components, while in 3-space the vectors $\vec{r}$, $\vec{r}_0$, and $\vec{m}$ have three components.

Example 1 a) Find a vector equation of the line, through the point $A(2,-1)$, that is parallel to the vector $\overrightarrow{(-5,3)}$.

b) Find a vector equation of the line, through the point $B(2,3,1)$, having $\vec{m} = \overrightarrow{(4,5,6)}$ as direction vector.

Solution For both a) and b) the vector equation of a line is $\vec{r} = \vec{r}_0 + k\vec{m}$.

a) Here, $\vec{r}_0 = \overrightarrow{OA} = \overrightarrow{(2,-1)}$ and $\vec{m} = \overrightarrow{(-5,3)}$

Thus $\vec{r} = \overrightarrow{(2,-1)} + k\overrightarrow{(-5,3)}$ ①

which is the required vector equation of the line.

b) Here, $\vec{r}_0 = \overrightarrow{OB} = \overrightarrow{(2,3,1)}$ and $\vec{m} = \overrightarrow{(4,5,6)}$

Thus $\vec{r} = \overrightarrow{(2,3,1)} + k\overrightarrow{(4,5,6)}$ is a vector equation of the line. ■

Note 1 Any vector parallel to $\overrightarrow{(-5,3)}$ could have been used for $\vec{m}$ in part a). If $\vec{m} = \overrightarrow{(-10,6)}$ the equation is $\vec{r} = \overrightarrow{(2,-1)} + s\overrightarrow{(-10,6)}$. ②

2 Any point on the line besides point A could be used for P_0 and hence for $\overrightarrow{OP_0} = \vec{r}_0$. Therefore, if $k = 1$ in $\vec{r} = \overrightarrow{(2,-1)} + k\overrightarrow{(-5,3)}$, $\vec{r} = \overrightarrow{(2,-1)} + (1)\overrightarrow{(-5,3)} = \overrightarrow{(-3,2)}$. Hence, $\overrightarrow{(-3,2)}$ could have been substituted for $\vec{r}_0$. Thus, another equation for the same line in a) would be $\vec{r} = \overrightarrow{(-3,2)} + t\overrightarrow{(-5,3)}$ ③

3 A comparison of equations ① ② and ③, all of which represent the same line, should help you to realize that it is not always easy to see at a glance whether two equations represent the same line.

Similar remarks could be made about part b) of Example 1.

Example 2

a) Find a vector equation of the line passing through the points $A(3,2)$ and $B(0,-5)$.

b) Find a vector equation of the line passing through the points $C(3,2,-1)$ and $D(0,-5,8)$.

Solution

For both a) and b) the vector equation of a line is $\vec{r} = \vec{r_0} + k\vec{m}$.

a) Here, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \overrightarrow{(0,-5)} - \overrightarrow{(3,2)}$

Thus $\overrightarrow{AB} = \overrightarrow{(-3,-7)}$
is a direction vector for the line.

Hence $\vec{m} = \overrightarrow{(-3,-7)}$.

Since both A and B lie on the line, either $\overrightarrow{OA}$ or $\overrightarrow{OB}$ can be used for $\vec{r_0}$.

Using $\vec{r_0} = \overrightarrow{OA} = \overrightarrow{(3,2)}$, a vector equation is
$\vec{r} = \overrightarrow{(3,2)} + k\overrightarrow{(-3,-7)}$.

b) Here, $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$= \overrightarrow{(0,-5,8)} - \overrightarrow{(3,2,-1)}$

Thus $\overrightarrow{CD} = \overrightarrow{(-3,-7,9)}$ is a direction vector for the line.

Hence $\vec{m} = \overrightarrow{(-3,-7,9)}$.

Since both C and D lie on the line, either $\overrightarrow{OC}$ or $\overrightarrow{OD}$ can be use for $\vec{r_0}$.

Using $\vec{r_0} = \overrightarrow{OC} = \overrightarrow{(3,2,-1)}$, a vector equation is
$\vec{r} = \overrightarrow{(3,2,-1)} + k\overrightarrow{(-3,-7,9)}$. ■

Example 3

Find a vector equation of the line passing through the point $E(3,0,2)$ that is perpendicular to vector $\vec{u} = \overrightarrow{(4,-1,2)}$, and also perpendicular to vector $\vec{v} = \overrightarrow{(1,0,-3)}$.

Solution

A vector equation of the line is $\vec{r} = \vec{r_0} + k\vec{m}$.

Here, $\vec{r_0} = \overrightarrow{OE} = \overrightarrow{(3,0,2)}$.

Since the line is perpendicular to both $\vec{u}$ and $\vec{v}$, the line is parallel to any vector that is perpendicular to both $\vec{u}$ and $\vec{v}$.

But $\vec{u} \times \vec{v}$ is a vector that is perpendicular to both $\vec{u}$ and $\vec{v}$.

cross product

Hence a direction vector $\vec{m} = \vec{u} \times \vec{v} = \overrightarrow{(4,-1,2)} \times \overrightarrow{(1,0,-3)} = \overrightarrow{(3,14,1)}$.

Substituting in $\vec{r} = \vec{r_0} + k\vec{m}$ gives

$$\vec{r} = \overrightarrow{(3,0,2)} + k\overrightarrow{(3,14,1)}$$

This is a vector equation for the line. ■

5.1 Exercises

1. For each of the following vector equations of lines, state the coordinates of a point on the line and a vector parallel to the line.
   a) $\vec{r} = \overrightarrow{(2,-1)} + k\overrightarrow{(4,2)}$
   b) $\vec{r} = \overrightarrow{(8,-3)} + t\overrightarrow{(5,-4)}$
   c) $\vec{r} = \overrightarrow{(3,-1,4)} + k\overrightarrow{(5,-2,1)}$
   d) $\vec{r} = t\overrightarrow{(1,0,-8)} + \overrightarrow{(-4,7,5)}$

2. For each of the following find a vector equation of the line passing through the point P_0 that has the given vector $\vec{m}$ as direction vector.
   a) $P_0\ (3,7)$ $\vec{m} = \overrightarrow{(1,5)}$
   b) $P_0\ (-2,0)$ $\vec{m} = \overrightarrow{(-9,-2)}$
   c) $P_0\ (6,9)$ $\vec{m} = \overrightarrow{(-2,4)}$
   d) $P_0\ (3,2,7)$ $\vec{m} = \overrightarrow{(1,5,3)}$
   e) $P_0\ (0,-2,0)$ $\vec{m} = \overrightarrow{(-9,-2,5)}$
   f) $P_0\ (2,4,-3)$ $\vec{m} = \overrightarrow{(0,0,6)}$

3. a) Find a vector equation of the line that passes through the points $A(4,-6)$ and $B(2,7)$.
   b) Find a vector equation of the line that passes through the points $A(4,-6,2)$ and $B(-1,2,7)$.

4. a) Find a vector equation of the line that passes through the point $C(-5,2)$ and is parallel to the line through the points $K(1,4)$ and $M(3,7)$.
   b) Find a vector equation of the line that passes through the point $C(3,-5,2)$ and is parallel to the line through the points $K(1,4,-2)$ and $M(3,7,4)$.

5. Find a vector equation of the line that passes
   a) through the point $A(3,-1)$ and is parallel to the line with equation $\vec{r} = \overrightarrow{(0,2)} + k\overrightarrow{(-3,2)}$
   b) through the point $A(3,-1,-5)$ and is parallel to the line with equation $\vec{r} = \overrightarrow{(0,2,0)} + k\overrightarrow{(0,-3,2)}$.

6. Find a vector equation of the line that passes through the point $A(3,-1)$ and is perpendicular to the line with equation $\vec{r} = \overrightarrow{(0,2)} + k\overrightarrow{(-3,2)}$.

7. Find a vector equation of the line passing through the point $A(3,0,2)$ that is perpendicular to vector $\vec{u} = \overrightarrow{(4,-1,2)}$ and is also perpendicular to vector $\vec{v} = \overrightarrow{(1,0,-3)}$.

8. Find a vector equation of the line that passes through the point $D(3,-1,2)$ and is perpendicular to the line with equation $\vec{r} = \overrightarrow{(4,0,2)} + k\overrightarrow{(5,-3,2)}$ and is also perpendicular to the line $\vec{r} = \overrightarrow{(1,1,2)} + s\overrightarrow{(-2,1,3)}$.

9. a) Find the value of t so that the two lines $\vec{r} = \overrightarrow{(1,2)} + k\overrightarrow{(3,-1)}$ and $\vec{r} = \overrightarrow{(4,1)} + s\overrightarrow{(4,t)}$ will be perpendicular.
   b) Find the value of t so that the two lines $\vec{r} = \overrightarrow{(1,8,2)} + k\overrightarrow{(-4,3,-1)}$ and $\vec{r} = \overrightarrow{(4,1,2)} + s\overrightarrow{(4,t,-3)}$ will be perpendicular.

10. Given the points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ with position vector $\vec{r}_1$ and position vector $\vec{r}_2$ respectively.
   a) Show that $\vec{r} = (1-k)\,\vec{r}_1 + k\vec{r}_2$ is a vector equation of the line through P_1 and P_2.
   b) Describe the position of points on the line in part a) for $0 < k < 1$.
   c) Repeat part b) for $k > 1$ and for $k < 0$.

11. a) Find a vector equation of the line, passing through the point $A(3,8)$, that is parallel to the x-axis.
   b) Find a vector equation of the line, passing through the point $A(3,8,1)$, that is parallel to the y-axis.
   c) Find a vector equation of the line, passing through the point $A(3,8,1)$, that is parallel to the z-axis.

5.2 Parametric Equations of a Line

Suppose $P(x,y)$ is any point on the line in 2-space, through the point $P_0(x_0,y_0)$, that is parallel to the vector $\vec{m} = \overrightarrow{(m_1,m_2)}$.

A vector equation for this line is

$$\vec{r} = \vec{r_0} + k\vec{m} \quad ①$$

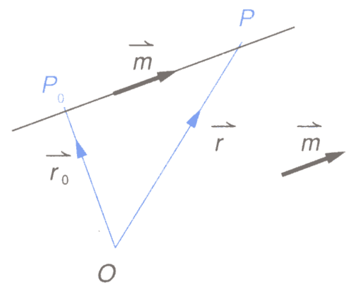
But $\vec{r} = \overrightarrow{OP} = \overrightarrow{(x,y)}$

$\vec{r_0} = \overrightarrow{OP_0} = \overrightarrow{(x_0,y_0)}$

Substituting in ① gives

$\overrightarrow{(x,y)} = \overrightarrow{(x_0,y_0)} + k\overrightarrow{(m_1,m_2)}$ or,

$\overrightarrow{(x,y)} = \overrightarrow{(x_0 + km_1, y_0 + km_2)}$



By the definition of equal vectors, corresponding components must be equal.

Thus $\begin{cases} x = x_0 + km_1 \\ y = y_0 + km_2 \end{cases}$

These equations for the components are called **parametric equations** of a line, through the point $P_0(x_0,y_0)$, that is parallel to the vector $\vec{m} = \overrightarrow{(m_1,m_2)}$.

The vector $\vec{m}$ determines the direction of the line. Hence m is called a **direction vector** of the line. The components m_1 and m_2, written with the x components first, are called **direction numbers** of the line.

In a similar manner parametric equations can be found for a line in 3-space.

If P and P_0 have 3-space coordinates (x,y,z) and (x_0,y_0,z_0) respectively, and $\vec{m} = \overrightarrow{(m_1,m_2,m_3)}$ then $\overrightarrow{OP} = \overrightarrow{(x,y,z)}$ and $\overrightarrow{OP_0} = \overrightarrow{(x_0,y_0,z_0)}$.

Therefore, equation ① becomes

$\overrightarrow{(x,y,z)} = \overrightarrow{(x_0,y_0,z_0)} + k\overrightarrow{(m_1,m_2,m_3)}$ or

$\overrightarrow{(x,y,z)} = \overrightarrow{(x_0 + km_1, y_0 + km_2, z_0 + km_3)}$

Thus, parametric equations of the line are

$$\begin{cases} x = x_0 + km_1 \\ y = y_0 + km_2 \\ z = z_0 + km_3 \end{cases}$$

Example 1 a) Find parametric equations of the line, through the point $P_0(2,3)$, having $\overrightarrow{m} = \overrightarrow{(4,5)}$ as direction vector.
b) Find parametric equations of the line, through the point $P_0(2,3,1)$, having $\overrightarrow{m} = \overrightarrow{(4,5,6)}$ as direction vector.

Solution A vector equation of a line is

a)
$$\overrightarrow{r} = \overrightarrow{r_0} + k\overrightarrow{m}.$$

Here, $\overrightarrow{r} = \overrightarrow{(x,y)}$, $\overrightarrow{r_0} = \overrightarrow{(2,3)}$ and $\overrightarrow{m} = \overrightarrow{(4,5)}$

Thus $\overrightarrow{(x,y)} = \overrightarrow{(2,3)} + k\overrightarrow{(4,5)}$

$\overrightarrow{(x,y)} = \overrightarrow{(2 + 4k,3 + 5k)}$ is a vector equation of the line.

Thus $\begin{cases} x = 2 + 4k \\ y = 3 + 5k \end{cases}$

are the required parametric equations.

b) A vector equation of a line is
$$\overrightarrow{r} = \overrightarrow{r_0} + k\overrightarrow{m}.$$

Here, $\overrightarrow{r} = \overrightarrow{(x,y,z)}$, $\overrightarrow{r_0} = \overrightarrow{(2,3,1)}$ and $\overrightarrow{m} = \overrightarrow{(4,5,6)}$

Thus $\overrightarrow{(x,y,z)} = \overrightarrow{(2,3,1)} + k\overrightarrow{(4,5,6)}$

$\overrightarrow{(x,y,z)} = \overrightarrow{(2 + 4k,3 + 5k,1 + 6k)}$ is a vector equation of the line.

Thus $\begin{cases} x = 2 + 4k \\ y = 3 + 5k \\ z = 1 + 6k \end{cases}$

are the required parametric equations. ■

Note 1 Both part a) and part b) could be done by substituting directly into the formula for the parametric equations of a line. But Example 1 shows that the parametric equation formulas need not be memorized.

2 The multipliers of the parameters, namely the numbers 4 and 5 (in part a) and 4, 5, and 6 (in part b) are the same numbers as the *direction numbers* of the line. *Direction numbers should always be given in the order*
x then y in 2-space and
x then y then z in 3-space.

3 If the vector equation of a line is written in column form as
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + k\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$
then the parametric equations are more easily recognised.

The following Examples 2 and 3 will be done for lines in 2-space. Similar solutions can be used for lines in 3-space.

Example 2 Given the line L with parametric equations $\begin{cases} x = 2 - 5k \\ y = -1 + 3k \end{cases}$

a) Determine the coordinates of three points on line L.

b) Find the value of k that corresponds to the point $(12,-7)$ that lies on line L.

c) Show that the point $(-3,4)$ does not lie on line L.

Solution

a) Each value of the parameter k gives the position vector of a point on line L. Let $k = 1$, then $x = 2 - 5(1) = -3$ and $y = -1 + 3(1) = 2$.
Hence $(-3,2)$ is a point on line L.
Similarily using, say, the values $k = 2$ and $k = 3$ you will obtain the points $(-8,5)$ and $(-13,8)$ on the line L.

b) Since the point $(12,-7)$ lies on line L, $x = 12$ and $y = -7$ may be substituted in the parametric equations for L.
Thus $12 = 2 - 5k$ and $-7 = -1 + 3k$. Each equation solves to give $k = -2$.

c) If the point $(-3,4)$ lies on line L, $x = -3$ and $y = 4$ may be substituted in the parametric equations for L.

Thus $-3 = 2 - 5k$ and $4 = -1 + 3k$
or $-5 = -5k$ and $5 = 3k$
or $k = 1$ and $k = \frac{5}{3}$, which is impossible.

Hence the point $(-3,4)$ does not lie on line L. ■

Example 3 Explain why the following parametric equations represent the same line.

① $\begin{cases} x = 1 + 3k \\ y = 5 + 2k \end{cases}$ ② $\begin{cases} x = 1 - 6t \\ y = 5 - 4t \end{cases}$

Solution From ①, a point on the line is $(1,5)$ and a direction vector is $\vec{u} = \overrightarrow{(3,2)}$.
From ②, a point on the line is $(1,5)$ and a direction vector is $\vec{v} = \overrightarrow{(6,4)}$.
Since $\vec{v} = \overrightarrow{(6,4)} = 2\overrightarrow{(3,2)} = 2\vec{u}$, the vectors $\vec{u}$ and $\vec{v}$ are parallel.
Because the two lines have a common point $(1,5)$ and have direction vectors that are parallel, therefore the two lines are the same line. ■

SUMMARY

$\vec{r} = \overrightarrow{OP}$ the position vector of *any* point P on the line,
$\vec{r}_0 = \overrightarrow{OP_0}$ the position vector of a *given* point P_0 on the line,
$\vec{m}$ is a vector parallel to the line,
k is any real number called a *parameter*.

	2-Space	*3-Space*
vector equation	$\vec{r} = \vec{r}_0 + k\vec{m}$	
parametric equations	$\begin{cases} x = x_0 + km_1 \\ y = y_0 + km_2 \end{cases}$	$\begin{cases} x = x_0 + km_1 \\ y = y_0 + km_2 \\ z = z_0 + km_3 \end{cases}$

5.2 Exercises

1. For each of the following parametric equations of lines, state the coordinates of a point on the line and a vector parallel to the line. Indicate whether the line is in 2-space or in 3-space.

a) $\begin{cases} x = 5 + 2k \\ y = 2 + 4k \end{cases}$

b) $\begin{cases} x = -3 + 8s \\ y = 1 - 5s \end{cases}$

c) $\begin{cases} x = -2 + a \\ y = 3a \end{cases}$

d) $\begin{cases} x = 6k - 4 \\ y = 1 \end{cases}$

e) $\begin{cases} x = 5 + 2k \\ y = 2 + 4k \\ z = 2 - 5k \end{cases}$

f) $\begin{cases} x = -3 + 8s \\ y = 1 - 5s \\ z = -2 + 5s \end{cases}$

g) $\begin{cases} x = -2 + a \\ y = 3a \\ z = 2a + 4 \end{cases}$

h) $\begin{cases} x = 6k - 4 \\ y = 1 \\ z = -7k \end{cases}$

2. State direction numbers for each of the lines in question 1.

3. a) Write the coordinates of three points on the line whose parametric equations are $\begin{cases} x = -6 + 4k \\ y = 3 - 2k \end{cases}$

b) Write three other sets of parametric equations representing the line in part a).

4. a) Write the coordinates of three points on the line whose parametric equations are $\begin{cases} x = -6 + 4k \\ y = 3 - 2k \\ z = 5 - k \end{cases}$

b) Write three other sets of parametric equations representing the line in part a).

5. Find a vector equation and parametric equations for each of the following lines.

a) through the point $A(-3,4)$ with direction vector $\overrightarrow{(5,1)}$

b) through the points $A(-3,4)$ and $B(7,2)$

c) through the point $A(-3,4)$ with direction numbers 6 and −2

d) through the point $B(7,2)$ parallel to the line $\vec{r} = \overrightarrow{(4,7)} + k\overrightarrow{(4,-5)}$

e) through the point $C(8,-3)$ and parallel to the x-axis

f) through the point $C(8,-3)$ and parallel to the y-axis

6. Find a vector equations and parametric equations for each of the following lines.

a) through the point $A(5,-3,4)$ with direction vector $\overrightarrow{(-6,2,1)}$

b) through the points $A(5,-3,4)$ and $B(7,2,-1)$

c) through the point $A(5,-3,4)$ with direction vector $\overrightarrow{(6,7,-2)}$

d) through the point $B(7,2,-1)$ parallel to the line $\vec{r} = \overrightarrow{(4,7,0)} + k\overrightarrow{(4,-5,1)}$

e) through the point $C(8,-3,4)$ and parallel to the x-axis

f) through the point $C(8,-3,4)$ and parallel to the y-axis

g) through the point $C(8,-3,4)$ and parallel to the z-axis

7. Given the line with parametric equations $\begin{cases} x = 5 - 3t \\ y = 2 + 4t \end{cases}$ determine whether or not the following points lie on this line.
$A(2,6)$ $B(-1,3)$ $C(1.5,4)$ $D(8,-2)$ $E(-3,4)$

8. Given the line with parametric equations $\begin{cases} x = 5 - 3t \\ y = 2 + 4t \\ z = -1 + 2t \end{cases}$ determine whether or not the following points lie on this line.
$A(2,6,1)$ $B(-1,3,3)$ $C(3.5,4,0)$ $D(8,-2,-3)$ $E(-3,4,0)$

9. In each of the following, determine whether or not the given point lies on the line with the given equation.

point	line equation
a) $(-2,-1,-6)$	$\vec{r} = \overrightarrow{(2,1,0)} + k\overrightarrow{(4,2,3)}$
b) $(10,17,2)$	$\vec{r} = \overrightarrow{(1+3k,-4+7k,5-k)}$
c) $(2,12,1)$	$\vec{r} = \overrightarrow{(2,3k,5)}$

10. Explain why each of the following equations represents the same line as the equation $\vec{r} = \overrightarrow{(1,2)} + k\overrightarrow{(4,6)}$

a) $\vec{r} = \overrightarrow{(1,2)} + k\overrightarrow{(8,12)}$

b) $\vec{r} = \overrightarrow{(1,2)} + w\overrightarrow{(-2,-3)}$

c) $\vec{r} = \overrightarrow{(5,8)} + t\overrightarrow{(4,6)}$

d) $\vec{r} = \overrightarrow{(-1,-1)} + s\overrightarrow{(2,3)}$

11. Explain why each of the following equations represents the same line as the equation $\vec{r} = \overrightarrow{(1,2,3)} + k\overrightarrow{(4,6,-2)}$

a) $\vec{r} = \overrightarrow{(1,2,3)} + k\overrightarrow{(8,12,-4)}$

b) $\vec{r} = \overrightarrow{(1,2,3)} + w\overrightarrow{(-2,-3,1)}$

c) $\vec{r} = \overrightarrow{(5,8,1)} + t\overrightarrow{(4,6,-2)}$

d) $\vec{r} = \overrightarrow{(-1,-1,4)} + s\overrightarrow{(2,3,-1)}$

12. Show that the equations $\vec{r} = \vec{r_0} + k\overrightarrow{(a,b)}$ and $\vec{r} = \vec{r_0} + t\overrightarrow{(sa,sb)}$ represent the same line, where a, b, k, t, and $s \in \mathbb{R}$.

13. Given the line with Cartesian equation $3x + 2y = 5$,

a) find parametric equations for the line using the parameter k by letting $x = k$ and solving for y in terms of k

b) find parametric equations for the line using the parameter t by letting $x = 3 + 2t$ and solving for y in terms of t.

14. a) Find parametric equations of the line, passing through the point $D(-3,2)$, that is parallel to the vector $\vec{w} = \overrightarrow{(-4,7)}$.

b) For each parametric equation found in part a), express the parameter in terms of x or y.

c) Eliminate the parameter from the equations found in b), hence obtain a Cartesian equation for the line.

15. a) Find parametric equations of the line, passing through the point $D(-3,2,1)$, that is parallel to the vector $\vec{u} = \overrightarrow{(2,-4,7)}$.

b) For each parametric equation found in part a), express the parameter in terms of x, y, or z.

c) Solve each equation found in b) for the parameter. Equate the three values of the parameter to obtain Cartesian equations for the line.

16. Given the points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ with position vector $\vec{r_1}$ and position vector $\vec{r_2}$ respectively. In question 10 of 5.1 Exercises you showed that a vector equation of this line is $\vec{r} = (1 - k)\,\vec{r_1} + k\vec{r_1}$.

Let $\overrightarrow{OQ} = \vec{q}$ be the position vector of the point that divides segment P_1P_2 internally in the ratio $a:b$.

a) Show that the value of k corresponding to point Q is $\dfrac{a}{a+b}$

b) Show that $\vec{q} = \dfrac{b}{a+b}\vec{r_1} + \dfrac{a}{a+b}\vec{r_2}$

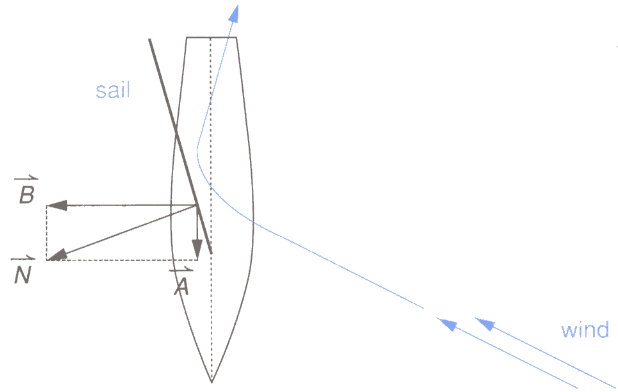
c) Use the results of part b) to find the coordinates of the point dividing the segment $P_1(3,1)$ to $P_2(5,4)$ internally in the ratio $7:2$.

17. Show that the lines
$L_1: \vec{r} = \overrightarrow{(2,0,0)} + k\overrightarrow{(0,1,3)}$ and
$L_2: \vec{r} = \overrightarrow{(1,2,3)} + t\overrightarrow{(2,1,0)}$ never intersect.

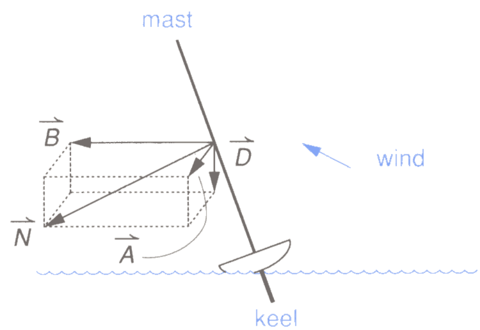
M A K I N G

Sailboards and Sailboats

A sailboat can use the power of the wind to travel 'upwind', as shown in the top-view diagram below. If the sail is properly set, the wind exerts a horizontal force $\vec{N}$ normal to the sail as shown. This force can be resolved into two rectangular components, $\vec{A}$ parallel to the keel (1) of the craft, and $\vec{B}$ perpendicular to it. The force $\vec{A}$ drives the boat forward, while the force $\vec{B}$ is counteracted by the keel.

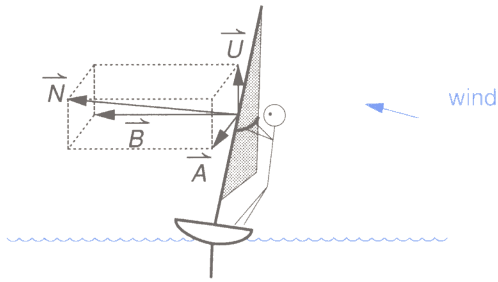


This model works well when the wind is light. However, the stronger the wind, the more another factor must be considered. The wind also causes the boat to lean sideways. Thus, the normal force $\vec{N}$ described above is not actually horizontal, but points partly downward as shown in this front-view diagram. $\vec{N}$ can thus be resolved in three directions, $\vec{A}$ driving the boat forward, $\vec{B}$ opposed by the keel, and $\vec{D}$ pushing the boat downward into the water, thus increasing its weight, and thereby its resistance to forward motion.



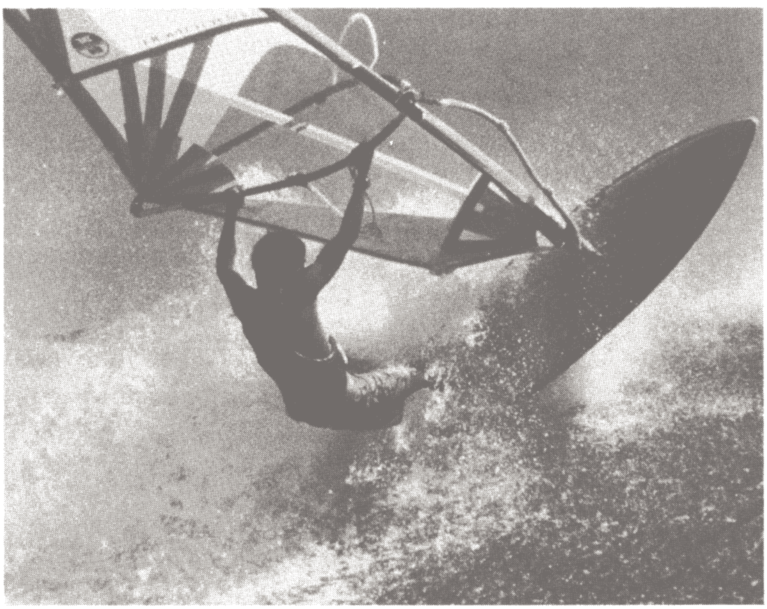
A sailboard is a device that has a free sail system. This means that the mast is attached to the board by a universal joint (rather than being fixed rigidly to it). Thus the mast, and the sail, are free to move in any direction.

A sailboard or sailboat acts in the same way in light winds, when the mast is approximately vertical. However, when the wind is strong, riders of sailboards must keep their balance by pulling the mast *toward* them. Thus, the normal force $\vec{N}$ points partly *upward*.



$\vec{N}$ can be resolved into the following forces. $\vec{A}$ drives the board forward. $\vec{B}$ is opposed by the daggerboard (2) and $\vec{U}$ *decreases* the weight of the craft in the water, and thereby its resistance to forward motion.

In fact, $\vec{U}$ can be strong enough to lift the entire sailboard off the surface of the water! When airborne, however, the rider has the problem of keeping both board and sail balanced so as to avoid a crash landing. Good luck if you try it!



(1) The keel is a plate, often weighted, under the boat and along its length, that helps the boat resist being pushed sideways by the wind.

(2) The daggerboard is a removable plate that when positioned beneath the sailboard helps a sailboard resist being pushed sideways by the wind.

5.3 Cartesian Equations of Lines

You are familiar with the Cartesian equation of a line in 2-space. A single equation such as $y = mx + b$ or $Ax + By + C = 0$ is sufficient to determine a line in 2-space. You should wonder what form the Cartesian equation of a line in 3-space will take. In this section you will learn how to find Cartesian equations of lines.

Example 1 Find a Cartesian equation of the line having the vector equation $\vec{r} = \overrightarrow{(2 + 3k, 4 + 5k)}$.

Solution Since a vector equation of the line is $\vec{r} = \overrightarrow{(2 + 3k, 4 + 5k)}$, parametric equations are

$$\begin{cases} x = 2 + 3k \\ y = 4 + 5k \end{cases}$$

Solving each equation for k gives

$$k = \frac{x - 2}{3} \text{ and } k = \frac{y - 4}{5}$$

Equating the two values for k gives the Cartesian equation

$\frac{x - 2}{3} = \frac{y - 4}{5}$, where (2,4) is a point on the line and 3,5 are direction numbers of the line. ■

This form of a Cartesian equation is called a **symmetric equation** of the line in 2-space. The word "symmetric" is used because x and y appear symmetrically in the equation.

The equation $\frac{x - 2}{3} = \frac{y - 4}{5}$ can be written $5(x - 2) = 3(y - 4)$ or $5x - 3y + 2 = 0$, the usual form for the Cartesian equation of a line in 2-space.

Example 2 Find Cartesian equations in symmetric form of the line with vector equation $\vec{r} = \overrightarrow{(2,3,1)} + k\overrightarrow{(4,5,6)}$.

Solution The equation $\vec{r} = \overrightarrow{(2,3,1)} + k\overrightarrow{(4,5,6)}$ may be written $\overrightarrow{(x,y,z)} = \overrightarrow{(2 + 4k, 3 + 5k, 1 + 6k)}$, giving the parametric equations

$$\begin{cases} x = 2 + 4k \\ y = 3 + 5k \\ z = 1 + 6k \end{cases}$$

Solving the equations for k gives

$x - 2 = 4k$, $y - 3 = 5k$, $z - 1 = 6k$ or

$$\frac{x - 2}{4} = k, \frac{y - 3}{5} = k, \frac{z - 1}{6} = k$$

Hence, Cartesian equations of the line in symmetric form are

$\frac{x - 2}{4} = \frac{y - 3}{5} = \frac{z - 1}{6} \ (=k)$, where (2,3,1) is a point on the line and 4,5,6 are direction numbers of the line. ■

Note 1 In 2-space, a single symmetric equation is sufficient to determine a line.
2 In 3-space, two symmetric equations are needed to determine a line.
3 Cartesian equations of lines are also called **scalar equations** of a line.
4 If one of the direction numbers equals 0 the symmetric equations take on a different form. For example, if the direction numbers are 3,2,0 and the line passes through the point (4,5,6), the symmetric equations are written

$$\frac{x-4}{3} = \frac{y-5}{2},\ z-6=0 \text{ rather than } \frac{x-4}{3} = \frac{y-5}{2} = \frac{z-6}{0}$$

SUMMARY

2-space
For a line in 2-space, through the point $P_0(x_0,y_0)$, having direction numbers m_1, and m_2,

$$\frac{x-x_0}{m_1} = \frac{y-y_0}{m_2} \qquad m_1, m_2 \neq 0$$

3-space
For a line in 3-space, through the point $P_0(x_0,y_0,z_0)$, having direction numbers m_1, m_2, and m_3,

$$\frac{x-x_0}{m_1} = \frac{y-y_0}{m_2} = \frac{z-z_0}{m_3} \qquad m_1, m_2, m_3 \neq 0$$

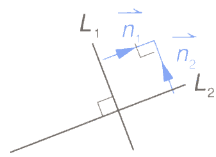
The remainder of this section deals with lines in 2-space and their Cartesian equations. Indeed, the familiar equation $Ax + By + C = 0$ for a 2-space line can be derived by using the fact that two non-zero vectors $\vec{u}$ and $\vec{v}$ are perpendicular if and only if $\vec{u} \cdot \vec{v} = 0$.

Recall that a vector $\vec{n}$ perpendicular to a vector $\vec{a}$ is called a vector *normal to* $\vec{a}$ or a *normal vector* for vector $\vec{a}$.

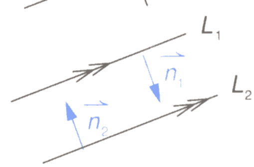
PROPERTY

The following are relationships among two lines in 2-space and the normal vectors to these lines.

1. Two lines are perpendicular if and only if their normal vectors are perpendicular.



2. Two lines are parallel if and only if their normal vectors are parallel.



Example 3

Find a Cartesian equation of the line that passes through the point $P_0(1,2)$ with $\vec{n} = \overrightarrow{(3,4)}$ as a normal vector.

Solution

Let $P(x,y)$ be any point on the line.
Since $\vec{n} = \overrightarrow{(3,4)}$ is a normal vector, then $\vec{n}$ is perpendicular to the line.
Thus, $\overrightarrow{P_0P} \perp \vec{n}$. Hence, $\overrightarrow{P_0P} \cdot \vec{n} = 0$.
But $\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0} = \overrightarrow{(x,y)} - \overrightarrow{(1,2)} = \overrightarrow{(x-1, y-2)}$ and $\vec{n} = \overrightarrow{(3,4)}$
Thus, $\overrightarrow{(x-1,y-2)} \cdot \overrightarrow{(3,4)} = 0$
or $(x-1)(3) + (y-2)(4) = 0$, that is, $3x + 4y - 11 = 0$.

Thus, a Cartesian equation of the line is $3x + 4y - 11 = 0$. ■

Notice that the coefficients of x and y in the Cartesian equations are the direction numbers of the normal vector $\vec{n} = \overrightarrow{(3,4)}$. The following shows this to be true for any Cartesian equation of a line written in the form $Ax + By + C = 0$

THEOREM

The line that passes through the point $P_0(x_0,y_0)$, with $\vec{n} = \overrightarrow{(A,B)}$ as normal vector, has Cartesian equation $Ax + By + C = 0$.

Proof: Let $P(x,y)$ be any point on this line.

Since $\vec{n} = \overrightarrow{(A,B)}$ is a normal vector, then $\vec{n}$ is perpendicular to the line.

Thus, $\overrightarrow{P_0P} \perp \vec{n}$

Hence, $\overrightarrow{P_0P} \cdot \vec{n} = 0$.

But $\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0} = \overrightarrow{(x,y)} - \overrightarrow{(x_0,y_0)} = \overrightarrow{(x - x_0, y - y_0)}$

and $\vec{n} = \overrightarrow{(A,B)}$

Thus, $\overrightarrow{(x - x_0, y - y_0)} \cdot \overrightarrow{(A,B)} = 0$

or $(x - x_0)A + (y - y_0)B = 0$

that is $Ax + By + (-Ax_0 - By_0) = 0$.

If the number $-Ax_0 - By_0$ is replaced by the constant C, this equation becomes the Cartesian equation $Ax + By + C = 0$.

PROPERTY

$\vec{n} = \overrightarrow{(A,B)}$ is a normal to the line with Cartesian equation $Ax + By + C = 0$.

Observe that: the slope of $Ax + By + C = 0$ is $-\frac{A}{B}$

the slope of a normal to $Ax + By + C = 0$ is $\frac{B}{A}$

Example 4 Find a Cartesian equation of the line, passing through the point $D(-6,2)$, that is perpendicular to the vector $\vec{u} = \overrightarrow{(5,-4)}$.

Solution Let the equation be $Ax + By + C = 0$. ①

Since $\vec{u} = \overrightarrow{(5,-4)}$ is perpendicular to the line, a normal vector $\overrightarrow{(A,B)} = \overrightarrow{(5,-4)}$

Substituting for A and B in ① gives

$$5x + (-4)y + C = 0.$$

But the point $D(-6,2)$ lies on the line,

Thus $5(-6) + (-4)(2) + C = 0$ or, $C = 38$.

Thus a Cartesian equation of the line is $5x - 4y + 38 = 0$. ■

Note: This equation could have been obtained from first principles by using the fact that $\overrightarrow{DP} \cdot \overrightarrow{(5,-4)} = 0$, where $\overrightarrow{DP} = \overrightarrow{(x + 6, y - 2)}$.

5.3 Exercises

1. For each of the following equations of a line, state the coordinates of a point on the line and direction numbers for the line.

   a) $\frac{x-3}{5} = \frac{y-2}{6}$

   b) $\frac{x+1}{2} = \frac{y-4}{-7}$

   c) $\frac{x-2}{-8} = \frac{y}{3}$

   d) $\frac{2x-6}{5} = \frac{3-y}{4}$

   e) $\frac{x-2}{3} = \frac{y-4}{8} = \frac{z-1}{5}$

   f) $\frac{x+3}{-3} = \frac{2y-4}{7} = \frac{1-z}{9}$

   g) $\frac{x-2}{3} = \frac{y-4}{5}, z-7=0.$

   h) $\frac{x-1}{-3} = \frac{z+2}{4}, y=2$

2. Find a vector equation for the lines in question 1, parts a) to h).

3. Find, where possible, symmetric equations for each of the lines having the given vector equations.

   a) $\vec{r} = \overrightarrow{(3-2k, 4+5k)}$

   b) $\vec{r} = \overrightarrow{(6+4t, -3-t)}$

   c) $\vec{r} = \overrightarrow{(2,1,0)} + k\overrightarrow{(4,2,3)}$

   d) $\vec{r} = \overrightarrow{(-3,4,2)} + k\overrightarrow{(-1,5,0)}$

   e) $\vec{r} = \overrightarrow{(2,3k,5)}$

   f) $\vec{r} = \overrightarrow{(-4k, 3k-1)}$

4. Find, where possible, symmetric equations for each of the following lines.

   a) through the point $A(5,-3,4)$, with direction numbers $-6,2,1$

   b) through the point $E(4,-1,0)$, with direction vector $\overrightarrow{(1,0,-3)}$

   c) through the points $A(5,-3,4)$ and $B(7,2,-1)$

   d) through the point $B(7,2,-1)$, parallel to the line $\vec{r} = \overrightarrow{(4,7,0)} + k\overrightarrow{(4,-5,1)}$

   e) through the point $C(8,-3,4)$ and parallel to the x-axis

   f) through the point $C(8,-3,4)$ and parallel to the y-axis

   g) through the point $C(8,-3,4)$ and parallel to the z–axis

5. Find a vector equation for the line with symmetric equation $\frac{x-1}{4} = \frac{y+3}{2} = \frac{z-5}{-3}$

6. Find the value of the variable t so that the line with symmetric equations $\frac{x-1}{6} = \frac{y+2}{5} = \frac{z-3}{-2}$ is perpendicular to the line with vector equation $\vec{r} = \overrightarrow{(0,3,1)} + k\overrightarrow{(2,t,11)}$.

7. Find the value of the variable s so that the line with symmetric equations $\frac{x+3}{2} = y-5 = \frac{z-5}{-8}$ is parallel to the line with vector equation $\vec{r} = \overrightarrow{(9,2,6)} + k\overrightarrow{(-1,s,4)}$.

8. Show that the following lines are the same line.

   L_1: $\frac{x-1}{5} = \frac{y+2}{3} = \frac{z-5}{-1}$ and

   L_2: $\frac{x-11}{10} = \frac{y-4}{6} = \frac{z-3}{-2}$

9. Find the value of the variable t so that the following lines are perpendicular.

$$L_1: \frac{x-4}{2t-1} = \frac{y+1}{3} = \frac{z+2}{4t-1} \text{ and}$$

$$L_2: \frac{x-5}{8} = \frac{y-1}{-2t} = \frac{z+3}{-2}.$$

10. State direction numbers for a normal to each of the following lines.
    a) $3x + 4y + 6 = 0$
    b) $5x - 2y = 3$
    c) $x - 5y = 1$
    d) $2x + 3 = 0$
    e) $y = 5$

11. For each of the following find a scalar equation of the line passing through the given point P_0 and having $\vec{n}$ as a normal vector.
    a) $P_0(3,-1)$ $\vec{n} = \overrightarrow{(-4,1)}$
    b) $P_0(0,-5)$ $\vec{n} = \overrightarrow{(6,4)}$
    c) $P_0(-4,7)$ $\vec{n} = \overrightarrow{(2,-3)}$
    d) $P_0(-2,-5)$ $\vec{n} = \overrightarrow{(-2,0)}$

12. Use normal vectors to decide which pairs of lines are parallel and which pairs of lines are perpendicular.
    a) $3x + 2y = 5$
       $6x + 4y = 1$
    b) $4x - 5y = 7$
       $5x + 4y = 1$
    c) $5x - 3y + 1 = 0$
       $3x + 5y + 2 = 0$
    d) $x - 13y = 18$
       $3x - 39y = 0$
    e) $2x + 1 = 0$
       $x = 10$
    f) $3y + 5 = 0$
       $2x = 3$

13. a) Show that a line with equation $Ax + By + C = 0$ has $\overrightarrow{(B,-A)}$ as a direction vector.
    b) Find a vector equation of the line in 2-space, passing through the point $P_0(3,5)$, that is parallel to the line $2x + 5y + 9 = 0$.

14. Given the line L with equation $4x - 7y + 9 = 0$ and the point $P_0(1,-3)$.
    a) Find a vector equation of the line through P_0 parallel to the line L.
    b) Find a vector equation of the line through P_0 perpendicular to the line L.

15. Find the value of the variable t so that the line with scalar equation $4x + 7y = 3$ is perpendicular to the line with vector equation $\vec{r} = \overrightarrow{(0,1)} + k\overrightarrow{(4,t)}$

16. Find the value of the variable s so that the line with scalar equation $5x + 2y = 8$ is parallel to the line with vector equation $\vec{r} = \overrightarrow{(2,1)} + k\overrightarrow{(3,s)}$

17. Write the symmetric equation $\frac{x-3}{5} = \frac{y-2}{6}$ in the scalar form $Ax + By + C = 0$.

18. Write the Cartesian equation $4x + 7y - 11 = 0$ in symmetric form.

19. a) Show that the vector $\vec{n} = \overrightarrow{(3,-1,2)}$ is a normal of the line
    $$\frac{x-a}{2} = \frac{y-b}{6}, z - c = 0.$$
    b) Are all normals to this line parallel to $\vec{n}$?

5.4 Direction Numbers and Direction Cosines of a Line

The direction of a line in 3-space can be given by many direction numbers or vectors. For the line with vector equation $\vec{r} = \overrightarrow{(1 + 2k, 4 - 3k, 5 + 7k)}$, the numbers 2, −3, and 7 are *direction numbers* and $\vec{a} = \overrightarrow{(2,-3,7)}$ is a *direction vector*. Any scalar multiple of $\vec{a}$ is parallel to $\vec{a}$ and thus is also a direction vector; for example, $\overrightarrow{(4,-6,14)}$, $\overrightarrow{(-6,9,-21)}$ and $\overrightarrow{(2t,-3t,7t)}$, $t \in \mathbb{R}$, are direction vectors of the line. Thus the ordered sets of numbers 4,−6,14, and −6,9,−21, and $2t,-3t,7t$ are also direction numbers of the line.

None of the these direction numbers in 3-space specify direction in the same way as *slope* does in 2-space. You need to compare the direction of the line with the directions of the *three coordinate axes*. This is done using a special set of direction numbers called **direction cosines**. These are associated with the cosines of the three angles a direction vector of a line makes with the x-axis, the y-axis, and the z-axis.

Example 1 A line has a direction vector $\vec{a} = \overrightarrow{(2,-3,7)}$. Calculate, to the nearest degree, the angle this line makes with the x-axis.

Solution Let the required angle be α. Then $\vec{a} \cdot \vec{i} = |\vec{a}||\vec{i}| \cos \alpha$

$$\Rightarrow \cos \alpha = \frac{\vec{a} \cdot \vec{i}}{|\vec{a}||\vec{i}|}$$

$$= \frac{\overrightarrow{(2,-3,7)} \cdot \overrightarrow{(1,0,0)}}{\sqrt{2^2 + (-3)^2 + 7^2}\sqrt{1^2 + 0^2 + 0^2}} = \frac{2}{\sqrt{62}\sqrt{1}} = 0.2540002\ldots$$

$$\Rightarrow \quad \alpha = 75.2856\ldots°$$

The angle between the line and the x-axis is 75°. ■

Example 1 can be generalized to find the angle a line makes with the three coordinate axes.

Let α, β and γ be the angles a direction vector $\vec{m} = \overrightarrow{(m_1,m_2,m_3)}$ makes with the x-axis, y-axis, and z-axis respectively.

Since the basis vector $\vec{i}$ lies along the x-axis, the angle between $\vec{m}$ and $\vec{i}$ must be α.

$$\text{Thus } \cos \alpha = \frac{\vec{m} \cdot \vec{i}}{|\vec{m}||\vec{i}|}, = \frac{\overrightarrow{(m_1,m_2,m_3)} \cdot \overrightarrow{(1,0,0)}}{|\vec{m}|} = \frac{m_1}{|\vec{m}|}$$

In a similar manner you can prove that $\cos \beta = \frac{m_2}{|\vec{m}|}$ and $\cos \gamma = \frac{m_3}{|\vec{m}|}$

PROPERTY The direction cosines of a line with direction vector $\vec{m} = \overrightarrow{(m_1,m_2,m_3)}$ are

$$\frac{m_1}{|\vec{m}|}, \frac{m_2}{|\vec{m}|}, \frac{m_3}{|\vec{m}|}$$

Similarily you can show that in 2-space the direction cosines of a line with direction vector $\overrightarrow{(m_1,m_2)}$ are $\frac{m_1}{|\vec{m}|}$, $\frac{m_2}{|\vec{m}|}$

Example 2 Given the line with vector equation $\vec{r} = \overrightarrow{(1,2,3)} + k\overrightarrow{(3,-4,5)}$

a) find the direction cosines of the line, correct to 4 decimal places, and the angles α, β, and γ the line makes with the coordinate axes, correct to the nearest degree

b) calculate $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma$.

Solution a) Direction numbers of the given lines are 3,−4,5.

Hence $m_1 = 3$, $m_2 = -4$, $m_3 = 5$.

Thus, $|\vec{m}| = \sqrt{3^2 + (-4)^2 + 5^2} = \sqrt{50}$

Therefore, the direction cosines are

$$\cos \alpha = \frac{3}{\sqrt{50}} \quad \cos \beta = -\frac{4}{\sqrt{50}} \quad \cos \gamma = \frac{5}{\sqrt{50}}$$

or, $\cos \alpha \doteq 0.424\ 264$ $\quad \cos \beta \doteq -0.565\ 854$ $\quad \cos \gamma \doteq 0.707\ 106$

Thus, to the nearest degree, $\alpha = 65°$, $\beta = 124°$ and $\gamma = 45°$.
Thus the direction cosines are 0.4243, −0.5659, and 0.7071.
The angles made with the coordinates axes are 65°, 124° and 45°.

Note: Since a line has two 'directions', $\vec{m}$ could have been replaced by $-\vec{m} = \overrightarrow{(-3,4,-5)}$. Thus $\cos \alpha$, $\cos \beta$ and $\cos \gamma$ would have been the negatives of the above. α, β, γ would then have been $180° - 65° = 115°$, $180° - 124° = 56°$ and $180° - 45° = 135°$.

b) $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma$

$$= \left(\frac{3}{\sqrt{50}}\right)^2 + \left(-\frac{4}{\sqrt{50}}\right)^2 + \left(\frac{5}{\sqrt{50}}\right)^2 = \frac{9}{50} + \frac{16}{50} + \frac{25}{50} = \frac{50}{50} \text{ or } 1.$$ ■

You will now see by calculating the length of the vector $\vec{u} = \overrightarrow{(\cos \alpha, \cos \beta, \cos \gamma)}$ that the result of Example 2 part b) is always true.

Now $|\vec{u}|^2 = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma$

$$= \left(\frac{m_1}{|\vec{m}|}\right)^2 + \left(\frac{m_2}{|\vec{m}|}\right)^2 + \left(\frac{m_3}{|\vec{m}|}\right)^2 = \frac{(m_1)^2 + (m_2)^2 + (m_3)^2}{|\vec{m}|^2} = \frac{|\vec{m}|^2}{|\vec{m}|^2} = 1.$$

Therefore, the vector $\overrightarrow{(\cos \alpha, \cos \beta, \cos \gamma)}$ is a *unit vector*. If $\vec{m}$ is a direction vector of the line, you can call the vector $\overrightarrow{(\cos \alpha, \cos \beta, \cos \gamma)} = \vec{e}_m$.

The following property follows directly.

PROPERTY If α, β, and γ are the angles the direction vector of a line makes with the x-axis, the y-axis, and the z-axis respectively, then $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

5.4 Exercises

1. Given a line with direction numbers 3,2,−4,
   a) state three other sets of direction numbers
   b) find the values of the direction cosines, correct to 4 decimal places
   c) find the angles a direction vector of the line makes with the coordinates axes, correct to the nearest degree.

2. Repeat question 1 parts b) and c) for the following sets of direction numbers
   a) −1,3,5
   b) 0,−5,8
   c) 6,−2,−1
   d) −1,0,2

3. Find direction cosines for each of the lines given by the following.
   a) $\vec{r} = \overrightarrow{(3,-1,4)} + k\overrightarrow{(5,-2,1)}$
   b) $\vec{r} = \overrightarrow{(-4,7,5)} + t\overrightarrow{(1,0,-8)}$
   c) $\begin{cases} x = 5 + 2k \\ y = 2 + 4k \\ z = 2 - 5k \end{cases}$
   d) $\begin{cases} x = -3 + 8s \\ y = 1 - 5s \\ z = -2 + 5s \end{cases}$
   e) $\dfrac{x-2}{3} = \dfrac{y-4}{8} = \dfrac{z-1}{5}$
   f) $\dfrac{x+3}{-3} = \dfrac{2y-4}{7} = \dfrac{1-z}{9}$
   g) $\dfrac{x-1}{-3} = y - 2$

4. Two of the direction cosines of a vector are 0.3 and 0.4. Find the other direction cosine, correct to 4 decimal places.

5. A direction vector of a line makes an angle of 30° with the x-axis and an angle of 70° with the y-axis. Find, correct to the nearest degree, the angle the direction vector makes with the z-axis.

6. A line through the origin of a 3-space coordinate system makes an angle of 45° with the y-axis and an angle of 80° with the z-axis. Find a vector equation of the line.

7. The direction cosines of two vectors are $\dfrac{1}{\sqrt{18}}, \dfrac{1}{\sqrt{18}}, \dfrac{4}{\sqrt{18}}$ and $0, \dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}$. Find the angle between the vectors, correct to the nearest degree.

8. The direction cosines of two intersecting lines are $\dfrac{1}{2}, \dfrac{\sqrt{2}}{2}, \dfrac{1}{2}$ and $-\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{\sqrt{2}}{2}$. Find the angle between the lines, correct to the nearest degree.

9. The line with vector equation $\vec{r} = \overrightarrow{(1,4,-6)} + k\overrightarrow{(-3,1,-2)}$ is perpendicular to the line with symmetric equations $\dfrac{x-2}{4} = \dfrac{y-4}{t} = \dfrac{z-1}{5}$. Find the direction cosines of each line.

10. Vectors $\vec{m}_1$ and $\vec{m}_2$ make angles $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ respectively with the coordinate axes.
    a) If $\vec{m}_1$ is perpendicular to $\vec{m}_2$ show that $\cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2 = 0$.
    b) If θ is the angle between $\vec{m}_1$ and $\vec{m}_2$ show that $\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2$

11. Find the direction cosines of a line perpendicular to both of the lines $\dfrac{x-1}{2} = \dfrac{y-1}{3} = \dfrac{z-1}{6}$ and $\dfrac{x}{3} = \dfrac{y}{8} = \dfrac{z}{12}$

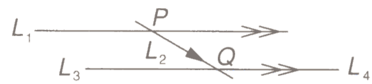
12. a) For a line in 2-space with direction vector $\vec{m} = \overrightarrow{(m_1,m_2)}$ show that the direction cosines are
    $$\cos\alpha = \frac{m_1}{|\vec{m}|},\ \cos\beta = \frac{m_2}{|\vec{m}|}$$
    b) For the line in part a) show that $\cos^2\alpha + \cos^2\beta = 1$.

13. A unit vector makes equal angles with each of the three coordinates axes. Find this unit vector.

5.5 The Intersection of Lines in 2-Space

You have studied intersections of lines in 2-space before, using Cartesian equations of lines. In this section you will investigate the intersection of lines using vector and parametric equations. You will also examine the various relationships that can occur among lines. Your understanding of the different ways lines can intersect in 2-space will help you to understand the intersection of lines in 3-space.

Consider the figure in which three lines L_1, L_2, and L_3 are drawn.
L_1 and L_2 intersect at point P. L_2 and L_3 intersect at point Q. L_1 and L_3 are parallel and distinct hence they do not intersect.
Two parallel lines L_3 and L_4 that have one point in common must have every point in common. The lines are the same line and are sometimes called *coincident* lines.



Situations like these are modelled in the following examples.

Example 1 Given the line L_1 with equation $3x + 2y = 8$. Determine whether or not L_1 is parallel to each of the following lines. If the lines are not parallel, find their point of intersection. If the lines are parallel, determine whether the lines are the same line, or distinct lines.
L_2: $4x - 5y = 3$ $\quad$ L_3: $12x + 8y = 32$ $\quad$ L_4: $6x + 4y = 2$

Solution *L_1 and L_2*

For L_1, $3x + 2y = 8$, so a normal is $\vec{n}_1 = \overrightarrow{(A,B)} = \overrightarrow{(3,2)}$.
For L_2, $4x - 5y = 3$, so a normal is $\vec{n}_2 = \overrightarrow{(4,-5)}$.
Since $\vec{n}_1$ is not a scalar multiple of $\vec{n}_2$, vectors $\vec{n}_1$ and $\vec{n}_2$ are not parallel.
Thus, L_1 and L_2 intersect. To find the point of intersection you must solve the system

$$3x + 2y = 8 \quad ①$$
$$4x - 5y = 3 \quad ②$$

Eliminating x, $4 \times ① - 3 \times ②$ gives $23y = 23$. Thus, $y = 1$.
Substituting $y = 1$ in ① or ② gives $x = 2$.
Hence, the point of intersection of L_1 and L_2 is $(2,1)$.

L_1 and L_3

For L_1, $3x + 2y = 8$, so a normal is $\vec{n}_1 = \overrightarrow{(3,2)}$.
For L_3, $12x + 8y = 32$, so a normal is $\vec{n}_3 = \overrightarrow{(12,8)}$.
Since $\vec{n}_3 = 4\vec{n}_1$, the normals $\vec{n}_3$ and $\vec{n}_1$ are parallel.
But the equation for L_3 is $12x + 8y = 32$ or $4(3x + 2y) = 4(8)$
Dividing by 4 gives $3x + 2y = 8$ which is the equation for L_1. Hence, every point $P(x,y)$ that is a solution for equation ① is also a solution for ③.
Thus, L_1 and L_3 are the same line.

L_1 and L_4

For L_1, $3x + 2y = 8$, so a normal is $\vec{n}_1 = \overrightarrow{(3,2)}$.

For L_4, $6x + 4y = 2$, so a normal is $\vec{n}_4 = \overrightarrow{(6,4)}$.

Since $\vec{n}_4 = 2\vec{n}_1$, the normals $\vec{n}_4$ and $\vec{n}_1$ are parallel.

But the equation for L_4 is $6x + 4y = 2$ or $2(3x + 2y) = 2(1)$

Dividing by 2 gives $3x + 2y = 1$.

Comparing this equation with the equation for L_1: $3x + 2y = 8$, you can see that no point $P(x,y)$ that is a solution for equation $3x + 2y = 8$ can be a solution for equation $3x + 2y = 1$ (nor for equation $6x + 4y = 2$).

Thus, L_1 and L_3 can not have a common point. Lines L_1 and L_3 are parallel and distinct. ■

If two linear equations in two variables have *one or more solutions*, then the system of equations is said to be *consistent*.

If a consistent system has *exactly one solution* the system is *independent*.

If a consistent system has *more than one solution* the system is *dependent*.

If two linear equations in two variables have *no solutions*, then the system of equations is said to be *inconsistent*.

SUMMARY

For a system of two equations in two variables

	number of solutions	*geometric description*
consistent and independent	one	two intersecting lines
consistent and dependent	infinite	same line
inconsistent	none	lines are parallel, distinct

Example 2

Given the line L_1 with vector equation $\vec{r} = \overrightarrow{(5 + 3k, -1 + 4k)}$. Determine whether or not L_1 is parallel to each of the following lines. If the lines are not parallel, find their point of intersection. If the lines are parallel, then determine whether the lines are distinct, or the same line.

L_2: $\vec{r} = \overrightarrow{(4 - 2t, 5 + t)}$ $\quad$ L_3: $\vec{r} = \overrightarrow{(-4 - 6s, 7 - 8s)}$ $\quad$ L_4: $\vec{r} = \overrightarrow{(8 + 3a, 3 + 4a)}$

Solution

L_1 and L_2

A direction vector for L_1 is $\vec{m}_1 = \overrightarrow{(3,4)}$.

A direction vector for L_2 is $\vec{m}_2 = \overrightarrow{(-2,1)}$.

Since no scalar k exists such that $\vec{m}_1 = k\vec{m}_2$, $\vec{m}_1$ and $\vec{m}_2$ are not parallel.

Hence, L_1 and L_2 are not parallel, so they intersect.

The system is consistent (a solution) and independent (only one solution).

Parametric equations for L_1 and L_2 are

$$\begin{cases} x = 5 + 3k \\ y = -1 + 4k \end{cases} \text{and} \begin{cases} x = 4 - 2t \\ y = 5 + t \end{cases}$$

At the point of intersection, (x,y) for L_1 equals (x,y) for L_2. Thus,

$$\begin{cases} 5 + 3k = 4 - 2t \\ -1 + 4k = 5 + t \end{cases} \text{ or } \begin{cases} 3k + 2t = -1 & ① \\ 4k - t = 6 & ② \end{cases}$$

Eliminating k: $4 \times ① - 3 \times ②$ gives $11t = -22$, or $t = -2$.
Substitute in ① or ② to obtain $k = 1$.
Substituting $k = 1$ and $t = -2$ in either the parametric equations for L_1, or for L_2, produces $x = 8$ and $y = 3$.
Thus, the point of intersection is $(8,3)$.

L_1 and L_3

A direction vector for L_1 is $\vec{m}_1 = \overrightarrow{(3,4)}$.
A direction vector for L_3 is $\vec{m}_3 = \overrightarrow{(-6,-8)}$.
Since $\vec{m}_3 = -2\vec{m}_1$, the vectors $\vec{m}_3$ and $\vec{m}_1$ are parallel.
Hence, L_1 and L_3 are parallel.
Are L_1 and L_3 the same line or distinct lines? If the lines have any one point in common they must have all points in common, and be the same line.
Test one point from L_1 in L_3. Now $(5,-1)$ is on L_1.
Parametric equations for L_3 are

$$\begin{cases} x = -4 - 6s \\ y = 7 - 8s \end{cases}$$

Substituting 5 for x and -1 for y gives
$5 = -4 - 6s$ and $-1 = 7 - 8s$

$$s = -\frac{9}{6} = -\frac{3}{2} \text{ or } s = \frac{-8}{-8} = 1 \neq -\frac{3}{2}.$$

Since the two values for s are different, $(5,-1)$ does not lie on L_3.
Hence, L_1 and L_3 are parallel and distinct.
The system is inconsistent (no solution).

L_1 and L_4

A direction vector for L_1 is $\vec{m}_1 = \overrightarrow{(3,4)}$.
A direction vector for L_4 is $\vec{m}_4 = \overrightarrow{(3,4)}$.
Since $\vec{m}_4 = \vec{m}_4$ the vectors $\vec{m}_4$ and $\vec{m}_1$ are parallel.
Hence, L_1 and L_4 are parallel.
To determine whether L_1 and L_4 are the same line or distinct lines one point from L_1 should be tested in L_4. Now $(5,-1)$ is on L_1.
Parametric equations for L_4 are

$$\begin{cases} x = 8 + 3a \\ y = 3 + 4a \end{cases}$$

Substituting 5 for x and -1 for y gives
$5 = 8 + 3a$ and $-1 = 3a + 4a$

$$a = -\frac{3}{3} = -1 \text{ or } a = -\frac{4}{4} = -1$$

Since both values of a are the same, $(5,-1)$ does lie on L_4.
Thus, L_1 and L_4 are the same line.
The system is consistent (a solution) and dependent (an infinite number of solutions). ■

5.5 Exercises

1. a) A system of two linear equations in two variables is inconsistent. Explain what this means both geometrically and with respect to the number of solutions of the system.
   b) Repeat part a) for a system that is consistent and dependent.
   c) Repeat part a) for a system that is consistent and independent.

2. For each of the following systems determine whether or not the corresponding lines are parallel. If the lines are parallel, indicate whether they are distinct, or the same line. If the lines are not parallel, find their point of intersection.

   a) $\begin{cases} 4x - 5y = 22 \\ 3x + 2y = 5 \end{cases}$ c) $\begin{cases} 7x + 3y = -30 \\ 21x + 9y = -90 \end{cases}$

   b) $\begin{cases} -2x + 6y = -16 \\ 3x - 9y = 15 \end{cases}$ d) $\begin{cases} 8x + 6y = 18 \\ -2x - 9y = -15 \end{cases}$

3. For each of the systems in question 2 indicate which of the following terms apply.
   a) consistent and independent
   b) consistent and dependent
   c) inconsistent

4. For each of the following pairs of lines in 2-space determine whether or not the lines are parallel. If the lines are parallel, then determine whether the lines are distinct or the same line. If the lines are not parallel, find their point of intersection.

   a) $\vec{r} = \overrightarrow{(3 + 2k, 7 + k)}$
   $\vec{r} = \overrightarrow{(-5 + 4t, 8 - 3t)}$

   b) $\vec{r} = \overrightarrow{(19 - 5k, 16 - 6k)}$
   $\vec{r} = \overrightarrow{(2 + s, 4 - 3s)}$

   c) $\vec{r} = \overrightarrow{(1 + 8a, 4 - 6a)}$
   $\vec{r} = \overrightarrow{(2 - 4b, -5 + 3b)}$

   d) $\vec{r} = \overrightarrow{(1 + t, 3 - 2t)}$
   $\vec{r} = \overrightarrow{(-1 + 4k, -7 + 6k)}$

5. Find the point of intersection of the line $\vec{r} = \overrightarrow{(1,-6)} + k\overrightarrow{(-3,5)}$ and the line through the points $A(6,-4)$ and $B(-10,2)$.

6. Find the point of intersection of the line $\vec{r} = \overrightarrow{(-3,1)} + k\overrightarrow{(4,7)}$ and the line $\frac{x - 11}{-2} = \frac{y}{5}$

7. Find the point of intersection of the lines $\frac{x + 3}{2} = \frac{y + 2}{4}$ and $\frac{x - 5}{-3} = \frac{y - 4}{-1}$

8. Given the triangle $A(1,4)$ $B(-5,6)$ $C(3,-2)$
   a) find a vector equation of the median through vertex A
   b) find a vector equation of the median through vertex B
   c) find a vector equation of the median through vertex C
   d) find the point of intersection of the medians through vertices A and B
   e) show that the point of intersection found in part d) also lies on the median through vertex C.

9. Given the parallelogram $O(0,0)$, $Q(5,0)$, $R(8,4)$, $S(3,4)$
   a) find a vector equation of the diagonals OR and QS
   b) find the point of intersection of the diagonals in part a)
   c) use your point in part b) to show that the diagonals of parallelogram $OQRS$ bisect each other.

10. Given the point $P(3,10)$ and the line L, $\vec{r} = \overrightarrow{(1 + 3k, 17 - 4k)}$
    a) find a vector equation of the line through P that is perpendicular to L
    b) find the point A of intersection of L and the line in part a)
    c) find the perpendicular distance from point P to the line L.

11. Find a vector equation of the line through the origin that passes through the point of intersection of the lines $\frac{x - 1}{-1} = \frac{y - 2}{2}$ and $\frac{x + 1}{3} = \frac{y - 1}{2}$

5.6 The Intersection of Lines in 3-Space

In the introduction to this chapter you learned of a 'power line' problem.

Read page 196 again.

Recall that two lines in 3-space can intersect or not intersect. If the two lines do not intersect, then the two lines may be parallel, or not parallel. Two lines in 3-space that are not parallel *and* do not intersect are called *skew lines*.

The following examples will give you the mathematics necessary to solve the 'power line' problem.

Example 1 Given the line L_1 with vector equation $\vec{r} = \overrightarrow{(3 + 2k, 4 - 3k, 5 + k)}$.
Determine whether or not L_1 is parallel to each of the following lines.
If the lines are not parallel, then determine whether the lines intersect or are skew. If the lines intersect, then find the point of intersection.

L_2: $\vec{r} = \overrightarrow{(-7 + 6t, 5 + 5t, -1 + 4t)}$

L_3: $\vec{r} = \overrightarrow{(-1 + 3a, 3 - a, 7 + 6a)}$

L_4: $\vec{r} = \overrightarrow{(5 - 4w, 2 + 6w, 4 - 2w)}$

Solution (Using elimination)

L_1 and L_2

A direction vector for L_1 is $\vec{m}_1 = \overrightarrow{(2,-3,1)}$.
A direction vector for L_2 is $\vec{m}_2 = \overrightarrow{(6,5,4)}$.
Since no scalar k exists such that $\vec{m}_1 = k\vec{m}_2$, $\vec{m}_1$ and $\vec{m}_2$ are not parallel.
Hence, L_1 and L_2 are not parallel.
To determine if the lines intersect, you will need their parametric equations.
Now the parametric equations for L_1 and L_2 are

$$L_1\begin{cases} x = 3 + 2k \\ y = 4 - 3k \\ z = 5 + k \end{cases} \qquad L_2\begin{cases} x = -7 + 6t \\ y = 5 + 5t \\ z = -1 + 4t \end{cases}$$

At the point of intersection, (x,y,z) for L_1 equals (x,y,z) for L_2. Thus,

$$\begin{cases} 3 + 2k = -7 + 6t \\ 4 - 3k = 5 + 5t \\ 5 + k = -1 + 4t \end{cases} \text{ or } \begin{cases} 2k - 6t = -10 & ① \\ -3k - 5t = 1 & ② \\ k - 4t = -6 & ③ \end{cases}$$

Solving ① and ② gives $k = -2$, $t = 1$.
Substituting in ③ you will obtain $(-2) - 4(1) = -6$, which is true.
For $k = -2$ and $t = 1$ substitution in either the parametric equations for L_1 or L_2 gives $x = -1$, $y = 10$, and $z = 3$.
Thus, the point of intersection is $(x,y,z) = (-1,10,3)$.
(Notice that the linear system for L_1 and L_2, consisting of equations ① ② and ③, is consistent and independent.)

L_1 and L_3

A direction vector for L_1 is $\vec{m}_1 = \overrightarrow{(2,-3,1)}$.

A direction vector for L_3 is $\vec{m}_3 = \overrightarrow{(3,-1,6)}$.

Since no scalar k exists such that $\vec{m}_1 = k\vec{m}_3$, $\vec{m}_1$ and $\vec{m}_3$ are not parallel.

Hence, L_1 and L_3 are not parallel.

To determine if the lines intersect, you will need their parametric equations.

Now parametric equations for L_1 and L_3 are

$$L_1\begin{cases} x = 3 + 2k \\ y = 4 - 3k \\ z = 5 + k \end{cases} \qquad L_3\begin{cases} x = -1 + 3a \\ y = 3 - a \\ z = 7 + 6a \end{cases}$$

At the point of intersection, (x,y,z) for L_1 equals (x,y,z) for L_3. Thus,

$$\begin{cases} 3 + 2k = -1 + 3a \\ 4 - 3k = 3 - a \\ 5 + k = 7 + 6a \end{cases} \text{ or } \begin{cases} 2k - 3a = -4 & ④ \\ -3k + a = -1 & ⑤ \\ k - 6a = 2 & ⑥ \end{cases}$$

Solving ④ and ⑤ gives $k = 1$, and $a = 2$.

Substituting in ⑥ you will obtain $(1) - 6(2) = 2$ or $-11 = 2$, which is not true.

Thus, the lines L_1 and L_3 do not intersect. Since L_1 and L_3 are also not parallel, L_1 and L_3 are skew lines.

(Notice that the linear system for L_1 and L_3, consisting of equations ④ ⑤ and ⑥, is inconsistent.)

L_1 and L_4

A direction vector for L_1 is $\vec{m}_1 = \overrightarrow{(2,-3,1)}$.

A direction vector for L_4 is $\vec{m}_4 = \overrightarrow{(-4,6,-2)}$.

Since $\vec{m}_4 = -2\vec{m}_1$, therefore $\vec{m}_1$ and $\vec{m}_4$ are parallel.

Hence, L_1 and L_4 are parallel.

Since the lines are parallel, they are either distinct lines or the same line.

To determine whether L_1 and L_4 are the same line or distinct lines, one point from L_1 should be tested in L_4.

Now $(3,4,5)$ is on L_1.

Parametric equations for L_4 are

$$\begin{cases} x = 5 - 4w \\ y = 2 + 6w \\ z = 4 - 2w \end{cases}$$

Substituting 3 for x, 4 for y, and 5 for z gives

$$\begin{cases} 3 = 5 - 4w \\ 4 = 2 + 6w \\ 5 = 4 - 2w \end{cases}$$

Solving each of the equations for w, gives $w = -\frac{1}{2}$, $w = \frac{1}{3}$, and $w = -\frac{1}{2}$.

But w cannot have different values at the same time. Thus, the point $(3,4,5)$ does not lie on line L_4. Hence, the lines can not have a common point. Hence the lines are parallel and distinct.

(Notice that the linear system for L_1 and L_4, obtained by equating components, will be inconsistent.) ■

The systems of equations for the parameters k and t of lines L_1 and L_2 and for k and a of lines L_1 and L_3 in Example 1 can also be solved using matrices as follows.

Alternate Solution for part of Example 1 using Matrices

L_1 and L_2

As in Example 1, the three equations are

$$\begin{cases} 2k - 6t = -10 & ① \\ -3k - 5t = 1 & ② \\ k - 4t = -6 & ③ \end{cases}$$

The augmented matrix for the three equations in the two variables k and t and the reduced form of the matrix follows.

$$\begin{bmatrix} 2 & -6 & -10 \\ -3 & -5 & 1 \\ 1 & -4 & -6 \end{bmatrix} \quad \begin{matrix} \\ 3 \times \text{row}① + 2 \times \text{row}② \\ \text{row}① - 2 \times \text{row}② \end{matrix} \begin{bmatrix} 2 & -6 & -10 \\ 0 & -28 & -28 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\begin{matrix} \\ \\ \text{row}② + 14 \times \text{row}③ \end{matrix} \begin{bmatrix} 2 & -6 & -10 \\ 0 & -28 & -28 \\ 0 & 0 & 0 \end{bmatrix}.$$

From row ② of the reduced matrix, $-28t = -28$. Thus, $t = 1$.
From row ① of the reduced matrix, $2k - 6t = -10$.

Substituting $t = 1$ in this equation gives $k = -2$.

L_1 and L_3

As in Example 1, the equations are

$$\begin{cases} 2k - 3a = -4 & ④ \\ -3k + a = -1 & ⑤ \\ k - 6a = 2 & ⑥ \end{cases}$$

The augmented matrix for three equations in the two variables k and a and the reduced form of the matrix follows.

$$\begin{bmatrix} 2 & -3 & -4 \\ -3 & 1 & -1 \\ 1 & -6 & 2 \end{bmatrix} \quad \begin{matrix} \\ 3 \times \text{row}① + 2 \times \text{row}② \\ \text{row}① - 2 \times \text{row}③ \end{matrix} \begin{bmatrix} 2 & -3 & -4 \\ 0 & -7 & -14 \\ 0 & 9 & -8 \end{bmatrix}$$

$$\begin{matrix} \\ \\ 9 \times \text{row}② - 7 \times \text{row}③ \end{matrix} \begin{bmatrix} 2 & -3 & -4 \\ 0 & -7 & -14 \\ 0 & 0 & -70 \end{bmatrix}$$

From the last line of the reduced matrix, $0a = -70$.

No value of a satisfies this equation so there is no solution to equations ④, ⑤, and ⑥. L_1 and L_3 do not intersect.

Because the direction vectors of L_1 and L_3 are not parallel, lines L_1 and L_3 are skew lines.

L_1 and L_4

The parametric equations for L_1 and L_4 are

$$L_1 \begin{cases} x = 3 + 2k \\ y = 4 - 3k \\ z = 5 + k \end{cases} \qquad L_4 \begin{cases} x = 5 - 4w \\ y = 2 + 6w \\ z = 4 - 2w \end{cases}$$

Equating the components for L_1 and L_4, then simplifying, gives the following system of equations.

$$2k + 4w = 2 \quad ⑦$$
$$-3k - 6w = -2 \quad ⑧$$
$$k + 2w = -1 \quad ⑨$$

The augmented matrix for this system is

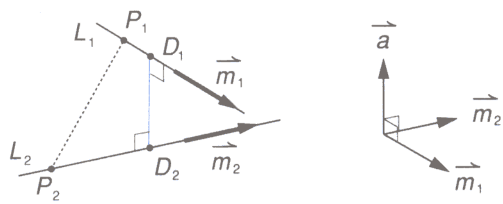
$$\begin{bmatrix} 2 & 4 & 2 \\ -3 & -6 & -1 \\ 1 & 2 & -1 \end{bmatrix} \qquad \begin{matrix} \\ 3 \times \text{row}① + 2 \times \text{row}② \\ \text{row}① - 2 \times \text{row}③ \end{matrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

From the last line of the reduced matrix, $0w = 4$.
Since this equation has no solution, the lines L_1 and L_4 do not intersect.
Because the direction vectors of L_1 and L_4 are parallel, lines L_1 and L_4 are parallel and distinct. ■

The Shortest Distance between Two Lines

If two lines are skew, they do not intersect. How close do they come to each other? In other words, what is the shortest distance between two skew lines? The following analysis will explain *how to find the shortest distance between two skew lines L_1 and L_3.*

The shortest distance between two skew lines L_1 and L_2 is the distance between the two points D_1 and D_2 on L_1 and L_2 respectively, where $D_1D_2 \perp L_1$ and $D_1D_2 \perp L_2$.



Suppose the vector $\vec{a} = \vec{m}_1 \times \vec{m}_2$, where $\vec{m}_1$ and $\vec{m}_2$ are direction vectors for L_1 and L_2 respectively. Then $\vec{a}$ is perpendicular to $\vec{m}_1$, and to $\vec{m}_2$, and hence to L_1 and to L_2. Hence, $\overrightarrow{D_1,D_2}$ is parallel to $\vec{a}$.
Let d = the shortest distance between L_1 and L_2.
Then $d = |\overrightarrow{D_1D_2}|$. Let P_1 and P_2 be points on L_1 and L_2 respectively.
Thus, $D_1D_2 \perp D_1P_1$ and $D_1D_2 \perp D_2P_2$.

Thus, $d = |\text{component of } \overrightarrow{P_1P_2} \text{ along } \overrightarrow{D_1D_2}|$

$= |\text{component of } \overrightarrow{P_1P_2} \text{ along } \vec{a}|$ $\qquad \vec{a} \parallel \overrightarrow{D_1D_2}$

$$\text{Thus } d = \frac{|\overrightarrow{P_1P_2} \cdot \vec{a}|}{|\vec{a}|}$$

FORMULA

The shortest distance d between two skew lines L_1 and L_2 is

$$d = \frac{|\overrightarrow{P_1P_2} \cdot \vec{a}|}{|\vec{a}|}$$

where P_1 is a point on L_1, P_2 is a point on L_2, L_1 has direction vector $\vec{m}_1$, L_2 has direction vector $\vec{m}_2$, and $\vec{a} = \vec{m}_1 \times \vec{m}_2$.

Example 2 a) Find the shortest distance, D_1D_2, between the skew lines L_1: $\vec{r} = \overrightarrow{(-1 + 2k,-2 + 2k,3k)}$ and L_2: $\vec{r} = \overrightarrow{(9 + 6a,3 - a,1 + 2a)}$

b) Find the coordinates of points D_1 and D_2.

Solution a) The shortest distance between the lines is given by the formula

$$d = \frac{|\overrightarrow{P_1P_2} \cdot \vec{a}|}{|\vec{a}|}$$

Here

$P_1 = (-1,-2,0)$, $P_2 = (9,3,1)$ so $\overrightarrow{P_1P_2} = \overrightarrow{(9,3,1)} - \overrightarrow{(-1,-2,0)} = \overrightarrow{(10,5,1)}$

Also, $\vec{a} = \vec{m}_1 \times \vec{m}_2 = \overrightarrow{(2,2,3)} \times \overrightarrow{(6,-1,2)} = \overrightarrow{(7,14,-14)}$

$$|\vec{a}| = \sqrt{7^2 + 14^2 + (-14)^2} = \sqrt{441} = 21$$

Thus $d = \dfrac{|\overrightarrow{(10,5,1)} \cdot \overrightarrow{(7,14,-14)}|}{21} = \dfrac{126}{21}$ or 6

The shortest distance between the lines is 6.

b) You can find the coordinates of points D_1 and D_2 by using the fact that $D_1D_2 \perp L_1$ and $D_1D_2 \perp L_2$

Since $\overrightarrow{OD_1} = \overrightarrow{(-1 + 2k,-2 + 2k,3k)}$ for some value of k, and

$\overrightarrow{OD_2} = (9 + 6a,3 - a,1 + 2a)$ for some value of a,

therefore $\overrightarrow{D_1D_2} = \overrightarrow{OD_2} - \overrightarrow{OD_1}$

$= \overrightarrow{(6a - 2k + 10,-a -2k + 5,2a -3k + 1)}$

For $D_1D_2 \perp L_1$

$\overrightarrow{D_1D_2} \cdot \vec{m}_1 = 0$

$\overrightarrow{(6a -2k + 10,-a - 2k + 5,2a - 3k + 1)} \cdot \overrightarrow{(2,2,3)} = 0$

$12a - 4k + 20 - 2a - 4k + 10 + 6a - 9k + 3 = 0$

$16a - 17k = -33$ ①

For $D_1D_2 \perp L_2$

$\overrightarrow{D_1D_2} \cdot \vec{m}_2 = 0$

$\overrightarrow{(6a -2k + 10,- a - 2k + 5,2a - 3k + 1)} \cdot \overrightarrow{(6,-1,2)} = 0$

$36a - 12k + 60 + a + 2k - 5 + 4a - 6k + 2 = 0$

$41a - 16k = -57$ ②

Solving ① and ② gives $k = 1$ and $a = -1$.

Hence the coordinates of D_1 are $(-1 + 2[1],-2 + 2[1],3[1]) = (1,0,3)$ and of D_2 are $(9 + 6[-1],3 - [-1],1 + 2[-1] = (3,4,-1))$. ■

5.6 Exercises

1. For each of the following pairs of lines in 3-space determine whether or not the lines are parallel.
If the lines are not parallel, determine whether they intersect, or are skew.
If the lines intersect, find the point of intersection.

 a) $\vec{r} = \overrightarrow{(5 + 2k, 8 + k, 13 + 3k)}$
 $\vec{r} = \overrightarrow{(3 + 4t, 2 - 3t, 2 - 2t)}$

 b) $\vec{r} = \overrightarrow{(10 - 5k, 4 - 6k, 3 + k)}$
 $\vec{r} = \overrightarrow{(3 + s, 4 - 3s, -6 + 5s)}$

 c) $\vec{r} = \overrightarrow{(1 - 3k, 2 + 5k, 4 + k)}$
 $\vec{r} = \overrightarrow{(3 + b, -3b, -2 + 6b)}$

 d) $\vec{r} = \overrightarrow{(1 + 8a, 4 - 6a, 2 + 2a)}$
 $\vec{r} = \overrightarrow{(2 - 4t, -5 + 3t, -3 - t)}$

 e) $\vec{r} = \overrightarrow{(1 + t, 3 - 2t, -14 + 5t)}$
 $\vec{r} = \overrightarrow{(-1 + 4k, -7 + 6k, -2 - 2k)}$

 f) $\vec{r} = \overrightarrow{(5 + a, -5 - 2a, 3 + 4a)}$
 $\vec{r} = \overrightarrow{(3 + 4k, -10 + k, -1 - 2k)}$

2. For each of the pairs of lines in question 1 indicate which of the following terms apply to the corresponding linear system.
 a) consistent and independent
 b) consistent and dependent
 c) inconsistent

3. Find the point of intersection of the line $\vec{r} = \overrightarrow{(-1,4,6)} + k\overrightarrow{(-3,5,1)}$ and the line through the points $A(-4,-3,7)$ and $B(8,1,3)$

4. Find the point of intersection of the line $\vec{r} = \overrightarrow{(-3,4,1)} + k\overrightarrow{(4,-1,7)}$ and the line
$$\frac{x - 11}{-2} = \frac{y - 5}{-1} = \frac{z}{5}$$

5. Find the point of intersection of the lines
$$L_1: \frac{x + 2}{5} = \frac{y + 3}{2} = \frac{z + 2}{4} \text{ and}$$
$$L_2: \frac{x + 5}{4} = \frac{y - 5}{-3} = \frac{z - 4}{-1}$$

6. Find the distance between the lines in questions 1 c) and 1 f).

7. a) Prove that the lines
 L_1: $\vec{r} = \overrightarrow{(1 + 5k, -2 + k, 4 - 3k)}$ and
 L_2: $\vec{r} = \overrightarrow{(2 + 3t, 6 - 2t, 7 - 4t)}$ are skew lines.

 b) What is the distance between the skew lines L_1 and L_2?

8. Given the lines
 L_1: $\vec{r} = \overrightarrow{(4 + 2k, 4 + k, -3 - k)}$ and
 L_2: $\vec{r} = \overrightarrow{(-2 + 3s, -7 + 2s, 2 - 3s)}$

 a) Prove that the lines are skew.

 b) Find the shortest distance between the lines.

 c) Find the coordinates of the point P_1 on L_1 and the point P_2 on L_2 such that P_1P_2 is the shortest distance of part b).

9. Given the lines $\vec{r} = \overrightarrow{(3 - k, 4 + 5k, 1 - 2k)}$ and $\vec{r} = \overrightarrow{(8 + 2w, 5 + 3w, -9 - 6w)}$.

 a) Prove that the lines intersect.

 b) Find the angle between the lines.

10. Prove that the lines
 $\vec{r} = \overrightarrow{(-5 + 3k, 2 + 2k, -7 + 6k)}$ and
 $\vec{r} = \overrightarrow{(s, -6 - 5s, -3 - s)}$ lie in the same plane.

11. Two power lines L_1 and L_2 from the point A described in the introduction to this chapter have equations
$\vec{r} = \overrightarrow{(1 + 5k, 11k, k)}$ and
$\vec{r} = \overrightarrow{(1 - s, s, -3s)}$ respectively.
A line L_3 from point B has equation
$\vec{r} = \overrightarrow{(3 - t, -4 - 5t, -5 - 2t)}$.
   a) Determine if either of the power lines from point A intersect with the line from point B.
   b) If either of the power lines from point A do not intersect with the line from point B, then find the shortest distance between the lines.
   c) For the lines in part b) find the points on the lines that give this shortest distance.

12. a) Find an equation of the line through the origin that intersects the lines
$\vec{r} = \overrightarrow{(1 - t, 2 + 2t, 3 + t)}$ and
$\vec{r} = \overrightarrow{(-1 + 3k, 1 + 2k, -1 - k)}$
   b) Find the point of intersection of the line found in part a) and the first of the given lines.

13. By definition, a diagonal of a cube joins a vertex of the cube to an opposite vertex not on a same face of the cube. A certain cube has sides of length d. Find the shortest distance between any diagonal of the cube and an edge that is skew to that diagonal.

14. Prove that the lines $\vec{r} = \vec{a} + k\vec{v}$ and $\vec{r} = \vec{b} + t\vec{v}$ lie in the same plane.

15. Determine whether or not the following pairs of equations represent the same line.
   a) $\vec{r} = \overrightarrow{(2,3,-1)} + k\overrightarrow{(-5,0,1)}$ and
   $\vec{r} = \overrightarrow{(-3,3,0)} + t\overrightarrow{(10,0,-2)}$
   b) $\vec{r} = \overrightarrow{(4,5,4)} + k\overrightarrow{(1,2,3)}$ and
   $\vec{r} = \overrightarrow{(4,5,4)} + t\overrightarrow{(3,2,1)}$
   c) $\vec{r} = \overrightarrow{(1,-3,7)} + k\overrightarrow{(1,1,0)}$ and
   $\vec{r} = \overrightarrow{(2,-1,7)} + t\overrightarrow{(1,1,0)}$

16. Given the points $D_1(-1,3,2)$ and $D_2(4,0,1)$.
   a) Find $\overrightarrow{D_1D_2}$.
   b) Prove that the line
   L_1: $\vec{r} = \overrightarrow{(-1,3,2)} + k\overrightarrow{(a,b,1)}$ is perpendicular to D_1D_2 provided $5a - 3b = 1$.
   c) Prove that the line
   L_2: $\vec{r} = \overrightarrow{(4,0,1)} + t\overrightarrow{(p,q,2)}$ is perpendicular to D_1D_2 provided $5p - 3q = 2$.
   d) Find an equation of any one line L_1, and for any one line L_2.
   e) Prove that the shortest distance between the lines L_1 and L_2 of d) is equal to the length of segment D_1D_2.
   f) Prove that the shortest distance between any line L_1 of b) and any line L_2 of c) equals the length of segment D_1D_2.

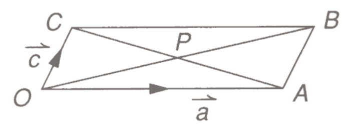
5.7 Geometric Proofs Using Vector Equations

Now that you have learned how to form vector equations of lines, you can apply this knowledge to some geometric proofs. You saw problems of this type already in section 2.6. Some of the examples will be the same as in section 2.6. You can use the method presented here as an alternative.

Recall that the vector equation of a line is $\vec{r} = \vec{r_0} + k\vec{m}$, where $\vec{r}$ is the position vector of any point on the line, $\vec{r_0}$ is the position vector of a given point on the line, $\vec{m}$ is a vector parallel to the line, and $k \in \mathbb{R}$ is a parameter.

Example 1 Prove that the diagonals of a parallelogram bisect each other.

Solution Let the parallelogram be $OABC$.
Let $\overrightarrow{OA} = \vec{a}$, and $\overrightarrow{OC} = \vec{c}$.



In order to find P, you need to pinpoint the intersection of lines OB and AC; thus you will require the equations of lines OB and AC.
Taking the point O as origin, you can use the vectors $\vec{a}$ and $\vec{c}$ to determine these equations as follows.
In general, $\vec{r} = \vec{r_0} + k\vec{m}$, $k \in \mathbb{R}$.

Equation of line OB:
Here, $\vec{r_0} = \vec{0}$, $\vec{m} = \overrightarrow{OB} = (\vec{a} + \vec{c})$.
Using $k \in \mathbb{R}$ as parameter,
$\vec{r} = \vec{0} + k\overrightarrow{OB}$ ①
or $\vec{r} = \vec{0} + k(\vec{a} + \vec{c})$ or $\vec{r} = k(\vec{a} + \vec{c})$

Equation of line AC:
Here, $\vec{r_0} = \overrightarrow{OA} = \vec{a}$, $\vec{m} = \overrightarrow{AC} = (\vec{c} - \vec{a})$.
Using $t \in \mathbb{R}$ as parameter,
$\vec{r} = \overrightarrow{OA} + t\overrightarrow{AC}$ ②
or $\vec{r} = \vec{a} + t(\vec{c} - \vec{a})$

These lines intersect at point P when $\vec{r} = \overrightarrow{OP}$ in equations ① and ②,

so $k(\vec{a} + \vec{c}) = \vec{a} + t(\vec{c} - \vec{a})$

$\Rightarrow$ $k\vec{a} + k\vec{c} = \vec{a} + t\vec{c} - t\vec{a}$

$\Rightarrow (k + t - 1)\,\vec{a} + (k - t)\,\vec{c} = \vec{0}$

But $\vec{a}$ and $\vec{c}$ are linearly independent, so

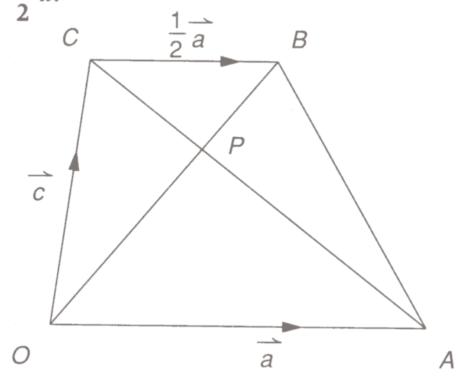
$k + t - 1 = 0$ and $k - t = 0$, giving $k = \frac{1}{2}$ and $t = \frac{1}{2}$

These are the values which give the point P on each line.

Thus, in equation ①, reaching P along line OB requires $\frac{1}{2}\overrightarrow{OB}$, and, in equation ②, reaching P along line AC requires $\frac{1}{2}\overrightarrow{AC}$. More simply, $\overrightarrow{OP} = \frac{1}{2}\overrightarrow{OB}$ and $\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AC}$; the diagonals do indeed bisect each other. ■

Example 2 Given a trapezoid $OABC$ in which $OA = 2CB$, $OA \| CB$, prove that the diagonals each intersect at $\frac{2}{3}$ of their lengths.

Solution To find P, the point of intersection of the diagonals, you need the equations of the lines OB and AC. Using O as origin, choose $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OC} = \vec{c}$, then $\overrightarrow{CB} = \frac{1}{2}\vec{a}$.



Equation of line OB: $\vec{r} = \vec{0} + k\overrightarrow{OB}$ ①

or $\vec{r} = k\left(\vec{c} + \frac{1}{2}\vec{a}\right)$

Equation of line AC: $\vec{r} = \overrightarrow{OA} + t\overrightarrow{AC}$ ②

or $\vec{r} = \vec{a} + t(\vec{c} - \vec{a})$

The lines intersect at P when $\vec{r} = \overrightarrow{OP}$ in both ① and ②.

Thus, $$k\left(\vec{c} + \frac{1}{2}\vec{a}\right) = \vec{a} + t(\vec{c} - \vec{a})$$

$$\Rightarrow \quad k\vec{c} + \frac{1}{2}k\vec{a} = \vec{a} + t\vec{c} - t\vec{a}$$

$$\Rightarrow \quad (k - t)\,\vec{c} + \left(\frac{1}{2}k + t - 1\right)\vec{a} = \vec{0}$$

But $\vec{a}$ and $\vec{c}$ are linearly independent, so $k - t = 0$ and $\frac{1}{2}k + t - 1 = 0$, which gives $k = \frac{2}{3}$ and $t = \frac{2}{3}$ at the point P.

Thus, in equation ①, reaching P along line OB requires $\frac{2}{3}\overrightarrow{OB}$, that is, $\overrightarrow{OP} = \frac{2}{3}\overrightarrow{OB}$.

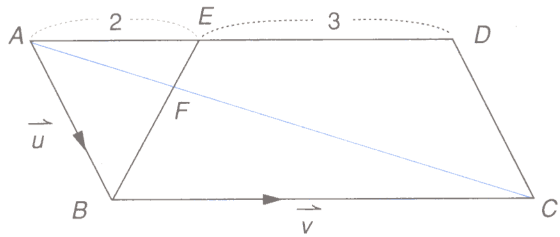
In equation ②, reaching P along line AC requires $\frac{2}{3}\overrightarrow{AC}$, that is, $\overrightarrow{AP} = \frac{2}{3}\overrightarrow{AC}$.

Hence, the required result is confirmed. ■

Example 3 (This is the same as section 2.6, Example 3, on page 104.)

In parallelogram $ABCD$, E divides AD in the ratio 2 : 3. BE and AC intersect at F. Find the ratio into which F divides AC.

Solution



To find the intersection of lines AC and BE, you need the equations of AC and BE.

To obtain these, choose any origin (say point A), and any two independent vectors (say $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{AD} = \overrightarrow{BC} = \vec{v}$). If E divides AD in the ratio 2 : 3, then $\overrightarrow{AE} = \frac{2}{5}\overrightarrow{AD}$, that is, $\overrightarrow{AE} = \frac{2}{5}\vec{v}$.

Equation of line AC: $\vec{r} = \vec{0} + k\overrightarrow{AC}$ ①

or $\vec{r} = k(\vec{v} + \vec{u})$

Equation of line BE: $\vec{r} = \overrightarrow{AB} + t\overrightarrow{BE}$ ②

or $\vec{r} = \vec{u} + t(\overrightarrow{AE} - \overrightarrow{AB})$

or $\vec{r} = \vec{u} + t\left(\frac{2}{5}\vec{v} - \vec{u}\right)$

The lines intersect at point F when $\vec{r} = \overrightarrow{AF}$ in both ① and ②.

Thus, $k(\vec{v} + \vec{u}) = \vec{u} + t\left(\frac{2}{5}\vec{v} - \vec{u}\right)$

$\Rightarrow \quad k\vec{v} + k\vec{u} = \vec{u} + \frac{2t}{5}\vec{v} - t\vec{u}$

$\Rightarrow \quad (k + t - 1)\,\vec{u} + \left(k - \frac{2t}{5}\right)\vec{v} = \vec{0}$

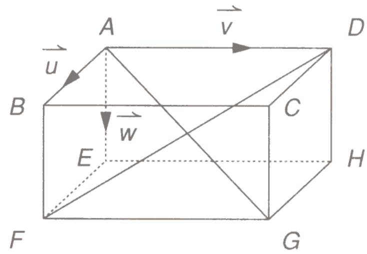
But $\vec{u}$ and $\vec{v}$ and are linearly independent, so $k + t - 1 = 0$ and $k - \frac{2t}{5} = 0$.

These equations give $k = \frac{2}{7}$ and $t = \frac{5}{7}$ at the point F. From equation ①, using $k = \frac{2}{7}$, $\overrightarrow{AF} = \frac{2}{7}\overrightarrow{AC}$, that is F divides AC in the ratio 2 : 5. ■

(Also, as a bonus, you can use $t = \frac{5}{7}$ in equation ② to find that $\overrightarrow{BF} = \frac{5}{7}\overrightarrow{BE}$, so F divides BE in the ratio 5 : 2.)

This method is readily extended to 3-space problems, as in the following example. The only difference is that three independent vectors are required to define the equations of the lines.

Example 4 Consider the rectangular box $ABCDHGFE$ shown. Prove that the diagonals AG and FD intersect, and determine where the point of intersection lies.



Solution First you need the equations of lines AG and FD.

To obtain these, use any origin (say point A), and any three independent vectors (say $\overrightarrow{AB} = \vec{u}$, $\overrightarrow{AD} = \vec{v}$, and $\overrightarrow{AE} = \vec{w}$).

Notice that $\overrightarrow{AB} = \overrightarrow{DC} = \overrightarrow{HG} = \overrightarrow{EF} = \vec{u}$, $\overrightarrow{AD} = \overrightarrow{BC} = \overrightarrow{FG} = \overrightarrow{EH} = \vec{v}$, and $\overrightarrow{AE} = \overrightarrow{BF} = \overrightarrow{CG} = \overrightarrow{DH} = \vec{w}$.

Equation of line AG: $\vec{r} = \vec{0} + k\overrightarrow{AG}$

or $\quad \vec{r} = k(\vec{u} + \vec{v} + \vec{w}) \quad ①$

Equation of line FD: $\vec{r} = \overrightarrow{AF} + t\overrightarrow{FD}$

or $\quad \vec{r} = (\vec{u} + \vec{w}) + t(\vec{v} - \vec{u} - \vec{w}) \quad ②$

If the lines intersect, then from ① and ②

$$k(\vec{u} + \vec{v} + \vec{w}) = (\vec{u} + \vec{w}) + t(\vec{v} - \vec{u} - \vec{w})$$

$$\Rightarrow \quad k\vec{u} + k\vec{v} + k\vec{w} = \vec{u} + \vec{w} + t\vec{v} - t\vec{u} - t\vec{w}$$

$$\Rightarrow (k + t - 1)\,\vec{u} + (k - t)\,\vec{v} + (k + t - 1)\,\vec{w} = \vec{0}$$

But $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly independent, so

$k + t - 1 = 0$

$k - t = 0$

$k + t - 1 = 0.$

These equations form a consistent system with solution $k = \frac{1}{2}$ and $t = \frac{1}{2}$.

Thus, the lines do intersect. If the point of intersection is P, then, from equations ① and ②,

$\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AG}$, and $\overrightarrow{FP} = \frac{1}{2}\overrightarrow{FD}$.

Thus, P is the midpoint of each diagonal. That is, the diagonals AG and FD bisect each other. ■

SUMMARY

You can use vector equations of lines to solve any geometric problem in which you are required to find the intersection of two well-defined lines, by choosing an origin and any known independent vectors to write your equations.

Equating the position vector of a general point on each line will yield the values of the parameters at the point of intersection.

5.7 Exercises

Use vector methods to solve the following problems.

1. $OBCD$ is a parallelogram. E is the midpoint of side OD. Segments OC and BE intersect at point F. Find the ratio into which OC divides BE.

2. $OBCD$ is a parallelogram. E is the point that divides side OD in the ratio $2:5$. Segments OC and BE intersect at point F. Find the ratio into which OC divides BE.

3. a) In $\triangle OAB$, medians AD and BE intersect at point G. Find the ratios into which G divides AD and BE.
   b) Show the medians of a triangle trisect each other.

4. In $\triangle OBC$, E is the midpoint of side OB. Point F is on side OC such that segment EF is parallel to side BC. Into what ratio does F divide side OC?

5. In $\triangle OBC$, E is the point that divides side OB into the ratio $1:2$. Point F is on side OC such that segment EF is parallel to side BC. Into what ratio does F divide side OC?

6. In $\triangle OBC$, E is the point that divides side OB into the ratio $1:k$, $k \neq 0$. Point F is on side OC such that segment EF is parallel to side BC. Into what ratio does F divide side OC?

7. In $\triangle ABC$, D divides AB in the ratio $1:2$ and E divides AC in the ratio $1:4$. BE and CD intersect at point F. Find the ratios into which F divides each of BE and CD.

8. In parallelogram $PQRS$, A divides PQ in the ratio $2:5$, and B divides SR in the ratio $3:2$. Segments PR and AB intersect at C. Find the ratio into which C divides segment PR.

9. $PQRS$ is a trapezoid with PQ parallel to SR. PR and QS intersect at point A. If A divides segment QS in the ratio $2:3$, then find the ratio into which A divides PR.

10. $ABCD$ is a parallelogram. E is the point that divides side AD in the ratio $1:k$, where $k \geqslant 0$. Segments AC and BE intersect at point F. Find the ratio into which point F divides AC.

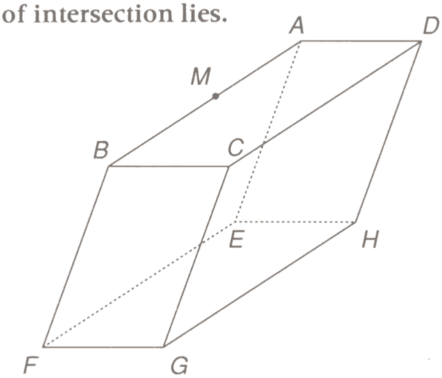
11. Let M be the midpoint of median AD of $\triangle ABC$. BM extended and AC intersect at K. Find the ratio into which K divides AC.

12. $ABCD$ is a trapezoid in which AD is parallel to BC. P and Y divide AB and DC respectively in the same ratio. Q is the point on diagonal AC such that PQ is parallel to BC. Prove that points P, Q, and Y are collinear.

13. In a tetrahedron, prove that the line segments joining a vertex to the centroid of the opposite face intersect at a point that divides the line segments in the ratio $1:3$. (The centroid of a triangle is the point of intersection of the medians. See also question 3.)

14. Show that the point found in question 13 is the same as the point of intersection of the line segments joining the midpoints of opposite edges of a tetrahedron.

15. The box shown, called a parallelepiped, is made up of three pairs of congruent parallelograms. Prove that the diagonals BH and EC intersect, and determine where the point of intersection lies.



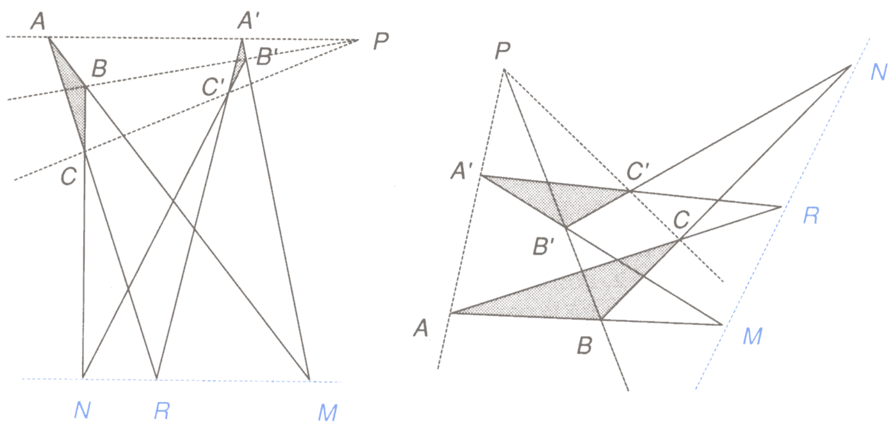
16. In the box shown, let M be the midpoint of AB. Prove that MG and FD do not intersect.

In Search of A Proof of Desargues' Theorem

The French geometer Girard Desargues (1591-1661) was one of the first mathematicians to initiate the formal study of projective geometry. Projective geometry studies the properties of geometric configurations that are invariant under a projection. An important theorem that he proved involves the relationship between two triangles whose vertices lie on three concurrent lines. The point of intersection of the lines is called the *centre of the projection* of one triangle on the other. The lines can be in 2-space or 3-space. Desargues' theorem is true in both spaces.

Desargues' theorem

If two triangles have corresponding vertices joined by concurrent lines, then the intersections of corresponding sides are collinear.



coplanar triangles   non-coplanar triangles

Given: $\triangle ABC$ and $\triangle A'B'C'$ such that lines AA', BB', and CC' intersect at point P.
Lines AB and $A'B'$ intersect at M.
Lines BC and $B'C'$ intersect at N.
Lines AC and $A'C'$ intersect at R.

Prove: Points M, N and R are collinear.

Analysis: You can prove M, N, and R are collinear by finding real numbers p and q such that $\overrightarrow{ON} = p\overrightarrow{OM} + q\overrightarrow{OR}$ where $p + q = 1$. ① (See page 98.)

Use the fact that P lies on each of the lines AA', BB', and CC' to determine a relationship among $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{a}'$, $\vec{b}'$, and $\vec{c}'$ where $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, etc.
Use these relationships to find an equation like ①.

Proof: *For Line AA′:*

An equation of line AA' is $\vec{r} = \vec{r_0} + k\vec{m}$ where $\vec{r_0} = \vec{a}$, and $\vec{m} = \overrightarrow{A'A} = \vec{a} - \vec{a'}$.

Thus, an equation for line AA' is $\vec{r} = \vec{a} + k(\vec{a} - \vec{a'})$

Similarly line BB' has equation $\vec{r} = \vec{b} + t(\vec{b} - \vec{b'})$

and line CC' has equation $\vec{r} = \vec{c} + s(\vec{c} - \vec{c'})$

Since point P lies on both of AA' and BB',

$\overrightarrow{OP} = \vec{a} + k(\vec{a} - \vec{a'})$ and $\overrightarrow{OP} = \vec{b} + t(\vec{b} - \vec{b'})$

Therefore, $\vec{a} + k(\vec{a} - \vec{a'}) = \vec{b} + t(\vec{b} - \vec{b'})$

or, $t\vec{b'} - k\vec{a'} = (1 + t)\vec{b} - (1 + k)\vec{a}$ ②

Dividing by $t - k$ gives

$$\frac{t}{t-k}\vec{b'} - \frac{k}{t-k}\vec{a'} = \frac{1+t}{t-k}\vec{b} - \frac{1+k}{t-k}\vec{a}$$

This statement implies there is a point Q such that

$$\overrightarrow{OQ} = \frac{t}{t-k}\vec{b'} - \frac{k}{t-k}\vec{a'}, \text{ where } \frac{t}{t-k} + \frac{-k}{t-k} = \frac{t-k}{t-k} = 1$$

Hence Q is collinear with A' and B'. Also

$$\overrightarrow{OQ} = \frac{1+t}{t-k}\vec{b} - \frac{1+k}{t-k}\vec{a}, \text{ where } \frac{1+t}{t-k} + \frac{-1-k}{t-k} = \frac{t-k}{t-k} = 1$$

Hence Q is collinear with points A and B.

Hence Q is the point M.

Thus, $$\overrightarrow{OM} = \frac{t}{t-k}\vec{b'} - \frac{k}{t-k}\vec{a'}$$

or $(t - k)\overrightarrow{OM} = t\vec{b'} - k\vec{a'}$ ③

Now the same argument can be repeated for points N and R giving

$(k - s)\overrightarrow{ON} = k\vec{a'} - s\vec{c'}$ ④

$(s - t)\overrightarrow{OR} = s\vec{c'} - t\vec{b'}$ ⑤

Adding equations ③, ④, and ⑤ gives the following.

$(t - k)\overrightarrow{OM} + (k - s)\overrightarrow{ON} + (s - t)\overrightarrow{OR} = \vec{0}$

or, $(k - s)\overrightarrow{ON} = -(t - k)\overrightarrow{OM} - (s - t)\overrightarrow{OR}$

that is, $$\overrightarrow{ON} = \frac{-t+k}{k-s}\overrightarrow{OM} + \frac{-s+t}{k-s}\overrightarrow{OR}$$

The sum of the coefficients on the $R.S.$ is

$$\frac{-t+k}{k-s} + \frac{-s+t}{k-s} = \frac{k-s}{k-s} = 1.$$

Thus, points M, N, and R are collinear, as required.

Activities

1. Is there any situation where you could not divide by any one of $t - k$, $k - s$, or $s - t$ in the proof?

2. The proof assumes that $AB \nparallel A'B'$, $BC \nparallel B'C'$, and $CA \nparallel C'A'$. If $AB \| A'B'$ and $BC \| B'C'$, then prove that $CA \| C'A'$.

Summary

- A vector that is collinear with (or parallel to) a line is called a *direction vector* of the line.
- The components of a direction vector are called *direction numbers*.
- Special direction numbers associated with the angles a direction vector makes with the coordinate axes are the *direction cosines* of a line.
- A vector that is perpendicular to a line is called a *normal vector* of the line.
- In the following table
  $\vec{r} = \overrightarrow{OP}$, the position vector of *any* point P on the line,
  $\vec{r}_0 = \overrightarrow{OP_0}$ the position vector of a *given* point P_0 on the line,
  $\vec{m}$ is a vector parallel to the line,
  k is any real number called a *parameter*.

Lines	*2-Space*	*3-Space*
vector equation	$\vec{r} = \vec{r}_0 + k\vec{m}$	
parametric equations	$\begin{cases} x = x_0 + km_1 \\ y = y_0 + km_2 \end{cases}$	$\begin{cases} x = x_0 + km_1 \\ y = y_0 + km_2 \\ z = z_0 + km_3 \end{cases}$
symmetric equations	$\dfrac{x - x_0}{m_1} = \dfrac{y - y_0}{m_2}$, $m_1, m_2, \neq 0$	$\dfrac{x - x_0}{m_1} = \dfrac{y - y_0}{m_2} = \dfrac{z - z_0}{m_3}$, $m_1, m_2, m_3 \neq 0$
direction numbers	m_1, m_2	m_1, m_2, m_3
direction cosines	$\cos\alpha = \dfrac{m_1}{\lvert\vec{m}\rvert}$, $\cos\beta = \dfrac{m_2}{\lvert\vec{m}\rvert}$ $\lvert\vec{m}\rvert = \sqrt{m_1^2 + m_2^2}$	$\cos\alpha = \dfrac{m_1}{\lvert\vec{m}\rvert}$, $\cos\beta = \dfrac{m_2}{\lvert\vec{m}\rvert}$, $\cos\gamma = \dfrac{m_3}{\lvert\vec{m}\rvert}$ $\lvert\vec{m}\rvert = \sqrt{m_1^2 + m_2^2 + m_3^2}$
angle relation	$\cos^2\alpha + \cos^2\beta = 1$	$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$
scalar equation	$Ax + By + C = 0$	none exists (applies to planes in 3-space)
slope	$\dfrac{m_2}{m_1}$ or $-\dfrac{A}{B}$	does not exist in 3-space
normal vector	$\overrightarrow{(-m_2, m_1)}$ or $\overrightarrow{(A, B)}$	not unique in 3-space

- In 2-space, two lines can be parallel and distinct, or be parallel and identical, or intersect.

- If a system of linear equations has *one or more solutions*, then the system is said to be *consistent*.
  If a consistent system has *exactly one solution* the system is *independent*.
  If a consistent system has *more than one solution* the system is *dependent*.
  If a system of linear equations has *no solutions*, then the system is said to be *inconsistent*.
  For a system of two equations in two variables

	number of solutions	*geometric description*
consistent and independent	one	two intersecting lines
consistent and dependent	infinite	same line
inconsistent	none	lines are parallel, distinct

- In 3-space, two lines can be parallel, or not parallel. If the lines are not parallel, then they may intersect, or not intersect.
  Two lines in 3-space that are not parallel *and* do not intersect are called *skew lines*.

- The shortest distance d between two skew lines
  L_1, with direction vector $\vec{m}_1$, and L_2, with direction vector $\vec{m}_2$, is the distance between the two points D_1 and D_2 on L_1 and L_2 respectively, where $D_1D_2 \perp L_1$ and $D_1D_2 \perp L_2$.

$$d = \frac{|\overrightarrow{P_1P_2} \cdot \vec{a}|}{|\vec{a}|}$$

where P_1 is a point on L_1, P_2 is a point on L_2, and $\vec{a} = \vec{m}_1 \times \vec{m}_2$.

Using Vectors in Euclidean Geometry

A method of solving geometric problems using vector equations of lines can be found on page 229.

Inventory

1. In the vector equation $\vec{r} = \vec{r}_0 + k\vec{m}$, $\vec{r}$ is _____, $\vec{r}_0$ is _____, $\vec{m}$ is _____, and k is _____.

2. In the vector equation $\vec{r} = \overrightarrow{(1 + 3k, 4 + 5k)}$, a point on the line is _____, a direction vector is _____, the parameter is _____.

3. In the vector equation $\vec{r} = \overrightarrow{(5 - 2t, -1 + 3t, -6t)}$, a point on the line is _____, a direction vector is _____, the parameter is _____.

4. In the parametric equations $x = -4 + 3s$, $y = 2 - 5s$, a point on the line is ____, a direction vector is ____, the parameter is ____, the slope is ____.

5. In the parametric equations $x = -6t$, $y = -1 + t$, $z = 4$, a point on the line is ____, a direction vector is ____, the parameter is ____.

6. In the symmetric equation $\frac{x-3}{4} = \frac{y+2}{-3}$, a point on the line is ____, a direction vector is ____.

7. In the symmetric equations $\frac{x+1}{4} = \frac{y}{4} = z - 3$, a point on the line is ____, a direction vector is ____.

8. For the line $4x + 3y + 7 = 0$, the slope is ____ and a normal vector is ____.

9. For direction cosines of a line, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma =$ ____, where α is ____, β is ____, and γ is ____.

10. A vector with direction cosines as components has a length of ____.

11. Two lines in 2-space that do not intersect must be ____.

12. Two lines in 3-space that do not intersect must be either ____ or ____.

13. For a system of two linear equations that is consistent and dependent, there is (are) ____ solution(s) of the system and the two lines have ____ point(s) of intersection.

14. For a system of two linear equations that is consistent and independent, there is (are) ____ solution(s) of the system and the two lines have ____ point(s) of intersection.

15. For a system of two linear equations that is inconsistent there is (are) ____ solution(s) of the system and the two lines have ____ point(s) of intersection.

16. In the formula $d = \frac{|\overrightarrow{P_1P_2} \cdot \vec{a}|}{|\vec{a}|}$

d is ____, P_1 is ____, P_2 is ____, and $\vec{a}$ is ____.

17. For points $P_1(3, 2, 1)$ and $P_2(5, 6, 8)$, the vector $\overrightarrow{P_1P_2}$ is ____.

Review Exercises

1. Find a vector equation and parametric equations for each of the following lines.
   a) through the point $A(2,-5)$ with direction vector $\overrightarrow{(3,2)}$
   b) through the points $A(-5,-4)$ and $B(1,-6)$
   c) through the point $A(-7,0)$ with direction numbers 2 and -5
   d) through the point $B(2,1)$ parallel to the line $\vec{r} = \overrightarrow{(3,-8)} + k\overrightarrow{(-3,-2)}$
   e) through the point $C(-4,1)$ and parallel to the x-axis
   f) through the point $C(-4,1)$ and parallel to the y-axis

2. Given the line with parametric equations
   $$\begin{cases} x = 4 - t \\ y = 3 + 2t \end{cases}$$
   determine whether or not the following points lie on this line.
   $A(3,5)$ $B(6,-1)$ $C(5,0)$ $D(2.4,4)$

3. Find a Cartesian equation for each of the lines having the given vector equations.
   a) $\vec{r} = \overrightarrow{(1 - 2k, 3 - 4k)}$
   b) $\vec{r} = \overrightarrow{(3 - 2t, t + 1)}$
   c) $\vec{r} = \overrightarrow{(-2 + s, -4 - s)}$

4. For each of the following, find a scalar equation of the line passing through the given point P_0 and having $\vec{n}$ as a normal vector.
   a) $P_0(-1,5)$ $\vec{n} = \overrightarrow{(4,-1)}$
   b) $P_0(4,0)$ $\vec{n} = \overrightarrow{(8,-2)}$
   c) $P_0(11,-5)$ $\vec{n} = \overrightarrow{(0,4)}$

5. Use normal vectors to decide which pairs of lines are parallel and which pairs of lines are perpendicular.
   a) $5x - 2y = 5$
      $2x + 5y = 9$
   b) $4x - 3y + 1 = 0$
      $-8x + 6y = 80$
   c) $2x + 3y = 0$
      $3x = 2y + 1$

6. There are an infinite number of vectors and hence lines that are coplanar with vectors $\vec{a} = \overrightarrow{(2,0,1)}$ and $\vec{b} = \overrightarrow{(-3,4,-5)}$. Find a vector equation of any one of these lines that passes through the point $A(-3,6,1)$.

7. $OBCD$ is a parallelogram with point O at the origin of a 2-space Cartesian coordinate system. $\overrightarrow{OB} = \vec{b}$ lies along the x-axis. $\overrightarrow{OD} = \vec{d}$.
   a) Show that $\overrightarrow{OC} = \vec{b} + \vec{d}$, and $\overrightarrow{BD} = \vec{d} - \vec{b}$.
   b) Find a vector equation for the line through the points O and C in terms of vectors $\vec{b}$ and $\vec{d}$.
   c) Find a vector equation for the line through the points B and D in terms of vectors $\vec{b}$ and $\vec{d}$.

8. a) A line is parallel to the vector $\vec{m} = \overrightarrow{(4,5)}$. Show that the slope of the line is $\frac{5}{4}$
   b) A line is parallel to the vector $\vec{m} = \overrightarrow{(m_1,m_2)}$. Show that the slope of the line is $\frac{m_2}{m_1}$
   c) Find a vector equation of the line passing through the point $A(3,4)$ that has slope equal to $\frac{2}{7}$

9. Given the line L with vector equation $\vec{r} = \overrightarrow{(1,2)} + k\overrightarrow{(-3,-1)}$ and the point $P_0(-4,3)$.
   a) Find a scalar equation of the line through P_0 parallel to the line L.
   b) Find a scalar equation of the line through P_0 perpendicular to the line L.

10. Find the value of the variable t so that the line with scalar equation $2x - 5y = 3$ is perpendicular to the line with vector equation $\vec{r} = \overrightarrow{(4,2)} + k\overrightarrow{(t,3)}$.

11. Find the value of the variable s so that the following lines are perpendicular.
    $\frac{x - 6}{4s + 5} = \frac{y + 1}{1}$ and $\frac{x - 3}{2} = \frac{y - 1}{-4s - 2}$

12. Find a vector equation, parametric equations, and symmetric equations for each of the following lines.
    a) through the point $A(2,3,0)$ with direction vector $\overrightarrow{(4,2,-2)}$
    b) through the points $A(-2,3,5)$ and $B(3,-1,6)$
    c) through the point $A(1,0,10)$ with direction numbers 1, 2, and -4
    d) through the point $B(5,6,10)$, parallel to the line $\vec{r} = \overrightarrow{(6,1,1)} + k\overrightarrow{(-2,1,3)}$
    e) through the point $C(-4,3,6)$ and parallel to the x-axis
    f) through the point $C(-4,3,6)$ and parallel to the z-axis

13. In each of the following, determine whether or not the given point lies on the line with the given equation.

	point	line equation
a)	$(-1,4,9)$	$\vec{r} = \overrightarrow{(1,3,4)} + k\overrightarrow{(-2,1,5)}$
b)	$(-10,-8,4)$	$\vec{r} = \overrightarrow{(0,1,0)} + k\overrightarrow{(5,4,-1)}$
c)	$(11,8,0)$	$\vec{r} = \overrightarrow{(5 + 2k,-4 + 4k,3 - k)}$

14. a) Find a vector equation of the line that passes through the points $A(3,-2,1)$ and $B(4,0,7)$.
    b) Show that the line with equation $\vec{r} = \overrightarrow{(5,2,13)} + t\overrightarrow{(1,2,6)}$ is the same as the line in part a).

15. Find a vector equation of the line that passes through the point $A(-1,-2,4)$ and is parallel to the line with equation $\vec{r} = \overrightarrow{(0,4,9)} + k\overrightarrow{(-3,1,4)}$.

16. Find a vector equation of the line passing through the point $A(4,1,5)$ that is perpendicular to vector $\vec{u} = \overrightarrow{(0,-1,7)}$ and also perpendicular to vector $\vec{v} = \overrightarrow{(2,4,-3)}$.

17. Find a vector equation of the line with symmetric equations
$$\frac{x-3}{4} = \frac{y+1}{5} = \frac{z+3}{-1}$$

18. Find the value of t so that the two lines $\vec{r} = \overrightarrow{(1,7,3)} + k\overrightarrow{(-2,3,5t)}$ and $\vec{r} = \overrightarrow{(4,1,2)} + s\overrightarrow{(t,1,-3)}$ will be perpendicular.

19. Find the value of the variable k so that the line with symmetric equations $\frac{x-3}{2} = \frac{y+1}{4} = \frac{z+3}{k}$ is perpendicular to the line with vector equation $\vec{r} = \overrightarrow{(5 + 2ks,3 - 4s,-8 - 12s)}$

20. Find the value of the variable t so that the following lines are perpendicular.
$$\frac{x-3}{3t+1} = \frac{y+6}{2} = \frac{z+3}{2t};\ \frac{x+7}{3} = \frac{y+8}{-2t} = \frac{z+9}{-3}$$

21. Which of the following triples of numbers can be the direction cosines of a line?
    a) $\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}$
    b) $\frac{5}{8}, \frac{1}{3}, \frac{1}{2}$
    c) $1, -\frac{1}{2}, \frac{1}{2}$
    d) $\cos A, \sin A, 0$

22. a) A line passing though the point $(1,-2,3)$ has direction cosines $\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}$ and $\cos \gamma$, where $\cos \gamma$ is positive. Find a vector equation of the line.
    b) Repeat part a) but take $\cos \gamma$ to be negative.

23. If vectors $\vec{a}$ and $\vec{b}$ are linearly dependent prove that their direction cosines are the same.

24. Given the vectors $\vec{a} = \overrightarrow{(1,2,3)}$ and $\vec{b} = \overrightarrow{(-2,0,1)}$ find the direction cosines of any vector coplanar with $\vec{a}$ and $\vec{b}$.

25. Find direction cosines for each of the lines given by the following.
    a) $\vec{r} = k\overrightarrow{(2,-3,4)}$
    b) $\begin{cases} x = 1 - 2a \\ y = 7 + 5a \\ z = 5 + a \end{cases}$
    c) $\frac{x}{2} = \frac{y-1}{-1} = \frac{z-2}{3}$

26. A direction vector of a line makes an angle of 40° with the x-axis and an angle of 70° with the z-axis. Find, correct to the nearest degree, the angle the direction vector makes with the y-axis.

27. A line through the origin of a 3-space coordinate system makes an angle of 25° with the x-axis and an angle of 85° with the z-axis. Find a vector equation of the line.

28. For each of the following systems determine whether or not the corresponding lines are parallel. If the lines are not parallel, find their point of intersection.

a) $3x - 2y = 4$
$4x + y = 9$

b) $-x + 6y = -1$
$3x - 18y = 5$

c) $4x - 3y = 5$
$8x - 6y = 10$

29. For each of the systems in the previous question, indicate which of the following terms apply.
consistent and independent
consistent and dependent
inconsistent

30. For each of the following pairs of lines in 2-space, determine whether or not the lines are parallel. If the lines are parallel, then determine whether the lines are distinct or the same line. If the lines are not parallel, find their point of intersection.

a) $\vec{r} = \overrightarrow{(4 + k, 5 - 3k)}$
$\vec{r} = \overrightarrow{(4 + 2t, 3 - 4t)}$

b) $\vec{r} = \overrightarrow{(11 - 6k, -5k)}$
$\vec{r} = \overrightarrow{(2 + 5s, 8 - s)}$

c) $\vec{r} = \overrightarrow{(1 + 3a, 4 - 2a)}$
$\vec{r} = \overrightarrow{(2 - 6b, -5 + 4b)}$

31. a) Find the point of intersection of the line $\vec{r} = \overrightarrow{(-4,2)} + k\overrightarrow{(1,6)}$ and the line $\frac{x+5}{3} = \frac{y-10}{4}$.

b) Find the angle between the given lines.

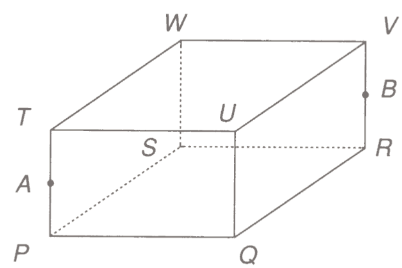
32. $OBCD$ is a parallelogram with point O at the origin of a 2-space Cartesian coordinate system. $\overrightarrow{OB} = \vec{b}$ lies along the x-axis. $\overrightarrow{OD} = \vec{d}$. Find a vector equation for the line through the points B and D in terms of vectors $\vec{b}$ and $\vec{d}$.

33. In the figure, A and B are the midpoints of the sides TP and VR respectively of a rectangular box. The corner S appears to lie on the line though A and B. The coordinates of five of the corners are $P(0,0,0)$, $Q(2,0,0)$, $R(2,4,0)$, $V(2,4,3)$, and $W(0,4,3)$.

a) Find the coordinates of the remaining three corners S, T, and U.

b) Find the coordinates of the points A and B.

c) Prove that the point S does not lie on the line through A and B.



34. For each of the following pairs of lines in 3-space determine whether or not the lines are parallel. If the lines are not parallel, determine whether the lines intersect or are skew. If the lines intersect, find the point of intersection.

a) $\vec{r} = \overrightarrow{(3 + k, 7 - 2k, 4 + 3k)}$
$\vec{r} = \overrightarrow{(-2 + 2t, 1 + 4t, 11 - 5t)}$

b) $\vec{r} = \overrightarrow{(2 + 3k, 1 + 4k, 4 + k)}$
$\vec{r} = \overrightarrow{(3 + b, 1 + 2b, -2 + b)}$

c) $\vec{r} = \overrightarrow{(-1 + a, 4 - 3a, -6 + a)}$
$\vec{r} = \overrightarrow{(2 - 3t, -5 + 9t, -3 - 3t)}$

d) $\vec{r} = \overrightarrow{(6 - 2w, -9 + 4w, 7 + w)}$
$\vec{r} = \overrightarrow{(7 - 5k, -3 + 2k, -3 - 4k)}$

35. For each of the pairs of lines in the previous question, indicate which of the following terms apply to the corresponding linear system.
consistent and independent
consistent and dependent
inconsistent

36. Find the point of intersection of the line $\vec{r} = \overrightarrow{(0,-8,4)} + k\overrightarrow{(2,7,-1)}$ and the line through the points $A(8,4,-1)$ and $B(0,8,5)$.

37. Find the point of intersection of the lines $\frac{x+1}{4} = \frac{y+2}{-2} = \frac{z}{3}$; $\frac{x+7}{5} = \frac{y+10}{3} = \frac{z-5}{-1}$.

38. a) Prove that the lines
$\vec{r} = \overrightarrow{(1+k, 4+3k, 3-2k)}$ and
$\vec{r} = \overrightarrow{(-3+3t, 6+2t, -3t)}$ are skew lines.
b) Find the distance between the given skew lines.

39. Given the lines $\vec{r} = \overrightarrow{(-1+k, 3+2k, 1-3k)}$ and $\vec{r} = \overrightarrow{(3+2w, 10+3w, -6-w)}$.
a) Prove that the lines intersect.
b) Find the angle between the lines.

40. Prove that the following three lines intersect and lie in the same plane.
L_1: $\vec{r} = \overrightarrow{(-1,0,2)} + k\overrightarrow{(3,1,1)}$
L_2: $\vec{r} = \overrightarrow{(-3,-4,1)} + k\overrightarrow{(-4,2,-1)}$
L_3: $\vec{r} = \overrightarrow{(-5,2,1)} + k\overrightarrow{(2,4,1)}$

41. Prove that the following three lines intersect and do not lie in the same plane.
L_1: $\vec{r} = \overrightarrow{(4,1,1)} + k\overrightarrow{(2,1,2)}$
L_2: $\vec{r} = \overrightarrow{(9,1,4)} + k\overrightarrow{(-1,2,1)}$
L_3: $\vec{r} = \overrightarrow{(-1,0,-1)} + k\overrightarrow{(3,1,2)}$

42. a) Find the shortest distance between the skew lines

$$x = 3, \ \frac{y}{-2} = \frac{z-5}{1} \text{ and}$$

$$\frac{x+2}{3} = \frac{y+1}{2} = \frac{z+1}{-2}$$

b) Find a symmetric equation of the line joining the points on the lines that are at this shortest distance.

43. Find symmetric equations for the following lines.
a) $x - 3y + 6 = 0 = x - 2z - 4$
b) $2x + z = 5 - y = y + z$

Use vector equations of lines to solve problems 44–48.

44. *ABCD* is a parallelogram. *E* is the point that divides side *AD* in the ratio 4 : 7. Segments *AC* and *BE* intersect at point *F*. Find the ratio into which *AC* divides *BE*.

45. In a parallelogram *ABCD, H* is the midpoint of *AD*, and *E* divides *BC* in the ratio 3 : 2. If *BH* and *AE* intersect at *M*, find the ratio *AM* : *ME*.

46. In $\triangle ABC$, *E* is the point that divides side *AB* into the ratio 3 : 2. Point *F* is on side *AC* such that segment *EF* is parallel to side *BC*. Into what ratio does *F* divide side *AC*?

47. In $\triangle ABC$, *D* divides *AB* in the ratio 3 : 2 and *E* divides *AC* in the ratio 5 : 4. *BE* and *CD* intersect at point *F*. Find the ratios into which *F* divides each of *BE* and *CF*.

48. In parallelogram *PQRS*, *A* divides *PQ* in the ratio 2 : 1, and *B* divides *SR* in the ratio 3 : 4. Segments *PR* and *AB* intersect at *C*. Find the ratio into which *C* divides segment *PR*.

49. The point *P* with coordinates (1,1,3) lies on the line *L* with equation $2x = y + 1 = z - 1$. Find the coordinates of two points Q_1 and Q_2 on *L* such that $PQ_1 = PQ_2 = 6$.

(87 H)

50. Two lines L_1 and L_2 have equations

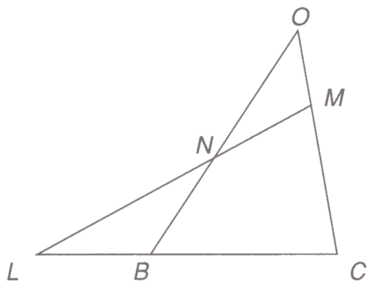
$$\frac{x+3}{3} = \frac{y+4}{2} = \frac{z-6}{-2} \text{ and}$$

$$\frac{x-4}{-3} = \frac{y+7}{4} = \frac{z+3}{-1} \text{ respectively.}$$

a) Find the coordinates of a point P_1 on the line L_1 and a point P_2 on the line L_2 such that the line (P_1P_2) is perpendicular to both the lines L_1 and L_2.
b) Show that the length of (P_1P_2) is 7.
(84 H)

51.i) The lines l_1 and l_2 have equations
$\vec{r} = \overrightarrow{(2,3,1)} + \lambda\overrightarrow{(1,2,2)}$ and
$\vec{r} = \overrightarrow{(5,-1,-13)} + \mu\overrightarrow{(-2,1,6)}$
respectively. Show that these lines intersect by finding the co-ordinates of their point of intersection.
Find, in the form $ax + by + cz + d$, the equation of the plane which contains both l_1 and l_2.

ii)



In the above diagram $\overrightarrow{OM} = \frac{1}{3}\overrightarrow{OC}$ and $\overrightarrow{ON} = \frac{2}{3}\overrightarrow{OB}$. The line (CB) intersects the line (MN) at L. If $\overrightarrow{LB} = \lambda\overrightarrow{BC}$ find the value of λ.

(82 H)

52. Find the position vector of the point of intersection of the lines with equations

$$\vec{r} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \text{ and}$$

$$\vec{r} = \begin{pmatrix} 5 \\ 6 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

(85 H)

53. The lines l_1 and l_2 have equations
$\frac{x+2}{1} = \frac{y-1}{-2} = \frac{z}{3}$ and
$\frac{x-2}{2} = \frac{y-1}{4} = \frac{z-7}{1}$ respectively. Which one of the following is true?

A. l_1 and l_2 are parallel.
B. l_1 and l_2 are perpendicular.
C. l_1 and l_2 have no common point.
D. l_1 and l_2 intersect at the point with co-ordinates $(-2,1,0)$.
E. l_1 and l_2 intersect at the point with co-ordinates $(0,-3,6)$.

(85 H)

54. Three points A, B and C are given whose coordinates in a rectangular Cartesian system are $(0,9)$, $(6,7)$ and $(8,3)$ respectively.

a) Find the equation of the line l_1 perpendicular to $\overrightarrow{BC}$ and passing through the point A.
b) Find the equation of the line l_2 perpendicular to $\overrightarrow{CA}$ and passing through the point B.
c) Find the coordinates of H, the point of intersection of the lines l_1 and l_2.
d) Verify that $\overrightarrow{HC} \cdot \overrightarrow{AB} = 0$.
e) What geometrical result has now been established for $\triangle ABC$?
f) Find the coordinates of the point S such that $\overrightarrow{SH} = \overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC}$.

(88 S)