VECTORS, MATRICES and COMPLEX NUMBERS

with International Baccalaureate questions

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CHAPTER EIGHT

TRANSFORMATIONS OF CONICS

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ISBN of Gage Edition 0-7715-3503-1

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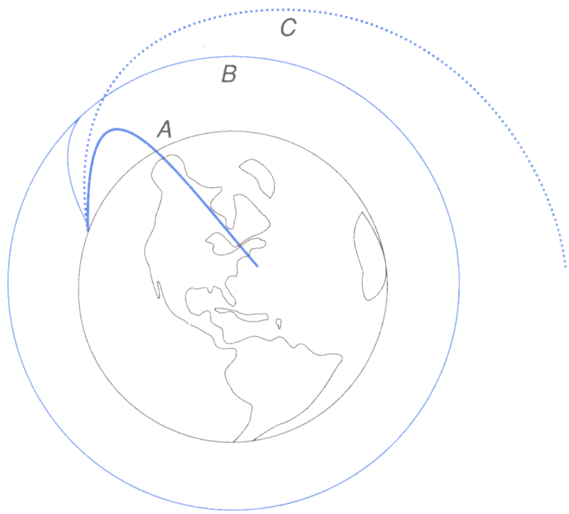
Transformations of Conics

Conics are all around you—the path of a ball thrown through the air; the shape of a cross section of the reflector used in a TV satellite dish or in an automobile headlight; the arch of a bridge; the path of the planets around the sun; the path of a space vehicle, etc.

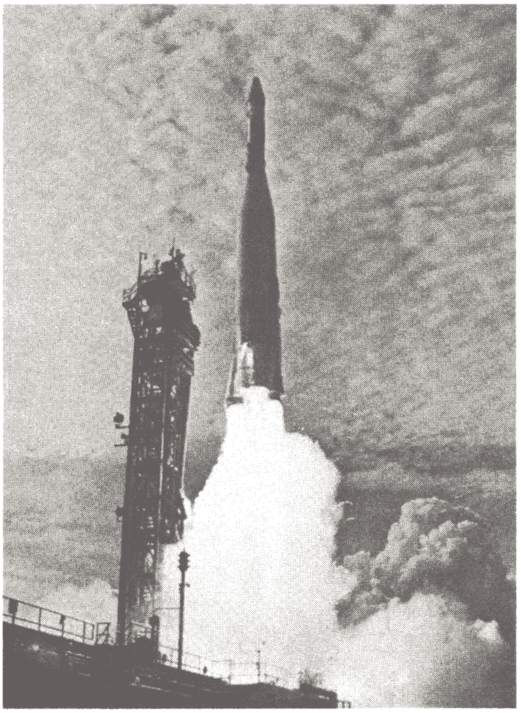
The diagram shows possible conical paths of a space vehicle launched from the surface of the earth.

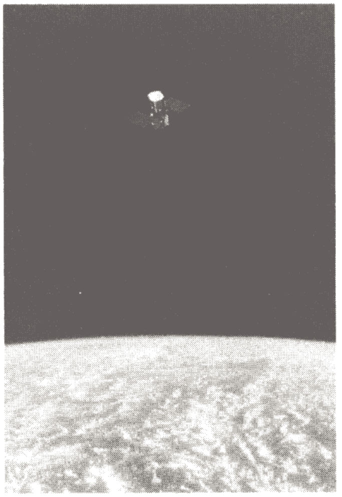


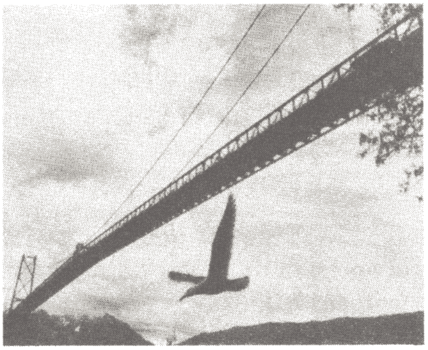
The parabolic path *A* is taken by a space vehicle that falls back to earth. The circular path *B* is followed by a space vehicle sent into orbit around the earth. Path *C*, which may be an ellipse, a hyperbola or a parabola, is that of a space vehicle with sufficient velocity to escape the gravity of the earth and go into orbit around the sun.

In actual space travel, any one of the conical paths *A*, *B*, or *C* may need to be *transformed* into another conical path by a translation, a rotation, a dilatation, or some combination of these transformations.

In this chapter you will study the transformation of conics under translations and rotations.

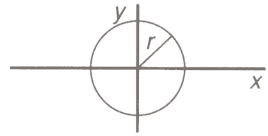
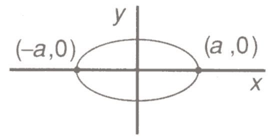
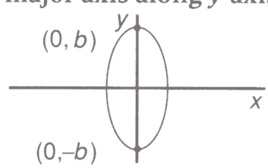
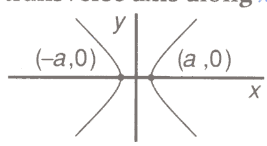
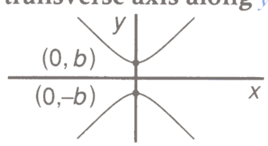
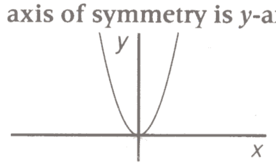
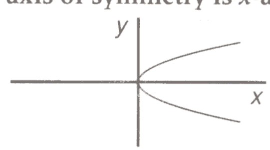






8.1 Conics in Standard Position

In previous grades you met the conic sections, namely, the circle, the ellipse, the hyperbola, and the parabola.
The following chart will remind you of some of the information that you learned about the conics.

conic	equation	centre	vertices	graph
circle	$x^2 + y^2 = r^2$	(0,0)	none	radius = r, $r > 0$ 
ellipse	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $a > b > 0$	(0,0)	$(a,0)(-a,0)$	major axis along x-axis 
	$b > a > 0$	(0,0)	$(0,b)(0,-b)$	major axis along y-axis 
hyperbola	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ $a > 0, b > 0$	(0,0)	$(a,0)(-a,0)$	transverse axis along x-axis 
	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$ $a > 0, b > 0$	(0,0)	$(0,b)(0,-b)$	transverse axis along y-axis 
parabola	$y = kx^2$ $k > 0$: opens up $k < 0$: opens down	none	(0,0)	axis of symmetry is y-axis 
	$x = ky^2$ $k > 0$: opens right $k < 0$: opens left	none	(0,0)	axis of symmetry is x-axis 

The circle, ellipses, and hyperbolas with equations given in the chart have their centres at the origin (0,0) and their major axis or transverse axis along the x-axis or y-axis. These conics are called **central conics**. The central conics and the parabolas with equations given in the chart (vertex at the origin (0,0) and axis of symmetry along the y-axis or x-axis) are said to be in **standard position**, and their equations in **standard form**.

Equations of conics do not have to be in standard form. They can also be in general form.

An equation of a conic written $ax^2 + by^2 + 2gx + 2fy + c = 0$ is in **general form**.

Some examples of standard form and general form follow.

conic	standard form	general form	a	b	g	f	c
ellipse	$\frac{x^2}{4} + \frac{y^2}{9} = 1$	$9x^2 + 4y^2 - 36 = 0$	9	4	0	0	−36
hyperbola	$\frac{x^2}{3} - \frac{y^2}{5} = 1$	$5x^2 - 3y^2 - 15 = 0$	5	−3	0	0	−15
ellipse	$\frac{x^2}{16} + \frac{y^2}{9} = 1$	$9x^2 + 16y^2 - 144 = 0$	9	16	0	0	−144
hyperbola	$\frac{x^2}{4} - \frac{y^2}{9} = -1$	$9x^2 - 4y^2 + 36 = 0$	9	−4	0	0	36
circle	$x^2 + y^2 = \frac{9}{4}$	$4x^2 + 4y^2 - 9 = 0$	4	4	0	0	−9
parabola	$y = 4x^2$	$4x^2 - y = 0$	4	0	0	$-\frac{1}{2}$	0
parabola	$x = -3y^2$	$3y^2 + x = 0$	0	3	$\frac{1}{2}$	0	0

Example Describe the graph of each of the following conics.

a) $9x^2 + 9y^2 = 16$

b) $16x^2 + 9y^2 = 144$

c) $25x^2 - 16y^2 = 400$

d) $4x^2 - y = 0$

Solution

a) $9x^2 + 9y^2 = 16$ or $x^2 + y^2 = \frac{16}{9}$ is an equation of a circle with centre (0,0) and radius $= \sqrt{\frac{16}{9}} = \frac{4}{3}$.

b) $16x^2 + 9y^2 = 144$ ①
can be divided by 144 to obtain

$$\frac{x^2}{9} + \frac{y^2}{16} = 1.$$

The conic is an ellipse with $a^2 = 9$, $a = 3$ and $b^2 = 16$, $b = 4$. Since $b > a$, the major axis is on the y-axis.

Alternatively, you can find the x-intercepts and y-intercepts and then sketch the graph of the ellipse.

x-intercepts: let $y = 0$ in ①	*y-intercepts*: let $x = 0$ in ①
$16x^2 = 144$	$9y^2 = 144$
$x^2 = 9$	$y^2 = 16$
$x = 3$ or $x = -3$	$y = 4$ or $y = -4$

The graph of $16x^2 + 9y^2 = 144$ is as shown below.

c) $25x^2 - 16y^2 = 400$ ②

Each side can be divided by 400 to obtain

$$\frac{x^2}{16} - \frac{y^2}{25} = 1.$$

The conic is a hyperbola with $a^2 = 16$, $a = 4$ and $b^2 = 25$, $b = 5$. Since this is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the transverse axis is on the x-axis.

Alternatively, you can find the x-intercepts and y-intercepts and then sketch the graph of the hyperbola.

x-intercepts: let $y = 0$ in ②	*y-intercepts*: let $x = 0$ in ②
$25x^2 = 400$	$-16y^2 = 400$
$x^2 = 16$	Since $y^2 > 0$ for $y \in \mathbb{R}$, no real value of
$x = 4$ or $x = -4$	y satisfies this equation.

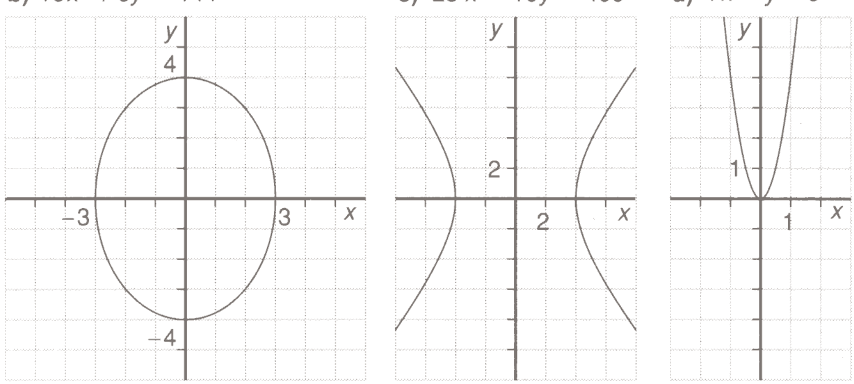
Hence, the x-intercepts are 4 and -4 but the hyperbola does not intersect the y-axis.

The graph of $25x^2 - 16y^2 = 400$ is as shown below.

d) $4x^2 - y = 0$ can be written $y = 4x^2$, which is an equation for a parabola with vertex at (0,0), axis of symmetry the y-axis, and opening upward.

The graph of $4x^2 - y = 0$ is as shown below.

b) $16x^2 + 9y^2 = 144$ **c)** $25x^2 - 16y^2 = 400$ **d)** $4x^2 - y = 0$



Note 1 Sketching a conic in this context means to find the x-intercepts and y-intercepts, if they exist, and to show the direction of opening for a hyperbola or a parabola.

2 If more points are desired, obtain points near the vertices. In part c) above, $x = 5$ gives $y = \pm 3.75$, and $x = -5$ also gives $y = \pm 3.75$, which produce the four points (5,3.75), (5,–3.75), (–5,3.75), and (–5,–3.75).

The general form of the equation of a conic is $ax^2 + by^2 + 2gx + 2fy + c = 0$.

(Do not confuse these a's and b's with those of, say, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The two sets are different.)

By comparing the standard form equation of a conic with the general form you can obtain some useful information.

circle

The equation $9x^2 + 9y^2 = 16$ from Example 1 can be rewritten

$$9x^2 + 9y^2 - 16 = 0.$$

Each term of the equation can be multiplied by any non-zero number k to obtain $9kx^2 + 9ky^2 - 16k = 0$.

Comparing this equation with $ax^2 + by^2 + 2gx + 2fy + c = 0$

gives $a = b = 9k$, $g = f = 0$, $c \neq 0$.

Note that $(a)(b) = (9k)(9k) = 81k^2$, a positive number.

You will learn that the important fact to remember about a circle is $ab > 0$, $a = b$.

ellipse

The equation $16x^2 + 9y^2 = 144$ from Example 1 can be rewritten

$$16x^2 + 9y^2 - 144 = 0.$$

Comparing this equation with $ax^2 + by^2 + 2gx + 2fy + c = 0$

gives $a = 16$, $b = 9$, $g = f = 0$, $c \neq 0$.

Note that $(a)(b) = (16)(9) = 144$, a positive number.

You will learn that the important fact to remember about an ellipse is $ab > 0$, $a \neq b$

hyperbola

The equation $25x^2 - 16y^2 = 400$ from Example 1 can be rewritten

$$25x^2 - 16y^2 - 400 = 0.$$

Comparing this equation with $ax^2 + by^2 + 2gx + 2fy + c = 0$

gives $a = 25$, $b = -16$, $g = f = 0$, $c \neq 0$.

Note that $(a)(b) = (25)(-16) = -400$, a negative number.

You will learn that the important fact to remember about a hyperbola is $ab < 0$.

parabola

The equation $4x^2 - y = 0$ from Example 1 is in general form.

Comparing this equation with $ax^2 + by^2 + 2gx + 2fy + c = 0$

gives $a = 4$, $b = 0$, $g = c = 0$, $f = -\frac{1}{2}$. Note that $(a)(b) = (4)(0) = 0$.

You will learn that the important fact to remember about a parabola is $ab = 0$.

You should check the truth of the following summary as you do Exercises 8.1 and 8.2.

SUMMARY

conic	$ax^2 + by^2 + 2gx + 2fy + c = 0$
circle	$ab > 0$, $a = b$
ellipse	$ab > 0$, $a \neq b$
hyperbola	$ab < 0$
parabola	$ab = 0$.

8.1 Exercises

1. Sketch the graph of each of the following conics.
   a) $x^2 + y^2 = 4$
   b) $x^2 + 9y^2 = 9$
   c) $4x^2 - y^2 = 4$
   d) $y = 4x^2$
   e) $x = 2y^2$

2. Sketch the graph of each of the following circles.
   a) $x^2 + y^2 - 16 = 0$
   b) $x^2 + y^2 - 9 = 0$
   c) $4x^2 + 4y^2 - 16 = 0$
   d) $3x^2 + 3y^2 - 15 = 0$
   e) $-2x^2 - 2y^2 + 8 = 0$
   f) $-5x^2 - 5y^2 + 10 = 0$

   Indicate the value of a, b, c, g, and f in the general equation for a conic $ax^2 + by^2 + 2gx + 2fy + c = 0$. In each case verify that $ab > 0$, $a = b$.

3. Sketch the graph of each of the following ellipses.
   a) $9x^2 + 16y^2 - 144 = 0$
   b) $25x^2 + 4y^2 - 100 = 0$
   c) $4x^2 + 25y^2 - 100 = 0$
   d) $-4x^2 - 9y^2 + 36 = 0$
   e) $-32x^2 - 18y^2 + 144 = 0$
   f) $-2x^2 - 3y^2 + 18 = 0$

   Indicate the value of a, b, c, g, and f in the general equation for a conic $ax^2 + by^2 + 2gx + 2fy + c = 0$. In each case verify that $ab > 0$, $a \neq b$.

4. Sketch the graph of each of the following hyperbolas.
   a) $9x^2 - 16y^2 - 144 = 0$
   b) $9x^2 - 16y^2 + 144 = 0$
   c) $4x^2 - y^2 + 4 = 0$
   d) $-8x^2 + 2y^2 + 8 = 0$
   e) $-x^2 + y^2 + 9 = 0$
   f) $3x^2 - 5y^2 - 30 = 0$

   Indicate the value of a, b, c, g, and f in the general equation for a conic $ax^2 + by^2 + 2gx + 2fy + c = 0$. In each case verify that $ab < 0$.

5. Sketch the graph of each of the following parabolas.
   a) $x^2 - y = 0$
   b) $4x^2 - y = 0$
   c) $3x^2 + y = 0$
   d) $-x^2 + 2y = 0$
   e) $4y^2 + x = 0$
   f) $5y^2 - x = 0$

   Indicate the value of a, b, c, g, and f in the general equation for a conic $ax^2 + by^2 + 2gx + 2fy + c = 0$. In each case verify that $ab = 0$.

6. Identify each of the following equations as representing a circle, an ellipse, a hyperbola or a parabola.
   a) $x^2 + y^2 = 25$
   b) $\frac{x^2}{9} - \frac{y^2}{4} = 1$
   c) $y = 9x^2$
   d) $4x^2 - 25y^2 + 100 = 0$
   e) $x^2 + y^2 - 16 = 0$
   f) $\frac{x^2}{9} + \frac{y^2}{4} = 1$
   g) $16x^2 - 9y^2 = 144$
   h) $3y^2 + x = 0$

   For each circle, state the coordinates of the centre and the length of the radius.
   For each ellipse, state the coordinates of the centre, the coordinates of the vertices and the name of the major axis.
   For each hyperbola, state the coordinates of the centre, the coordinates of the vertices and the name of the transverse axis.
   For each parabola, state the coordinates of the vertex and name the axis of symmetry.

7. For each of the following equations of conics, compare the equation with the general equation
   $ax^2 + by^2 + 2gx + 2fy + c = 0$
   and determine the values of a and b. Calculate the value of ab and use this value to determine the type of conic.
   a) $x^2 + 7y^2 - 8 = 0$
   b) $3x^2 - 5y^2 - 1 = 0$
   c) $4x^2 + 5y = 0$
   d) $8x^2 + 8y^2 - 5 = 0$
   e) $3y^2 - 4x = 0$
   f) $3x^2 - 16y^2 + 1 = 0$
   g) $5x^2 + 6y^2 - 2 = 0$
   h) $tx^2 + ty^2 - 14 = 0$, $t \neq 0$
   i) $tx^2 + my^2 - 14 = 0$, $t > 0$, $m < 0$
   j) $tx^2 + my^2 - 14 = 0$, $t \neq m$, $t > 0$, $m > 0$

8.2 Translations of Conics

In section 1.1 you observed the relationship between vectors and translations. Each vector $\vec{a} = \overrightarrow{(h,k)}$ defines the *translation* that maps each point P with coordinates (x,y) into the point P' with coordinates $(x + h, y + k)$, that is,

$$\text{point } P(x,y) \rightarrow \text{point } P'(x + h, y + k).$$

In earlier grades you learned that a translation is an **isometry**, that is, every line segment maps into a congruent line segment, which means that any figure is congruent to its image figure. A special property of a translation is that a line L and its image line L' are parallel.

In this section you will learn how the equation of a conic changes when a conic is translated from its standard position.

Example 1 Given the points $P(2,3)$, $Q(-1,5)$, and $R(0,-2)$

a) find the image of $\triangle PQR$ under the translation defined by the vector $\vec{a} = \overrightarrow{(4,-2)}$

b) sketch $\triangle PQR$ and its image $\triangle P'Q'R'$.

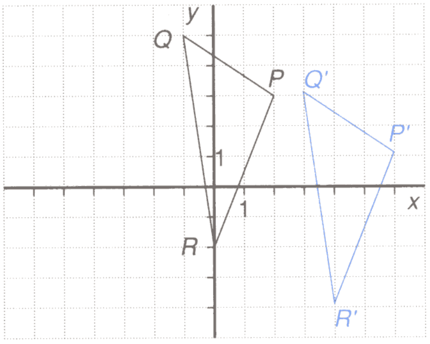
Solution For the translation defined by vector $\vec{a} = \overrightarrow{(4,-2)}$, the point (x,y) maps into the point $(x + 4, y - 2)$, that is,

$$(x,y) \rightarrow (x + 4, y - 2).$$

Thus,

$$P(2,3) \rightarrow P'(2 + 4, 3 - 2) = (6,1)$$
$$Q(-1,5) \rightarrow Q'(-1 + 4, 5 - 2) = (3,3)$$
$$R(0,-2) \rightarrow R'(0 + 4, -2 - 2) = (4,-4)$$

b) $\triangle PQR$ and $\triangle P'Q'R'$ are graphed as shown.



Note: $\triangle PQR \cong \triangle P'Q'R'$,
$PQ \parallel P'Q'$,
$PR \parallel P'R'$,
$QR \parallel Q'R'$.

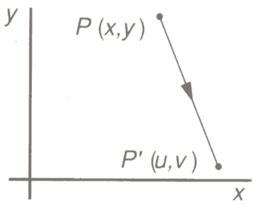
Example 2 Given the ellipse E: $9x^2 + 4y^2 = 36$.

a) Find, in general form, an equation for E', the image of E, under the translation $(x,y) \rightarrow (x + 2, y - 5)$.

b) Sketch a graph of the ellipse E and its image ellipse E'.

c) By comparing the given equation with the general equation $ax^2 + by^2 + 2gx + 2fy + c = 0$, find the values of a, b, c, g, f. Calculate ab for both the equation of E and the equation of E', and note the sign in each case.

Solution a) To avoid confusing the coordinates of a point P on the ellipse E and a point P' on the image ellipse E', a point P on E will be called (x,y) while a point P' on E' will be denoted (u,v).



Thus, under the translation, $P(x,y) \rightarrow P'(u,v)$

But under this translation $(x,y) \rightarrow (x + 2, y - 5) = (u,v)$

Therefore, $u = x + 2$ and $v = y - 5$ ①

To find the relationship between u and v on E' you must solve ① for x and y in terms of u and v, then substitute these values into the equation for E, namely, into $9x^2 + 4y^2 = 36$ ②

From ①, $x = u - 2$, and $y = v + 5$

Substituting into ② gives

$$9(u - 2)^2 + 4(v + 5)^2 = 36$$

or $9(u^2 - 4u + 4) + 4(v^2 + 10v + 25) - 36 = 0$

or $$E': 9u^2 + 4v^2 - 36u + 40v + 100 = 0.$$

Because it customary to write the equation of a conic using x and y you should replace u by x and v by y to obtain the following equation for E'

$$E': 9x^2 + 4y^2 - 36x + 40y + 100 = 0.$$

b) To sketch image ellipse E' you should find the images of the centre, and of the points of intersection of E: $9x^2 + 4y^2 = 36$ with the x-axis and y-axis.

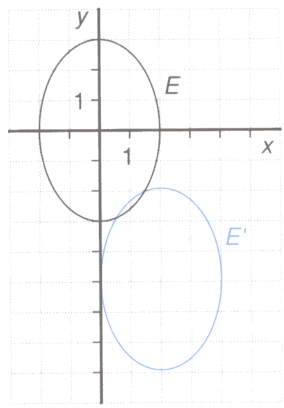
x-intercepts: let $y = 0$
$9x^2 + 0 = 36$
$x^2 = 4$
$x = 2$ or $x = -2$

y-intercepts: let $x = 0$
$0 + 4y^2 = 36$
$y^2 = 9$
$y = 3$ or $y = -3$

Under the translation $(x,y) \rightarrow (x + 2, y - 5)$

ellipse E		image ellipse E'
centre (0,0)	→	centre (2,−5)
(2,0)	→	(4,−5)
(−2,0)	→	(0,−5)
(0,3)	→	(2,−2)
(0,−3)	→	(2,−8)

The graphs of E and E' are as shown.



c) Comparing the general equation $ax^2 + by^2 + 2gx + 2fy + c = 0$ with each equation gives

for E: $9x^2 + 4y^2 - 36 = 0$

$a = 9$, $b = 4$,
$g = f = 0$,
$c = -36$
$ab = 36$ is positive

for E': $9x^2 + 4y^2 - 36x + 40y + 100 = 0$

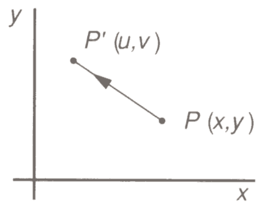
$a = 9$, $b = 4$,
$2g = -36$, so $g = -18$; $2f = 40$, so $f = 20$,
$c = 100$
$ab = 36$ is positive ■

Example 3 a) Given the hyperbola $H: x^2 - 16y^2 = 16$, find, in general form, an equation for H', the image of H, under the translation $(x,y) \to (x - 3, y + 2)$.

b) Sketch a graph of the hyperbola H and its image hyperbola H'.

c) By comparing the given equation with the general equation $ax^2 + by^2 + 2gx + 2fy + c = 0$ find the values of a, b, c, g, f. Calculate ab for both the equation of H and the equation of H', and note the sign in each case.

Solution a) As in Example 2, a point P on H will be called (x,y) while a point P' on H' will be denoted (u,v).



Thus, under the translation, $P(x,y) \to P'(u,v)$

But under this translation $(x,y) \to (x - 3, y + 2) = (u,v)$

Therefore, $u = x - 3$ and $v = y + 2$ ③

As in Example 2, solve ③ for x and y, then substitute into

$x^2 - 16y^2 = 16$ ④

From ③, $x = u + 3$, and $y = v - 2$

Substituting into ④ gives

$(u + 3)^2 - 16(v - 2)^2 = 16$

or $u^2 + 6u + 9 - 16(v^2 - 4v + 4) - 16 = 0$

or $H': u^2 - 16v^2 + 6u + 64v - 71 = 0$.

As in Example 2, replace u by x and v by y to obtain the following equation for H'.

$H': x^2 - 16y^2 + 6x + 64y - 71 = 0$.

b) To sketch image hyperbola H' you should find the images of the centre of $H: x^2 - 16y^2 = 16$ and of the points where H intersects the x-axis and the y-axis.

x-intercepts: let $y = 0$
$x^2 + 0 = 16$
$x = 4$ or $x = -4$

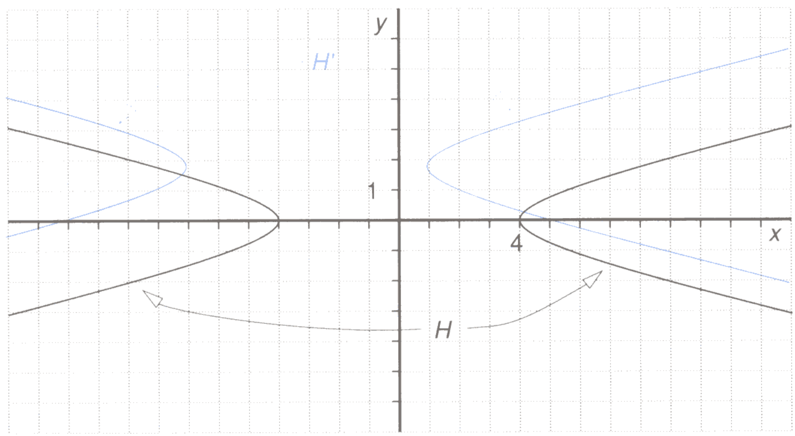
y-intercepts: let $x = 0$
$0 - 16y^2 = 16$
$-16y^2 = 16$ which has no solution in real numbers.

The x-intercepts are 4 and -4. The hyperbola does not intersect the y-axis. Hence, the hyperbola opens along the positive x-axis and the negative x-axis.

Under the translation $(x,y) \rightarrow (x - 3, y + 2)$

hyperbola H		image hyperbola H'
centre $(0,0)$	$\rightarrow$	centre $(-3,2)$
$(4,0)$	$\rightarrow$	$(1,2)$
$(-4,0)$	$\rightarrow$	$(-7,2)$

The graphs of H and H' are as shown.



c) Comparing each equation with the general equation $ax^2 + by^2 + 2gx + 2fy + c = 0$ gives

for H: $x^2 - 16y^2 - 16 = 0$

$a = 1, b = -16,$
$g = f = 0,$
$c = -16$
$ab = -16$ is negative

for H': $x^2 - 16y^2 + 6x + 64y - 71 = 0$

$a = 1, b = -16,$
$2g = 6$, so $g = 3$; $2f = 64$, so $f = 32$,
$c = 71$
$ab = -16$ is negative ■

From Example 2 and Example 3 you can see that, under a translation, the image of an ellipse or hyperbola in standard position has an equation of the form

$$ax^2 + by^2 + 2gx + 2fy + c = 0$$

Note: As you saw in Section 8.1, for an ellipse $ab > 0$ with $a \neq b$, and for a hyperbola $ab < 0$. In 8.2 Exercises, you will find $ab > 0$ with $a = b$ for a circle, and $ab = 0$ for a parabola.

8.2 Exercises

1–10

1. Find the image of the given point under the given translation.
   a) point (3,5)
   translation $(x,y) \to (x + 3, y + 6)$
   b) point (−2,4)
   translation $(x,y) \to (x - 1, y + 7)$
   c) point (8,0)
   translation $(x,y) \to (x - 3, y - 2)$
   d) point (−4,−5)
   translation $(x,y) \to (x + 4, y + 5)$

2. Given the ellipse E: $25x^2 + 9y^2 = 225$.
   a) Find, in general form, an equation for E', the image of E, under the translation $(x,y) \to (x + 3, y + 1)$.
   b) Sketch a graph of the ellipse E and its image ellipse E'.
   c) Find the values of a, b, c, g, f, and the sign of ab for both the equation of E and the equation of E'.

3. Repeat the previous question for the following ellipses and translations.
   a) ellipse $9x^2 + 16y^2 = 144$
   translation $(x,y) \to (x + 3, y + 6)$
   b) ellipse $x^2 + 4y^2 = 16$
   translation $(x,y) \to (x - 1, y + 7)$
   c) ellipse $8x^2 + 200y^2 = 1600$
   translation $(x,y) \to (x - 3, y - 2)$
   d) ellipse $4x^2 + 9y^2 = 36$
   translation $(x,y) \to (x + 4, y + 5)$

4. a) Given the hyperbola H: $x^2 - 16y^2 = 16$, find, in general form, an equation for H', the image of H, under the translation $(x,y) \to (x - 3, y + 2)$.
   b) Sketch a graph of the hyperbola H and its image hyperbola H'.
   c) Find the values of a, b, c, g, f, and the sign of ab for both the equation of H and the equation of H'.

5. Repeat the previous question for the following hyperbolas and translations.
   a) hyperbola $9x^2 - 16y^2 = 144$
   translation $(x,y) \to (x - 3, y + 5)$
   b) hyperbola $x^2 - 4y^2 = -16$
   translation $(x,y) \to (x + 1, y + 7)$
   c) hyperbola $8x^2 - 50y^2 = 800$
   translation $(x,y) \to (x - 3, y - 2)$
   d) hyperbola $x^2 - y^2 = 36$
   translation $(x,y) \to (x + 4, y - 3)$

6. Given the parabola P: $4x^2 - y = 0$.
   a) Find, in general form, an equation for P', the image of P, under the translation $(x,y) \to (x + 3, y + 1)$.
   b) Sketch a graph of the parabola P and its image parabola P'.
   c) Find the values of a, b, c, g, f, and the value of ab for both the equation of P and the equation of P'.

7. Repeat the previous question for the following parabolas and translations.
   a) parabola $x^2 - y = 0$
   translation $(x,y) \to (x - 1, y + 6)$
   b) parabola $4y^2 - x = 0$
   translation $(x,y) \to (x - 1, y - 3)$
   c) parabola $8y^2 + x = 0$
   translation $(x,y) \to (x + 2, y - 2)$
   d) parabola $4x^2 + y = 0$
   translation $(x,y) \to (x - 1, y + 3)$

8. Given the circle C: $x^2 + y^2 = 25$.
   a) Find, in general form, an equation for C', the image of C, under the translation $(x,y) \to (x - 3, y + 1)$.
   b) Sketch a graph of the circle C and its image circle C'.
   c) Find the values of a, b, c, g, f, and the sign of ab for both the equation of C and the equation of C'.

9. Repeat the previous question for the following circles and translations.
   a) circle $x^2 + y^2 = 144$
   translation $(x,y) \to (x + 3, y - 1)$
   b) circle $x^2 + y^2 = 25$
   translation $(x,y) \to (x - 5, y - 2)$
   c) circle $8x^2 + 8y^2 = 16$
   translation $(x,y) \to (x + 3, y - 2)$
   d) circle $-4x^2 - 4y^2 = -16$
   translation $(x,y) \to (x + 4, y + 5)$

10. Examine your results of questions 2–9 to show that for any particular conic the value of ab and hence the sign of ab is invariant under a translation. That is, observe the value of ab before and after translation for each conic to see the following are true.
for an ellipse, $ab > 0$ and $a \neq b$
for a hyperbola, $ab < 0$
for a parabola, $ab = 0$
for a circle, $ab > 0$ and $a = b$

11. Given the curve C: $4x^2 - y + 3 = 0$ and the translation $(x,y) \rightarrow (x,y - 3)$.
  a) Find an equation of its image under the given translation.
  b) Name the type of curve.
  c) Sketch the *image* curve C'. Then sketch the given curve C by using the inverse translation to the given translation.

12. Repeat the previous question for the following curves and translations.
  a) curve $x^2 + y^2 + 4x + 6y - 12 = 0$
  translation $(x,y) \rightarrow (x + 2,y + 3)$
  b) curve $2x^2 + 8x + y = 0$
  translation $(x,y) \rightarrow (x + 2,y - 8)$
  c) curve $9x^2 + 4y^2 + 72x + 24y + 36 = 0$
  translation $(x,y) \rightarrow (x + 4,y + 3)$
  d) curve $4x^2 - y^2 + 16x - 6y + 3 = 0$
  translation $(x,y) \rightarrow (x + 2,y + 3)$

13. a) Show by factoring, that the conic $4x^2 - y^2 = 0$ is a **degenerate conic** representing a pair of straight line.
  b) Find an equation of the image of this degenerate conic under the translation $(x,y) \rightarrow (x + 5,y + 2)$
  c) Graph the degenerate conic and its image under the translation.
  d) Why is this conic called a degenerate conic?

14. a) Find the equation of the image of the conic $ax^2 + by^2 + t = 0$ under the translation $(x,y) \rightarrow (x + h,y + k)$.
  b) Use your results of part a) to show that the sign of ab is invariant under a translation.

15. Find the translation under which the curve $x^2 + y^2 + 4x + 6y - 3 = 0$ maps into the circle $x^2 + y^2 = 16$.

16. Given the conic C: $ax^2 + by^2 + c = 0$, where $a, b \neq 0$;
and the translation $(x,y) \rightarrow (x + h,y + k)$,

show that the image of C under the translation is the conic
C': $Ax^2 + By^2 + 2Gx + 2Fy + C = 0$, where
$A = a$, $B = b$, $G = -2ah$, $F = -2bk$,
$C = ah^2 + bk^2 + c$.

17. a) The conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is translated under the translation $(x,y) \rightarrow (x + h,y + k)$.

  Show that the image conic has the equation

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1.$$

  b) Give the coordinates of the centre of the image curve.
  c) State the lengths of the major axis and the minor axis for the image conic.

18. a) The conic $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is translated under the translation $(x,y) \rightarrow (x + h,y + k)$. State an equation for the image curve in a form similar to that of question 17.
  b) Repeat part a) for the curve $y = ax^2$.

19. You will see in a later section that the equation $ax^2 + 2hxy + by^2 + c = 0$ defines an ellipse or a hyperbola for $a \neq 0$, $b \neq 0$, and $a \neq b$. Suppose the conic maps into $Ax^2 + 2Hxy + By^2 + C = 0$ under a translation. Prove the following
  a) $a + b = A + B$
  b) $b^2 - 4ac = B^2 - 4AC$

20. Prove that a translation is an isometry by showing that the length of the line segment joining $P_1(x_1,y_1)$ to $P_2(x_2,y_2)$ equals the length of the line segment joining the images of P_1 and P_2.

8.3 Translating Conics into Standard Position

The equation $ax^2 + by^2 + 2gx + 2fy + c = 0$ represents some conic.

If $g = 0$ and $f = 0$, the conic is a circle, an ellipse, or a hyperbola, *in standard position*: for example, $x^2 + y^2 - 9 = 0$ represents a circle, $4x^2 + 9y^2 - 36 = 0$ an ellipse, or $16x^2 - 25y^2 - 400 = 0$ a hyperbola.

If $a = f = c = 0$, or $b = g = c = 0$, the conic is a parabola *in standard position*: for example, $4y^2 + x = 0$ or $9x^2 - y = 0$.

For conics in standard position, graphs can be drawn as in section 8.1.

If the coefficients indicate that the conic is *not* in standard position, then the conic is graphed by first translating it into standard position, then using the *inverse* translation on the standard position graph.

You will remember from section 8.1 that terms in x and y in non-standard equations arise from perfect squares such as $(x + 3)^2 = x^2 + 6x + 9$. To reverse the process of translating back into standard position you will need to be able to *complete a square*.

Recall that $x^2 + mx$ becomes a perfect square by the addition of $\left(\frac{m}{2}\right)^2$.

Then the following is true.

$$x^2 + mx + \left(\frac{m}{2}\right)^2 = \left(x + \frac{m}{2}\right)^2$$

For example:

$$x^2 + 8x + \left(\frac{8}{2}\right)^2 = \left(x + \frac{8}{2}\right)^2 = (x + 4)^2$$

$$x^2 - 12x + \left(\frac{-12}{2}\right)^2 = \left(x + \frac{-12}{2}\right)^2 = (x - 6)^2$$

$$x^2 + 7x + \left(\frac{7}{2}\right)^2 = \left(x + \frac{7}{2}\right)^2$$

Recall that, in standard position or not, the following is the relationship among a, b, and the conic.

The Graph of $ax^2 + by^2 + 2gx + 2fy + c = 0$

circle	$ab > 0, a = b$
ellipse	$ab > 0, a \neq b$
hyperbola	$ab < 0$
parabola	$ab = 0$

The following example will demonstrate a method of graphing a conic with equation $ax^2 + by^2 + 2gx + 2fy + c = 0$, that is, when the conic is not in standard position.

Example 1 Given the conic C: $9x^2 + 4y^2 + 36x - 8y + 4 = 0$

a) Name the type of conic.

b) Determine the translation that changes the equation into standard form.

c) State an equation for the image curve C'.

d) Graph the image conic C' and given conic C.

Solution

a) Here $a = 9$, $b = 4$ and $ab = 36$.
Hence $ab > 0$ and $a \neq b$, and so the conic is an ellipse.

b) $g \neq 0, f \neq 0$, so the conic is not in standard position. The translation can be obtained by completing the squares of the terms in x and also of the terms in y.

$9x^2 + 4y^2 + 36x - 8y + 4 = 0$ can be written

$9(x^2 + 4x) + 4(y^2 - 2y) = -4$, or

$9(x^2 + 4x + 4) + 4(y^2 - 2y + 1) = -4 + 36 + 4$ add $9(\frac{4}{2})^2$ and $4(\frac{2}{2})^2$ to both sides of equation

or $9(x + 2)^2 + 4(y - 1)^2 = 36$

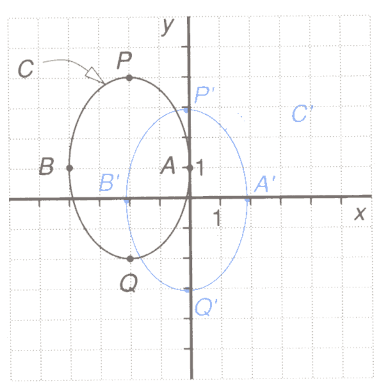
Replacing $x + 2$ by u and $y - 1$ by v (to avoid confusing a point on C and a point on C') gives an equation in standard form, namely
$9u^2 + 4v^2 = 36$
Thus, the translation moving $9x^2 + 4y^2 + 36x - 8y + 4 = 0$ into standard position is
$(x,y) \rightarrow (u,v)$ or
$(x,y) \rightarrow (x + 2, y - 1) = (u,v)$

c) The equation $9u^2 + 4v^2 = 36$ of the image conic C' should be rewritten using x and y, as $9x^2 + 4y^2 = 36$.

d) The curve $9x^2 + 4y^2 = 36$ intersects the x-axis at $A'(2,0)$ and $B'(-2,0)$ and the y-axis at $P'(0,3)$ and $Q'(0,-3)$. The graph of C' is shown below.

To obtain the equation of C', you have applied the translation $(x,y) \rightarrow (x + 2, y - 1)$. Hence, to graph C, you must apply the *inverse* translation $(x,y) \rightarrow (x - 2, y + 1)$ to the points A', B', P' and Q' on the image ellipse C'.

Under this inverse translation
$(x,y) \rightarrow (x - 2, y + 1)$
$A'(2,0) \rightarrow A(0,1)$
$B'(-2,0) \rightarrow B(-4,1)$
$P'(0,3) \rightarrow P(-2,4)$
$Q'(0,-3) \rightarrow Q(-2,-2)$
The graph of C is shown on the same axes as the graph of C'.



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Alternate Solution for parts b) and c) of Example 1

b) Let the required translation be $(x,y) \rightarrow (x + h, y + k)$.

If (x,y) is a point on C and (u,v) is the image point on C' then
$u = x + h \quad v = y + k$
Hence,
$x = u - h \quad y = v - k$
But $9x^2 + 4y^2 + 36x - 8y + 4 = 0$.
Substituting for x and y gives
$9(u - h)^2 + 4(v - k)^2 + 36(u - h) - 8(v - k) + 4 = 0$
or $9(u^2 - 2hu + h^2) + 4(v^2 - 2kv + k^2) + 36u - 36h - 8v + 8k + 4 = 0$
or $9u^2 + 4v^2 + u(-18h + 36) + v(-8k - 8) + 9h^2 + 4k^2 - 36h + 8k + 4 = 0$*

This equation will be in standard form if the coefficients of u and v are 0.
Hence, $-18h + 36 = 0$ and $-8k - 8 = 0$,
that is, $h = 2$ and $k = -1$.

c) To obtain the equation of E' you should substitute $h = 2$, $k = -1$ into equation* giving
$9u^2 + 4v^2 + u(0) + v(0) + 9(2)^2 + 4(-1)^2 - 36(2) + 8(-1) + 4 = 0$
or $9u^2 + 4v^2 - 36 = 0$.

Example 2

Given the conic C: $4x^2 - y^2 - 8x - 6y - 9 = 0$.

a) Name the type of conic.
b) Determine the translation that changes the equation into standard form.
c) State an equation for the image curve C'.
d) Graph the image conic C' and given conic C.

Solution

a) Here $a = 4$, $b = -1$ and $ab = -4$.
Hence $ab < 0$, so the conic is a hyperbola.

b) The translation can be obtained by completing the squares of the terms in x and also of the terms in y.
$4x^2 - y^2 - 8x - 6y - 9 = 0$ can be written
$4(x^2 - 2x) - (y^2 + 6y) = 9$ or
$4(x^2 - 2x + 1) - (y^2 + 6y + 9) = 9 + 4 - 9$ add $4(\frac{-2}{2})^2$ and $-(\frac{6}{2})^2$ to both sides of the equation
or $4(x - 1)^2 - (y + 3)^2 = 4$

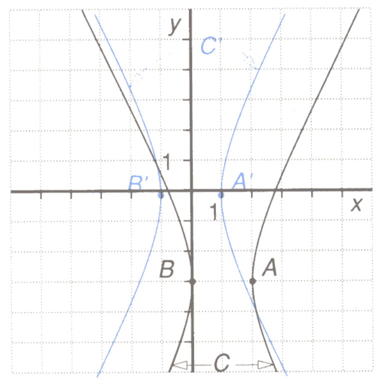
Replacing $x - 1$ by u and $y + 3$ by v (to avoid confusing a point on C and a point on C') gives an equation in standard form, namely
$4u^2 - v^2 = 4$
Thus, the translation moving $4x^2 - y^2 - 8x + 6y - 9 = 0$ into standard position is
$(x,y) \rightarrow (u,v)$ or
$(x,y) \rightarrow (x - 1, y + 3)$

c) The equation $4u^2 - v^2 = 4$ of the image conic C' should be rewritten using x and y, as $4x^2 - y^2 = 4$.

d) The curve $4x^2 - y^2 = 4$ intersects the x-axis at $A'(1,0)$ and $B'(-1,0)$ and does not intersect the y-axis. Since the hyperbola C' is in standard position and intersects the x-axis, the conic opens right along the positive x-axis and left along the negative y-axis. The graph of C' is as shown below.

To graph C, you must apply the *inverse* translation $(x,y) \rightarrow (x + 1, y - 3)$ to the points A' and B' on the image hyperbola C'.

Under this inverse translation
$A'(1,0) \rightarrow A(2,-3)$
$B'(-1,0) \rightarrow B(0,-3)$
The graph of C is shown on the same axes as the graph of C'.



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Example 3

Given the conic C: $3x^2 - 24x - y + 46 = 0$.

a) Name the type of conic.

b) Determine the translation that changes the equation into standard form.

Solution

a) Here $a = 3$, $b = 0$.
Since $ab = 0$, the conic is a parabola.

b) Note that there are no y^2 terms, so that the translation can be obtained by completing the square for the terms in x.

$3x^2 - 24x - y + 46 = 0$
can be written
$3(x^2 - 8x) = y - 46$
$3(x^2 - 8x + 16) = y - 46 + 48$ add $3(\frac{8}{2})^2$ to both sides of the equation
$3(x - 4)^2 = y + 2$
Replacing $x - 4$ by u and $y + 2$ by v gives an equation in standard form, namely $3u^2 = v$.
Thus the translation changing $3x^2 - 24x^2 - y + 46 = 0$ into standard form is
$(x,y) \rightarrow (u,v)$ or
$(x,y) \rightarrow (x - 4, y + 2)$ ■

8.3 Exercises

1. For each of the following conics, indicate whether the conic is an ellipse, a circle, a hyperbola or a parabola.
   a) $3x^2 + 2y^2 - 4x - 7y - 10 = 0$
   b) $5x^2 - 2y^2 - 3x - 2y - 3 = 0$
   c) $4x^2 - 3x - 7y - 1 = 0$
   d) $4x^2 + 4y^2 - 8x + 7y + 3 = 0$
   e) $x^2 + 4y^2 - 3x - 8y - 10 = 0$
   f) $x^2 - y^2 - 2x - 3 = 0$

2. Given the conic
   $C: 4x^2 + y^2 + 8x - 6y + 9 = 0$.
   a) Name the type of conic.
   b) Determine the translation that changes the equation into standard form.
   c) Find an equation for the image curve C'.
   d) Graph the image conic C' and given conic C.

3. Repeat the previous question for the following conics.
   a) $4x^2 - 9y^2 - 8x - 18y - 41 = 0$
   b) $x^2 + y^2 - 6x + 10y - 2 = 0$
   c) $4x^2 + 25y^2 - 48x + 200y + 444 = 0$
   d) $x^2 + 8x - 16y + 32 = 0$
   e) $9x^2 + y^2 + 18x - 6y + 9 = 0$
   f) $4x^2 - 4y^2 + 8x - 16y + 13 = 0$
   g) $2x^2 + 2y^2 + 8x + 12y + 1 = 0$
   h) $y^2 + 4x - 12y + 4 = 0$

4. Given the conic $C: 4x^2 - y^2 - 8x - 4y = 0$.
   a) Determine the translation that changes the equation into standard form.
   b) By factoring the image equation show that the image curve C' consists of two lines that intersect at the origin.
   c) Find equations for the two lines that make up the conic C.
   d) Explain why C is called a degenerate hyperbola.

5. Show that the sign of ab is invariant when the conic $ax^2 + by^2 + 2gx + 2fy + c = 0$ is translated under the translation $(x,y) \rightarrow (x + h, y + k)$.

6. a) Use translations to show that the conic $x^2 + y^2 + 2gx + 2fy + c = 0$ is a circle with centre $(-g,-f)$ and radius $\sqrt{g^2 + f^2 - c}$
   b) Under what conditions on g, f, and c will the circle have a non-real radius? Such a circle is called an *imaginary circle.*
   c) Under what conditions on g, f, and c will the circle have a radius that is zero? Such a circle is called a *point circle.*

7. Given the conic
   $C: ax^2 + by^2 + 2gx + 2fy + c = 0$.
   a) Show that C maps into $ax^2 + by^2 = k$ under the translation
   $$(x,y) \rightarrow \left(x + \frac{g}{a}, y + \frac{f}{a}\right)$$
   b) Show that C is an ellipse if
   $$\frac{g^2}{a} + \frac{f^2}{b} - c > 0 \text{ and } ab > 0.$$
   c) Show that C is a hyperbola if
   $$\frac{g^2}{a} + \frac{f^2}{b} - c \neq 0 \text{ and } ab < 0.$$

8. Given the conic
   $C: ax^2 + by^2 + 2gx + 2fy + c = 0$
   where $b = 0$.
   a) What type of conic is C?
   b) Show that the conic is translated into standard position by the translation
   $$(x,y) \rightarrow \left(x + \frac{g}{a}, y + \frac{c}{2f} - \frac{g^2}{2af}\right)$$

9. Show that the equation
   $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, $a > b$, determines an ellipse with centre (h,k) and having major axis $y = k$ and minor axis $x = h$.

10. Show that the equation
   $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1$ determines an hyperbola with centre (h,k) and transverse axis $y = k$.

11. Show that the equation $x - h = m(y - k)^2$ defines a parabola with vertex (h,k) and axis of symmetry $y = k$.

8.4 Matrices, their Transposes and the Central Conics

So far you have studied the effect of a translation on conics. Next you will consider what happens to the equation of a central conic (an ellipse or hyperbola) under a rotation.

In section 7.4 you discovered that a rotation transformation can be described using a matrix. One way to study the effect of a rotation transformation on an ellipse or a hyperbola will be to write such conic equations using matrices.

Consider the equation of ellipse $4x^2 + 9y^2 = 36$, and the matrices
$T = [x \quad y]$, $M = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$, $V = \begin{bmatrix} x \\ y \end{bmatrix}$, and $K = [36]$

Because matrix multiplication is associative, the matrix product TMV can be performed either as $(TM)V$ or $T(MV)$. Using the latter,

$$\begin{aligned} TMV &= [x \quad y]\left(\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right) \\ &= [x \quad y]\begin{bmatrix} 4x + 0 \\ 0 + 9y \end{bmatrix} \\ &= [4x^2 + 9y^2] \end{aligned}$$

Hence, the conic equation $4x^2 + 9y^2 = 36$ is equivalent to the matrix equation $[4x^2 + 9y^2] = [36]$ which is written $TMV = K$.

When a central conic in standard position is rotated about the point (0,0) you will see (in section 8.5) that the conic has an equation with an xy term. The form of such an equation is $ax^2 + 2hxy + y^2 = k$. You will now learn how to express such an equation using matrices.

Example 1 Show that the quadratic expression $ax^2 + 2hxy + by^2$ is equivalent to the matrix product TMV where $T = [x \quad y]$, $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$, $V = \begin{bmatrix} x \\ y \end{bmatrix}$

Solution The product $TMV = T(MV)$

$$\begin{aligned} &= [x \quad y]\left(\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right) \\ &= [x \quad y]\begin{bmatrix} ax + hy \\ hx + by \end{bmatrix} \\ &= [ax^2 + 2hxy + by^2] \end{aligned}$$

Hence, the quadratic expression $ax^2 + 2hxy + by^2$ is equivalent to the matrix product $TMV = [x \quad y]\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ ■

Note: The determinant of M, det(M), equals $ab - h^2$. You will learn that the value of this determinant is significant in determining the type of central conic represented by the matrix equation $TMV = K$.

If $K = [k]$, then Example 1 shows you that the quadratic equation $ax^2 + 2hxy + by^2 = k$ is equivalent to the matrix equation

$TMV = K$, or

$$[x \quad y]\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [k]$$

Observe that the matrix T and the matrix V contain the same elements but the row in T is a column in V.
T is called the *transpose* of matrix V and is denoted V^t.

PROPERTY

The quadratic expression $ax^2 + 2hxy + by^2 = k$ is equivalent to the matrix equation $V^tMV = K$ where $V = \begin{bmatrix} x \\ y \end{bmatrix}$, $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$, $K = [k]$, and $V^t = [x \quad y]$

DEFINITION

The transpose A^t of an $m \times n$ matrix A is the $n \times m$ matrix formed by interchanging the elements in rows and columns, that is,
$A = [a_{ij}] \Rightarrow A^t = [a_{ji}]$

The following are some examples of a matrix and its transpose.

matrix A	$\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$	$\begin{bmatrix} m & k \\ k & c \end{bmatrix}$	$\begin{bmatrix} 5 & 4 & 1 \\ 3 & 2 & 6 \\ 7 & 8 & 9 \end{bmatrix}$	$\begin{bmatrix} 6 & 5 & 4 \\ 7 & 8 & 9 \end{bmatrix}$
transpose A^t	$\begin{bmatrix} 5 & 3 \\ 4 & 2 \end{bmatrix}$	$\begin{bmatrix} m & k \\ k & c \end{bmatrix}$	$\begin{bmatrix} 5 & 3 & 7 \\ 4 & 2 & 8 \\ 1 & 6 & 9 \end{bmatrix}$	$\begin{bmatrix} 6 & 7 \\ 5 & 8 \\ 4 & 9 \end{bmatrix}$

Note: The symmetric matrix $\begin{bmatrix} m & k \\ k & c \end{bmatrix}$ equals its transpose.

One important property of the transpose of a matrix is the following.

PROPERTY

If A and B are matrices such that AB exists then $(AB)^t = B^tA^t$.
The proof of this property for 2×2 matrices follows.

Proof: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} m & t \\ r & s \end{bmatrix}$

Then $A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and $B^t = \begin{bmatrix} m & r \\ t & s \end{bmatrix}$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} m & t \\ r & s \end{bmatrix} = \begin{bmatrix} am + br & at + bs \\ cm + dr & ct + ds \end{bmatrix}$$

$$B^tA^t = \begin{bmatrix} m & r \\ t & s \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} am + br & cm + dr \\ at + bs & ct + ds \end{bmatrix} = (AB)^t$$

Hence, $(AB)^t = B^tA^t$.

In the next section you will be performing rotations on conics about the point (0,0) using the rotation matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

In 8.4 Exercises you will be asked to prove the following property of the rotation matrix R and its transpose R^t.

PROPERTY

If R is the rotation matrix and I is the 2×2 unit matrix, then $RR^t = R^tR = I$.

This property means that R^{-1}, the inverse of R, and R^t, the transpose of R, are equal matrices.

Example 2

a) Write the Cartesian form of the matrix equation $V^tMV = K$ where $V = \begin{bmatrix} x \\ y \end{bmatrix}$, $M = \begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}$ and $K = [7]$.

b) Write the matrix equation for the Cartesian equation $3x^2 - 5xy - 7y^2 = 10$.

Solution

a) *Method 1*

From the first property on page 362, the matrix form of the equation $ax^2 + 2hxy + by^2 = k$ is the matrix equation $V^tMV = k$ where $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$, $V = \begin{bmatrix} x \\ y \end{bmatrix}$, and $K = [k]$.

But $M = \begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}$ so $a = 2$, $h = -3$ and $b = 6$. Also, $K = [7]$, so $k = 7$.

Hence, the matrix equation $V^tMV = K$ becomes the Cartesian equation $ax^2 + 2hxy + by^2 = k$ or $2x^2 - 6xy + 6y^2 = 7$.

Method 2

The matrix equation $V^tMV = K$ becomes

$$[x \quad y]\begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [7]$$

$$\left([x \quad y]\begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = [7]$$

$$[2x - 3y \quad -3x + 6y]\begin{bmatrix} x \\ y \end{bmatrix} = [7]$$

$$[2x^2 - 3xy - 3xy + 6y^2] = [7]$$

$$[2x^2 - 6xy + 6y^2] = [7]$$

Hence, the Cartesian equation is $2x^2 - 6xy + 6y^2 = 7$.

b) From the first property on page 362, the matrix M for the Cartesian equation $ax^2 + 2hxy + y^2 = k$ is $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$, and $K = [k]$.

But for $3x^2 - 5xy - 7y^2 = 10$, $a = 3$, $h = -\frac{5}{2}$ or -2.5, $b = -7$, $k = 10$.

Thus the Cartesian equation $3x^2 - 5xy - 7y^2 = 10$ becomes the matrix equation $V^tMV = K$. That is, $[x \quad y]\begin{bmatrix} 3 & -2.5 \\ -2.5 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [10]$. ■

8.4 Exercises

1. Calculate the matrix product.

a) $[7 \quad 8]\begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

2. For each of the following matrices M, write the Cartesian equation of the conic whose matrix equation is $V^tMV = K$.

a) $\begin{bmatrix} 7 & 8 \\ 8 & 9 \end{bmatrix}$ d) $\begin{bmatrix} -2 & -4 \\ -4 & 3 \end{bmatrix}$

b) $\begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}$ e) $\begin{bmatrix} 12 & 10 \\ 10 & 2 \end{bmatrix}$

c) $\begin{bmatrix} 4 & -5 \\ -5 & 2 \end{bmatrix}$ f) $\begin{bmatrix} -2 & 9 \\ 9 & 1 \end{bmatrix}$

3. For each of the following conics $ax^2 + 2hxy + by^2 = k$, write the matrix M and the matrix K for its matrix equation $V^tMV = K$.

a) $4x^2 + 6xy + 5y^2 = 3$
b) $7x^2 - 8xy + 3y^2 = 1$
c) $-2x^2 + y^2 = 9$
d) $5x^2 - 3xy - 7y^2 = 8$
e) $9x^2 - 11y^2 = -5$
f) $x^2 - xy + 5y^2 = 3$

4. Write the matrix equation $V^tMV = K$ for each of the conics in question 3.

5. Write the transpose of each of the following matrices.

a) $\begin{bmatrix} 7 & 8 \\ 8 & 9 \end{bmatrix}$ d) $\begin{bmatrix} -2 & -4 \\ -4 & 3 \end{bmatrix}$

b) $\begin{bmatrix} 6 & 7 \\ 0 & -1 \end{bmatrix}$ e) $\begin{bmatrix} 12 & 0 \\ 10 & 2 \end{bmatrix}$

c) $\begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix}$ f) $\begin{bmatrix} 1 & 4 & 3 \\ 7 & 1 & 2 \end{bmatrix}$

6. Given $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

a) Calculate the matrix product AB.
b) Write $(AB)^t$.
c) Write the matrix A^t and the matrix B^t and then calculate B^tA^t.
d) Use your results of parts b) and c) to verify that $(AB)^t = B^tA^t$.

7. Repeat the previous question using the following pairs of matrices

a) $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} -4 & 3 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}$

c) $A = \begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 4 & 1 \end{bmatrix}$

8. For each of the following values of θ, write the rotation matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for the rotation of θ about (0,0). Find the value of each element, correct to 2 decimal places.

a) 30° b) 20° c) 140° d) 90°

9. For each of the following values of θ, write the rotation matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for the rotation of θ about (0,0). Find the exact value of each element using the table on page 543.

a) 30° b) 45° c) 60°

10. Suppose that R is the rotation matrix corresponding to a rotation of $\theta = 40°$ about the point (0,0), and $M = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$

Calculate the matrix product RM. State the value of each element of RM, correct to 2 decimal places.

11. Repeat the previous question for the following values of M and θ.

a) $M = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}$ and $\theta = 30°$

b) $M = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ and $\theta = 60°$

12. Repeat question 11 but find exact values for each element of RM.

13. Given that R is the rotation matrix

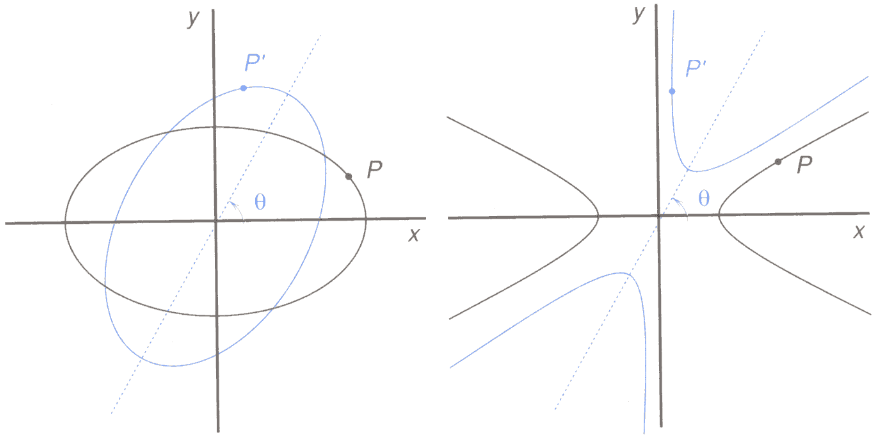
$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

and I is the 2×2 unit matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

a) Prove that $R^tR = RR^t = I$.
b) Explain why you can conclude that $R^t = R^{-1}$.

8.5 Rotations of Central Conics

Suppose a central conic C (an ellipse or a hyperbola) in standard position, with its axes of symmetry along the x-axis and y-axis, is rotated through an angle θ about the origin (0,0) as in the figure.



Then a point P on the conic C maps into a point P' on the image conic C'. If $\overrightarrow{OP} = V$ and $\overrightarrow{OP'} = U$, then under the rotation matrix R, $V \to U$, where

$RV = U$ ①

$R^t(RV) = R^tU$	(multiplying *on the left side* both sides by R^t)
$(R^tR)V = R^tU$	(matrix multiplication is associative)
Thus, $IV = R^tU$	(from section 8.4, $R^tR = I$)
Or, $V = R^tU$	(for any 2×2 matrix W, $WI = IW = W$)
Also, $V^t = (R^tU)^t$	(taking the transpose of the matrices on each side)
or $V^t = U^tR$	($(AB)^t = B^tA^t$, and the transpose of A^t is A)

Thus, under a rotation θ about the point (0,0),
V becomes R^tU, and V^t becomes U^tR.

Hence, upon applying the rotation transformation,
the matrix equation $V^tMV = K$ becomes
the matrix equation $(U^tR)\,M\,(R^tU) = K$ or $U^t(RMR^t)U = K$.

Once this equation has been obtained it is customary to replace U by V and U^t by V^t. Hence, you now have the following property.

PROPERTY

Under the rotation defined by the rotation matrix R
the conic $V^tMV = K$ maps into the conic $V^tM'V = K$,
where $M' = RMR^t$, and

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},\ V = \begin{bmatrix} x \\ y \end{bmatrix},\ \text{and } M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$$

Example 1 Given the ellipse E: $4x^2 + 9y^2 = 36$.

a) Write the corresponding matrix equation $V^tMV = K$.

b) Find the matrix equation for E', the image of E under a rotation of $20°$ about the point (0,0). Round off numbers to 1 decimal place.

c) Write the Cartesian equation for the image of E.

d) Sketch a graph of E and E'.

Solution

a) For the curve $ax^2 + 2hxy + by^2 = k$, $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$

For E: $4x^2 + 9y^2 = 36$, $a = 4$, $b = 9$ and $h = 0$. Thus, $M = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$

Therefore the matrix equation for E is

$$V^tMV = K \text{ or}$$

$$[x \quad y]\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [36]$$

b) The matrix equation for E' is $V^tM'V = K$, where $M' = RMR^t$. But the rotation angle about (0,0) is $20°$, thus

the rotation matrix $R = \begin{bmatrix} \cos 20° & -\sin 20° \\ \sin 20° & \cos 20° \end{bmatrix}$ and,

$$R^t = \begin{bmatrix} \cos 20° & \sin 20° \\ -\sin 20° & \cos 20° \end{bmatrix}$$

Thus, $M' = RMR^t$

$$= \begin{bmatrix} \cos 20° & -\sin 20° \\ \sin 20° & \cos 20° \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} \cos 20° & \sin 20° \\ -\sin 20° & \cos 20° \end{bmatrix}$$

$$= \left(\begin{bmatrix} \cos 20° & -\sin 20° \\ \sin 20° & \cos 20° \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}\right)\begin{bmatrix} \cos 20° & \sin 20° \\ -\sin 20° & \cos 20° \end{bmatrix}$$

$$= \begin{bmatrix} 4\cos 20° & -9\sin 20° \\ 4\sin 20° & 9\cos 20° \end{bmatrix}\begin{bmatrix} \cos 20° & \sin 20° \\ -\sin 20° & \cos 20° \end{bmatrix}$$

$$= \begin{bmatrix} 4\cos^2 20° + 9\sin^2 20° & 4\cos 20°\sin 20° - 9\sin 20°\cos 20° \\ 4\sin 20°\cos 20° - 9\cos 20°\sin 20° & 4\sin^2 20° + 9\cos^2 20° \end{bmatrix}$$

$$= \begin{bmatrix} 4.6 & -1.6 \\ -1.6 & 8.4 \end{bmatrix} \text{ correct to 1 decimal place.}$$

Thus, the matrix equation for E' is

$$V^tM'V = K, \text{ or } [x \quad y]\begin{bmatrix} 4.6 & -1.6 \\ -1.6 & 8.4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [36].$$

c) For the image ellipse E',

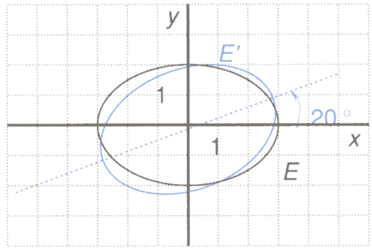
$M' = \begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix}$, thus $a' = 4.6$, $b' = 8.4$ and $h' = -1.6$

Since the equation of E' is $a'x^2 + 2h'xy + b'y^2 = k$, the Cartesian equation for the image ellipse E' is

$$4.6x^2 - 3.2xy + 8.4y^2 = 36$$

d) For $4x^2 + 9y^2 = 36$: the x-intercepts are 3 and -3, while the y-intercepts are 2 and -2. The graph of E is shown in the diagram.

The graph of E' is the graph of E rotated $20°$ counterclockwise about (0,0), shown on the same axes as that of E.



■

Note: Once you have had some practice using the method of Example 1, the equation of the rotated conic can be found simply by calculating the matrix $M' = RMR^t = \begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix}$ and then substituting into the image equation $a'x^2 + 2h'xy + b'y^2 = k$.

PROPERTY

Under a rotation of θ about the point (0,0) with rotation matrix R, the conic $ax^2 + 2hxy + by^2 = k$ maps into the conic $a'x^2 + 2h'xy + b'y^2 = k$, where $\begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix} = RMR^t$.

Example 2

The hyperbola H: $x^2 - 16y^2 = 5$ is rotated through an angle of 30°. Find an equation of H', the image of H after rotation. Use exact values for the sine and cosine of 30°.

Solution

The image H' has equation $a'x^2 + 2h'xy + b'y^2 = k$, where $\begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix} = RMR^t$ and $k = 5$.

For H: $x^2 - 16y^2 = 5$, so $a = 1$, $h = 0$, $b = -16$. Also $\theta = 30°$. Thus,

$M = \begin{bmatrix} 1 & 0 \\ 0 & -16 \end{bmatrix}$, $R = \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix}$ and $R^t = \begin{bmatrix} \cos 30° & \sin 30° \\ -\sin 30° & \cos 30° \end{bmatrix}$

Thus $M' = \left(\begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -16 \end{bmatrix} \right) \begin{bmatrix} \cos 30° & \sin 30° \\ -\sin 30° & \cos 30° \end{bmatrix}$

$= \begin{bmatrix} \cos 30° & 16 \sin 30° \\ \sin 30° & -16 \cos 30° \end{bmatrix} \begin{bmatrix} \cos 30° & \sin 30° \\ -\sin 30° & \cos 30° \end{bmatrix}$

$= \begin{bmatrix} \cos^2 30° - 16 \sin^2 30° & \cos 30° \sin 30° + 16 \sin 30° \cos 30° \\ \sin 30° \cos 30° + 16 \cos 30° \sin 30° & \sin^2 30° - 16 \cos^2 30° \end{bmatrix}$

$= \begin{bmatrix} -\frac{13}{4} & \frac{17\sqrt{3}}{4} \\ \frac{17\sqrt{3}}{4} & -\frac{47}{4} \end{bmatrix}$ using the table on page 543

Thus, $a' = -\frac{13}{4}$, $b' = -\frac{47}{4}$ and $h' = \frac{17\sqrt{3}}{4}$

Hence, the equation of H' is $-\frac{13}{4}x^2 + \frac{34\sqrt{3}}{4}xy - \frac{47}{4}y^2 = 5$ or

$13x^2 - 34\sqrt{3}xy + 47y^2 = -20$. ■

In Examples 1 and 2 above, examine $ab - h^2$, the value of the determinant M, for the original curves E and H. Compare these values with the value of the determinant of $M' = a'b' - h'^2$ for the image curves E' and H'.

for E: $ab - h^2 = (4)(9) - 0^2 = 36$;
for E': $a'b' - h'^2 = (4.6)(8.4) - (\frac{3.2}{2})^2 \doteq 36$
Thus $ab - h^2$ and $a'b' - h'^2$ are *both positive in value.*
for H: $ab - h^2 = (1)(-16) - 0^2 = -16$;
for H': $a'b' - h'^2 = (-\frac{13}{4})(-\frac{47}{4}) - (\frac{17\sqrt{3}}{4})^2 = -\frac{256}{16} = -16$.
Thus $ab - h^2 = a'b' - h'^2$, and *this value is negative.*
This demonstrates, but does not prove, that the value, and hence the sign, of $ab - h^2$ is invariant for rotations. For an ellipse, $ab - h^2 > 0$. For a hyperbola, $ab - h^2 < 0$.

Even though the value of $ab - h^2$ is invariant (provided neither conic equation has been multiplied by a constant), all that you need to remember is that the *sign of $ab - h^2$ does not change under a rotation.*

In 8.5 Exercises you will be given the opportunity to check that the sign of $ab - h^2$ is invariant under a rotation. You should observe that this invariant sign is the same as for translations, that is, the sign of $ab - h^2$ when $h = 0$.

In Search of a Proof that $ab - h^2$ is Invariant Under a Rotation

Consider the conic $ax^2 + 2hxy + by^2 = k$. This conic has the matrix equation $V^tMV = K$, where $V = \begin{bmatrix} x \\ y \end{bmatrix}$, $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$, and $K = [k]$.
The determinant of matrix M is $ab - h^2$, that is $\det(M) = ab - h^2$.
Under a rotation of θ about the point (0,0), this conic maps into the conic $V^tM'V = K$ where $M' = RMR^t$.
If this new conic has equation $a'x^2 + 2h'xy + b'y^2 = k$, then the matrix $M' = RMR^t$ would equal $\begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix}$. Hence, $\det(M') = a'b' - h'^2$.
To prove that the value, and therefore the sign, of $ab - h^2$ is invariant, you need to prove that $\det(M) = \det(M')$

$$\text{Now } \det(R) = \det\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
$$= \cos^2\theta - (-\sin^2\theta) = \cos^2\theta + \sin^2\theta = 1 \text{ (see page 542)}$$

Similarly, $\det(R^t) = 1$
From chapter 7, for matrices A, B, and C and the matrix product ABC, $\det(ABC) = \det(A) \times \det(B) \times \det(C)$.
Thus, $\det(M') = \det(RMR^t) = \det(R) \times \det(M) \times \det(R^t)$
$= 1 \times \det(M) \times 1$
Thus, $\det(M') = \det(M)$.
Thus the value of $ab - h^2$, and hence its sign, is invariant under a rotation.

8.5 Exercises

Round off all numbers to 1 decimal place where appropriate.

1. Given the ellipse E: $x^2 + 4y^2 = 4$.
   a) Write the corresponding matrix equation $V'^tM'V = K$.
   b) Use the fact that $M' = RMR^t$ to find the matrix equation $V^tMV = K$, for E', the image of E under a rotation of 30° about the point (0,0).
   c) Write a Cartesian equation for the image of E.
   d) Sketch a graph of E and E'.

2. a) The ellipse E: $4x^2 + y^2 = 16$ is rotated through an angle of 45° about the point (0,0). Find an equation of E', the image of E after rotation.
   b) Sketch a graph of E and E'.

3. Repeat question 2 for the following ellipses and rotation angles θ.
   a) $x^2 + 9y^2 = 9$, $\theta = 120°$
   b) $4x^2 + 25y^2 = 100$, $\theta = 250°$
   c) $25x^2 + 9y^2 = 200$, $\theta = -40°$
   d) $3x^2 + y^2 = 4$, $\theta = 60°$

4. For each of the ellipses in questions 2 and 3 check to see that the value of $ab - h^2$ is invariant under the given rotation and that its sign is positive.

5. Repeat question 2 for the following ellipses and rotation angles θ. Use exact values for the sines and cosines of θ, from the table on page 543.
   a) $4x^2 + y^2 = 16$, $\theta = 45°$
   b) $9x^2 + y^2 = 9$, $\theta = 120°$
   c) $3x^2 + y^2 = 4$, $\theta = 60°$

6. a) The hyperbola H: $9x^2 - y^2 = 9$ is rotated through an angle of 30° about the point (0,0). Find an equation of H', the image of H after rotation.
   b) Sketch a graph of H and H'.

7. Repeat question 6 for the following hyperbolas and rotation angles θ.
   a) $x^2 - 4y^2 = -4$, $\theta = 30°$
   b) $16x^2 - 25y^2 = 400$, $\theta = 120°$
   c) $x^2 - y^2 = -1$, $\theta = -40°$
   d) $x^2 - 3y^2 = -3$, $\theta = 60°$

8. For each of the hyperbolas in questions 6 and 7 check to see that the value of $ab - h^2$ is invariant under the given rotation and that its sign is negative.

9. Repeat question 6 for the following hyperbolas and rotation angles θ. Use exact values for the sines and cosines of θ, from the table on page 543.
   a) $9x^2 - y^2 = 9$, $\theta = 30°$
   b) $16x^2 - 25y^2 = 400$, $\theta = 120°$
   c) $x^2 - 3y^2 = -3$, $\theta = 60°$.

10. Find an equation of each of the following after rotation of 90° about the point (0,0).
   a) $4x^2 + 9y^2 = 36$
   b) $x^2 - 9y^2 = 9$
   c) $4x^2 + y^2 = 16$

11. a) A rotation of 90° about (0,0) is equivalent to interchanging the x-axis and the y-axis. Explain.
   b) Make use of part a) to write the image equations for the previous question, *without making any calculations*.

12. Given the curve C: $17x^2 + 16xy + 17y^2 = 225$ and the rotation of 45° about the point (0,0).
   a) Name the type of curve.
   b) Find an equation of C', the image of C under the given rotation.
   c) Sketch the *image* curve C'. Then sketch the given curve by using the inverse rotation to the given rotation.

13. Repeat the previous question for the curve $x^2 + 3.5xy - y^2 = 1$ and a rotation of 60°.

14. The method of this section can not be applied to a parabola in standard position. Why not?

Earth Satellites

Each flight of a satellite about the earth is achieved by launching a rocket vertically. Eventually the trajetory of the rocket must be tilted. This is necessary so that flight of the rocket is parallel to the surface of the earth at the time that the orbital velocity at the desired altitude is reached. The space vehicle attached to the final stage of the rocket is then in free fall about the earth.

Communication satellites and meterological satellites work best if they remain fixed above one place on the surface of the earth. This will occur when the time for the satellite to move about the earth is equal to the time for the earth to make one complete rotation about its axis. In this case the satellite appears to be stationary in the sky. Such an orbit is called *geostationary* or *geosynchronous*.

What is the height h km above the earth of a satellite in a geosynchronous orbit?

Newton's first law of motion indicates that a body will continue in its state of rest or uniform motion in a straight line unless acted upon by some external force. Thus a rocket launched in a straight line would continue forever along that line, unless some force caused it to change its direction. The force that is continually changing the direction of the rocket and the satellite, causing them to make a circular orbit about the earth, is the gravitational pull of the earth on the rocket and satellite.

Now the force required continually to change the direction of a satellite of mass m (causing the satellite to move in a circular orbit) is called *centripetal* force. For a satellite moving along a circle of radius r with a tangential velocity v, this force equals $\frac{mv^2}{r}$.

But Newton showed that the gravitational pull of the earth that provides this centripetal force equals $\frac{kmE}{r^2}$

where E is the mass of the earth, r is the distance between the centre of the earth and the centre of the satellite, k is a constant number whose value depends only on the units chosen for E, r and m.

Thus, centripetal force = gravitational force, or $\frac{mv^2}{r} = \frac{kmE}{r^2}$, or $v^2r = kE$.

Since k and E do not change, v^2r = constant.

This formula is true for all earth satellites. Since the value of v and r are known for the moon, which is an earth satellite, the value of this constant can be calculated.

Now the moon rotates about the earth every 27.322 days, along a circle of approximate radius 3.844×10^5 km.

Thus the tangential velocity of the satellite

$$= \frac{\text{distance travelled}}{\text{time taken to travel this distance}}$$

$$= \frac{\text{circumference of the orbit of the moon}}{\text{time for one orbit of the moon}}$$

$$= \frac{2\pi \times 3.844 \times 10^5}{27.322 \times 24} \text{ km/h or } 3.683\ 32 \times 10^3 \text{ km/h}$$

Thus, for the moon, $v^2 r = (3.683\ 32 \times 10^3)^2 \times 3.844 \times 10^5$

$$= 5.2151 \times 10^{12}$$

Thus, for any earth satellite, and in particular, a geosynchronous satellite, $v^2 r = 5.2151 \times 10^{12}$ ①

But for a geosynchronous satellite the speed is

$$v = \frac{\text{circumference of orbit}}{24} \text{ km/h or } \frac{2\pi r}{24} \text{ km/h}$$

For a satellite at a height h km above the earth, the distance from the centre of the satellite to the centre of the earth equals

r = the mean radius of the earth + h, so $r = 6371 + h$.

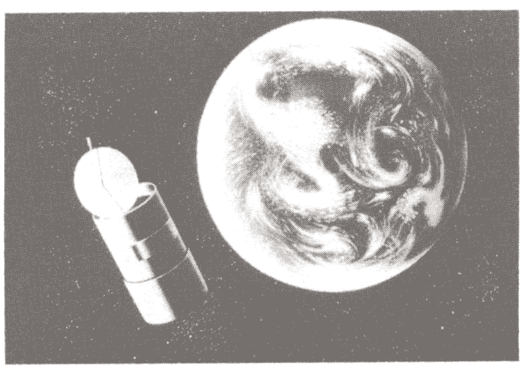
Substituting into ① gives

$$\left[\frac{2\pi(6731 + h)}{24}\right]^2 \times (6371 + h) = 5.2151 \times 10^{12} \text{ or}$$

$$6731 + h = \left(\frac{5.2151 \times 10^{12} \times 24^2}{(2\pi)^2}\right)^{\frac{1}{3}}$$

Thus, $6371 + h = 42\ 375$, or $h = 36\ 004$ km.

Thus, a geosynchronous satellite must be about 36 000 km above the surface of the earth.



8.6 Rotations that Eliminate *xy* Terms

In section 8.5 and 8.5 Exercises you saw that the rotation of an ellipse or hyperbola in standard position changed the equation from the form $ax^2 + by^2 = k$ to $a'x^2 + 2h'xy + b'y^2 = k$ where $h' \neq 0$, thus introducing a term in xy.

In this section you will determine the rotation that you must apply to $a'x^2 + 2h'xy + b'y^2 = k$ to return the equation to the form $ax^2 + by^2 = k$, that is, the form $ax^2 + 2hxy + by^2 = k$, where $h = 0$.

If the conic C' with equation $a'x^2 + 2h'xy + b'y^2 = k$, with $h' \neq 0$, maps into the conic C with equation $ax^2 + 2hxy + by^2 = k$, with $h = 0$, then the matrix equation $V^tM'V = K$ where $M' = \begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix}$ with $h' \neq 0$ must become the matrix equation $V^tMV = K$ where $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ with $h = 0$.

If C' maps into C under the rotation defined by matrix R, then C will have matrix equation $V^tMV = K$ where $M = RM'R^t$.

To determine the rotation that will eliminate the xy term in $ax^2 + 2hxy + by^2 = k$ you only need find the element h in the first row and second column of M and decide what value of θ will make that term zero.

$$\text{Now } M = RM'R^t = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a'\cos\theta - h'\sin\theta & h'\cos\theta - b'\sin\theta \\ a'\sin\theta + h'\cos\theta & h'\sin\theta + b'\cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\text{must become } \begin{bmatrix} a & h \\ h & b \end{bmatrix}, \text{ where } h = 0.$$

The element h in the first row and second column of this product is the dot product of the elements in the first row of the left hand matrix with the elements in the second column of the right hand matrix.

$$\begin{aligned}\text{That is, } h &= (a'\cos\theta - h'\sin\theta,\ h'\cos\theta - b'\sin\theta)\cdot(\sin\theta, \cos\theta) \\ &= a'\cos\theta\sin\theta - h'\sin^2\theta + h'\cos^2\theta - b'\sin\theta\cos\theta \\ &= (a' - b')\sin\theta\cos\theta + h'(\cos^2\theta - \sin^2\theta)\end{aligned}$$

But $\sin\theta\cos\theta = \dfrac{\sin 2\theta}{2}$, and $\cos^2\theta - \sin^2\theta = \cos 2\theta$

Thus, $h = (a' - b')\dfrac{\sin 2\theta}{2} + h'\cos 2\theta$

$h = 0$ means that

$(a' - b')\dfrac{\sin 2\theta}{2} + h'\cos 2\theta = 0$, that is, $(a' - b')\dfrac{\sin 2\theta}{2} = -h'\cos 2\theta$

Thus $\dfrac{\sin 2\theta}{\cos 2\theta} = \dfrac{-2h'}{a' - b'}$

or $\tan 2\theta = \dfrac{2h'}{b' - a'}$ $\qquad$ $\tan\phi = \dfrac{\sin\phi}{\cos\phi}$ and $(a' - b') = -(b' - a')$

This formula assumes that $b' \neq a'$. If $b' = a'$, then $\tan 2\theta$ is undefined, so that 2θ is 90° or 270° (or any angle having the same initial arm and terminal arm with either one). This means that θ is 45° or 135°.
In using this formula it is customary to ignore the primes on a', b' and h', obtaining the following.

FORMULA

To remove the xy term from the conic equation $ax^2 + 2hxy + by^2 = k$, rotate the conic through an angle θ about the point (0,0) where

if $a \neq b$, then $\tan 2\theta = \dfrac{2h}{b-a}$ and if $a = b$, then $\theta = 45°$ or $135°$.

When $a \neq b$, θ can be found using your calculator by keying $\dfrac{2h}{b-a}$ then using the inverse tangent function key. This keying will give 2θ, so you must divide this number by 2. If $\dfrac{2h}{b-a} > 0$, then $\theta > 0$. If $\dfrac{2h}{b-a} < 0$, then $\theta < 0$.
These are not the only rotation angles that will eliminate the xy term. Any one of the infinite number of solutions of $\tan 2\theta = \dfrac{2h}{b-a}$ can be divided by 2 to give a suitable angle of rotation. In particular, if your calculator gives you $m°$ for 2θ, another value for 2θ is $180° + m°$.
One rotation angle brings the major or transverse axis to the x-axis while the other brings it to the y-axis.

Example 1

Determine, correct to 1 decimal place, an angle of rotation about the point (0,0) needed to remove the xy term from each of the following.

a) $9x^2 - 4xy + 6y^2 = 35$ **b)** $x^2 + xy + y^2 = 6$

Solution

a) For the conic $9x^2 - 4xy + 6y^2 = 35$, $a = 9$, $h = \dfrac{-4}{2} = -2$, $b = 6$.

Thus, $\tan 2\theta = \dfrac{2h}{b-a} = \dfrac{2(-2)}{6-9} = \dfrac{4}{3}$

Using your calculator you obtain $2\theta = 53.1301°$.
Hence, $\theta = 26.565° = 26.6°$, correct to 1 decimal place. Hence, an angle of rotation of 26.6° about the point (0,0) will eliminate the xy term.

b) For the conic $x^2 + xy + y^2 = 6$, $a = 1$, $h = \dfrac{1}{2}$, $b = 1$.

Since $a = b$, a rotation about the point (0,0) of $\theta = 45°$ or $\theta = 135°$ will eliminate the xy term. ■

Note: In part a), 2θ could be $180° + 53.1301° = 233.1301°$. Thus another angle of rotation would be 116.6°.

Before you eliminate an xy term it will be helpful to know the form of the standard equation of the conic to help you to detect errors in your calculations. Hence, you should make use of the following information that you verified in section 8.5 and in 8.5 Exercises.

The Graph of $ax^2 + 2hxy + by^2 = k$

ellipse: $ab - h^2 > 0$ hyperbola: $ab - h^2 < 0$

Example 2 Given the conic C: $-5x^2 + 8xy + y^2 = 21$.

a) Determine the type of conic.

b) Find an angle of rotation about the point (0,0) to eliminate the xy term.

c) Find an equation of the image curve C'.

d) Sketch the graph of the image curve C', then of the original curve C.

Solution

a) Here $a = -5$, $h = \dfrac{8}{2} = 4$, and $b = 1$.

Thus, $ab - h^2 = (-5)(1) - 16 = -21 < 0$. The conic is a hyperbola.

b) Since $a \neq b$, the angle of rotation about (0,0) that will eliminate the xy term is a solution of $\tan 2\theta = \dfrac{2h}{b-a} = \dfrac{(2)(4)}{1-(-5)} = \dfrac{8}{6}$ or $\dfrac{4}{3}$

Using your calculator, $2\theta = 53.130\ 102\ldots^\circ$ Thus, $\theta = 26.565\ 051\ldots^\circ$

c) The image C' has equation $a'x^2 + 2h'xy + b'y^2 = k$ where

$$M' = \begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix} = RMR^t, \text{ and } k = 21$$

But here, $M = \begin{bmatrix} -5 & 4 \\ 4 & 1 \end{bmatrix}$ and $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Thus $M' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} -5 & 4 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

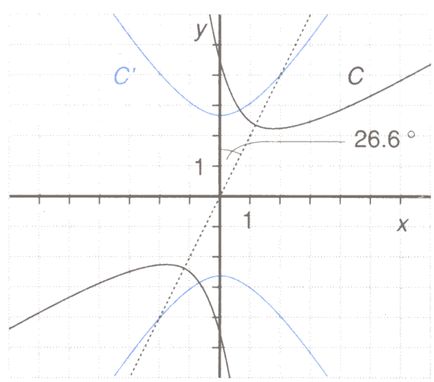
$$= \begin{bmatrix} -5\cos\theta - 4\sin\theta & 4\cos\theta - \sin\theta \\ -5\sin\theta + 4\cos\theta & 4\sin\theta + \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} -5\cos^2\theta - 8\sin\theta\cos\theta + \sin^2\theta & -4\sin^2\theta + 4\cos^2\theta - 6\sin\theta\cos\theta \\ -4\sin^2\theta + 4\cos^2\theta - 6\sin\theta\cos\theta & -5\sin^2\theta + 8\sin\theta\cos\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 \\ 0 & 3 \end{bmatrix}$$ use the memory key for θ

Hence the equation of C' is $-7x^2 + 3y^2 = 21$ or $\dfrac{x^2}{3} - \dfrac{y^2}{7} = -1$.

d) The image curve C' is a hyperbola with centre at (0,0) that intersects the y-axis at $(0,\sqrt{7})$ and $(0,-\sqrt{7})$.



Since C maps into C' under a *counterclockwise* rotation of approximately 26.6° about (0,0), C' will map back into C under a *clockwise* rotation of approximately 26.6° about (0,0).

Note: The angle $\theta = 26.6^\circ$ rotates the transverse axis of C onto the y-axis. The other solution, $2\theta = 180^\circ + 53.13 = 233.13^\circ$ gives $\theta \doteq 116.6^\circ$. This rotation maps the transverse axis of C onto the x-axis. ■

In Search of a Method of Eliminating the *xy* Term using Characteristic Values

On page 333, when looking for invariant lines, you learned about characteristic values of a matrix. Recall that the characteristic values k of the matrix M are obtained by solving $\det(M - kI) = 0$.

You also proved the following facts in the activities on page 335.

1. A symmetric matrix always has real characteristic values.
2. The characteristic vectors corresponding to each characteristic value of a symmetric matrix are orthogonal.

Using these facts, you will now prove a theorem about diagonalizing a symmetric matrix, that is, changing the elements of the matrix, so that all *non-zero* elements are along the leading diagonal. An application of this theorem will allow you to eliminate the xy term in a quadratic form, remarkably quickly.

THEOREM

Given the symmetric matrix $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ that has characteristic values c and d.

Prove that this matrix can be diagonalized as $\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$

Proof: Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be a unit characteristic vector associated with the value c. (That is, $\vec{u}$ is such that $M\vec{u} = c\vec{u}$.)

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be a unit characteristic vector associated with the value d. (That is, $\vec{v}$ is such that $M\vec{v} = d\vec{v}$.)

Let $H = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$. Then $H^t = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$.

Matrix M can be diagonalised by computing H^tMH.

You will show that $H^tMH = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$

First note that $M\vec{u} = c\vec{u} \Rightarrow \begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$\Rightarrow au_1 + hu_2 = cu_1$ ①

and $hu_1 + bu_2 = cu_2$ ②

Also, $M\vec{v} = d\vec{v}$, so $\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = d\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$\Rightarrow av_1 + hv_2 = dv_1$ ③

and $hv_1 + bv_2 = dv_2$ ④

$$\begin{aligned}
\text{Now } H^tMH &= \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \\
&= \begin{bmatrix} au_1 + hu_2 & hu_1 + bu_2 \\ av_1 + hv_2 & hv_1 + bv_2 \end{bmatrix}\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \\
&= \begin{bmatrix} cu_1 & cu_2 \\ dv_1 & dv_2 \end{bmatrix}\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \quad \text{using ①, ②, ③ and ④} \\
&= \begin{bmatrix} cu_1^2 + cu_2^2 & cu_1v_1 + cu_2v_2 \\ du_1v_1 + du_2v_2 & dv_1^2 + dv_2^2 \end{bmatrix} \\
&= \begin{bmatrix} c|\vec{u}|^2 & c(\vec{u}\cdot\vec{v}) \\ d(\vec{u}\cdot\vec{v}) & d|\vec{v}|^2 \end{bmatrix} \\
&= \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}
\end{aligned}$$

since $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors (because M is symmetric).

Application to the Rotation of a Conic

In section 8.6 you learned that the conic $ax^2 + 2hxy + by^2 = k$ can have its xy term eliminated ($h = 0$) under a rotation of θ about the origin O, where

$$\tan 2\theta = \frac{2h}{b - a}$$

If the equation of the rotated conic is $cx^2 + dy^2 = k$, then

the matrix RMR^t equals $\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$,

where R is the rotation matrix for a rotation θ about the origin O.

But from the theorem, $H^tMH = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$

This implies that the matrix H and the matrix R^t are the same matrix.

Thus, $H = R^t = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$

(since $\sin(-\theta) = -\sin\theta$, and $\cos(-\theta) = \cos\theta$)

Hence H corresponds to a rotation of $-\theta$ about the origin O, where

$$\tan 2\theta = \frac{2h}{b - a}$$

Example

a) Eliminate the xy term from $41x^2 - 24xy + 34y^2 = 100$, and describe the conic.

b) Find the rotation that maps the conic to its new position.

Solution

a) The defining matrix is $M = \begin{bmatrix} 41 & -12 \\ -12 & 34 \end{bmatrix}$

Thus the characteristic values are the solutions of $\begin{vmatrix} 41 - c & -12 \\ -12 & 34 - c \end{vmatrix} = 0$

$$(41-c)(34-c)-(-12)^2=0$$
$$c^2-75c+1250=0$$
$$(c-25)(c-50)=0$$
$$c=25 \text{ or } c=50.$$

Hence the conic can be rotated so that its equation without the xy term is exactly $25x^2 + 50y^2 = 100$

or $\quad x^2 + 2y^2 = 4$

The rotated conic is an ellipse which intersects the axes at $(\pm 2,0)$, $(0,\pm\sqrt{2})$.

b) To find the rotation, you need unit characteristic vectors of the matrix M.

$$M\vec{u} = 25\vec{u} \Rightarrow \begin{bmatrix} 41 & -12 \\ -12 & 34 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 25\begin{bmatrix} x \\ y \end{bmatrix}$$
$$\Rightarrow \quad 41x - 12y = 25x$$
$$\text{and } -12x + 34y = 25y$$

that is, $4x - 3y = 0$ so $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\vec{e}_u = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$

$$M\vec{v} = 50\vec{v} \Rightarrow \begin{bmatrix} 41 & -12 \\ -12 & 34 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 50\begin{bmatrix} x \\ y \end{bmatrix}$$
$$\Rightarrow \quad 41x - 12y = 50x$$
$$\text{and } -12x + 34y = 50y,$$

that is, $3x + 4y = 0$ so $\vec{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ and $\vec{e}_v = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$

The columns of the rotation matrix R are $\vec{e}_u$ and $\vec{e}_v$, that is $R = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$.

(Notice that if the columns were interchanged, you would not have a rotation matrix.)

This is the matrix of a rotation through $-\theta$,

where $\sin(-\theta) = \frac{4}{5}$ and $\cos(-\theta) = \frac{3}{5}$

Therefore $-\theta \doteq 53.1°$, so that $\theta \doteq -53.1°$ ■

Note: Using the formula $\tan 2\theta = \dfrac{2h}{b-a} = \dfrac{-24}{-7}$ gives $\theta \doteq 36.9°$ or $\theta \doteq -53.1°$.
$-53.1°$ rotates the major axis of the ellipse onto the x-axis, while $36.9°$ rotates the major axis onto the y-axis.

8.6 Exercises

Round off all numbers to 1 decimal place where appropriate.

1. Given the conic $x^2 - 2xy + 3y^2 = 1$.
   a) Name the type of conic.
   b) Determine an angle of rotation about (0,0) needed to remove the xy term.

2. Repeat question 1 for the following conics.
   a) $4x^2 - 3xy - 2y^2 = 8$
   b) $x^2 - xy + y^2 = 1$
   c) $3x^2 - 4xy + 4y^2 = 3$
   d) $2x^2 + 9xy + y^2 = 4$
   e) $2x^2 - 2\sqrt{6}xy + y^2 = 4$
   f) $4x^2 + 3xy + 5y^2 = 8$

3. Given the conic C: $5x^2 + 6xy + 5y^2 = 8$.
   a) Determine the type of conic.
   b) Find an angle of rotation about (0,0) that will eliminate the xy term.
   c) Find an equation of the image curve C'.
   d) Sketch the graph of the image curve C', then the graph of C, the original curve.

4. Repeat question 3 for the following conics.
   a) $23x^2 - 16xy + 42y^2 = 180$
   b) $1175x^2 - 1472xy + 325y^2 = -500$
   c) $170x^2 - 1970xy - 170y^2 = -1000$
   d) $225x^2 - 135xy + 65y^2 = 1000$
   e) $200x^2 - 346xy = 300$
   f) $184x^2 - 158xy + 156y^2 = 2000$

5. Given the conic C: $17x^2 + 16xy + 17y^2 = 225$.
   a) Determine the type of conic.
   b) Find an angle of rotation about (0,0) that will eliminate the xy term.
   c) Using exact values for sines and cosines from the table on page 543, find an equation of the image curve C'.
   d) Sketch the graph of the image curve C', then the graph of C, the original curve.

6. Repeat question 5 for the following conics.
   a) $x^2 + 2\sqrt{3}xy - y^2 = -2$
   b) $13x^2 + 10xy + 13y^2 = 8$
   c) $5x^2 + 2\sqrt{3}xy + 3y^2 = 36$

7. In section 8.6 you showed that the conic $a'x^2 + 2h'xy + b'y^2 = k$ maps into the conic $ax^2 + 2hxy + by^2 = k$, with $h = 0$, under a rotation θ about (0,0) where
   $h = (a' - b')\dfrac{\sin 2\theta}{2} + h' \cos 2\theta$.
   a) Show that
   $$a = \left(\frac{a' + b'}{2}\right) + \left(\frac{a' - b'}{2}\right) \cos 2\theta - h' \sin 2\theta$$
   and that
   $$b = \left(\frac{a' + b'}{2}\right) - \left(\frac{a' - b'}{2}\right) \cos 2\theta + h' \sin 2\theta$$
   b) Use your results of part a) to verify that $a'b' - h'^2 = ab - h^2$.

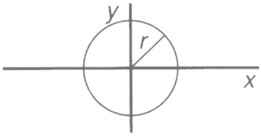
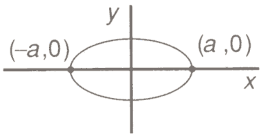
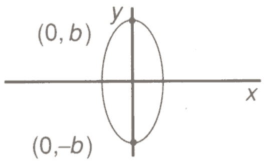
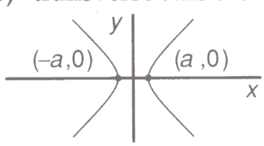
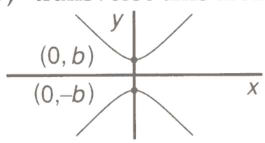
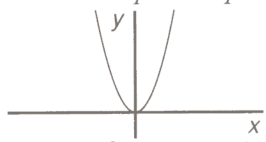
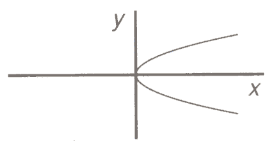
8. Given the conic C: $130x^2 + 100xy + 130y^2 + 481x + 28y = 0$.
   a) Determine the type of conic.
   b) Find an angle of rotation about (0,0) that will eliminate the xy term.
   c) Find an equation of the image curve C' under this rotation.
   d) Translate C' so that the equation of the image C'' of C' is in standard form.
   e) Sketch the graph of the final image curve C''.
   f) Use the inverse translation to that in part d) to graph C'.
   g) Use the inverse rotation to that in part b) to graph C.

9. Repeat the previous question for the following conic.
   $5x^2 - 6xy + 5y^2 + 5.7x - 17y + 8 = 0$

10. Repeat question 8 for the conic $5x^2 - 6xy + 5y^2 + 4\sqrt{2}x - 12\sqrt{2}y + 8 = 0$, using the exact values for the sines and cosines from the table on page 543.

11. Write a computer program to transform a conic C: $ax^2 + 2hxy + by^2 = k$ into the standard conic C': $a'x^2 + b'y^2 = k$. You may wish to use the results of question 7.

Summary

conic	equation	centre	vertices	graph
circle	$x^2 + y^2 = r^2$	(0,0)	none	radius = r, $r > 0$ 
ellipse	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $a > b > 0$	(0,0)	$(a,0)(-a,0)$	major axis along x-axis 
	$b > a > 0$	(0,0)	$(0,b)(0,-b)$	major axis along y-axis 
hyperbola	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ $a > 0, b > 0$	(0,0)	$(a,0)(-a,0)$	transverse axis along x-axis 
	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$ $a > 0, b > 0$	(0,0)	$(0,b)(0,-b)$	transverse axis along y-axis 
parabola	$y = kx^2$ $k > 0$: opens up $k < 0$: opens down	none	(0,0)	axis of symmetry is y-axis 
	$x = ky^2$ $k > 0$: opens right $k < 0$: opens left	none	(0,0)	axis of symmetry is x-axis 

- The circles, ellipses, and hyperbolas with equations given in the chart have their centres at the origin (0,0) and their major axis or transverse axis along the x-axis or y-axis. Also, the parabolas with equations given in the chart have their vertices at the origin (0,0) and their axes of symmetry along the y-axis or x-axis. Such conics are said to be in *standard position*, and their equations in *standard form*.
- An equation written $ax^2 + by^2 + 2gx + 2fy + c = 0$ is in *general form*.
- Any vector $\vec{a} = \overrightarrow{(h,k)}$ defines the *translation* that maps each point P with coordinates (x,y) into the point P' with coordinates $(x + h, y + k)$, that is, point $P(x,y) \rightarrow$ point $P'(x + h, y + k)$.

The Graph of $ax^2 + by^2 + 2gx + 2fy + c = 0$

ellipse	$ab > 0,\ a \neq b$
circle	$ab > 0,\ a = b$
hyperbola	$ab < 0$
parabola	$ab = 0$

- A conic that has an equation $ax^2 + by^2 + 2gx + 2fy + c = 0$ with at least one of g and f not zero is not in standard position. To translate the conic into standard position you should complete the square of the terms in x and y.
- The expression $x^2 + mx$ becomes a perfect square by the addition of $\left(\frac{m}{2}\right)^2$, then $x^2 + mx + \left(\frac{m}{2}\right)^2 = \left(x + \frac{m}{2}\right)^2$
- The *transpose* A^t of an $m \times n$ matrix A is the $n \times m$ matrix formed by interchanging the elements in rows and columns, that is, $A = [a_{ij}]_{m\times n} \Rightarrow A^t = [a_{ji}]_{n\times m}$
- If A and B are matrices such that AB exists, then $(AB)^t = B^tA^t$.
- The conic $ax^2 + 2hxy + by^2 = k$ has matrix equation $V^tMV = K$ where $V = \begin{bmatrix} x \\ y \end{bmatrix}$, $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ and $K = [k]$.
- Under the rotation defined by the rotation matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ the conic $V^tMV = K$ maps into the conic $V^t(M')V = K$, where $M' = RMR^t$.
- Under a rotation of θ about the point (0,0) with rotation matrix R, the conic $ax^2 + 2hxy + by^2 = k$ having $M = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ maps into the conic $a'x^2 + 2h'xy + b'y^2 = k$ where $\begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix} = M' = RMR^t$.
- If R is a rotation matrix and I is the 2×2 unit matrix, then $RR^t = R^tR = I$.

The Graph of $ax^2 + 2hxy + by^2 = k$

ellipse: $ab - h^2 > 0$ hyperbola: $ab - h^2 < 0$

- To remove the xy term from the conic equation $ax^2 + 2hxy + by^2 = k$, rotate the conic through an angle θ about the point (0,0) where if $a \neq b$, $\tan 2\theta = \dfrac{2h}{b - a}$ and if $a = b$, $\theta = 45°$ or $135°$.

Inventory

1. The conic $x^2 + y^2 = 16$ is a(n) ____ with ____ at (0,0) and ____ equal to 4.

2. The conic $9x^2 + 4y^2 = 36$ is a(n) ____ with ____ at (0,0) and x-intercepts ____, and y-intercepts ____.

3. The conic $9x^2 - 4y^2 = 36$ is a(n) ____ with ____ at (0,0) and x-intercepts ____, and y-intercepts ____.

4. The conic $9x^2 - 4y^2 = -36$ is a(n) ____ with ____ at (0,0) and x-intercepts ____, and y-intercepts ____.

5. The conic $9x^2 = y$ is a(n) ____ with ____ at (0,0) and axis of symmetry along ____.

6. The conic $4y^2 = x$ is a(n) ____ with ____ at (0,0) and axis of symmetry along ____.

7. To transform the conic with equation $4x^2 + 9y^2 + 6x + 10y + 1 = 0$ into standard position you must perform a transformation that is a ____. This transformation can be obtained by completing the ____ of the terms $4x^2 +$ ____ and also of the terms ____ $+ 10y$.

8. If $x^2 + 6x + w = (x + p)^2$, then $p =$ ____ and $w =$ ____.

9. If the conic C maps into the conic C' under the translation $(x,y) \rightarrow (x + 2, y + 3)$, then the conic C' maps back into the conic C under the translation $(x,y) \rightarrow$ ____.

10. If the conic $ax^2 + 2hxy + by^2 = k$ is an ellipse, then ____ is ____ 0. If the conic is a hyperbola, then ____ is ____ 0.

11. To transform the conic with equation $4x^2 + 6xy + 5y^2 = 10$ into standard position you must perform a transformation that is a ____. This transformation will eliminate the ____ term.

12. A rotation of θ about (0,0) that will eliminate the xy term from the equation $ax^2 + 2hxy + by^2 = k$ is given by the formula tan____ = ____, if a____b. If a____b, then $\theta = 45°$ or ____°.

13. The rotation matrix for a rotation of θ about the point (0,0) is $R =$ ____. The transpose of R, that is $R^t =$ ____.

14. For the conic $4x^2 + 6xy + 5y^2 = 7$, $a =$ ____, $b =$ ____, and h ____. The matrix equation is $V^tMV = K$ where V____, M____, and K____.

15. Under a rotation of 20° about the point (0,0) the conic $5x^2 + 6xy + 7y^2 = 2$ maps into the conic $a'x^2 + 2h'xy + b'y^2 = k$ where $\begin{bmatrix} a' & h' \\ h' & b' \end{bmatrix} = RMR^t$. For this conic, $R =$ ____, and $M =$ ____.

Review Exercises

Round off all numbers to 1 decimal place where appropriate.

1. Sketch the graph of each of the following conics.
   a) $x^2 + y^2 = 4$
   b) $4x^2 - 25y^2 = -100$
   c) $9x^2 - y^2 = 36$
   d) $4x^2 - y^2 = 4$
   e) $16x^2 + y^2 = 16$
   f) $y = 8x^2$
   g) $9x^2 + 9y^2 = 25$
   h) $x = -2y^2$

2. Given the ellipse E: $25x^2 + y^2 = 25$.
   a) Find, in general form, an equation for E', the image of E, under the translation $(x,y) \rightarrow (x + 3, y + 1)$.
   b) Sketch a graph of the ellipse E and its image ellipse E'.

3. Repeat the previous question for the following conics and translations.

	conic	translation
a)	$9x^2 - 16y^2 = 144$	$(x,y) \rightarrow (x - 3, y - 1)$
b)	$x^2 + y^2 = 16$	$(x,y) \rightarrow (x - 2, y + 3)$
c)	$y = 4x^2$	$(x,y) \rightarrow (x - 3, y - 2)$

4. For each of the following, indicate whether the conic is an ellipse, a circle, a hyperbola or a parabola.
   a) $2x^2 - 3y^2 - 5x - 7y - 10 = 0$
   b) $5x^2 + 5y^2 + 3x - 2y - 3 = 0$
   c) $8x^2 + 3x - 7y - 1 = 0$
   d) $4x^2 + 9y^2 - 3x + 7y + 3 = 0$
   e) $8x^2 - 4y^2 - 6x + 8y - 0 = 0$

5. Given the conic C: $x^2 + 8y^2 + 8x - 16y + 9 = 0$.
   a) Name the type of conic.
   b) Determine the translation that changes the equation into standard form.
   c) State an equation for C', the image of C.
   d) Graph the image conic C' and then the given conic C.

6. Repeat the previous question for the following conics.
   a) $x^2 + y^2 + 6x + 12y - 2 = 0$
   b) $x^2 - 12x + 4y + 4 = 0$
   c) $9x^2 - 4y^2 - 18x - 8y - 31 = 0$
   d) $4x^2 + y^2 + 8x - 10y + 13 = 0$
   e) $4x^2 - 4y^2 + 16x - 8y + 13 = 0$
   f) $x^2 + 25y^2 - 4x - 200y + 4 = 0$
   g) $y^2 - 8x + 4y - 36 = 0$
   h) $x^2 + y^2 - 8x + 6y - 1 = 0$

7. For each of the following conics $ax^2 + 2hxy + by^2 = k$, write the matrix M and the matrix K for its matrix equation $V^tMV = K$.
   a) $5x^2 - 6xy + 4y^2 = 7$
   b) $3x^2 - 6xy - y^2 = 1$
   c) $2x^2 + 3xy - y^2 = 9$
   d) $3x^2 + 2xy - 5y^2 = 8$
   e) $4x^2 - 12y^2 = 15$
   f) $5x^2 - 4xy + 5y^2 = 13$

8. Write the matrix equation $V^tMV = K$ for each of the conics in question 7.

9. For each of the following matrices M, write the Cartesian equation of the conic whose matrix equation is $V^tMV = K$.
   a) $\begin{bmatrix} 4 & -3 \\ -3 & 9 \end{bmatrix}$
   b) $\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$
   c) $\begin{bmatrix} -7 & -4 \\ -4 & 3 \end{bmatrix}$
   d) $\begin{bmatrix} 17 & 10 \\ 10 & 7 \end{bmatrix}$

10. Write the transpose of each of the following matrices.
   a) $\begin{bmatrix} 4 & 5 \\ -3 & 9 \end{bmatrix}$
   b) $[-4 \quad 3]$

11. Given $A = \begin{bmatrix} 2 & x \\ 0 & y \end{bmatrix}$,
   a) write A^t
   b) find values of x and y so that $A = A^t$.

12. Given the ellipse E: $9x^2 + 4y^2 = 36$.
 a) Write the corresponding matrix equation $V^tMV = K$.
 b) Find the matrix RMR^t for E', the image of E under a rotation of 30° about the point (0,0).
 c) Write a Cartesian equation for the image of E.
 d) Sketch a graph of E and E'.

13. Repeat the previous question for the following conics and rotation angles θ.
 a) $x^2 + 4y^2 = 9$, $\theta = 50°$
 b) $4x^2 - 9y^2 = 36$, $\theta = 100°$
 c) $x^2 + 9y^2 = 36$, $\theta = -20°$
 d) $3x^2 - y^2 = 4$, $\theta = 60°$

14. a) Given the ellipse E: $9x^2 + 4y^2 = 36$, find the image of E under a rotation of 30° about (0,0). Use exact values, from the table on page 543.
 b) Sketch the graph of E and E'.

15. a) A circle $x^2 + y^2 = 25$ is rotated about the point (0,0). Explain why an xy term will not be introduced into the image equation.
 b) Will an xy term be introduced in the image equation if the circle $x^2 + y^2 - 4x + 8y - 1 = 0$ is rotated about the point (0,0)? Explain.

16. For the rotation matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

and $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ show that

$$RMR^t = \begin{bmatrix} a + (b-a)\sin^2\theta & \frac{a-b}{2}\sin 2\theta \\ \frac{a-b}{2}\sin 2\theta & a + (b-a)\cos^2\theta \end{bmatrix}$$

17. Given the conic $x^2 + 2xy + 5y^2 = 4$.
 a) Name the type of conic.
 b) Determine an angle of rotation about (0,0) needed to remove the xy term.

18. Repeat the previous question for the following conics.
 a) $5x^2 + 2xy - 3y^2 = 4$
 b) $3x^2 - 2xy + y^2 = 2$
 c) $x^2 - 3xy + 4y^2 = 4$
 d) $3x^2 + 4xy + 3y^2 = 1$
 e) $3x^2 - 16xy + 5y^2 = 4$
 f) $6x^2 - 3xy + 2y^2 = 8$

19. Given the conic C: $17x^2 - 15xy + 17y^2 = 32$.
 a) Determine the type of conic.
 b) Find an angle of rotation about the point (0,0) that will eliminate the xy term.
 c) Find an equation of the image curve C'.
 d) Sketch the graph of the image curve C', then the graph of C, the original curve.

20. Repeat question 19 for the following conics.
 a) $x^2 - xy + y^2 = 2$
 b) $100x^2 - 173xy = 600$
 c) $20x^2 - 69xy - 20y^2 = 40$
 d) $30x^2 - 17.3xy + 50y^2 = 360$
 e) $29x^2 + 6xy + 21y^2 = 120$
 f) $22x^2 - 12xy + 17y^2 = 65$
 g) $5x^2 + 18xy + 5y^2 = 26$
 h) $9x^2 + 16xy + 21y^2 = 50$
 i) $47x^2 - 24xy + 57y^2 = 195$
 j) $x^2 - 16xy - 11y^2 = -30$
 k) $x^2 - 24xy - 6y^2 = -30$
 l) $4x^2 - 2xy + 4y^2 = 15$

21. Repeat question 19 for the following conics. Use exact values of the sines and cosines of the rotation angles, from the table on page 543.
 a) $17x^2 - 15xy + 17y^2 = 32$
 b) $13x^2 + 34\sqrt{3}xy + 47y^2 = 20$

22. Given the conic C: $10x^2 + 20xy + 10y^2 - 28x + 28y = 0$.
 a) Find an angle of rotation about (0,0) that will eliminate the xy term.
 b) Find an equation of the image curve C'.
 c) Determine the type of conic.
 d) Sketch the graph of the image curve C', then the graph of C, the original curve.