VECTORS, MATRICES
and
COMPLEX NUMBERS

with
International Baccalaureate
questions

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CHAPTER SEVEN

MATRICES AND LINEAR TRANSFORMATIONS

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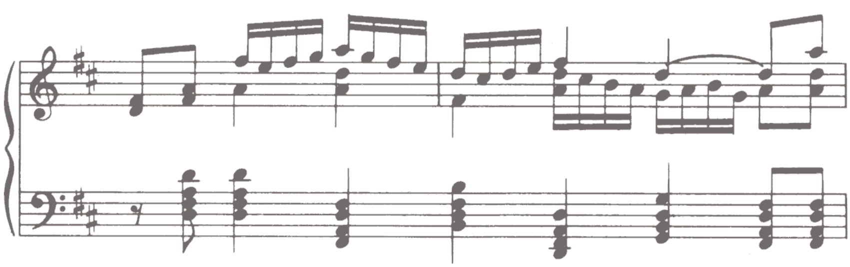
Matrices and Linear Transformations

Many of the introductions to television programs and movies include visual displays of transformations such as rotations and dilatations. Some well-known examples of this are the introductions to "The National" and "The Journal" on CBC television, or the text that appears to move into outer space in the introduction and credits of the "Star Wars" movies.

Transformations are a part of our life in a modern society. Indeed, transformations are involved in any form of representation—be it drawing or painting, sculpture, playing music, etc.



Transformations have an increasingly wide application in design technology. Computer programs are now available to architects as an aid in the design of elaborate structures. Engineers use similar programs to help them develop new projects.

In this chapter, you will be introduced to a new mathematical object called a **matrix** (plural matrices) that can be used as an operator to effect various transformations.

Matrices provide a very compact way of expressing transformations.

Recall that many vector equations can be applied to 2-space, 3-space, and even to spaces of higher dimensions. Matrix equations also have this same universality.

This chapter will provide you with an extensive study of transformations of 2-space, but the principles that you learn will be readily applicable to transformations of 3-space.

7.1 Matrices

In this section, you will be taking an elementary look at matrices, their properties, and some of their algebra. The main purpose of this chapter is the study of matrices as operators that transform vectors. That study will begin in the next section.

DEFINITION

A matrix is a rectangular array of numbers.

For example, $A = \begin{bmatrix} 2 & 1 & -1 \\ 5 & 6 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & -4 \\ -1 & 7 \end{bmatrix}$ are matrices.

The numbers composing the matrix are called **elements**. (They are also known as **entries** or **components**.)

The Dimension of a Matrix

A matrix can be described by its **rows** or its **columns** of elements. For example, in A,

the 1st row is $\quad 2 \quad 1 \quad -1$

the 2nd row is $\quad 5 \quad 6 \quad 0$

the 1st column is $\begin{matrix} 2 \\ 5 \end{matrix}$; the 2nd column is $\begin{matrix} 1 \\ 6 \end{matrix}$; the 3rd column is $\begin{matrix} -1 \\ 0 \end{matrix}$

Since the matrix A has 2 rows and 3 columns,
A is known as a 2×3 matrix (read "2 by 3 matrix").
Alternatively, A is said to have **dimension** 2×3 (or **order** 2×3, or **shape** 2×3).

DEFINITION

A matrix that has m rows and n columns is known as an $m \times n$ matrix.

Example 1 What are the dimensions of the matrices B and C above?

Solution The matrix B above is a 3×1 matrix.
The matrix C above is a 2×2 matrix. ■

Subscript Notation

Given any matrix A, it is often useful to specify its elements in the following way.

The element in the ith row, jth column is represented by a_{ij}.

You can also abbreviate A to $A = [a_{ij}]$.

Thus, if A is a 2×2 matrix, it can be written as

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Equality

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if and only if *all* their corresponding elements are respectively equal.
That is, $a_{ij} = b_{ij}$ for all values of i and j.
(Thus only matrices that have the *same dimensions* can be equal.)

Example 2 Given $M = \begin{bmatrix} 3 & -2 \\ -z & 5 \end{bmatrix}$, $N = \begin{bmatrix} x & 2y \\ 11 & 5 \end{bmatrix}$, and $M = N$, find the values of x, y, and z.

Solution The elements in the 1st row, 1st column, are 3 and x.
Thus $x = 3$.
The elements in the 1st row, 2nd column, are -2 and $2y$.
Thus $y = -1$.
The elements in the 2nd row, 1st column, are $-z$ and 11.
Thus $z = -11$. ■

Square Matrices

DEFINITION

A square matrix of order p is a matrix of dimension $p \times p$.

The properties and operations discussed in the rest of this section will be devoted entirely to 2×2 matrices, that is, square matrices of order 2. This is to prepare you for the 'matrices as operators' that you will be using in the rest of the chapter. Whenever the term "matrix" is used, it will mean "2×2 matrix".

The Algebra of 2×2 Matrices

Addition and Subtraction

Given $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $N = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$, the sum $M + N$ is defined as follows:

$M + N = \begin{bmatrix} a + w & b + x \\ c + y & d + z \end{bmatrix}$ Corresponding elements are added.

The difference is defined as follows:

$M - N = \begin{bmatrix} a - w & b - x \\ c - y & d - z \end{bmatrix}$ Corresponding elements are subtracted.

The Zero Matrix

The 2×2 zero matrix has each element equal to zero. It will be represented by $0_{2\times2}$.

Thus $0_{2\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

The zero matrix is called the **neutral element** for the addition of 2×2 matrices because, for any matrix M,
$M + 0_{2\times2} = M$ and $0_{2\times2} + M = M$.

The zero matrix is also known as the **null matrix**.

Commutativity of Matrix Addition

The addition of matrices is commutative.
That is, for any matrices M and N, $M + N = N + M$.

Proof: Given $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $N = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$,

$$M + N = \begin{bmatrix} a+w & b+x \\ c+y & d+z \end{bmatrix}$$

and

$$N + M = \begin{bmatrix} w+a & x+b \\ y+c & z+d \end{bmatrix}$$

Since the addition of numbers is commutative,
$a + w = w + a$, $b + x = x + b$, $c + y = y + c$, and $d + z = z + d$.
Hence
$M + N = N + M$.

Associativity of Matrix Addition

The addition of matrices is associative. That is, given any matrices L, M and N,
$(L + M) + N = L + (M + N)$.
Thus, brackets are not required when adding matrices. The expression $L + M + N$ can be used to mean $(L + M) + N$.
You will have an opportunity to prove this property in the exercises.

The Negative of a Matrix

Given the matrix $A = [a_{ij}]$, then the matrix $[-a_{ij}] = -A$.
$-A$ is called the **negative** of A.
$-A$ is also called the **additive inverse** of A, because
$A + (-A) = 0_{2\times2} = (-A) + A$.

Multiplication of a Matrix by a Scalar

Since matrix addition is associative, it seems natural to write
$M + M + M = 3M$.
This operation is accepted, and is called
multiplication of a matrix by a scalar.

Given a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and a real number k,

then $kM = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$

Example 3 Given $M = \begin{bmatrix} -4 & 3 \\ -2 & 0 \end{bmatrix}$ and $N = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$, calculate

a) $-1M$ b) $4N$ c) $2M + 3N$

Solution

a) $-1M = -1\begin{bmatrix} -4 & 3 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -1(-4) & -1(3) \\ -1(-2) & -1(0) \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 2 & 0 \end{bmatrix}$

Notice that the matrix $-1M = -M$.

b) $4N = 4\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4(2) & 4(-1) \\ 4(0) & 4(1) \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ 0 & 4 \end{bmatrix}$

c) $2M + 3N = 2\begin{bmatrix} -4 & 3 \\ -2 & 0 \end{bmatrix} + 3\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -8 & 6 \\ -4 & 0 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -4 & 3 \end{bmatrix}$ ■

Note: A 2×2 matrix is in fact an ordered quadruple of numbers. Thus, it can be considered a "four-dimensional vector". The properties listed above go a long way towards showing that 2×2 matrices, together with the operations of matrix addition, and of multiplication of a matrix by a scalar, form a **vector space.** You will have an opportunity to prove this in the exercises.

Properties of 2 × 2 Matrices

2×2 Matrices form a vector space $\mathbb{V}_{2\times2}$, that is, the following properties hold.

PROPERTIES

Matrix Addition

A1. $\mathbb{V}_{2\times2}$ is closed under addition: $M, N \in \mathbb{V}_{2\times2}$ implies $M + N \in \mathbb{V}_{2\times2}$

A2. Addition is associative: $L + (M + N) = (L + M) + N$

A3. There is a $0_{2\times2} \in \mathbb{V}_{2\times2}$ such that for all $M \in \mathbb{V}_{2\times2}$, $M + 0_{2\times2} = M$

A4. If $M \in \mathbb{V}_{2\times2}$, then there exists $-M \in \mathbb{V}_{2\times2}$ such that $M + (-M) = 0_{2\times2}$

A5. Addition is commutative: $M + N = N + M$

(These properties mean that $\mathbb{V}_{2\times2}$ is a commutative group with respect to addition.)

PROPERTIES

Multiplication of a Matrix by a Scalar

M1. If $M \in \mathbb{V}_{2\times2}$, $k \in \mathbb{R}$, then $kM \in \mathbb{V}_{2\times2}$

M2. $(kp)M = k(pM)$, $k, p \in \mathbb{R}$

M3. $k(M + N) = kM + kN$

M4. $(k + p)M = kM + pM$

M5. There exists $1 \in \mathbb{R}$ such that $1M = M$

7.1 Exercises

1. Give the dimensions of the following matrices.

$$A = \begin{bmatrix} 2 & 6 & 8 & 1 \\ -1 & 0 & 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 5 & -1 \\ 6 & 6 \\ 4 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 7 \\ 2 \\ 5 \\ -2 \end{bmatrix}$$

2. $A = [a_{ij}] = \begin{bmatrix} 3 & 1 & 0 & 1 \\ -2 & 4 & 5 & 2 \\ -1 & 0 & 8 & -6 \end{bmatrix}$

a) State the values of $a_{11}, a_{14}, a_{23}, a_{33}$.

b) Use the a_{ij} notation to describe the positions of the zero elements in this matrix.

3. Calculate the values of the variables in the following.

a) $\begin{bmatrix} x & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 4 & y \\ z & w \end{bmatrix}$

b) $\begin{bmatrix} 3x & 4y \\ 5 & w \end{bmatrix} = \begin{bmatrix} 9 & -8 \\ w - x & z \end{bmatrix}$

4. Calculate the values of the variables in the following.

a) $\begin{bmatrix} 2 & 3 \\ 5 & z \end{bmatrix} + \begin{bmatrix} x & 6 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & y \\ w & 0 \end{bmatrix}$

b) $\begin{bmatrix} a & 1 \\ 8 & -7 \end{bmatrix} - \begin{bmatrix} 9 & b \\ c & 5 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 8 & d \end{bmatrix}$

c) $2\begin{bmatrix} a & 1 \\ 8 & -7 \end{bmatrix} - 3\begin{bmatrix} 9 & b \\ c & 5 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 8 & d \end{bmatrix}$

d) $\begin{bmatrix} x - y & 4 \\ z & x \end{bmatrix} + \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 3 & x \end{bmatrix}$

5. Given $A = \begin{bmatrix} -5 & 0 \\ 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix}$,

$C = \begin{bmatrix} 7 & 9 \\ -4 & 6 \end{bmatrix}$ calculate the following.

a) $-B$ **d)** $A + B + C$ **g)** $2B + 2C$

b) $3A$ **e)** $\frac{1}{2}A - 2C$ **h)** $3(2A)$

c) $3A - B$ **f)** $2(B + C)$ **i)** $6A$

6. Using the matrices A, B, C given in question 5, calculate the matrix X in the following cases.

a) $2X = A$ **c)** $B - X = A$

b) $X - 3B = 0_{2\times2}$ **d)** $5X + C = 3X - A$

7. Show that no real values of x and y exist such that

$$\begin{bmatrix} 3x & x + y \\ 2y & x - y \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 10 & 4 \end{bmatrix}$$

8. Using as examples any of the matrices given in question 5, illustrate the following.

a) the commutative property of matrix addition

b) the associative property of matrix addition

For the following questions, in which you will be asked to prove various properties of 2×2 matrices belonging to a vector space, use the matrices

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, N = \begin{bmatrix} e & f \\ g & h \end{bmatrix}, L = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

9. Prove property A1: if M and N are both 2×2 matrices, then $M + N$ is a 2×2 matrix.

10. Prove property A2: matrix addition is associative, that is, $L + (M + N) = (L + M) + N$.

11. Prove property M1: if $k \in \mathbb{R}$, and M is a 2×2 matrix, then kM is a 2×2 matrix.

12. Prove property M2: if $k, p \in \mathbb{R}$, then $(kp)M = k(pM)$. (This was illustrated in parts h) and i) of question 5.)

13. Prove property M3: $k(M + N) = kM + kN$. (This was illustrated in parts f) and g) of question 5.)

14. Prove property M4: $(k + p)M = kM + pM$.

Your solutions to questions 9–14, together with the proofs supplied in the text of section 7.1, show that $\mathbb{V}_{2\times2}$ is a vector space.

7.2 Matrices and Linear Transformations

In this section you will be looking at transformations of vectors in $\mathbb{V}_2$ using 2×2 matrices as operators. It will be necessary to use a different notation when expressing vectors in component form.

The notation $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ will replace the notation $\vec{v} = \overrightarrow{(x,y)}$.

The former notation is called a **column vector** or a 2×1 matrix.

A **transformation** of a vector $\vec{v}$ is a function or mapping that changes $\vec{v}$ into another vector $\vec{v'}$.

For example,

$F: \vec{v} \rightarrow \vec{v'}$ where $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{v'} = \begin{bmatrix} 3x \\ -y \end{bmatrix}$ is a transformation.

It can be equally well described in the following ways.

$F: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 3x \\ -y \end{bmatrix}$ or $F\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ -y \end{bmatrix}$ or $\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{F} \begin{bmatrix} 3x \\ -y \end{bmatrix}$ or $F(\vec{v}) = \vec{v'}$

The vector $\vec{v'}$ is known as the **image** of $\vec{v}$ under F.

You can also say that $\vec{v}$ is **mapped** onto $\vec{v'}$ by F.

DEFINITION

Linear Transformations

A linear transformation T of a vector space is a transformation that has *both* of the following properties.

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
2. $T(k\vec{v}) = k[T(\vec{v})]$

where $\vec{u}$, $\vec{v}$ are vectors and k is a real number.

Example 1

Check if the transformation F as defined above, namely $F: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 3x \\ -y \end{bmatrix}$, is a linear transformation.

Solution

Let $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} w \\ z \end{bmatrix}$, then $\vec{u} + \vec{v} = \begin{bmatrix} x + w \\ y + z \end{bmatrix}$

$F(\vec{u}) = \begin{bmatrix} 3x \\ -y \end{bmatrix}$ $F(\vec{v}) = \begin{bmatrix} 3w \\ -z \end{bmatrix}$ $F(\vec{u} + \vec{v}) = \begin{bmatrix} 3(x + w) \\ -(y + z) \end{bmatrix} = \begin{bmatrix} 3x + 3w \\ -y - z \end{bmatrix}$

Now $F(\vec{u}) + F(\vec{v}) = \begin{bmatrix} 3x \\ -y \end{bmatrix} + \begin{bmatrix} 3w \\ -z \end{bmatrix} = \begin{bmatrix} 3x + 3w \\ -y - z \end{bmatrix} = F(\vec{u} + \vec{v})$

Thus the first property holds.

$F(k\vec{v}) = F(k\begin{bmatrix} w \\ z \end{bmatrix}) = F\begin{bmatrix} kw \\ kz \end{bmatrix} = \begin{bmatrix} 3kw \\ -kz \end{bmatrix}$

$k[F(\vec{v})] = k\begin{bmatrix} 3w \\ -z \end{bmatrix} = \begin{bmatrix} 3kw \\ -kz \end{bmatrix} = F(k\vec{v})$

Thus the second property also holds.

Hence, F is a linear transformation. ■

Example 2 Check if the following are linear transformations.

a) $G: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+2 \\ 5x \end{bmatrix}$ b) $H: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+2y \\ 3x-y \end{bmatrix}$

Solution Let $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} w \\ z \end{bmatrix}$, then $\vec{u} + \vec{v} = \begin{bmatrix} x+w \\ y+z \end{bmatrix}$

a) $G(\vec{u}) = \begin{bmatrix} x+2 \\ 5x \end{bmatrix} \quad G(\vec{v}) = \begin{bmatrix} w+2 \\ 5w \end{bmatrix}$

$G(\vec{u}+\vec{v}) = \begin{bmatrix} (x+w)+2 \\ 5(x+w) \end{bmatrix} = \begin{bmatrix} x+w+2 \\ 5x+5w \end{bmatrix}$

Now $G(\vec{u}) + G(\vec{v}) = \begin{bmatrix} x+2 \\ 5x \end{bmatrix} + \begin{bmatrix} w+2 \\ 5w \end{bmatrix} = \begin{bmatrix} x+w+4 \\ 5x+5w \end{bmatrix} \neq G(\vec{u}+\vec{v})$

Thus the first property does *not* hold. The transformation G is *not* linear.

b) $H(\vec{u}) = \begin{bmatrix} x+2y \\ 3x-y \end{bmatrix} \quad H(\vec{v}) = \begin{bmatrix} w+2z \\ 3w-z \end{bmatrix}$

$H(\vec{u}+\vec{v}) = \begin{bmatrix} (x+w)+2(y+z) \\ 3(x+w)-(y+z) \end{bmatrix} = \begin{bmatrix} x+2y+w+2z \\ 3x-y+3w-z \end{bmatrix}$

$H(\vec{u}) + H(\vec{v}) = \begin{bmatrix} x+2y+w+2z \\ 3x-y+3w-z \end{bmatrix} = H(\vec{u}+\vec{v})$

Thus the first property holds.

$H(k\vec{v}) = H(k\begin{bmatrix} w \\ z \end{bmatrix}) = H\begin{bmatrix} kw \\ kz \end{bmatrix} = \begin{bmatrix} kw+2kz \\ 3kw-kz \end{bmatrix}$

$k[H(\vec{v})] = k\begin{bmatrix} w+2z \\ 3w-z \end{bmatrix} = \begin{bmatrix} kw+2kz \\ 3kw-kz \end{bmatrix} = H(k\vec{v})$

Thus the second property also holds.

Hence, H is a linear transformation. ■

A general linear transformation in $\mathbb{V}_2$ has the following form.

$T: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ ①

where a, b, c and d are real numbers.

You will have an opportunity in the exercises to prove that T thus defined is indeed a linear transformation.

Matrix Notation for Transformations

It is customary to use matrix notation to indicate the relationship between $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$.

DEFINITION $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

This definition of a matrix multiplying a column vector can be remembered in the following way.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

This is called a "row-column" multiplication process. It is the basis of multiplication of matrices in general, as you shall see in the forthcoming sections.

You can also think of the result of this multiplication in the following way.

$$\begin{bmatrix} \text{dot product of the first row with } \begin{bmatrix} x \\ y \end{bmatrix} \\ \text{dot product of the second row with } \begin{bmatrix} x \\ y \end{bmatrix} \end{bmatrix}$$

Also, note that the product (2×2 matrix) $\times$ (2×1 matrix) gives a (2×1 matrix).

Thus the transformation ① can be written in **matrix form**, as follows,

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad ②$$

or even more compactly like this

$$\vec{v} \rightarrow M\vec{v} = \vec{v}', \quad ③$$

where $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\vec{v}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$, and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Here, the matrix M is an operator called the **transformation matrix** of T.

Note: The above discussion implies that a 2×2 matrix will *always* express a linear transformation of $\mathbb{V}_2$.

Example 3 Find the images of each of the following vectors under the transformation matrix $M = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

$\vec{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

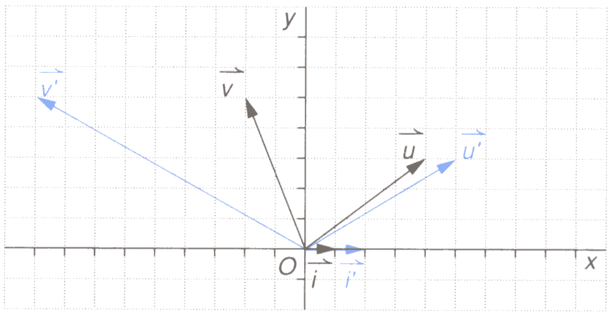
Sketch (as position vectors) $\vec{u}, \vec{v}, \vec{i}$ and their images $\vec{u'}, \vec{v'}, \vec{i'}$.

Solution

$$\vec{u'} = M\vec{u} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} (2)(4) + (-1)(3) \\ (0)(4) + (1)(3) \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\vec{v'} = M\vec{v} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} (2)(-2) + (-1)(5) \\ (0)(-2) + (1)(5) \end{bmatrix} = \begin{bmatrix} -9 \\ 5 \end{bmatrix}$$

$$\vec{i'} = M\vec{i} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (2)(1) + (-1)(0) \\ (0)(1) + (1)(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

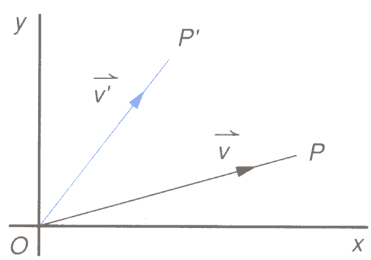


■

Note: The transformation affects the *entire plane*. The vectors $\vec{u}, \vec{v}$, and $\vec{i}$ are position vectors of just *some* of the points that are transformed.

Indeed, under a transformation described by matrix M, every vector maps onto a vector $M\vec{v}$.

Suppose $\vec{v}$ is the position vector $\overrightarrow{OP}$ of the point P,
and $M\vec{v}$ is the position vector of the point P'.
Then you can say that
P maps onto point P' under the transformation defined by the matrix M, or
$P \xrightarrow{M} P'$.



The Image of a Line

One of the most important properties of a linear transformation is that it transforms a straight line into another straight line, as the following example will show.

Example 4 Given a linear transformation whose matrix is T, find the image under T of the straight line L whose vector equation is $\vec{r} = \vec{r_0} + k\vec{m}$.

Solution Recall that $\vec{r}$ is the position vector of any point of L.

Let $\vec{r'}$ be the position vector of any point of the *image* of L.

Then $\vec{r'} = T(\vec{r})$

$= T(\vec{r_0} + k\vec{m})$

$= T(\vec{r_0}) + T(k\vec{m})$ — linear transformation, property 1

$= T(\vec{r_0}) + k[T(\vec{m})]$ — linear transformation, property 2

$= \vec{r_0'} + k\vec{m'}$,

where $\vec{r_0'}$ and $\vec{m'}$ are the images of $\vec{r_0}$ and $\vec{m}$ respectively.

Thus the image of L has vector equation $\vec{r'} = \vec{r_0'} + k\vec{m'}$.

This is the equation of a straight line. ■

PROPERTY

Thus, *straight lines are transformed into straight lines by linear transformations.*

The following example uses the above property to show how a diagram can portray the effect of a linear transformation.

Example 5 Consider the unit square S whose vertices are $O(0,0)$, $A(1,0)$, $B(1,1)$, $C(0,1)$. Describe the image of S under the transformation defined by matrix

$$M = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$

Solution Use the following notation.

points	O	A	B	C		images	O'	A'	B'	C'
position vectors	$\vec{0}$	$\vec{a}$	$\vec{b}$	$\vec{c}$		images	$\vec{0'}$	$\vec{a'}$	$\vec{b'}$	$\vec{c'}$

$$M\vec{0} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0'} = \vec{0}$$

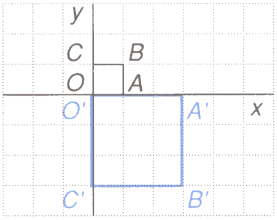
$$M\vec{a} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3+0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \vec{a'} = \overrightarrow{OA'}$$

$$M\vec{b} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3+0 \\ 0-3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \vec{b'} = \overrightarrow{OB'}$$

$$M\vec{c} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 0-3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \vec{c'} = \overrightarrow{OC'}$$

Thus the point O remains at O,
the point $A(1,0)$ has image $A'(3,0)$,
the point $B(1,1)$ has image $B'(3,-3)$,
and the point $C(0,1)$ has image $C'(0,-3)$.

Since straight lines are mapped onto straight lines, you know that OA', $A'B'$, $B'C'$ and $C'O$ are straight lines.
Thus, the transformation is as shown in this diagram.



■

The unit square seems to have been enlarged and reflected in the x-axis. Recall that the transformation affects the entire plane, not just the unit square.

7.2 Exercises

1. Check if the following are linear transformations.

$F: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 3x \\ 4y \end{bmatrix}$ $\quad R: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2 \\ x \end{bmatrix}$

$G: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+y \\ x-y \end{bmatrix}$ $\quad S: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x^2 \\ 2y \end{bmatrix}$

$H: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+1 \\ y \end{bmatrix}$ $\quad T: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 3y \\ -x \end{bmatrix}$

2. State the matrix of each of the linear transformations found in question 1.

3. Prove that the transformation $T: \vec{v} \to \overrightarrow{Mv}$ defined by the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a linear transformation.

4. Calculate the following products.

a) $\begin{bmatrix} -1 & 2 \\ 6 & -4 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

5. Calculate the values of the variables in the following.

a) $\begin{bmatrix} 5 & x \\ 4 & y \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

b) $\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

c) $\begin{bmatrix} 2x & y \\ -y & x \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

d) $\begin{bmatrix} -x & 3 \\ y & z \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \end{bmatrix}$

6. Show that there are no real values of x and y such that $\begin{bmatrix} x & y \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

7. Find the images of the following vectors under the transformation matrix

$M = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$.

$\vec{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ $\quad \vec{w} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Sketch as position vectors $\vec{u}, \vec{v}, \vec{w}$ and their images $\vec{u'}, \vec{v'}, \vec{w'}$.

8. For $M = \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix}$, $N = \begin{bmatrix} 3 & 5 \\ -7 & 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, find

a) $M\vec{v}$ b) $N\vec{v}$ c) $M(N\vec{v})$ d) $N(M\vec{v})$

Draw conclusions.

9. For the following, use the matrix $M = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

$\vec{0}$ is the zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\vec{i}$ and $\vec{j}$ are the standard basis vectors, that is, $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Calculate the following.

a) $M\vec{0}$ b) $M\vec{i}$ c) $M\vec{j}$

Draw conclusions.

10. The straight line L has vector equation $\vec{r} = \vec{r_0} + k\vec{m}$, where

$\vec{r} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\vec{r_0} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\vec{m} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

a) Find the vector equation of the image of L under the transformation of matrix

$A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

b) Graph L and its image on the same set of axes.

11. a) Find the coordinates of two points A and B on the line L whose Cartesian equation is $y = 2x + 1$.

b) Find the image A' of A, and the image B' of B, under the transformation of matrix $M = \begin{bmatrix} -3 & 5 \\ 2 & 0 \end{bmatrix}$.

c) Find the Cartesian equation of the line L' that passes through A' and B'.

d) What is the image of L under M?

12. a) Consider the points $O(0,0)$, $A(3,1)$ and $B(1,4)$. If $OABC$ is a parallelogram, calculate the coordinates of the point C. Sketch $OABC$ on a grid.

b) Find the image $O'A'B'C'$ of the parallelogram $OABC$ under the transformation of matrix $M = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}$

c) Sketch $O'A'B'C'$ on the same grid. Describe the nature of $O'A'B'C'$.

7.3 The Effect of a Linear Transformation

Since any linear transformation T of $\mathbb{V}_2$ can always be represented by its matrix, M, the expression "the transformation M" can clearly replace the expression "the transformation whose matrix is M".

As you have seen, linear transformations are defined for vector spaces. You can observe their effect on points in $\mathbb{R}^2$ as follows. By considering the vectors of $\mathbb{V}_2$ as position vectors of points in $\mathbb{R}^2$, you can examine the effect of a linear transformation of their tips.

Thus, you can speak of linear transformations 'of a plane'.

Some questions of the last exercises illustrated properties of matrices as transformations. These properties will be demonstrated here.

Image of $\vec{0}$ by a Linear Transformation

Consider the general linear transformation $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$M\vec{0} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (a)(0) + (b)(0) \\ (c)(0) + (d)(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

Thus the image of $\vec{0}$ is always $\vec{0}$.

Images of Parallel Lines by a Linear Transformation

Consider the two parallel lines L_1 and L_2 whose vector equations are

L_1: $\vec{r} = \vec{a} + k\vec{m}$

L_2: $\vec{s} = \vec{b} + q\vec{m}$

where L_1 contains a point A whose position vector is $\vec{a}$,

L_2 contains a point B whose position vector is $\vec{b}$,

$\vec{r}$ and $\vec{s}$ are position vectors of a general point on L_1 and L_2 respectively, and k, q are scalars.

(Notice that the lines both have the same direction vector $\vec{m}$. The lines are thus parallel.)

As you saw in Example 4 of the previous section (page 299), the images of L_1 and L_2 under a linear transformation T will be

L_1': $\vec{r'} = \vec{a'} + k\vec{m'}$

L_2': $\vec{s'} = \vec{b'} + q\vec{m'}$.

Thus the image lines will have the same direction $\vec{m'}$, that is, they will be parallel.

These conditions, together with the examples and exercises seen so far in this chapter, lead to the following generalization.

A general linear transformation of a plane may pull, push, turn, stretch, or compress the plane, in any directions, with the two following provisos.

PROPERTIES

1. The origin does not move.
2. Parallelism is preserved.

One thing that a linear transformation *cannot* perform is a *translation*. Indeed, a translation of the plane would violate the first proviso.

Reading and Writing a Matrix

Consider the effect of a general transformation $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ on the standard basis vectors $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$M\vec{i} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \text{ and } M\vec{j} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

Thus, the first column of a matrix M is the image of $\vec{i}$ under M and the second column of a matrix M is the image of $\vec{j}$ under M.

This beautiful property allows you to do *two* very useful things.

1. *WRITING A MATRIX*
   Given a transformation, you can state its matrix as follows.
   Write the first column as the image of $\vec{i}$,
   and the second column as the image of $\vec{j}$.
2. *READING A MATRIX*
   Given a matrix, you can determine the linear transformation it represents as follows.
   Its first column is the image of $\vec{i}$,
   and its second column is the image of $\vec{j}$.

The following examples show how this knowledge can be applied.

Example 1

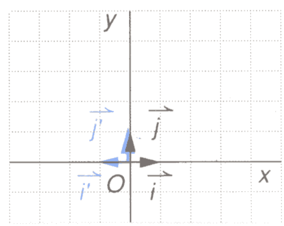
State the matrix M_y of the transformation that reflects the plane in the y-axis.

Solution

If all vectors are reflected in the y-axis then

$\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 \\ 0 \end{bmatrix}$; this is the first column of M_y.

$\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; this is the second column of M_y.



Thus the matrix $M_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ ■

Example 2

Describe the transformation whose matrix is $D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Solution

Read the columns of D.

The image of $\vec{i}$ is the first column, $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Thus $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

The image of $\vec{j}$ is the second column, $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Thus $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Thus, both $\vec{i}$ and $\vec{j}$ are doubled in magnitude, or 'enlarged by a factor of 2'.

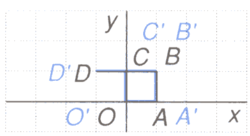
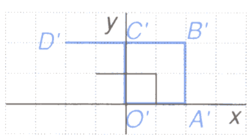
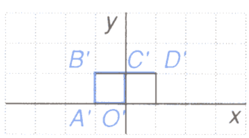
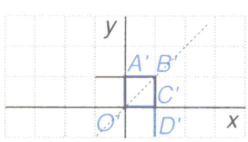
Hence, the transformation D can be described as an enlargement or a dilatation of factor 2. ■

(See a diagram of the effect of "D_2", in the second of the following illustrations.)

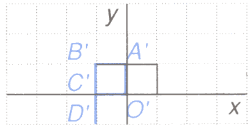
SUMMARY

To summarize this section, you will observe the effects of some common linear transformation matrices, as illustrated on the figure *OABCD*, which is the unit square *OABC* with *BC* produced to *D* so that *BC* = *CD*. (The use of a non-symmetric figure such as *OABCD* gives a clearer idea of the transformation in some cases.)

The image figure can be obtained in each case by calculating the image of each of the points *O*, *A*, *B*, *C*, and *D*. You will have an opportunity to verify these illustrations in the exercises.

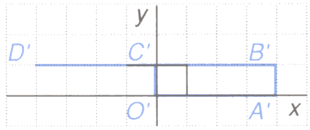
Matrix	Effect	Transformation
$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$		identity
$D_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$		dilatation, of factor 2
$M_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$		reflection in y-axis
$M_{45} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$		reflection in line $y = x$

$$R_{90} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



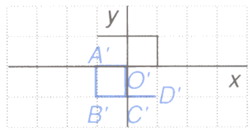
counterclockwise rotation about O, through 90°

$$S_{x4} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$



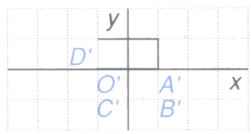
one-way stretch $\|$ x-axis, of factor 4

$$D_{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$



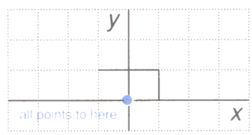
dilatation of factor −1, or reflection in O

$$P_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



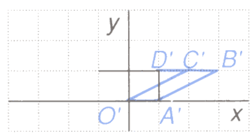
projection onto x-axis

$$0_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$



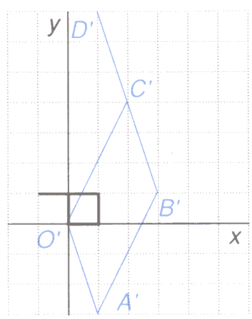
null transformation

$$H_{x2} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$



shear $\|$ x-axis, of factor 2

$$G = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$



general linear transformation

When a linear transformation acts on a plane,
1. the origin does not move,
2. parallelism is preserved.

The matrix M of a linear transformation is such that its first column is the image of $\vec{i}$ under M, and its second column is the image of $\vec{j}$ under M.

7.3 Exercises

In the following,
O shall refer to the origin (0,0).

1. Justify the diagrams of pages 304–305, at the end of section 7.3, that show the effect of each of the following transformations.
I, D_2, M_y,
M_{45}, R_{90}, S_{x4},
D_{-1}, P_x, $0_{2\times 2}$,
H_{x2}, and G.

2. Describe in words the effect of the following transformations, shown on pages 304–305.
the identity transformation $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
the null transformation $0_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3. Write the matrix that corresponds to the given transformation by finding the image of $\vec{i}$ and the image of $\vec{j}$.
   a) a dilatation of factor 3
   b) reflection in the x-axis
   c) counterclockwise rotation about O through 270°
   d) counterclockwise rotation about O through 180°
   e) projection onto the y-axis
   f) reflection in the line $y = -x$

4. By finding the images of the points O, $P(1,0)$, $Q(1,1)$, $R(0,1)$ sketch the effect of each of the transformations of question 3 on the unit square.

5. Compare your answer to question 3d) with the matrix $D_{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, listed in section 7.3 as a "dilatation of factor −1" or a "reflection in O".
Draw conclusions.

6. By reading the columns of the following matrices, describe the associated transformation in each case.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $F = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$ $G = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ $H = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$

$D = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$E = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ $J = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$

7. By finding the images of the points O, $P(1,0)$, $Q(1,1)$, $R(0,1)$ sketch the effect of each of the transformations of question 6 on the unit square.

8. What is the image of the point O under the transformation matrix $M = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$?

9. The transformation matrix $S = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is known as a horizontal shear of factor 2. Some of the properties of a shear will appear as answers to the following questions.
   a) Find the images of (1,0) and $(a,0)$, where $a \in \mathbb{R}$.
   b) Describe how any point of the x-axis is transformed.
   c) Find the images of (0,1) and $(a,1)$, where $a \in \mathbb{R}$.
   d) Describe how any point on the line $y = 1$ is transformed.
   e) Find the image of the point $(0,b)$, where $b \in \mathbb{R}$.
   f) Describe how any point on the y-axis is transformed.

10. Show the effect of $R = \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix}$ on the unit square.
Describe the transformation associated with the matrix R.

11. Write the matrix of the following transformations.

a) a stretch parallel to the x-axis of factor 5

b) a dilatation of factor $\frac{1}{4}$

c) a two-way stretch, of factor 2 parallel to the x-axis, and of factor 3 parallel to the y-axis

d) a perpendicular projection onto the line $y = x$

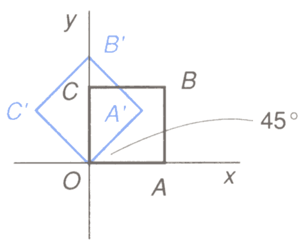
e) a shear parallel to the x-axis of factor -1

f) a shear parallel to the y-axis of factor 5

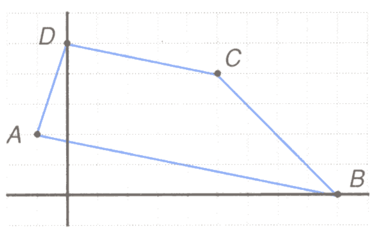
g) a reflection in the y-axis followed by a dilatation of factor 4

12. Sketch the effects of each of the transformations of question 11 on the unit square.

13. Write the matrix of a counterclockwise rotation about the origin, through 45°.



14. a) Verify that the following four points determine the vertices of a trapezoid. $A(-1,2)$, $B(9,0)$, $C(5,4)$, $D(0,5)$



b) Transform this trapezoid with the matrix $M = \begin{bmatrix} 2 & -3 \\ 2 & 5 \end{bmatrix}$

c) Show that the images of the parallel sides of the trapezoid remain parallel under the transformation.

15. Consider the straight lines L_1 and L_2 with the following vector equations.

$$L_1\colon \vec{r} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + k\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$L_2\colon \vec{s} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + p\begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

a) Explain why the lines L_1 and L_2 are parallel, and graph them on the same set of axes.

b) Find the vector equations of the images of L_1 and L_2 under the transformation $M = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$

c) Show that these images are also parallel.

16. Given the matrix $S = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, where k is a positive scalar.

a) Find the image under S of the unit square.

b) Calculate the image under S of a general vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

c) Describe the image under S of the entire plane.

17. Repeat question 16 using the matrix $S = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$

18. In question 5 you determined that the following transformations in 2-space are equivalent.

A. a dilatation of factor -1

B. a reflection in O

Consider also another transformation

C. a rotation through 180° about O

Is this transformation equivalent to **A** and to **B**?

Discuss whether or not these transformations are also equivalent in 3-space.

7.4 Rotations and Reflections

In this section you will learn about two important linear transformations, namely rotations, that you will be using in chapter 8, and reflections. You will also make further observations on linear transformations in general. These should help you to solve transformation problems with more assurance.

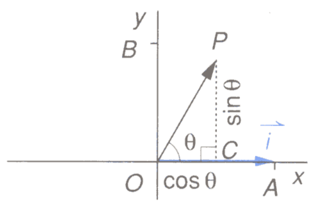
Example 1 Find the matrix R_θ of the transformation that rotates the plane counterclockwise about the origin O, through an angle of θ.

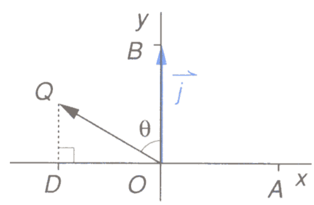
Solution If all vectors are rotated counterclockwise about O, through θ, then this will also be true for $\vec{i}$ and $\vec{j}$.

Let $\vec{i} = \overrightarrow{OA}$, and $\vec{j} = \overrightarrow{OB}$.

Let the image of $\vec{i}$ be $\overrightarrow{OP}$, that is, let $\overrightarrow{OA} \rightarrow \overrightarrow{OP}$. Under a rotation, lengths are invariant, so $|\overrightarrow{OP}| = |\vec{i}| = 1$.

Thus $\overrightarrow{OP} = \begin{bmatrix} OC \\ CP \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$





Similarly, let the image of $\vec{j}$ be $\overrightarrow{OQ}$, that is, let $\overrightarrow{OB} \rightarrow \overrightarrow{OQ}$.

Then the angle between $\overrightarrow{OQ}$ and $\vec{i}$ is $(90^\circ + \theta)$.

Note also that $|\overrightarrow{OQ}| = |\vec{j}| = 1$.

$$\text{Thus } \overrightarrow{OQ} = \begin{bmatrix} OD \\ DQ \end{bmatrix} = \begin{bmatrix} \cos(90^\circ + \theta) \\ \sin(90^\circ + \theta) \end{bmatrix} = \begin{bmatrix} \sin[90^\circ - (90^\circ + \theta)] \\ \cos[90^\circ - (90^\circ + \theta)] \end{bmatrix}$$

$$= \begin{bmatrix} \sin(-\theta) \\ \cos(-\theta) \end{bmatrix}$$

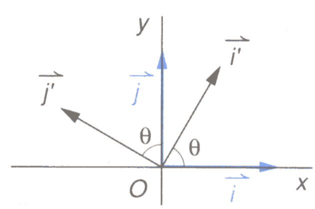
$$= \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

(The formulas on page 541 have been used.)

Thus, $\vec{i} \rightarrow \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\vec{j} \rightarrow \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

PROPERTY

Hence the matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$



This is known as a *positive* rotation through θ. (A counterclockwise rotation is a positive rotation.) ■

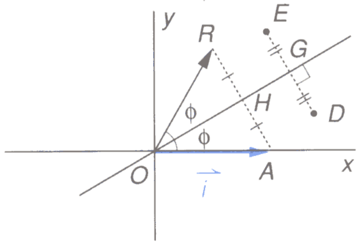
Example 2 Find the matrix M_ϕ of the transformation that reflects the plane in the line L, where L passes through the origin and makes an angle ϕ with the positive x-axis.

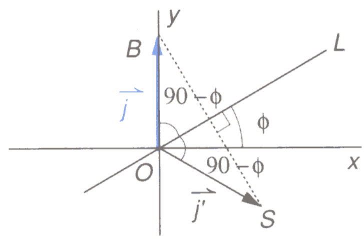
Solution If all vectors are reflected in L, then this will also be true for $\vec{i}$ and $\vec{j}$.
To obtain the image of any point D under a reflection in L, construct the perpendicular DG from D to the line L. Extend this perpendicular on the other side of L so that $DG = GE$. E is then the image of D.

Let the image of $\vec{i}$ be $\overrightarrow{OR}$, that is, let $\overrightarrow{OA} \to \overrightarrow{OR}$. Under a reflection, lengths are invariant, so $|\overrightarrow{OR}| = |\vec{i}| = 1$. Also, the right triangle OAH, where H is the foot of the perpendicular from A to the line L, is congruent to the triangle ORH. Hence, the angle between L and OR is the same as the angle between OA and L, namely ϕ.

Thus, the angle between OR and $\vec{i}$ is 2ϕ.

Hence, $\overrightarrow{OR} = \begin{bmatrix} \cos 2\phi \\ \sin 2\phi \end{bmatrix}$





Similarly, let the image of $\vec{j}$ be $\overrightarrow{OS}$, that is, let $\overrightarrow{OB} \to \overrightarrow{OS}$.
Then the angle between L and OS is equal to the angle between OB and L, namely $90° - \phi$.

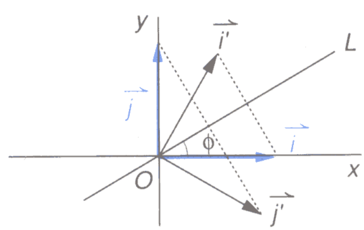
Hence, the angle between OS and $\vec{i}$ is $-([90 - \phi] - \phi) = (2\phi - 90°)$.

$$\text{Thus } \overrightarrow{OS} = \begin{bmatrix} \cos(2\phi - 90°) \\ \sin(2\phi - 90°) \end{bmatrix} = \begin{bmatrix} \cos[-(90° - 2\phi)] \\ \sin[-(90° - 2\phi)] \end{bmatrix} = \begin{bmatrix} \cos[90° - 2\phi] \\ -\sin[90° - 2\phi] \end{bmatrix} = \begin{bmatrix} \sin 2\phi \\ -\cos 2\phi \end{bmatrix}$$

(The formulas on page 541 have been used.)

Thus, $\vec{i} \to \begin{bmatrix} \cos 2\phi \\ \sin 2\phi \end{bmatrix}$ and $\vec{j} \to \begin{bmatrix} \sin 2\phi \\ -\cos 2\phi \end{bmatrix}$

Hence the matrix $M_\phi = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}$



Notice that the slope of the L is $\tan\phi$.
Thus, L has a Cartesian equation $y = (\tan\phi)x$. ■

Note: The matrix $M_\phi = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}$ is very similar to the rotation matrix of Example 1, that is, $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. You must be careful not to confuse them. A reflection matrix, like M_ϕ, is a *symmetric matrix*. Its elements are symmetric about the leading diagonal.

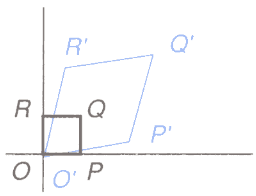
That is, it is of the form $\begin{bmatrix} a & h \\ h & b \end{bmatrix}$ leading diagonal

The observations that follow should help you to make this distinction.

The Size and Orientation of a Transformed Figure

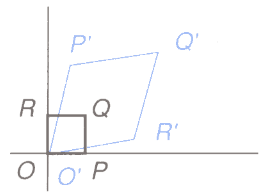
Consider the effect of the general linear transformation $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ on the unit square $OPQR$.

Recall that under M, parallelism is preserved and O does not move. Thus the image of the square $OPQR$ will be a parallelogram $OP'Q'R'$, where P' is the image of P, Q' is the image of Q, and R' is the image of R.



$O'P'Q'R'$ has *same* orientation as $OPQR$

or



$O'P'Q'R'$ has *opposite* orientation to $OPQR$

Reading the columns of M gives $\overrightarrow{OP'} = \begin{bmatrix} a \\ c \end{bmatrix}$, $\overrightarrow{OR'} = \begin{bmatrix} b \\ d \end{bmatrix}$

Recall that the area of a parallelogram whose adjacent sides represent the vectors $\vec{u}$ and $\vec{v}$ is $|\vec{u} \times \vec{v}|$.

However, the cross product is not defined in $\mathbb{V}_2$. In order to calculate the area of the parallelogram $OP'Q'R'$, you must imagine that it lies in the xy-plane of a 3-space coordinate system.

Then the vector $\overrightarrow{OP'} = (a,c,0)$, and the vector $\overrightarrow{OR'} = (b,d,0)$.

Thus the area of $OP'Q'R'$ is

$|\overrightarrow{OP'} \times \overrightarrow{OR'}| = |\overrightarrow{(a,c,0)} \times \overrightarrow{(b,d,0)}| = |ad - bc|$

Since the area of the unit square is 1, the area of the transformed figure is changed by a scale factor $|ad - bc|$.

DEFINITION

The quantity $(ad - bc)$ is known as the **determinant** of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, that is, the determinant of M, or $\det(M)$, or $|M|$.

Det(M) can also be denoted by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Notice that the original unit square $OPQR$ is read *counterclockwise*. Also, from the diagram, the slope of OP' is $\frac{c}{a}$, and the slope of OR' is $\frac{d}{b}$.

If OP' and OR' are in the first quadrant, and $\frac{d}{b} > \frac{c}{a}$, then $OP'Q'R'$ will be read *counterclockwise*.

Figure $OPQR$ and figure $O'P'Q'R'$ have the *same orientation*.

Observe that $\frac{d}{b} > \frac{c}{a} \Rightarrow ad > bc \Rightarrow ad - bc > 0 \Rightarrow \det(M) > 0$.

Similarly, you can see that if $\det(M) < 0$, $OP'Q'R'$ will be read *clockwise*. Figure $OPQR$ and figure $O'P'Q'R'$ have *opposite orientation*.

This can be extended to all four quadrants.

Example 3 Calculate the area and describe the orientation of the image of the unit square under each of the following transformations (taken from the examples of the last section).

$$D_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad M_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad P_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution Note that the area of the unit square is 1.

You can find the area scale factors by calculating the determinants.

$$\det(D_2) = (2)(2) - (0)(0) = 4$$

The image is enlarged 4 times.
Since $4 > 0$, the image retains the original orientation.

(D_2 is a dilatation of factor 2.)

$$\det(M_y) = (-1)(1) - (0)(0) = -1.$$

The area of the image is unchanged.

Since $-1 < 0$, the image has the opposite orientation.

(M_y is a reflection in the y-axis.)

$$\det(P_x) = (1)(0) - (0)(0) = 0.$$

The image has zero area. Orientation is not defined for a figure of zero area.

(P_x is a projection onto the x-axis.) ■

Example 4 Calculate the area scale factor and describe the orientation of the transformations described by the following.

a) the rotation matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

b) the reflection matrix $M_\theta = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}$

Solution To simplify the determinants in this question, use the Pythagorean trigonometric identity of page 542.

a) $\det(R_\theta) = \cos^2\theta - (-\sin^2\theta) = \cos^2\theta + \sin^2\theta = 1$. Note that $1 > 0$. Thus a rotation does not alter the area or orientation of any figure.

b) $\det(M_\phi) = -\cos^2 2\phi - \sin^2 2\phi = -(\cos^2 2\phi + \sin^2 2\phi) = -1$. Note that $-1 < 0$. Thus a reflection leaves the area invariant, but reverses the orientation of a figure. ■

SUMMARY

These results can be summarized as follows.

If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant, $\det(M) = ad - bc$.

If a figure of area S is transformed by matrix M, the area of the image figure is $|\det(M)|S$.

$|\det(M)|$ is called the area scale factor of matrix M.

If $\det(M) > 0$, the image retains the original orientation;
if $\det(M) < 0$, the image acquires the opposite orientation.

7.4 Exercises

In the following, all rotations are about the origin O, counterclockwise through the indicated angle, unless specified otherwise.

1. Write the matrix of the rotation through the given angle in each case, giving entries correct to 2 decimal places.
   a) 40° c) 90° e) 200°
   b) 80° d) 110° f) 342°

2. Write the matrix of the reflection in the line $y = (\tan \phi)x$ for the following values of ϕ, giving entries correct to 2 decimal places.
   a) 20° b) 45° c) 100°

3. a) Describe the similarities and differences in your answers to 1a) and 2a).
   b) By calculating the determinant of each of these two matrices, show how you can distinguish between a rotation and a reflection.

4. Repeat question 3 by comparing your answers to 1e) and 2c).

5. Write the matrix of the rotation through the given angle in each case, giving exact answers. (Use the trigonometric tables on page 543.)
   a) 45° b) 60° c) 120°

6. Write the matrix of the reflection in the line $y = (\tan \phi)x$ for the following values of ϕ, giving exact answers. (Use the trigonometric tables on page 543.)
   a) 22.5° b) 60° c) 150°

7. Given that R_θ represents a counterclockwise rotation about O of $\theta°$, compare R_{300} and R_{-60}. Explain.

8. Given that M_ϕ represents a reflection in the line $y = (\tan \phi°)x$, compare M_{30} and M_{210}. Explain.

9. Calculate the determinant of each of the following matrices. Hence describe the area scale factor and the orientation of the associated transformations.

   the identity transformation $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   the null transformation $0_{2\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

10. The following are the matrices whose effect you described in questions 6 and 7 of 7.3 Exercises. Calculate the determinant in each case, to describe the area scale factor and orientation of each transformation.

    $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $F = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

    $B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$ $G = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

    $C = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ $H = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$

    $D = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

    $E = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ $J = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$

11. Find k if the determinant of $\begin{bmatrix} 2 & 3 \\ 1 & k \end{bmatrix}$ equals
    a) 5 b) −1 c) 0

12. The transformation matrix $S = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, where k is any real number, is a horizontal shear of factor k.
    Calculate det(S). Hence, describe the area scale factor and the orientation of a figure transformed by a shear.

13. Given $M = \begin{bmatrix} -k & 1 \\ k^2 & -k \end{bmatrix}$, find det($M$). What is the area of a figure transformed by matrix M?

14. Sketch on a grid the right triangle whose vertices are O, $P(3,0)$, $Q(0,2)$.

a) Calculate the area of the triangle OPQ.

b) Transform O, P, Q into their images O', P', Q' by the matrix $M = \begin{bmatrix} -4 & 3 \\ 5 & -2 \end{bmatrix}$

c) Calculate the area of $O'P'Q'$.

d) Compare the orientation of OPQ and $O'P'Q'$.

15. Use the information of question 14 for the following.

a) By calculating the dot product $\overrightarrow{OP}\cdot\overrightarrow{OQ}$, prove that the angle POQ is 90°.

b) Calculate the angle $P'O'Q'$ by finding the dot product $\overrightarrow{O'P'}\cdot\overrightarrow{O'Q'}$.

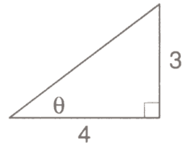
c) Are right angles preserved under linear transformations?

16. a) Show that the *clockwise* rotation through α about O is represented by the matrix

$$C = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

b) Verify that $\det(C) = 1$.

17. In the right triangle shown, $\tan\theta = \frac{3}{4}$.



Give all your answers to the following in fractional form.

a) Calculate $\sin\theta$ and $\cos\theta$.

b) Write the matrix of the rotation through angle θ.

c) Write the matrix of the reflection in the line $y = \left(\tan \frac{1}{2}\theta\right)x$.

18. Describe the following transformations.

$$K = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} \quad L = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$M = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & -\frac{4}{5} \end{bmatrix} \quad N = \begin{bmatrix} -\frac{5}{13} & \frac{12}{13} \\ -\frac{12}{13} & -\frac{5}{13} \end{bmatrix}$$

7.5 Inverse Transformations

So far you have learned how to carry out various linear transformations by using their matrices.

Is it possible to find a transformation that returns the plane to its original status after it has been transformed? That is, if $\vec{v} \xrightarrow{M} \vec{v}'$, can a transformation M^{-1} be found such that $\vec{v}' \xrightarrow{M^{-1}} \vec{v}$? Does a transformation always have an inverse? If an inverse exists, what does its matrix look like?
In this section, these questions will be investigated.
First observe the following examples.

Example 1 Given $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$, try to find a matrix D to reverse the effect of A.

Solution

Consider the vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ and its image under A, $\vec{v}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$

Then $A\vec{v} = \vec{v}'$.
Now if D exists, it will transform $\vec{v}'$ back to $\vec{v}$,
thus $D\vec{v}' = \vec{v}$ or $\vec{v} = D\vec{v}'$.
Thus, if you can find $\begin{bmatrix} x \\ y \end{bmatrix}$ in terms of $\begin{bmatrix} x' \\ y' \end{bmatrix}$,
and express your result as a product $D\begin{bmatrix} x' \\ y' \end{bmatrix}$,
you will have the answer.

Now $A\vec{v} = \vec{v}' \Rightarrow \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\Rightarrow \begin{cases} 3x - y = x' & ① \\ x + y = y' & ② \end{cases}$$

Eliminating y: ① + ② gives $4x = x' + y'$ so $x = \frac{1}{4}x' + \frac{1}{4}y'$

Substituting into ② gives $\frac{1}{4}x' + \frac{1}{4}y' + y = y'$ or $y = -\frac{1}{4}x' + \frac{3}{4}y'$

Thus $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$. Hence the matrix $D = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$ ■

D is called the **inverse** of A and is written A^{-1}.
Notice that $\det(A) = (3)(1) - (-1)(1) = 4$, and that the inverse can also be written $A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$; $\det(A^{-1}) = \frac{1}{4}$.

Observing this form of A^{-1} demonstrates the following facts, that you will prove later.

1. The numbers along the *leading diagonal* $\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$ of A are *exchanged* in A^{-1}.
2. The numbers along the *second diagonal* $\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$ of A are *multiplied by* (-1) in A^{-1}.
3. The *scalar coefficient* of A^{-1} is $\dfrac{1}{\det(A)}$. Also, $\det(A^{-1}) = \dfrac{1}{\det(A)}$

Example 2

Given the matrix $B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$, try to find a matrix B^{-1} that reverses the effect of B.

Solution

Proceeding as in Example 1, try to obtain $\begin{bmatrix} x \\ y \end{bmatrix}$ in terms of $\begin{bmatrix} x' \\ y' \end{bmatrix}$

$$B\vec{v} = \vec{v}' \Rightarrow \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2x + 4y = x' & ① \\ x + 2y = y' & ② \end{cases}$$

Eliminating y: ① $-\ 2 \times$ ② gives $0x = x' - 2y'$.
But x' and y' might be any numbers so that, in general, $x' \neq 2y'$. Hence no value exists for x. Thus no matrix B^{-1} exists. ■

In this case, matrix B is called **non-invertible**, or **singular**.
Its transformation is also called singular.

Note: The determinant of B is $(2)(2) - (4)(1) = 0$, that is $\det(B) = 0$.

FORMULA

The matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse M^{-1} if and only if $\det(M) \neq 0$;

then the inverse of M is $M^{-1} = \dfrac{1}{\det(M)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

If $\det(M) = 0$ the matrix M (and its transformation) are known as singular; no inverse exists.

You will have an opportunity to derive the general formula for M^{-1} in the exercises.

Example 3 Find the inverse, if it exists, of each of the matrices

a) $A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$ b) $B = \begin{bmatrix} \frac{1}{2} & -1 \\ -3 & 6 \end{bmatrix}$

Solution a) $\det(A) = (3)(6) - (5)(4) = -2 \neq 0.$
Thus A has an inverse A^{-1}.

By the formula, $A^{-1} = \frac{1}{-2}\begin{bmatrix} 6 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -3 & \frac{5}{2} \\ 2 & -\frac{3}{2} \end{bmatrix}.$

b) $\det(B) = \left(\frac{1}{2}\right)(6) - (-1)(-3) = 0.$

Thus, B is a singular matrix. No inverse for B exists. ■

Geometric Significance of Singular Transformations

The previous discussion leads to the fact that a matrix M is singular, that is, non-invertible, if its determinant is zero.

Since the area scale factor of a linear transformation of matrix M is $|\det(M)|$, *the area of any figure transformed by a singular matrix is zero.*

Look back at the examples of common transformations in section 7.3 (pages 304–305). The following transformations have zero determinants.

P_x (projection onto the x-axis)

$0_{2\times2}$ (null transformation)

You can see from the diagrams that both these transformations 'squash' the plane onto a single line or a single point. $\mathbb{V}_2$ is said to 'lose some dimension' by a singular transformation. When $\mathbb{V}_2$ loses some dimension by a transformation, that transformation cannot be reversed.

SUMMARY

$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse M^{-1} if and only if $\det(M) \neq 0$;

then the inverse of M is $M^{-1} = \frac{1}{\det(M)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

If $\det(M) = 0$ the matrix M (and its transformation) are known as singular; no inverse exists.

The area of any plane figure transformed by a singular 2×2 matrix is zero.

7.5 Exercises

1. Which of the following are singular?

$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $H = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ $J = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix}$

$D = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ $K = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{bmatrix}$

$E = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

$F = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ $L = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$

$G = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

2. Find the inverse of each of the invertible matrices of question 1.

3. By finding the images of the points O, $P(1,0)$, $Q(1,1)$, $R(0,1)$, sketch the effect on the unit square of each of the inverses of the transformations of question 1 (when they exist).

4. Calculate the area scale factor and describe the orientation for each of the inverses of the transformations of question 1 (when they exist).

5. a) Which of the matrices in question 1 describe rotations?
   b) Conjecture a formula for the inverse of the rotation matrix

$$R_0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

6. a) Write the matrix R of a rotation, counterclockwise about O, of $\theta°$.
   b) Find R^{-1}.
   c) Explain your answer to b) in terms of a rotation about O of $-\theta°$. (That is, a *clockwise* rotation of $\theta°$.)

7. Find the value of k if the matrix $M = \begin{bmatrix} 5 & -2 \\ k & -1 \end{bmatrix}$ is singular.

8. Find the inverse of the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Explain your answer.

9. Find the inverse of $S = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (the horizontal shear of factor 2). Describe this new transformation.

10. Consider the matrix $M = \begin{bmatrix} 7 & 10 \\ 2 & 3 \end{bmatrix}$.
    a) Find the inverse matrix M^{-1}.
    b) Find the image $\vec{v}'$ of the vector $\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ under M.
    c) Verify that $M^{-1}\vec{v}' = \vec{v}$.

11. Repeat question 10 with matrix $M = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix}$

12. By following the procedure of Example 1 in the text, page 315, prove that the inverse of the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $M^{-1} = \frac{1}{\det(M)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
What happens if $\det(M) = 0$?

13. M is an invertible matrix. Calculate $\det(M) \times \det(M^{-1})$.

14. Use $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $\det(M) \neq 0$, to prove the following statement.

If $\vec{v}$ is any vector, then $M\vec{v} = \vec{0} \Rightarrow \vec{v} = \vec{0}$. (This proves that *only* the zero vector is transformed by an invertible matrix into the zero vector.)

15. The matrix $S = \begin{bmatrix} 3 & -1 \\ 12 & -4 \end{bmatrix}$ maps the entire plane onto a single straight line.
    a) What is the image under S of a general vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$?
    b) What is the Cartesian equation of the image line?
    c) Does S have an inverse? Explain.

7.6 Composition of Transformations and Multiplication of Matrices

To 'compose' two transformations means to make one of the transformations follow the other. You then have a "composite transformation". The same term is used for functions. In this first example, you will observe the effect of the composition of two transformations. Later you will see how this composition is linked to the multiplication of matrices.

Example 1 Consider the two transformations whose matrices are

$P = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

and the general vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

a) Describe each transformation.

b) Calculate $P\vec{v} = \vec{v_1}$, then $Q(P\vec{v}) = \vec{v_2}$.

c) Calculate $Q\vec{v} = \vec{v'}$, then $P(Q\vec{v}) = \overrightarrow{v''}$.

d) Describe the transformations that take $\vec{v}$ to $\vec{v_2}$, and $\vec{v}$ to $\overrightarrow{v''}$.

Solution

a) Read the columns of P: $\vec{i} \rightarrow \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \vec{j} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Thus $\vec{i}$ goes to $-\vec{i}$, and $\vec{j}$ does not move.
Both $\vec{i}$ and $\vec{j}$ are reflected in the y-axis.

P therefore represents a reflection in the y-axis.

Read the columns of Q: $\vec{i} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{j} \rightarrow \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Thus $\vec{i}$ goes to $\vec{j}$, and $\vec{j}$ goes to $-\vec{i}$.
Both $\vec{i}$ and $\vec{j}$ are rotated counterclockwise through 90°.

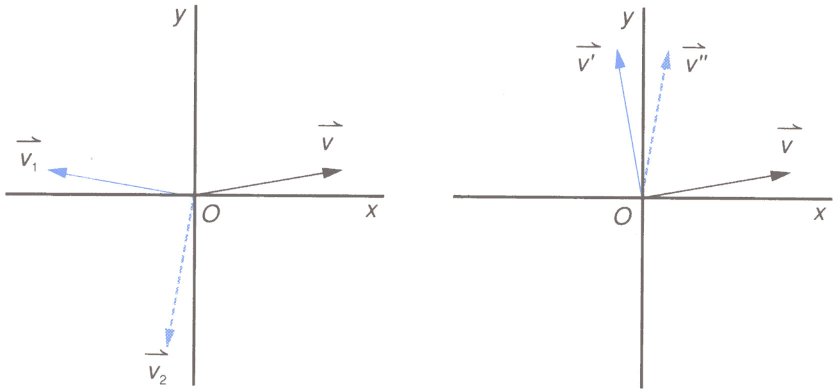
Q therefore represents a counterclockwise rotation about O through 90°. (This will be abbreviated to "rotation of 90°".)

b) $P\vec{v} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} = \vec{v_1}$

$Q(P\vec{v}) = Q(\vec{v_1}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix} = \vec{v_2}$

c) $Q\vec{v} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} = \vec{v'}$

$P(Q\vec{v}) = P(\vec{v'}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} = \overrightarrow{v''}$



d) Observe the figure.

In b), $\vec{v}$ is reflected in the y-axis to $\vec{v}_1$.
$\vec{v}_1$ is then rotated by 90° to $\vec{v}_2$.
Thus, $\vec{v}$ is reflected in the y-axis THEN rotated by 90° to $\vec{v}_2$.
In c), $\vec{v}$ is rotated by 90° to $\vec{v}'$.
$\vec{v}'$ is then reflected in the y-axis to $\vec{v}''$.
Thus, $\vec{v}$ is rotated by 90° THEN reflected in the y-axis to $\vec{v}''$. ■

Note 1 Doing P first, then Q, gives a different result from doing Q first, then P.
2 Writing $Q(P\vec{v})$ means that P acts first, then Q.

This example shows that the composition of transformations is non-commutative. (Beware! "Non-commutative" does not mean "never commutative". There are examples of transformations that do commute, as you shall see later.)

If you now compute a 'matrix product' QP by the same dot product 'row-column' process you used to multiply a 2×2 matrix and a 2×1 matrix, you have

$$QP = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \overset{\text{(row 1)}\cdot\text{(col 1)}}{(0)(-1) + (-1)(0)} & \overset{\text{(row 1)}\cdot\text{(col 2)}}{(0)(0) + (-1)(1)} \\ \underset{\text{(row 2)}\cdot\text{(col 1)}}{(1)(-1) + (0)(0)} & \underset{\text{(row 2)}\cdot\text{(col 2)}}{(1)(0) + (0)(1)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = R$$

You obtain a 2×2 matrix R that will transform $\vec{v}$ *directly* to $\vec{v}_2$, as follows.

$$R\vec{v} = (QP)\vec{v} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix} = \vec{v}_2$$

Note: The calculations $Q(P\vec{v})$ and $(QP)\vec{v}$ lead to the same result.
That is, $Q(P\vec{v}) = (QP)\vec{v}$

This is one of the manifestations of the associativity of matrix multiplication, which will be discussed later in this section.

Similarly, PQ can be computed.

$$PQ = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = S$$

S will transform $\vec{v}$ *directly* to $\vec{v''}$, as follows.

$$S\vec{v} = (PQ)\vec{v} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} = \vec{v''}$$

Recall that $\vec{v_2} \neq \vec{v''}$, that is, the image by QP was different from the image by PQ.
Hence it is not surprising that the matrix $QP \neq$ matrix PQ.

The Multiplication of 2 × 2 Matrices

DEFINITION

Given a matrix $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and a matrix $Q = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$, then the product PQ is defined as follows.

$$PQ = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

(row 1 of P)·(column 1 of Q) → $aw + by$ (row 2 of P)·(column 1 of Q) → $ax + bz$

$$= \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

(row 1 of P)·(column 2 of Q) → $cw + dy$ (row 2 of P)·(column 2 of Q) → $cx + dz$

The matrix PQ represents a transformation that is the result of doing *Q first, then P*.

Matrix multiplication is non-commutative.

However, there are cases of matrices that commute.

For example, recall that $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix.

The transformation I leaves the plane unchanged.
Thus, if A is any matrix, then $AI = IA = A$.
Hence, I commutes with any matrix.
Also, since any invertible matrix M has an inverse M^{-1} that 'undoes' the effect of M,
$MM^{-1} = M^{-1}M = I$.

Hence, a matrix commutes with its inverse.

You will have an opportunity to verify these properties in the exercises.

Example 2 Consider the matrices $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

a) Describe the transformations whose matrices are M and D.
b) Calculate the product MD, and describe its transformation.
c) Calculate the product DM, and describe its transformation.
d) State whether or not the transformations M and D commute.

Solution

a) For $M: \vec{i} \to \vec{j}$ and $\vec{j} \to \vec{i}$.
Thus M is a reflection in the line $y = x$.

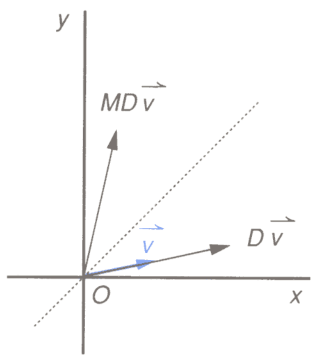
For D: $\vec{i} \to 2\vec{i}$ and $\vec{j} \to 2\vec{j}$.
Thus D is a dilatation of factor 2.

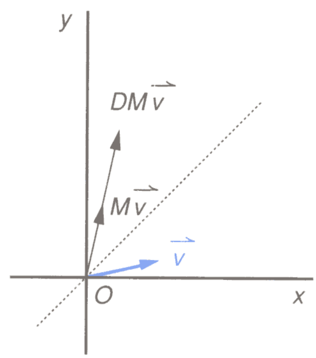
b) $MD = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

The plane is *dilated first, then reflected.*

c) $DM = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

The plane is *reflected first, then dilated.*





d) The order in which the transformations M and D are performed does *not* alter the final image. These transformations, as well as their matrices, *do* commute. ■

The Associativity of Matrix Multiplication

The associative property for matrix multiplication holds. That is, given any 2×2 matrices A, B, and C,
$(AB)C = A(BC)$.

You will be asked to prove this property in the exercises.

Multiplication of Matrices of Different Dimension

Recall that for any 2×2 matrices P and Q, and any column vector (that is, a 2×1 matrix) $\vec{v}$,

PROPERTY $Q(P\vec{v}) = (QP)\vec{v}$.

This seems to indicate that matrix multiplication is associative even in cases when the matrices are not all of equal dimension.

Indeed, it is valid to 'multiply' any two matrices for which the dot product 'row-column' process is possible. This will be true whenever the number of elements in the *rows* of the first matrix is equal to the number of elements in the *columns* of the second matrix. This leads to the following general results, which will not be proven.

PROPERTY

The product AB of two matrices is defined only if A and B have dimension as follows.
A has dimension $m \times n$, B has dimension $n \times p$.
Then the product AB has dimension $m \times p$.

Whenever the product of matrices is defined, the associative property holds.

The Determinant of a Composite Transformation

PROPERTY

The determinant of a product of matrices is equal to the product of the determinants, that is, given matrices P and Q,
$\det(PQ) = \det(P) \times \det(Q)$

In the exercises, you will have an opportunity to prove this property. An important consequence of this property is demonstrated in the following example.

Example 3 Calculate the area scale factor of AB where

$$A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -4 & 3 \\ 5 & 1 \end{bmatrix}$$

Solution Note that $\det(A) = (2)(-2) - (1)(-4) = 0$.
Hence, $\det(A) \times \det(B) = 0$, or $\det(AB) = 0$.
The transformation AB is singular.
Thus the area scale factor is zero. ■

Example 3 shows that
a singular transformation composed with any other transformation gives a singular transformation.

SUMMARY

Given a matrix $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and a matrix $Q = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$, then the product PQ is defined as follows.

$$PQ = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

The matrix PQ represents a transformation that is the result of doing *Q first, then P*.
(This composition of transformations is non-commutative.)

In general, $QP \neq PQ$ (Matrix multiplication is non-commutative.)

$(PQ)R = P(QR)$ (Matrix multiplication is associative.)

$\det(PQ) = \det(P) \times \det(Q)$

7.6 Exercises

1. Given
$L = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, M = \begin{bmatrix} 4 & -1 \\ 5 & 7 \end{bmatrix}, N = \begin{bmatrix} 6 & 5 \\ -2 & 0 \end{bmatrix},$
calculate
LM, ML, LN, NL, MN, NM.

The following matrices are to be used in answering questions 2–10.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$E = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

$G = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$J = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix}$

$K = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix}$

2. List which of the transformations defined by the above matrices fit the following.
   a) singular
   b) identity
   c) dilatation
   d) shear
   e) rotation
   f) area scale factor = 1

3. a) Calculate the products *IB, IC, IK*.
   b) Explain your results.

4. a) Calculate the products *CE* and *EC*.
   b) Show the effect of *CE* and of *EC* on the unit square, and describe each of these composite transformations.
   c) Calculate the area scale factor of *CE* and of *EC*.

5. a) Calculate the products *BJ* and *JB*.
   b) Describe each composite transformation. What is special about these?

6. a) Calculate the product *JK* and *KJ*.
   b) Describe each composite transformation. What is special about these?

7. a) Calculate the products *AK* and *GK*.
   b) Calculate the area scale factor in each case. What is special about these composite transformations?

8. Given a matrix *M*, the notation
$M^n = \underbrace{M \times M \times M \times \ldots\ldots \times M}_{n \text{ times}}$
   a) Calculate I^2 and A^2.
   b) Explain your results.

9. a) Calculate B^2 and E^2.
   b) Show the effect of B^2 and E^2 on the unit square, and describe each composite transformation.

10. a) Write *J* using exact values (see page 543). Calculate J^2, J^3, J^6.
    b) Explain your results.

11. $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called
the 2×2 identity matrix, or unit matrix.
    a) Show that for any vector $\vec{v}$, $I\vec{v} = \vec{v}$.
    b) Show that, for any 2×2 matrix *A*, $AI = IA = A$.

12. Consider the matrix $M = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$
    a) Find the inverse matrix M^{-1}.
    b) Calculate MM^{-1} and $M^{-1}M$.
    c) Draw conclusions.

13. Given $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and its inverse
$M^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$,
show that MM^{-1} and $M^{-1}M = I$.
(*I* is the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)

14. Consider the matrices $A = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix}$ and $B = \frac{1}{17}\begin{bmatrix} 2 & 1 \\ -5 & 6 \end{bmatrix}$
By calculating AB and BA, show that A and B are inverse matrices.

15. Repeat question 14 for $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$

16. Given three matrices
$K = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, L = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, M = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$

a) Calculate the following products.
$KL \quad LM \quad (KL)M \quad K(LM)$

b) Hence show that matrix multiplication is associative.

17. a) Using the matrices of question 16, calculate the following.
$\det(K)$; $\det(L)$; $\det(KL)$.

b) Hence show that
$\det(KL) = \det(K) \times \det(L)$

18. Using the matrix K of question 16, where $\det(K) \neq 0$,

a) calculate K^{-1},

b) show that $KK^{-1} = I$ and $K^{-1}K = I$.

c) Do the matrices K and K^{-1} commute?

19. A transformation has matrix $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

a) Show the effect of M on the unit square, and describe M.

b) According to your description, what would happen if M operated twice? three times? four times?

c) Confirm your answer to b) by calculating M^2, M^3, and M^4.

20. Repeat question 19 with the matrix
$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

21. The matrix $T = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

represents a reflection in the line $y = (\tan \theta)x$.

a) What would be the effect of applying T twice?

b) Confirm your answer to a) by finding T^2.

22. The matrix $R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

represents a counterclockwise rotation of angle θ about O.

a) What would be the effect of applying R twice?

b) By calculating R^2, obtain expressions for $\cos 2\theta$ and $\sin 2\theta$ in terms of $\cos \theta$ and $\sin \theta$.

23. Given the matrices

$P = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$ and

$Q = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

a) Describe each transformation, and hence describe the composite transformation PQ.

b) Confirm your answer by calculating the product PQ.

c) What is the inverse of matrix P? of matrix Q?

24. $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

a) Calculate AB and BA.

b) Calculate the inverse matrices A^{-1} and B^{-1}.

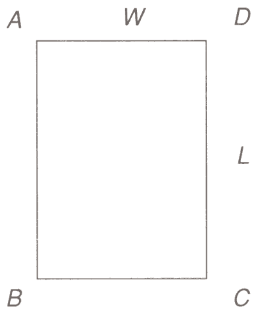
c) Calculate $A^{-1}B^{-1}$.

d) Calculate $(A^{-1}B^{-1})(AB)$ and $(A^{-1}B^{-1})(BA)$.

e) What is the inverse of $(A^{-1}B^{-1})$?

Similarity and Folding

The *golden ratio*, or *golden section*, is a well-known ratio that occurs in more than one branch of mathematics. It has also been used traditionally in architecture as the 'perfect rectangle'.



A rectangle *ABCD* is said to be 'golden' if its sides form the golden ratio, that is, if

$$\frac{AB}{AD} = \frac{AD}{AB - AD}$$

or $\frac{L}{W} = \frac{W}{L - W}$, where the length $AB = L$, and the width $AD = W$.

Thus, $L(L - W) = W^2$ or $L^2 - LW - W^2 = 0$,

which leads by the quadratic formula to $\frac{L}{W} = \frac{1 + \sqrt{5}}{2} = 1.618\ldots$

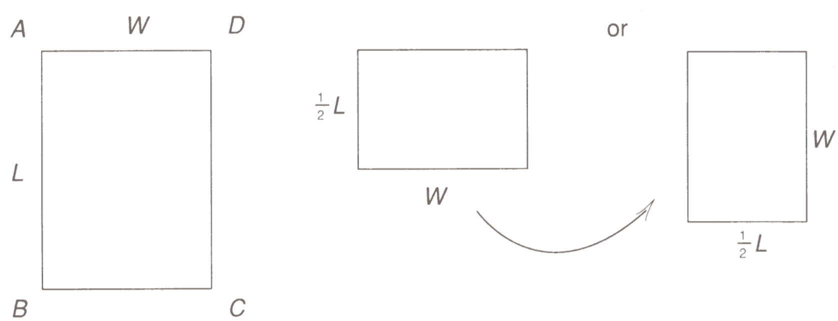
A less well known shape is the following, called the *folding section*.

Consider a sheet of paper of length L and width W.

If the sheet is folded once, its new dimensions will be $\frac{L}{2}$ and W.

If the half-sheet is to have the same shape as the original, then

$\frac{L}{W} = \frac{W}{\frac{1}{2}L}$ or $L^2 = 2W^2$ or $\frac{L}{W} = \sqrt{2} = 1.414\ldots$



Notice that when the sheet is folded again and again, the same shape is always retained. This idea was adopted to create the 'A' series of international paper sheet formats, as follows.

The base sheet size, called A0, has area 1 m^2 or 10 000 cm^2.

Thus $(W)(W\sqrt{2}) = 10\,000$

$$\Rightarrow \qquad W^2 = \frac{10\,000}{\sqrt{2}}$$

$$\Rightarrow \qquad W = \frac{100}{\sqrt[4]{2}} \doteq 84.089\ldots \text{ cm}$$

This leads to the following 'A*n*' sizes in cm, where *n* represents the number of folds.

A0	84.089	×	118.921
A1	59.46	×	84.089
A2	42.04	×	59.46
A3	29.7	×	42.0
A4	21.0	×	29.7
A5	14.8	×	21.0
A6	10.5	×	14.8
A7	7.4	×	10.5
	etc...		

This series of formats has been adopted by most countries.

The most frequently used paper size is A4. Note that any subsequent size can be obtained merely by folding.

Thus, if envelopes are manufactured in sizes marginally larger than these, any paper size can be put into any envelope merely by folding it the required number of times.

7.7 Properties of Matrix Multiplication and Matrix Equations

At this point, a multiplicative algebra of matrices has been established alongside the additive algebra described in section 7.1.
The set S of **invertible** 2×2 matrices together with the operation of matrix multiplication is said to form a **non-commutative group.**

The following properties hold.
M1. S is closed under multiplication: $M, N \in S \Rightarrow MN \in S$
M2. Multiplication is associative: $L(MN) = (LM)N$
M3. There exists $I \in S$ such that for all $M \in S$, $IM = MI = M$
M4. If $M \in S$, then there exists $M^{-1} \in S$ such that $MM^{-1} = M^{-1}M = I$

There remains only one property to be proven to allow you to solve matrix equations in a manner similar to that which you use to solve ordinary algebraic equations in $\mathbb{R}$.

The Distributivity of Matrix Multiplication over Matrix Addition

PROPERTY

Given any 2×2 matrices L, M and N, then
$$L(M + N) = LM + LN$$
and $(M + N)L = ML + NL$
You will have an opportunity to prove this property in the exercises.

Notice that the order of the letters is crucial. Since matrix multiplication is not commutative, it is *not valid* to replace, say, LM by ML.

You must continually be aware of the non-commutativity of matrix multiplication when working with matrices. With this proviso, you can solve matrix equations. In the first example, you will find the inverse of a composite transformation.

Example 1

Given matrices A and B, find the inverse of the product AB.

Solution

Let the image of $\vec{v}$ under AB be $\vec{v'}$.
To find $(AB)^{-1}$, you must express $\vec{v}$ in terms of $\vec{v'}$.

Now	$(AB)\vec{v} = \vec{v'}$	
thus	$(A^{-1})(AB)\vec{v} = (A^{-1})\vec{v'}$	
therefore	$(A^{-1}A)(B)\vec{v} = A^{-1}\vec{v'}$	property M2
or	$(I)(B)\vec{v} = A^{-1}\vec{v'}$	property M4
or	$B\vec{v} = A^{-1}\vec{v'}$	property M3
thus	$(B^{-1})(B\vec{v}) = (B^{-1})(A^{-1})\vec{v'}$	
therefore	$(B^{-1}B)(\vec{v}) = (B^{-1}A^{-1})\vec{v'}$	property M2
so	$I\vec{v} = (B^{-1}A^{-1})\vec{v'}$	property M4
and finally	$\vec{v} = (B^{-1}A^{-1})\vec{v'}$	

PROPERTY

Hence, the inverse of (AB) is $(B^{-1}A^{-1})$. ■

Using Matrix Equations to Solve Linear Systems

If M is an invertible matrix, $M\vec{v} = \vec{u}$

$$M^{-1}(M\vec{v}) = M^{-1}\vec{u}$$
$$(M^{-1}M)\vec{v} = M^{-1}\vec{u}$$
$$I\vec{v} = M^{-1}\vec{u}$$
$$\vec{v} = M^{-1}\vec{u}$$

Hence, solving $M\vec{v} = \vec{u}$ yields $\vec{v} = M^{-1}\vec{u}$.

Thus, if $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\vec{u} = \begin{bmatrix} p \\ q \end{bmatrix}$,

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$ or $\begin{cases} ax + by = p \\ cx + dy = q \end{cases}$ which yields $\begin{bmatrix} x \\ y \end{bmatrix} = \dfrac{1}{\det(M)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix}$ or

FORMULAS

$$x = \frac{dp - bq}{ad - bc} \text{ and } y = \frac{-cp + aq}{ad - bc}, \quad \text{or} \quad x = \frac{pd - bq}{ad - bc} \text{ and } y = \frac{aq - pc}{ad - bc}$$

These results are known as **Cramer's rule** for the solution of a system of two first-degree equations in two unknowns.

Example 2

Solve the system $2x - 3y = 7$
$4x - y = 1$

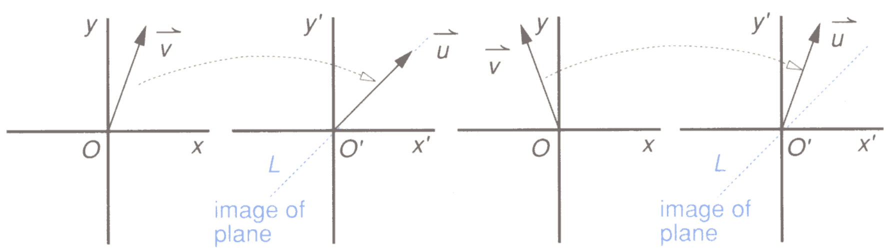
Solution

Substituting directly into the formulas,

$$x = \frac{(7)(-1) - (-3)(1)}{(2)(-1) - (-3)(4)} = -0.4, \quad y = \frac{(2)(1) - (7)(4)}{(2)(-1) - (-3)(4)} = -2.6 \quad ■$$

However, if M is *not* invertible, then Cramer's rule cannot be used. Recall that the plane will lose some dimension under the effect of a singular transformation. The plane will be 'squashed' into at most a single line, say L. Since the origin O never moves under a linear transformation, L will contain O.

Thus, if M is singular, there will be two possibilities when $M\vec{v} = \vec{u}$.



EITHER $\vec{u}$ is the position vector of a point of L. In that case, solutions for v' must exist.

OR no point of L has u' as position vector. In that case, no solutions for $\vec{v}$ are possible.

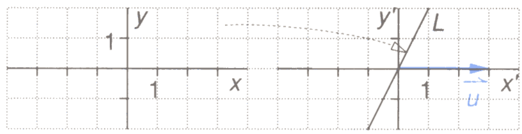
Example 3 Given $M = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, find the vectors $\vec{v}$ that satisfy the equation $M\vec{v} = \vec{u}$ if

a) $\vec{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ b) $\vec{u} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

Solution Notice that $\det(M) = (1)(4) - (2)(2) = 0$, so M is singular. The effect of M on a general vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ is

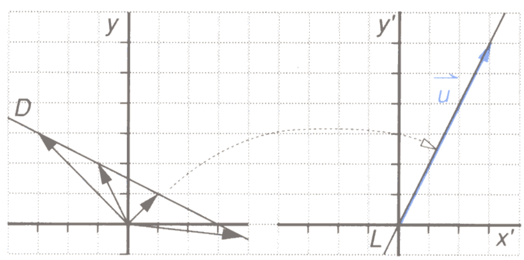
$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + 4y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2(x + 2y) \end{bmatrix} = \begin{bmatrix} a \\ 2a \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ where } a = x + 2y.$$

Thus the entire plane is squashed onto a line L, through the origin, of direction vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.



a) Since no a exists such that $a\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, the point whose position vector is $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ is not on L. Thus the transformation $M\vec{v} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ is impossible. No vector $\vec{v}$ can be found.

b) Since $a\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ for $a = 3$, the point whose position vector is $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ is on L. Thus, solutions exist for $\vec{v}$ given $M\vec{v} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$. Since $a = x + 2y$ and $a = 3$, then $x + 2y = 3$. Hence all the position vectors of points of the line D with equation $x + 2y = 3$ are mapped onto $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$



■

A vector equation of the line D can be found by introducing a parameter k, as follows. Let $x = k$. Then, from $x + 2y = 3$, $y = \dfrac{-k + 3}{2}$

$$\text{Thus } \vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ \dfrac{-k + 3}{2} \end{bmatrix} = \begin{bmatrix} 0 + k \\ \dfrac{3}{2} - \dfrac{k}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{3}{2} \end{bmatrix} + k\begin{bmatrix} 1 \\ -\dfrac{1}{2} \end{bmatrix}$$

Or, with the notation of chapter 5, $\overrightarrow{(x,y)} = \overrightarrow{\left(0,\frac{3}{2}\right)} + k\overrightarrow{\left(1,-\frac{1}{2}\right)}$ which represents the line through the point $\left(0,\frac{3}{2}\right)$, with direction vector $\overrightarrow{\left(1,-\frac{1}{2}\right)}$.

7.7 Exercises

1. $L = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $M = \begin{bmatrix} 5 & -4 \\ -1 & 1 \end{bmatrix}$

   I is the identity matrix.

   a) Calculate L^{-1} and M^{-1}.
   b) Calculate LM, $L^{-1}M^{-1}$ and $M^{-1}L^{-1}$.
   c) What is the inverse of LM?
   d) Verify that $(LM)(M^{-1}L^{-1}) = I$ and $(LM)(L^{-1}M^{-1}) \neq I$.

2. Given $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
   Show that $B^{-1}A^{-1} = A^{-1}B^{-1}$ in this case. Explain.

3. Solve the following systems by writing each system as a matrix equation.

   a) $5x - 6y = -1$
      $-3x + 4y = 2$
   b) $2x - 7y = 4$
      $4x - 14y = -6$
   c) $2x - 7y = 4$
      $4x - 14y = 8$
   d) $x + 2y = 5$
      $3x + 6y = 8$
   e) $-x + 4y = 7$
      $2x - 8y = -14$
   f) $2x + 3y = 6$
      $5x - y = -1$

4. Solve the systems of question 3 by using Cramer's rule, where possible.

5. Given $M = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$, find the vectors $\vec{v}$ that satisfy the equation $M\vec{v} = \vec{u}$ in the following cases.

   a) $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
   b) $\vec{u} = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$
   c) $\vec{u} = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$
   d) $\vec{u} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

6. Show that the matrix $M = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ maps all the points of the line $2x + y - 2 = 0$ to the point $P(2,4)$.

7. Given three matrices
   $K = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $L = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, $M = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$

   a) Calculate
      $L + M$, KL, KM, $K(L + M)$,
      LK, MK, $(L + M)K$.
   b) Hence show that matrix multiplication is distributive over matrix addition. That is, show that $K(L + M) = KL + KM$ and $(L + M)K = LK + MK$.

8. a) Calculate the product $\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix}$
   b) If A and B are two matrices such that $AB = 0_{2\times2}$, is it necessarily true that $A = 0_{2\times2}$ or $B = 0_{2\times2}$? Explain.

9. Three non-zero matrices P, Q and R are such that $PQ = RQ$. Is it necessarily true that $P = R$? Discuss this in the two cases
   a) Q is invertible   b) Q is singular.

10. Show that, for any 2×2 matrices A and B, $(A + B)^2 = A^2 + AB + BA + B^2$.
    Is it possible to simplify the expression on the right side?

11. The matrix T is said to be a square root of the matrix A, if $T^2 = A$.
    Find two different square roots of
    $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

12. Matrix A is such that $A - A^2 = I$, where I is the unit matrix.
    a) Prove that A is invertible.
    b) Prove that $A^3 = -I$.
    c) If X is a matrix such that $AX = I + A$, find the real numbers p and q such that $X = pI + qA$.

13. a) If $P = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$ and $Q = \begin{bmatrix} a & b \\ 4b & 3b + a \end{bmatrix}$, $a, b \in \mathbb{R}$, prove that $PQ = QP$.
    b) It is also known that for non-zero vectors $\vec{u}$ and $\vec{v}$, $P\vec{u} = \vec{u}$ and $P\vec{v} = 6\vec{v}$. Find the matrix Q such that $PQ = QP$, $Q\vec{u} = -\vec{u}$, and $Q\vec{v} = 4\vec{v}$.

14. a) State the matrix R that rotates the plane counterclockwise through an angle θ, and the matrix M that reflects the plane in the line $y = x$.

b) Solve for θ the equation $RM = MR$, $0° \leqslant \theta < 360°$.

15. An **orthogonal matrix** is one whose columns represent perpendicular unit vectors. Show that the following are orthogonal.

a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b) $\begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

c) the matrix of counterclockwise rotation through an angle θ

d) the matrix of reflection in the line $y = (\tan \alpha)x$.

16. Given $R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ and $M = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}$, show that the following are orthogonal matrices. (See question 15.)

a) R^2

b) M^2

c) RM

(Use the formulas on page 542.)

17. $S = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ is known as a **symmetric matrix.**

a) Verify that $\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$ is a symmetric matrix.

b) Calculate S^2, and show that it is symmetric also.

c) Calculate S^3, and show that it is symmetric also.

18. $A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 5 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 8 & -19 \\ 1 & -4 & 10 \\ 1 & -3 & 7 \end{bmatrix}$

a) Show that $DA = AD = I$, where

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(I is the 3×3 identity matrix. Thus, you have shown here that A and D are inverse 3×3 matrices.)

b) By writing the following system of equations as a matrix equation $A\vec{v} = \vec{b}$, solve the system for x, y and z.

$$\begin{aligned} 2x + y + 4z &= 2 \\ 3x + 5y + z &= 1 \\ x + 2y &= -5 \end{aligned}$$

19. To find the invariant lines of a transformation M, you look for vectors $\vec{v}$ whose images under M are *collinear* with $\vec{v}$. That is, you look for real numbers k and non-zero vectors $\vec{v}$ that satisfy $M\vec{v} = k\vec{v}$.

a) Given $M = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, show that $M\vec{v} = k\vec{v}$ will have non-zero solutions for x and y if and only if $k = 4$ or $k = -1$.

(4 and -1 are known as the **characteristic values** of the matrix M.)

b) If $k = 4$, show that $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ satisfies the equation $M\vec{v} = k\vec{u}$ and if $k = -1$, show that $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ satisfies the equation $M\vec{v} = k\vec{u}$.

(These are known as **characteristic vectors** of the matrix M.)

c) Hence find the Cartesian equations of two lines through O that are invariant under the matrix M.

In Search of Invariant Lines and Characteristic Vectors

Invariant Lines

The algebraic properties of matrices will also allow you to discover whether or not a transformation has any **invariant lines.**

Recall that when a matrix transforms a plane, the origin does not move. For some transformations, there may be entire lines that do not move. Such a line, called an invariant line, must therefore contain the origin.

A line L through the origin is completely determined by any direction vector $k\vec{v}$, where $\vec{v} \neq 0$ is any vector parallel to this line.

Now a line through (0,0) with direction vector $\vec{v}$ is invariant under a transformation M if $\vec{v}$ maps into some vector parallel to itself. In that case, $M\vec{v} = k\vec{v}$ for some $k \in \mathbb{R}$.

An example should help you to understand the general case.

Example 1 Find the invariant lines of the transformation $M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Solution If a line of direction vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ is invariant, the image of $\vec{v}$ is parallel to $\vec{v}$. Hence $M\vec{v} = k\vec{v}$ for some $k \in \mathbb{R}$.

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = k\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2x + y \\ x + 2y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$$

$$\begin{cases} (2 - k)x + y = 0 \\ x + (2 - k)y = 0 \end{cases}$$

$$\begin{bmatrix} 2 - k & 1 \\ 1 & 2 - k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad ①$$ in matrix form

or $C\vec{v} = \vec{0}$.

An invertible matrix sends only $\vec{0}$ to $\vec{0}$. Thus, matrix C must be singular. That is, $\det(C) = 0$.

$$\det(C) = (2 - k)(2 - k) - (1)(1) = 0$$
$$4 - 4k + k^2 - 1 = 0$$
$$k^2 - 4k + 3 = 0$$
$$(k - 1)(k - 3) = 0$$
$$k = 3 \text{ or } k = 1.$$

Thus, the direction vectors $\vec{v}$ of invariant lines are obtained for $k = 3$ or $k = 1$ (the numbers 3 and 1 are called the **characteristic values** of M).

When $k = 3$ $\quad \begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -x + y = 0 \\ x - y = 0 \end{cases}$$

and both these equations lead to $y = x$.
Thus one invariant line has equation $y = x$.

When $k = 1$ $\quad \begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x + y = 0 \\ x + y = 0 \end{cases}$$

and these equations are equivalent to $y = -x$.
Thus the other invariant line has equation $y = -x$. ■

Invariant Lines in the General Case—Characteristic Vectors

$\vec{v}$ will be a direction vector of a line that is invariant under the transformation M if $\vec{v}$ is mapped into a vector parallel to $\vec{v}$.

Hence $\quad M\vec{v} = k\vec{v}$ for some $k \in \mathbb{R}$.

Thus $\quad M\vec{v} = k(I\vec{v})$ since $I\vec{v} = \vec{v}$.

$$M\vec{v} - (kI)\vec{v} = \vec{0}$$

$$(M - kI)\vec{v} = \vec{0} \quad ②$$

Since $(M - kI)$ sends a non-zero vector to $\vec{0}$, this matrix *must be singular*.
Thus, $\det(M - kI) = 0$.

Now $M - kI = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} a-k & b \\ c & d-k \end{bmatrix}$

Thus $\det(M - kI) = (a - k)(d - k) - bc = 0 \quad ③$

Equation ③, a quadratic equation in k, is called the **characteristic equation** of matrix M.
The values of k that are the roots of equation ③ are the **characteristic values** of matrix M.

If the characteristic values are real, the direction vectors of the invariant lines can be obtained by substituting the characteristic values into equation ②. These vectors, defining invariant lines for M, are the **characteristic vectors** of matrix M.

If the characteristic values are *not* real, then the transformation has no invariant lines.

Note: Characteristic values and characteristic vectors are sometimes known as **eigenvalues** and **eigenvectors** (from the German "eigen" meaning "proper").

Example 2 Find the invariant lines of the transformation $R = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

Solution The characteristic equation of the matrix R is $(a - k)(d - k) - bc = 0$, where $a = 3, b = -4, c = 4, d = 3$.

Therefore $(3 - k)(3 - k) - (-4)(4) = 0$

$$9 - 6k + k^2 + 16 = 0$$

$$k^2 - 6k + 25 = 0.$$

The discriminant for this quadratic equation is $(-6)^2 - 4(1)(25) = -64 < 0$. Hence, the roots of this equation are not real. Thus, the transformation R has no invariant lines. ■

Note: R could be written $5\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

Thus, R represents a rotation through θ, where $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{4}{5}}{\frac{3}{5}} = \dfrac{4}{3}$, together with a dilatation of factor 5. Since every line in the plane is rotated by θ (approximately 53°), no line can be invariant.

Activities

1. a) Find the characteristic values of $M = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$
   b) Find the characteristic vectors of M (if they exist).
   c) Find the equations of the invariant lines of transformation M (if they exist).
2. Repeat question 1 for the following: $M = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$, and $M = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$
3. a) Show that matrix $M = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ has the characteristic equation $k^2 - 3k - 4 = 0$.
   b) Show that the matrix M satisfies its own characteristic equation, that is, show that $M^2 - 3M - 4I = 0_{2\times 2}$.
   (This is known as the **Cayley-Hamilton theorem.**)
4. $S = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ is known as a *symmetric matrix.*
   a) Prove that the characteristic values of a symmetric matrix are always real.
   b) Find a condition on a, b, h so that the characteristic values of S are equal.
5. A symmetric matrix S (see activity 4) for which $a \neq b$ always has two distinct real characteristic values p and q, associated with characteristic vectors $\vec{u}$ and $\vec{v}$ as follows.

$$S\vec{u} = p\vec{u} \qquad \text{and} \qquad S\vec{v} = q\vec{v}$$

   a) Show that $(S\vec{u})\cdot\vec{v} = (S\vec{v})\cdot\vec{u}$
   b) Hence show that $(p\vec{u})\cdot\vec{v} = (q\vec{v})\cdot\vec{u}$
   c) Hence show that the characteristic vectors of a symmetric matrix must be orthogonal.

Summary

- A matrix with m rows and n columns is said to have dimension $m \times n$.
- The element in the ith row, jth column is represented by a_{ij}.
- Two matrices are equal if and only if all their corresponding elements are respectively equal.
- Two matrices *that have the same dimensions* are added by adding their corresponding elements.

(In the following, all matrices are 2×2 unless specified otherwise.)

- Matrices form a vector space $\mathbb{V}_{2\times2}$, that is, the following properties hold.

Matrix Addition

A1. $V_{2\times2}$ is closed under addition: $M, N \in \mathbb{V}_{2\times2}$ implies $M + N \in \mathbb{V}_{2\times2}$

A2. Addition is associative: $L + (M + N) = (L + M) + N$

A3. There is a $0_{2\times2} \in \mathbb{V}_{2\times2}$ such that for all $M \in \mathbb{V}_{2\times2}$, $M + 0_{2\times2} = M$

A4. If $M \in \mathbb{V}_{2\times2}$, then there exists $-M \in \mathbb{V}_{2\times2}$ such that $M + (-M) = 0_{2\times2}$

A5. Addition is commutative: $M + N = N + M$

(These properties mean that $\mathbb{V}_{2\times2}$ is a commutative group with respect to addition.)

Multiplication of a Matrix by a Scalar

M1. If $M \in \mathbb{V}_{2\times2}$, $k \in \mathbb{R}$, then $kM \in \mathbb{V}_{2\times2}$

M2. $(kp)M = k(pM)$, $k, p \in \mathbb{R}$

M3. $k(M + N) = kM + kN$

M4. $(k + p)M = kM + pM$

M5. There exists $1 \in \mathbb{R}$ such that $1M = M$

Linear Transformations

- A linear transformation T of a vector space $\mathbb{V}_2$ is such that

$$\left.\begin{array}{ll} 1. & T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \\ 2. & T(k\vec{v}) = k[T(\vec{v})] \end{array}\right\} \text{ where } \vec{u}, \vec{v} \in \mathbb{V}_2 \text{ and } k \in \mathbb{R}$$

- A general linear transformation of $\mathbb{V}_2$ has the following form.

$$T: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

or $\vec{v} \to M\vec{v} = \vec{v}'$,

where $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\vec{v}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$, and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- When a linear transformation acts on a plane,
  1. the origin does not move,
  2. parallelism is preserved.
- The matrix M of a linear transformation is such that its first column is the image of $\vec{i}$ by M and its second column is the image of $\vec{j}$ by M.
- The matrix of a counterclockwise rotation about O through an angle θ is

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- For various types of linear transformations, see the summary of section 7.3, pages 304–305.

Determinants

- If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant, $\det(M) = ad - bc$.
- If a figure of area S is transformed by matrix M, the area of the image figure is $|\det(M)|\, S$.
- If $\det(M) > 0$, the image retains the orientation of the original figure; if $\det(M) < 0$, the image acquires the opposite orientation.

Inverse Matrices

- $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $\det(M) \neq 0$; then the inverse of M is

$$M^{-1} = \frac{1}{\det(M)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- If $\det(M) = 0$ the matrix M (and its transformation) are known as singular; no inverse exists.
- $\mathbb{V}_2$ 'loses some dimension' by a singular transformation.

Multiplication of Matrices

- Given a matrix $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and a matrix $Q = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$, then the product PQ is defined as follows.

$$PQ = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

- The matrix PQ represents a transformation that is the result of doing *Q first, then P.*
- In general, $QP \neq PQ$. (Matrix multiplication is non-commutative.)
- $(PQ)R = P(QR)$. (Matrix multiplication is associative.)
- Matrix multiplication is distributive over addition: $L(M + N) = LM + LN$ and $(M + N)L = ML + NL$
- The inverse of (AB) is $(B^{-1}A^{-1})$.
- A singular transformation composed with any other transformation gives a singular transformation.

General Matrix Products

- The product AB of two matrices is defined only if A and B have dimension as follows.
  A has dimension $m \times n$, B has dimension $n \times p$.
  The product AB then has dimension $m \times p$.
- Whenever the product of matrices is defined, the product has the associative property.

Cramer's rule

- Cramer's rule for the solution of $\begin{cases} ax + by = p \\ cx + dy = q \end{cases}$

$$x = \frac{pd - bq}{ad - bc} \quad \text{and} \quad y = \frac{aq - pc}{ad - bc} \qquad (ad - bc \neq 0)$$

Inventory

1. In the matrix $A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 4 \end{bmatrix}$, the element a_{23} is ________.

2. A matrix that has p rows and q columns has dimension ________.

3. A square matrix of order n has ________ rows and ________ columns.

4. $\begin{bmatrix} 1 & a \\ b & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 3 & c \end{bmatrix} \Rightarrow a =$ ________, $b =$ ________, $c =$ ________.

5. If T is a linear transformation of $\mathbb{V}_2$, then $T(\vec{u} + \vec{v}) =$ ________ and $T(k\vec{v}) =$ ________.

6. Under a linear transformation, the origin moves to ________.

7. ________ is always preserved under a linear transformation.

8. The first column of the matrix M of a linear transformation is the image of ________ by ________.
   The second column is the ________.

9. θ is an angle such that $\cos\theta = 0.6$ and $\sin\theta = 0.8$. The matrix of the counterclockwise rotation about O through the angle θ is ________.

10. The matrix product AB represents a transformation that is the result of doing ________ first, then ________.

11. The composition of transformations is in general *non-*________.

12. If A has dimension $p \times q$, and B has dimension $r \times s$, then the matrix product AB is defined if and only if ________ = ________. In that case, the dimension of AB is ________.

13. The inverse of the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ exists if and only if ________.
    Then $M^{-1} =$ ________.

14. A matrix M represents a singular transformation if ________.

15. A singular transformation cannot be ________.

16. A singular transformation composed with any other transformation gives a ________.

17. The area scale factor of matrix M is ________.

18. A figure is transformed into a figure of the same orientation if ________; and into a figure of the opposite orientation if ________.

Review Exercises

1. Given
$A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 0 \\ -2 & -3 \end{bmatrix}$, $C = \begin{bmatrix} 5 & -1 \\ -4 & 7 \end{bmatrix}$
calculate the following.
a) $-1A$ d) $A + B + C$ g) $-B + 6C$
b) $3B$ e) $A - 4C$ h) $-4(2A)$
c) $5A - B$ f) $\frac{1}{2}(B + C)$ i) $-8A$

2. Using the matrices A, B, C given in question 1, calculate the matrix X in the following cases.
a) $3X = B$ c) $C + X = A$
b) $X - 2A = 0_{2\times2}$ d) $X + 2B = 5X - A$

3. Calculate the values of the variables in the following.
a) $\begin{bmatrix} 2 & x \\ 1 & y \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -8 \end{bmatrix}$
b) $\begin{bmatrix} -x & 2y \\ y & x \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

4. Find the images of the following vectors under $M = \begin{bmatrix} 1 & -2 \\ 5 & 2 \end{bmatrix}$
$\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$ $\vec{w} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$
Sketch as position vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ and their images $\vec{u'}$, $\vec{v'}$, $\vec{w'}$.

5. Write the matrix that corresponds to the given transformation in each case.
a) a dilatation of factor $\frac{1}{4}$
b) reflection in the y-axis
c) rotation through $-90°$
d) projection onto the x-axis
e) reflection in the line $y = x$

6. By finding the images of the points O, $P(1,0)$, $Q(1,1)$, $R(0,1)$ sketch the effect of each of the transformations of question 5 on the unit square.

7. The straight line L has vector equation $\vec{r} = \vec{r_0} + k\vec{m}$, where
$\vec{r} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\vec{r_0} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$, $\vec{m} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
a) Find the vector equation of the image of L under the transformation of matrix
$A = \begin{bmatrix} 2 & 5 \\ -1 & 0 \end{bmatrix}$
b) Graph L and its image on the same set of axes.

8. By reading the columns of the following matrices, describe the associated transformation in each case.
$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $D = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$
$B = \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}$ $E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
$C = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ $F = \begin{bmatrix} \cos 65° & -\sin 65° \\ \sin 65° & \cos 65° \end{bmatrix}$

9. By finding the images of the points O, $P(1,0)$, $Q(1,1)$, $R(0,1)$ sketch the effect of each of the transformations of question 8 on the unit square.

10. Use the matrices of question 8 to answer the following.
a) For each transformation, describe the orientation and the area of a transformed figure.
b) Which matrices are singular?
c) Find the inverse of each of the invertible matrices.

11. Use the matrices of question 8 to calculate the following.
a) AB c) DE
b) CD d) E^2

12. Write the matrix of the rotation counterclockwise about the origin through the given angle. In each case, give entries correct to 2 decimal places.
a) 50° b) 125° c) 180°

13. Write the matrix of the reflection in the line $y = (\tan \phi)x$ for the following values of ϕ, giving entries correct to 2 decimal places.
a) 70° **b)** 132° **c)** 90°

14. Given that R_θ represents a counterclockwise rotation (about the origin) of $\theta°$, compare R_{225} and R_{-135}. Explain.

15. Show that the counterclockwise rotation (about the origin) of $(180° - \alpha)$ is represented by the matrix
$$\begin{bmatrix} -\cos\alpha & -\sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$$

16. Consider the straight lines L_1 and L_2 with the following vector equations.
$$L_1: \vec{r} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + k\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad L_2: \vec{s} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} + p\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
a) Explain why the lines L_1 and L_2 are perpendicular, and graph them on the same set of axes.
b) Find the vector equations of the images of L_1 and L_2 under the transformation
$$M = \begin{bmatrix} 1 & 0 \\ 6 & -5 \end{bmatrix}$$
c) Are the images also perpendicular?

17. Describe the following transformations, where $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, and $x^2 + y^2 = 1$.
$$M = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \qquad N = \begin{bmatrix} x & y \\ y & -x \end{bmatrix}$$

18. a) Write the matrix M_θ of the reflection in the line $y = (\tan\theta)x$.
b) Find M_θ^{-1}.
c) Describe the transformation M_θ^{-1}.

19. Given $M = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$, describe the vectors $\vec{v}$ such that $M\vec{v} = \vec{0}$.

20. a) Find the inverse of each of the following matrices.
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$C = \begin{bmatrix} \cos 80° & \sin 80° \\ \sin 80° & -\cos 80° \end{bmatrix}$$
b) Explain your answers by describing the transformations involved.

21. Let M^{-1} be the inverse of a matrix M.
a) What is the effect of applying M then M^{-1}?
b) What is $M M^{-1}$?

22. $A = \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 7 & 3 \end{bmatrix}$
a) Calculate AB and BA.
b) Calculate $\det(A)$, $\det(B)$, $\det(AB)$ and $\det(AB)$.
c) Verify that $\det(AB) = \det(A)\det(B)$.

23. Given the 2×2 matrices A and B, discuss the statement $AB = 0_{2\times2} \Rightarrow BA = 0_{2\times2}$.

24. Consider a matrix M, and a non-zero vector $\vec{u}$. Discuss whether or not a vector $\vec{v}$ such that $M\vec{v} = \vec{u}$ can always be found in the following cases.
a) M is invertible.
b) M is singular.

25. Consider the matrix $M = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, where none of the elements a, b, or c are zero. It is required to find a non-zero matrix X such that $MX + XM = 0_{2\times2}$.

Find the general solution in terms of a parameter t.

26. Prove that the set S of invertible 2×2 matrices forms a non-commutative group by verifying each of the following properties.

M1. S is closed under multiplication: $M, N \in S \Rightarrow MN \in S$

M2. Multiplication is associative: $L(MN) = (LM)N$

M3. There exists $I \in S$ such that for all $M \in S$, $IM = MI = M$

M4. If $M \in S$, then there exists $M^{-1} \in S$ such that $MM^{-1} = M^{-1}M = I$

27. The matrix M and the vectors $\vec{v}$ and $\vec{w}$ are given by
$M = \begin{bmatrix} 2 & p \\ 4 & 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} q \\ 5 \end{bmatrix}$, where $p, q \in \mathbb{R}$.

a) Given that the matrix N is the inverse of M,
   i) write down N, and
   ii) state the value of p if N is singular.

b) Prove that $M(\vec{v} - \vec{w}) = M(\vec{v}) - M(\vec{w})$.

c) Given $M(\vec{v} - \vec{w}) = \vec{v} + \vec{w}$, calculate the values of p and q.

d) Given $\det(M^2) = 64$, calculate the values of p.

(86 SMS)

28. $\vec{u}$, $\vec{v}$ and $\vec{w}$ are three vectors in a two-dimensional rectangular Cartesian coordinate system with origin O. $\vec{i}$ and $\vec{j}$ are unit vectors in the direction of the coordinate axes. A, B and C are three points such that

$$\overrightarrow{OA} = \vec{u} = 2\vec{i} - \vec{j}$$
$$\overrightarrow{OB} = \vec{v} = \vec{i} + 3\vec{j}$$
$$\overrightarrow{OC} = \vec{w} = 7\vec{i} + \frac{9}{2}\vec{j}$$

a) Show A, B and C on a diagram.

b) Find the values of λ and μ such that $\vec{w} = \lambda\vec{u} + \mu\vec{v}$.

c) Prove that $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are perpendicular.

d) Find the coordinates of D so that $ABCD$ is a rectangle and determine the magnitudes of the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

e) The rectangle $ABCD$ is transformed to the quadrilateral $A'B'C'D'$ under the transformation with matrix $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.
   i) Calculate the coordinates of the points A', B', C' and D'.
   ii) Prove that $A'B'C'D'$ is a parallelogram.
   iii) Calculate the value of α, $\alpha \in \mathbb{R}$, such that

$$\overrightarrow{BB'} = \alpha\overrightarrow{BC'}.$$

(83 SMS)

29. Given that P^T denotes the transpose of the matrix P,* which one of the following statements concerning 2×2 matrices and their determinants may be false?

A. $\det 2P = 4 \det P$
B. $\det(PQ) = \det P \det Q$
C. $\det(P + Q) = \det P + \det Q$
D. $\det P^T = \det P$
E. $(PQ)^T = Q^T P^T$.

(83 H)

*see page 362

30. A linear transformation L is such that $L\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $L\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Find $L\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

(85 H)

31. Oxy is a 2-dimensional rectangular Cartesian coordinate system. Points of the system $P(x,y)$ are mapped onto points $P'(x',y')$ by a linear transformation T represented by the (2×2) matrix M, in such a way that the coordinates obey the relation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = M\begin{bmatrix} x \\ y \end{bmatrix},$$

where

$$M = \begin{bmatrix} 1.4 & -0.2 \\ 0.8 & 0.6 \end{bmatrix}.$$

a) Find the images under T of the points O, A, B, C whose coordinates are $(0,0)$, $(1,2)$, $(3,1)$ and $(2,-1)$ respectively.

b) Draw a sketch on squared paper showing the figure $OABC$ and its image $O'A'B'C'$.

c) Prove that all points on the line with equation

$$y = 2x$$

are invariant under T.

d) Describe fully the geometrical effect of the linear transformation T.

e) Determine the images of the points A and C under the linear transformation T^{-1}.

(88 S)