VECTORS, MATRICES and COMPLEX NUMBERS

with
International Baccalaureate
questions

John EGSGARD
and
Jean-Paul GINESTIER

CHAPTER NINE

MATHEMATICAL INDUCTION

e-mails johnegs@rogers.com
& jean-paul.ginestier@uwc.net

ISBN of Gage Edition 0-7715-3503-1

For orders contact:
Jean-Paul Ginestier
Red Cross Nordic UWC
N-6968 FLEKKE
Norway
Tel +47 5773 7000
Fax +47 5773 7001
e-mail jean-paul.ginestier@uwc.net

CHAPTER NINE

Mathematical Induction

There is a story told of a prince who had been disobedient. The prince was taken to a large room with a lock on the door. Inside that room was a second room and a key that opened the lock on the door of the second room. Inside this second room was a third room and a key that opened the lock on the door of the third room. Inside the third room was a fourth room and a key that opened the lock on the door of the fourth room. These rooms continued forever, with each room containing the key to open the lock on the door of the next room. His punishment was to open lock after lock and open door after door forever. Would the prince be able to continue in this way if he lived forever?

Another story is told of a queen who enjoyed playing with dominoes. She would place one domino on its end, then place a second on its end beside it, then a third beside the second and so on. Each domino was positioned so that it would knock over the next domino if it fell. The queen would push over the first domino which would push over the second which would push over the third, which would push over the fourth, and so on, until all the dominoes were knocked over. She continually added more and more dominoes in order to see them fall over. Then the queen wondered if she could continue to add dominoes forever so that her push on the first would cause all of the others to fall over. The queen offered a prize of 100 gold pieces to whomever could answer this question.

A third story is told of a cow who wanted to reach the moon using a very long ladder. She observed that she could get on the first rung of the ladder. She also realized that once she was on any rung she knew how to climb to the next rung. She wondered if this was enough to ensure that she could climb the ladder to the moon, and perhaps beyond the moon forever.

The topic of mathematical induction which you will study in this chapter will help you to solve the problems introduced in these stories.

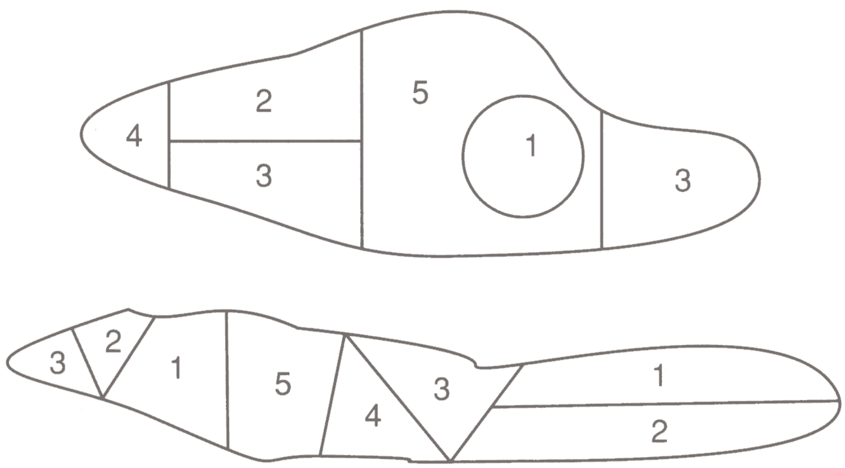
9.1 Making Conjectures

Proof is a very important part of mathematics. But, in the order of time, proof is generally the final event in a mathematical discovery. Mathematicians spend much of their energy trying to discover new mathematical truths. They make guesses or conjectures about what seems to be true then try to prove or disprove their conjectures. Some of these proofs or disproofs come immediately after the conjecture. Other conjectures are shown to be true or false a long time after their discovery. Some conjectures are still awaiting proof or disproof. Some examples of such conjectures are presented in the following paragraphs.

The Four Colour Problem

What is the minimum number of colours needed to colour the map of countries on a surface so that no two countries with the same boundary will be coloured with the same colour?

The following diagrams show some maps that have been coloured with five colours. Each of the five colours is indicated with a different number. Can you colour either map using fewer than five colours?



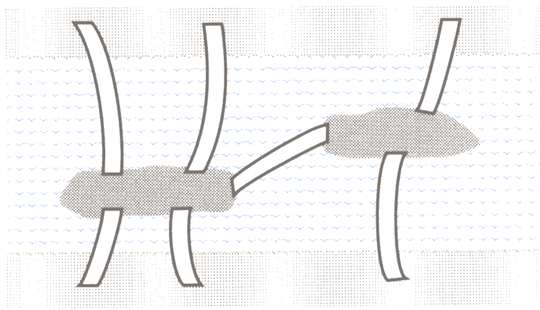
The Four Colour Problem goes back to October 23, 1852 when Francis Gutherie posed it to his teacher, De Morgan, who wrote to W.R. Hamilton. [reference: Stein: *The Man-Made Universe*, page 220] In 1890, P.J. Heawood proved that five colours were sufficient to colour any map. Most mathematicians conjectured that four colours were enough. Indeed, no one was able to draw a map that needed more than four colours. Nevertheless, it was not until 1976 that Kenneth Appel and Wolfgang Haken of the United States proved, using a computer, that four colours are sufficient.

The Koenigsberg Bridge Problem

In 1735 the Swiss mathematician Leonhard Euler described this problem as follows.

In the town of Koenigsberg there is an island called Kneiphof, with two branches of the river Pregel flowing around it. There are seven bridges crossing the two branches. The question is whether a person can walk in such a way that he will cross these bridges once but not more than once.

Here is a diagram of the seven bridges of Koenigsberg. Can a person plan a walk that will take the person across each bridge exactly once?



The problem had been around a long time before Euler. The townspeople used to spend their Sunday afternoons on such a walk, wondering if they could cross all of the bridges without repeating any bridges. They never succeeded in doing so. Euler showed in the same year that such a walk was impossible.

Fermat's Last Theorem

You know that the Pythagorean theorem states that for any right triangle with hypotenuse c and other two sides a, and b, that $a^2 + b^2 = c^2$.

Mathematicians wondered if a similar fact were true for any other power. For example, they tried to discover natural numbers a, b, and c such that $a^3 + b^3 = c^3$. The French mathematician Pierre de Fermat (1601-1665) wrote about this problem in the margin of a book he was reading. He said that he had discovered a truly wonderful proof that the equation $a^n + b^n = c^n$ does not have a solution in integer values of a, b and c for $n \geqslant 3$. He wrote that the margin of the book he was reading was too small to contain the proof. Since that time mathematicians have tried to rediscover Fermat's proof. But *no one* has been able to prove or disprove Fermat's statement!

Note how the simplicity of the statement of these problems gives no clue as to their difficulty.

In this section you will be given the opportunity to make your own conjectures about various mathematical ideas. You will not prove your guesses in this section. You will learn how to prove some of them in the next section 9.2.

Example 1 Conjecture a formula for the sum of the first n odd numbers, that is, for the sum $S_n = 1 + 3 + 5 + 7 + \ldots + (2n-1)$.

Solution To help find a pattern you should list and examine the values of S_1, S_2, S_3, S_4.

$S_1 = 1 = 1$

$S_2 = 1 + 3 = 4$

$S_3 = 1 + 3 + 5 = 9$

$S_4 = 1 + 3 + 5 + 7 = 16$

Observe that $S_1 = 1^2$

$S_2 = 2^2$

$S_3 = 3^2$

$S_4 = 4^2$

One conjecture or guess would be $S_n = n^2$

The conjecture should now be checked for other values of n.

Let $n = 6$.

$S_6 = 1 + 3 + 5 + 7 + 9 + 16 = 36 = 6^2$

as it should be, according to the conjecture. ■

Note: It is important for you to realize that the fact that a formula checks for particular values of n does *not* mean that the formula is true for all values of n.

For the series $1 + 2 + 3 + 4 + 5 + \ldots$

$S_1 = 1$, $S_2 = 3$, $S_3 = 6$, $S_4 = 10$.

The formula $S_n = \dfrac{n(n+1)}{2} + (n-1)(n-2)(n-3)$

produces the following values

$S_1 = 1$, $S_2 = 3$, $S_3 = 6$, $S_4 = 16$.

The formula checks for $n = 1$, 2, and 3 but *not* for $n = 4$.

Example 2 Guess a formula for the sum S_n of the series

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \ldots + \frac{1}{n(n+1)}$$

Solution To try to guess this sum you might list the partial sums as follows.

$$S_1 = \frac{1}{1 \times 2} = \frac{1}{2}$$

$$S_2 = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

$$S_3 = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} = \frac{2}{3} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4}$$

$$S_4 = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} = \frac{3}{4} + \frac{1}{20} = \frac{16}{20} = \frac{4}{5}$$

These values for S_n suggest the formula $S_n = \dfrac{n}{n+1}$

Checking the conjecture for $n = 5$, S_n should equal $\dfrac{5}{5+1}$ or $\dfrac{5}{6}$.

$$S_5 = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \frac{1}{5 \times 6} = \frac{4}{5} + \frac{1}{30} = \frac{25}{30} = \frac{5}{6},$$

as required. ■

Example 3 Make a conjecture about the values of $n \in \mathbb{N}$, for which $2^n < n!$, where $n! = n(n-1)(n-2)(n-3)\ldots(3)(2)(1)$ and $0! = 1$ ($n!$ is read "n factorial").

Solution Try values of $n \in \mathbb{N}$ beginning with $n = 1$.

Let $n = 1$: $L.S. = 2^1 = 2$, and $R.S. = 1! = 1$
Since $L.S. > R.S.$, the statement is false.

Let $n = 2$: $L.S. = 2^2 = 4$, and $R.S. = 2! = (2)(1) = 2$
Since $L.S. > R.S.$, the statement is false.

Let $n = 3$: $L.S. = 2^3 = 8$, and $R.S. = 3! = (3)(2)(1) = 6$
Since $L.S. > R.S.$, the statement is false.

Let $n = 4$: $L.S. = 2^4 = 16$, and $R.S. = 4! = (4)(3)(2)(1) = 24$
Since $L.S. < R.S.$, the statement is true.

Let $n = 5$: $L.S. = 2^5 = 32$, and $R.S. = 5! = (5)(4)(3)(2)(1) = 120$
Since $L.S. < R.S.$, the statement is true.

The statement appears to be true for $n \geq 4$.
Try further values of n to check the conjecture that $2^n < n!$, for $n \geq 4$. ■

9.1 Exercises

1. Conjecture a formula for the sum of n terms of the series
$$\frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \ldots - \frac{1}{2^n}$$

2. Conjecture a formula for the sum of n terms of each of the following series.

   a) $2 + 4 + 8 + 16 + 32 + \ldots + 2^n$, for $n > 1$

   b) $1 + 2 + 4 + 8 + 16 + \ldots + 2^{n-1}$

3. Given the series $1 + 2 + 3 + \ldots + n$.

   a) List S_1, S_2, S_3, and S_4.

   b) List $\frac{2S_1}{1}, \frac{2S_2}{2}, \frac{2S_3}{3}$, and $\frac{2S_4}{4}$

   c) Guess a formula for S_n.

4. Conjecture a formula for the sum of n terms of the series
$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \ldots + \frac{1}{(2n-1)(2n+1)}$$

5. Conjecture a formula for the sum of n terms of each of the following series.

   a) $\frac{1}{1 \times 5} + \frac{1}{5 \times 9} + \frac{1}{9 \times 13} + \ldots + \frac{1}{(4n-3)(4n+1)}$

   b) $\frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \ldots + \frac{1}{(n+1)(n+2)}$

   c) $\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n}$

6. a) Evaluate the 2-term product
$$(1+1)\left(1+\frac{1}{2}\right)$$

   b) Evaluate the 3-term product
$$(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)$$

   c) Evaluate the 4-term product
$$(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)$$

   d) Evaluate the 5-term product
$$(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)$$

   e) Conjecture a formula for value of the n-term product
$$(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\ldots\left(1+\frac{1}{n}\right)$$

7. Conjecture a formula for the value of each of the following n-term products.

   a) $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\left(1-\frac{1}{25}\right)\ldots\left(1-\frac{1}{n^2}\right)$, $n > 1$

   b) $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\left(1+\frac{9}{16}\right)\ldots\left(1+\frac{2n+1}{n^2}\right)$

8. Given $f(n) = \frac{n^3 + 3n^2 + 2n}{3}$

   a) Evaluate $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

   b) Make a conjecture concerning the values of $n \in \mathbb{N}$ for which $f(n)$ is a natural number.

9. Make a conjecture concerning the values of $n \in \mathbb{N}$ for which
$$t_n = \frac{5^n - 2^n}{3}$$
is a natural number.

10. Make a conjecture concerning the values of $n \in \mathbb{N}$, for which each of the following is a natural number.

   a) $\frac{n(n+1)}{2}$

   b) $\frac{n(n+1)(n+2)}{6}$

   c) $\frac{n^3 - n}{6}$

   d) $\frac{n^5 - n}{5}$

   e) $\frac{n^2 + 2n}{8}$

   f) $\frac{n^3 + 20n}{48}$

11. Make a conjecture concerning the values of $n \in \mathbb{N}$, for which
$2^n \geqslant n^2$

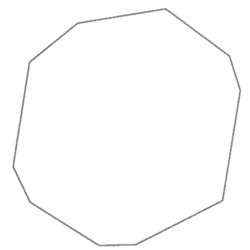
12. Make a conjecture concerning the values of $n \in \mathbb{N}$ for which each of the following inequalities is true.
   a) $3^n < n!$
   b) $3^n > 2^{n+1}$
   c) $\left(\frac{5}{6}\right)^n < \frac{5}{n}$

13. Given the series $S_n = 1 + 8 + 27 + \ldots + n^3$
   a) List $S_1, S_2, S_3, S_4, S_5, S_6$, and S_7
   b) List $\sqrt{S_1}, \sqrt{S_2}, \sqrt{S_3}, \sqrt{S_4}, \sqrt{S_5}, \sqrt{S_6}$, and $\sqrt{S_7}$
   c) Compare your answers to part b) with $S_1, S_2, S_3, S_4, S_5, S_6$, and S_7 of question 3.
   d) Guess a formula for S_n.

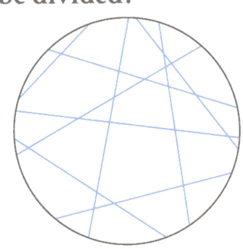
14. a) Mark 3 non-collinear points on a paper. How many different lines can you draw joining two of the points?
   b) Mark another point on the paper not collinear with the first 3 points of part a). You now have 4 non-collinear points on the paper. How many different lines can you draw joining two of the points?
   c) Mark another point on the paper not collinear with the 4 points of part b). You now have 5 non-collinear points on a paper. How many different lines can you draw joining two of the points?
   d) Conjecture a formula for the number of lines you can draw joining any two points among n non-collinear points, $n \in \mathbb{N}$. (If you need a hint read part e) of this question.)
   e) Completing the following table should help you to guess a formula for part d).

number of points (n)	number of lines	n^2	$n^2 - n$
1	0	1	0
2	1	4	2
3	3	9	6
4	*	*	*
5	*	*	*

15. a) Draw any quadrilateral and its diagonals. How many diagonals does a quadrilateral have?
   b) Draw any 5-gon, that is, a closed figure with 5 sides, and its diagonals. How many diagonals does a 5-gon have?
   c) Repeat part b) for a 6-gon and a 7-gon.
   d) Conjecture a formula for the number of diagonals that an n-gon has.



16. Draw a circle and any one chord of the circle. The interior of the circle is divided into 2 non-overlapping regions.
   a) Draw a second chord in the circle intersecting the first chord. What is the maximum number of non-overlapping regions into which the interior of the circle can be divided?



   b) Draw a third chord in the circle, intersecting the first and second chord. What is the maximum number of non-overlapping regions into which the interior of the circle can be divided?
   c) Draw a fourth chord in the circle, intersecting the three previous chords. What is the maximum number of non-overlapping regions into which the interior of the circle can be divided?
   d) Conjecture a formula for the maximum number of non-overlapping regions into which the interior of the circle can be divided by n chords.

9.2 The Principle of Mathematical Induction

In the introduction you met three stories, each of which involved starting an activity and trying to continue the activity forever. In each case the activity was repeated over and over. It was assumed that any one activity was followed immediately by the next one. You are now seeking a proof that these activities can indeed continue indefinitely.

Examine the story of the dominoes. Suppose the dominoes are numbered consecutively using the set of natural numbers $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. Suppose further that D is the set of numbers corresponding to the dominoes that will fall over after the queen has knocked over the first domino. Since the queen pushed over the first domino, you know that the number 1 is in D. Thus, $D = \{1, \ldots\}$. Now the dominoes are positioned in such a way that the kth domino falling over will push over the next, that is, the $(k + 1)$th domino. In other words, if the kth domino falls over the $(k + 1)$th will also fall over. Thus, if the number k is in D then the number $k + 1$ is also in D.

But the dominoes are infinite in number and marked with the natural numbers $1, 2, 3, 4, \ldots$. Thus, if you can show that D is actually the set $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ then you will know that all the dominoes have their numbers in D. Hence all the dominoes will fall over.

Now the set of natural numbers $\mathbb{N}$ has a very special property called the **inductive property of** $\mathbb{N}$. This property is as follows.

PROPERTY

Let T be a subset of the natural numbers $\mathbb{N}$. Then T is the *entire* set $\mathbb{N}$, if and only if *both* of the following are true.

a) 1 is a member of T.

b) If k is a member of T, then $k + 1$ is also a member of T.

But both of these are true for set D.

a) 1 is in D because the queen knocked over the first domino.

b) If the kth domino falls over it will push over the next domino, that is, the $(k + 1)$th domino. Thus, k being in D implies that $k + 1$ is also in D.

Hence by the inductive property of $\mathbb{N}$, the set D and the set $\mathbb{N}$ are the same set.

This process of using the inductive property of the set of natural numbers to prove something is true is called a **proof using mathematical induction.**

A proof using mathematical induction shows that a statement involving natural numbers is true. To help understand the method of proof, the inductive property of $\mathbb{N}$ is restated as **the principle of mathematical induction.**

PRINCIPLE

A statement involving the natural number n is true for every $n \in \mathbb{N}$ provided the following are true.

a) The statement is true for $n = 1$.

b) The truth of the statement for $n = k$ implies the statement is true for $n = k + 1$.

The principle of mathematical induction can be derived from the inductive property of $\mathbb{N}$. Suppose S is the set of natural numbers for which a statement is true. Then a) implies that $1 \in S$. But b) shows that $k \in S$ implies that $k + 1 \in S$. The inductive property of $\mathbb{N}$ allows the conclusion that $S = \mathbb{N}$.

METHOD

In practice, you should use three steps in a proof by mathematical induction.

Step 1: Show the statement is true for $n = 1$.
Step 2: Assume that the statement is true for $n = k$.
Step 3: Prove the statement is true for $n = k + 1$, *using the result of step 2*.

The principle of mathematical induction can only be used to prove a given formula is true. The principle does not help you to obtain such a formula. In Examples 1 and 2 and in Exercises 9.2 and 9.3 you will be given the opportunity to prove some of the conjectures you made in section 9.1 and in 9.1 Exercises.

Example 1

Use mathematical induction to prove the following formula for $n \in \mathbb{N}$.
$1 + 3 + 5 + \ldots + (2n - 1) = n^2$.

Solution

Step 1: Prove the statement is true for $n = 1$.
For $n = 1$, $L.S. = 1$, $R.S. = 1^2 = 1$
Since $L.S. = R.S.$, the statement is true for $n = 1$.

Step 2: Assume the formula is true for $n = k$. That is,
assume $1 + 3 + 5 + \ldots + (2k - 1) = k^2$. (*)

Step 3: Prove the formula is true for $n = k + 1$. That is, you have to prove that $1 + 3 + 5 + \ldots + (2k + 1) = (k + 1)^2$.

$L.S. = 1 + 3 + 5 + \ldots\ldots\ldots\ldots + (2k + 1)$
$= 1 + 3 + 5 + \ldots\ldots + (2k - 1) + (2k + 1)$ — the same as $L.S.$ (*) with $(2k + 1)$ added
$= [1 + 3 + 5 + \ldots\ldots + (2k - 1)] + (2k + 1)$
$= [k^2] + (2k + 1)$ — by step 2
$= k^2 + 2k + 1$
$= (k + 1)^2 = R.S.$

Thus, by the principle of mathematical induction,
$1 + 3 + 5 + \ldots + (2n - 1) = n^2$, for all $n = \in \mathbb{N}$. ■

Example 2 Prove, by mathematical induction

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

Solution

Step 1: For $n = 1$, $L.S. = \frac{1}{1\times 2}=\frac{1}{2}$, $R.S. = \frac{1}{1+1}=\frac{1}{2}=L.S.$

Step 2: Assume $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$ (*)

Step 3: Prove $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$

$$\begin{aligned}
L.S. &= \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\ldots\ldots\ldots\ldots\ldots+\frac{1}{(k+1)(k+2)}\\
&= \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\ldots\ldots\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}\\
&= \left[\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\ldots+\frac{1}{k(k+1)}\right]+\frac{1}{(k+1)(k+2)}\\
&= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \quad \text{using step 2}\\
&= \frac{k(k+2)+1}{(k+1)(k+2)}\\
&= \frac{k^2+2k+1}{(k+1)(k+2)}\\
&= \frac{(k+1)(k+1)}{(k+1)(k+2)}\\
&= \frac{k+1}{k+2} = R.S.
\end{aligned}$$

This is the *L.S.* of (*) with $\frac{1}{(k+1)(k+2)}$ added.

Thus, by the principle of mathematical induction, the formula is true. ■

Example 3 Use mathematical induction to prove the following formula for $n \in \mathbb{N}$.

$$\sum_{k=1}^{n}[2k-1]=n^2$$

Solution First write the sum explicitly by letting k equal successively $1, 2, 3, 4, \ldots, n$.

$\sum_{k=1}^{n}[2k-1]=n^2$ becomes

$[2(1)-1]+[2(2)-1]+[2(3)-1]+[2(4)-1]+\ldots+[2(n)-1]=n^2$
or $1+3+5+7+\ldots+[2n-1]=n^2$.

But this now is exactly the same problem as that of Example 1, so the solution is the same. This solution will not be repeated here. ■

9.2 Exercises

1. State the three steps in a proof using mathematical induction.

2. Prove the following statement using mathematical induction, where $n \in \mathbb{N}$.
$\frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \ldots - \frac{1}{2^n} = \frac{1}{2^n}$

3. Prove the following statements using mathematical induction, where $n \in \mathbb{N}$.
   a) $2 + 4 + 8 + 16 + 32 + \ldots + 2^n = 2^{n+1} - 2$
   [You will need to use the fact that $2^{k+1} + 2^{k+1} = 2^{k+2}$ Why is this true?]
   b) $1 + 2 + 4 + 8 + 16 + \ldots + 2^{n-1} = 2^n - 1$
   c) $6 + 12 + 18 + \ldots + 6n = 3n(n + 1)$
   d) $3 + 5 + 7 + \ldots + (2n + 1) = (n + 1)^2 - 1$

4. Prove the following statements using mathematical induction, where $n \in \mathbb{N}$.
   a) $\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \ldots + \frac{1}{(2n - 1)(2n + 1)} = \frac{n}{2n + 1}$
   b) $\frac{1}{1 \times 5} + \frac{1}{5 \times 9} + \ldots + \frac{1}{(4n - 3)(4n + 1)} = \frac{n}{4n + 1}$

5. All three steps are essential in a proof by mathematical induction, as the following will demonstrate.
A certain mathematician thought that he had a formula that produced prime numbers. (A prime number has exactly two divisors, 1 and the number itself.) He said that $n^2 + n + 41$ is always a prime number for $n \in \mathbb{N}$. He demonstrates the proof of his formula for $n = 1, 2, 3, \ldots 40$.
   a) Verify that the statement is true for $n = 1$, $n = 2$, $n = 3$, and $n = 4$.
   b) If you have the inclination you can show that the statement is true for all n from 1 to 40, inclusive. Nevertheless, do prove $n^2 + n + 41$ does not produce a prime number for $n = 41$.
   c) Which step(s) in a proof by mathematical induction is (are) missing in the demonstration given by the mathematician?

6. A friend tells you that the formula $7 + 9 + 11 + \ldots + (2n - 1) = n^2$ is true. He demonstrates this with the following argument.
Assume the formula is true for $n = k$, that is,
$7 + 9 + 11 + \ldots + (2k - 1) = k^2$ ①
Prove the formula is true for $n = k + 1$, that is, prove that
$7 + 9 + 11 + \ldots + (2[k + 1] - 1) = (k + 1)^2$ *
L.S. of * $= [7 + 9 + 11 + \ldots + (2k - 1)] + 2k + 1$
(the *L.S.* of ①, with $2k + 1$ added)
$= [k^2] + 2k + 1 = (k + 1)^2 =$ *R.S.* of *
(from ①)
   a) Which step(s), if any, in a proof by mathematical induction are missing in your friend's proof?
   b) Is your friend's formula true for all $n \in \mathbb{N}$?

7. Prove the following formulas by mathematical induction.
   a) $\sum_{s=1}^{n} s = \frac{n(n + 1)}{2}$
   b) $\sum_{s=1}^{n} s^2 = \frac{n(n + 1)(2n + 1)}{6}$
   c) $\sum_{s=1}^{n} s^3 = \left(\frac{n(n + 1)}{2}\right)^2$

8. Use mathematical induction to prove that the sum of n terms of an arithmetic series with first term a and common difference d
is $S_n = \frac{n}{2}[2a + (n - 1)d]$

9. Use mathematical induction to prove that the sum of n terms of a geometric series with first term a and common ratio r
is $S_n = \frac{a(r^n - 1)}{r - 1}$

9.3 Using Mathematical Induction

In section 9.1 and in 9.1 Exercises you made conjectures about formulas for the sum of series. In section 9.2 you learned how to use mathematical induction to prove your true conjectures. But in 9.1 Exercises you made conjectures about products, inequalities and geometrical conclusions. In this section you will learn how your true conjectures can be proven using mathematical induction.

The first example will deal with question 6 of 9.1 Exercises.

Example 1 Use mathematical induction to prove that

$$(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\dots\left(1+\frac{1}{n}\right)=n+1$$

Solution *Step 1*: Prove the statement is true for $n = 1$.

For $n = 1$, $L.S. = (1 + 1) = 2$, $R.S. = 1 + 1 = 2$

Since $L.S. = R.S.$, the statement is true for $n = 1$.

Step 2: Assume the statement is true for $n = k$. That is, assume

$$(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\dots\left(1+\frac{1}{k}\right)=k+1 \quad \circledast$$

Step 3: Prove the statement is true for $n = k + 1$. That is, prove

$$(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\dots\left(1+\frac{1}{k+1}\right)=(k+1)+1$$

$$L.S. = (1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\dots\dots\dots\left(1+\frac{1}{k+1}\right)$$

$$= (1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\dots\left(1+\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right)$$

the same as the $L.S.$ of $\circledast$ multiplied on the right by $\left(1+\frac{1}{k+1}\right)$

$$= \left[(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\dots\left(1+\frac{1}{k}\right)\right]\left(1+\frac{1}{k+1}\right)$$

$$= [k+1]\left(1+\frac{1}{k+1}\right) \quad \text{by step 2}$$

$$= \left([k+1]+[k+1]\frac{1}{k+1}\right) = (k+1+1) = k+2 = R.S.$$

Thus, by the principle of mathematical induction,

$$(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\dots\left(1+\frac{1}{n}\right)=n+1,$$

for all $n \in \mathbb{N}$. ■

In Example 2 you will prove a true conjecture about question 8 of 9.1 Exercises.

Example 2 Use mathematical induction to prove that

$$f(n) = \frac{n^3 + 3n^2 + 2n}{3} \text{ is a natural number for all } n \in \mathbb{N}.$$

Solution

Step 1: Prove the statement is true for $n = 1$.

For $n = 1$, $f(1) = \frac{(1)^3 + 3(1)^2 + 2(1)}{3} = \frac{6}{3} = 2$, which is a natural number. Thus, the statement is true for $n = 1$.

Step 2: Assume the statement is true for $n = k$. That is,

assume $f(k) = \frac{k^3 + 3k^2 + 2k}{3}$ is a natural number.

Step 3: Prove the statement is true for $n = k + 1$. That is,

prove $f(k + 1) = \frac{(k + 1)^3 + 3(k + 1)^2 + 2(k + 1)}{3}$ is a natural number.

$$\text{But} \quad f(k + 1) = \frac{k^3 + 3k^2 + 3k + 1 + 3(k^2 + 2k + 1) + 2k + 2}{3}$$

$$= \frac{k^3 + 3k^2 + 2k}{3} + \frac{3k^2 + 9k + 6}{3} = f(k) + k^2 + 3k + 2$$

From step 2 you know that $f(k)$ is natural number. Also, because k is a natural number, $k^2 + 3k + 2$ is also a natural number.

Thus, $f(k) + k^2 + 3k + 2 = f(k + 1)$ is a natural number.

Thus, by the principle of mathematical induction,

$f(n) = \frac{n^3 + 3n^2 + 2n}{3}$ is a natural number for all $n \in \mathbb{N}$. ■

The next example shows how to prove your true conjectures about inequalities by studying the inequality of Example 3, section 9.1.

Example 3 Use mathematical induction to prove that $2^n < n!$ for $n \geqslant 4$, $n \in \mathbb{N}$.

Solution This statement is not true for $n = 1, 2$, and 3 as you can easily check. So step 1 must begin with $n = 4$.

Step 1: Prove the statement is true for $n = 4$.
For $n = 4$, $L.S. = 2^4 = 16$, $R.S. = 4! = (4)(3)(2)(1) = 32$.
Therefore $L.S. < R.S.$, so the statement is true for $n = 4$.

Step 2: Assume the statement is true for $n = k$. That is, assume $2^k < k!$

Step 3: Prove the statement is true for $n = k + 1$. That is, prove $2^{k+1} < (k + 1)!$
But from step 2 you know that $2^k < k!$
Multiplying both sides by 2 gives $2^{k+1} < 2(k!)$
Since $k \geqslant 4$, $2 < k + 1$, so $2(k)! < (k + 1)(k!) = (k + 1)!$
Thus, $2^{k+1} < (k + 1)!$, as required by step 3.

Thus, by the principle of mathematical induction, $2^n < n!$ for $n \geqslant 4$, $n \in \mathbb{N}$. ■

Note: If the statement to be proved is not true for the first few natural numbers, then step 1 must be done for the first number for which the statement is true.

9.3 Exercises

1. Use mathematical induction to prove that
$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\left(1+\frac{9}{16}\right)\ldots\left(1+\frac{2n+1}{n^2}\right)=(n+1)^2$$

2. Use mathematical induction to prove that for $n \geqslant 2$
$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\left(1-\frac{1}{25}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$

3. Use mathematical induction to prove that
$$\left(1-\frac{4}{1}\right)\left(1-\frac{4}{9}\right)\left(1-\frac{4}{25}\right)\ldots\left(1-\frac{4}{(2n-1)^2}\right)=\frac{2n+1}{2n-1}$$

4. Use mathematical induction to prove that each of the following is a natural number for all $n \in \mathbb{N}$.
   a) $\dfrac{n(n+1)}{2}$
   b) $\dfrac{n(n+1)(n+2)}{6}$
   c) $\dfrac{n^3-n}{6}$
   d) $\dfrac{n^5-n}{5}$

5. Use mathematical induction to prove that each of the following is a natural number, provided n is an even natural number. (hint: let $n = 2m$, $m \in \mathbb{N}$)
   a) $\dfrac{n^2+2n}{8}$
   b) $\dfrac{n^3+20n}{48}$

6. Use mathematical induction to prove that $2^n > n^2$ for $n > 4$, $n \in \mathbb{N}$.

7. Prove the following where $n \in \mathbb{N}$.
   a) $3^n < n!$ for $n \geqslant 7$
   b) $3^n > 2^{n+1}$ for $n \geqslant 2$
   c) $\left(\frac{5}{6}\right)^n < \frac{5}{n}$ for all n

8. Prove that $(1+x)^n > 1 + nx$ for $x > 0$ and $n > 1$, $x \in \mathbb{R}$, $n \in \mathbb{N}$.

9. Prove that $(1+x)^n > 1 + nx + nx^2$ for $x > 0$ and $n > 2$, $x \in \mathbb{R}$, $n \in \mathbb{N}$.

10. a) Show that $5^{k+1} = 5 \times 5^k = 3 \times 5^k + 2 \times 5^k$.
    b) Use mathematical induction to prove that $\dfrac{5^n - 2^n}{3}$ is a natural number for all $n \in \mathbb{N}$.

11. Given a set of n points, no three of which are collinear, prove that the number of line segments that can be drawn joining these points in pairs is $\dfrac{n(n-1)}{2}$
[See question 14, 9.1 Exercises.]

12. Use mathematical induction to prove that an n-gon has $\dfrac{n(n-3)}{2}$ diagonals.
[See question 15, 9.1 Exercises.]

13. Given a circle and a set of n chords of this circle, show that the maximum number of non-overlapping regions into which the circle can be divided is $\dfrac{n^2+n+2}{2}$
[See question 16, 9.1 Exercises.]

14. Prove that $\dfrac{n}{2} \leqslant \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \ldots + \dfrac{1}{2^n} < n$

15. Prove that $\displaystyle\sum_{k=1}^{n-1} k^3 < \frac{n^4}{4} < \sum_{k=1}^{n} k^3$

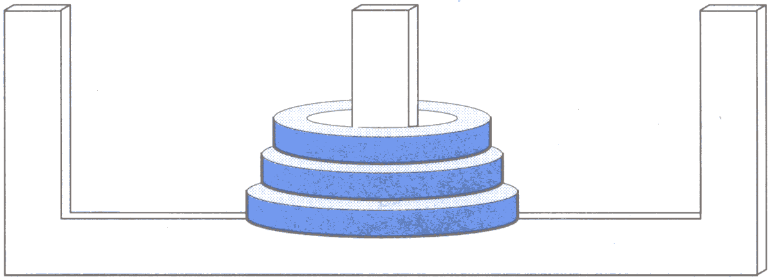
16. Use mathematical induction to prove the following.
    a) $\displaystyle\sum_{k=1}^{n} \frac{k^2}{(2k-1)(2k+1)} = \frac{n(n+1)}{2(2n+1)}$
    b) $\displaystyle\sum_{k=1}^{n} k^4 = \frac{6n^5+15n^4+10n^3-n}{30}$

17. Prove the following about matrix multiplication.
    a) $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$, $k \in \mathbb{N}$
    b) $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & kx \\ 0 & 1 \end{bmatrix}$

18. Prove $\displaystyle\frac{n!}{(n-k)!k!} = \sum_{i=k-1}^{n-1} \frac{i!}{(i-k+1)!(k-1)!}$

In Search of A Solution to the Tower of Hanoi Problem

There is an interesting and challenging puzzle called the *Tower of Hanoi*. The puzzle consists of three pegs and a set of graduated discs, as shown in the figure.



The problem posed is to transfer the discs from any one peg to another peg under the following rules.

1. Only one disc may be transferred at a time from one peg to another peg.
2. A larger disc may never be placed upon a smaller disc.

This problem can be solved using the principle of mathematical induction. Indeed, you can use this principle to calculate the minimum number of moves that would be needed for a given number n of discs.

Examine the problem for one disc, then for two discs, and finally for three discs to get some idea of the pattern involved.

One disc It is clear that one disc can be transferred in one move.

Two discs First transfer one disc leaving a peg free for the second disc. You then transfer the second disc. Finally cover the second disc with the first disc. This takes three moves.

Three discs First transfer the top two discs as above in three moves. This leaves a peg free for the third disc which is moved in one more move. Then the top two discs can be transferred onto the third disc in three moves, as above for two discs. This gives a total of seven moves.

The pattern for moving any number of discs is now clear. If you can transfer k discs you can easily transfer $k + 1$ discs. First you transfer the k discs leaving the $(k + 1)$th disc free to move to a new peg. Then the top k discs can be moved onto the $(k + 1)$th disc. Thus, the problem can be solved for any number of discs.

To determine the minimum number of moves needed to transfer n discs, observe that no disc can be moved until all of the discs above it have been transferred. Then a space is left free to which you can move that disc.

Suppose the minimum number moves for k discs is $m(k)$. To move the $(k + 1)$th disc, you will need $m(k)$ moves to the transfer the discs *above it* to another peg. Then you can transfer the $(k + 1)$th disc to the free peg. Now, to move the k discs over, to be on top of the $(k + 1)$th disc, will again take $m(k)$ moves.
Thus the total number of moves to transfer $k + 1$ discs is $m(k) + 1 + m(k)$, or $2m(k) + 1$. That is, $m(k + 1) = 2m(k) + 1$.

To use mathematical induction to determine the minimum number of moves for n discs you must now try to guess a formula for $m(n)$.

The following table gives values of $m(n)$ for n from 1 to 5.

n	1	2	3	4	5
$m(n)$	1	3	7	15	31

Adding a disc appears to 'double' the number of moves, so that this sequence of numbers should be compared with the doubling sequence 1, 2, 4, 8, 16, 32. It appears that $m(n) = 2^n - 1$.

A proof of this formula follows.
Step 1. The formula is true for $n = 1$ because $2^1 - 1 = 1 = m(1)$.
Step 2 Assume the formula is true for $n = k$.
Thus the minimum number of moves for k discs is $2^k - 1$.
Step 3 Prove the formula is true for $n = k + 1$, that is show that the minimum number of moves for $k + 1$ discs is $2^{k+1} - 1$.

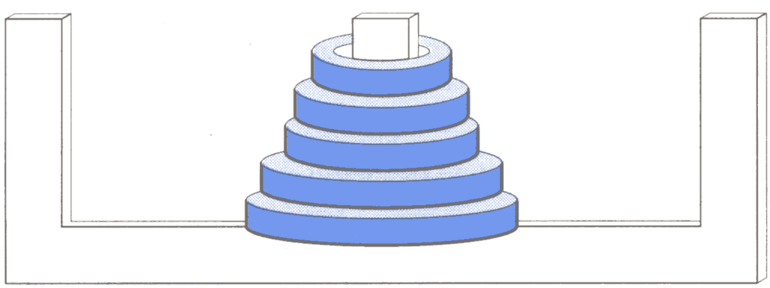
Proof: You showed above that
$m(k + 1) = 2m(k) + 1$.
Using step 2, you can say that

$$m(k + 1) = 2(2^k - 1) + 1 = 2^{k+1} - 2 + 1 = 2^{k+1} - 1$$

Hence, by the principle of mathematical induction, the minimum number of moves needed to transfer n discs is $2^n - 1$.

Activity

Try the puzzle with three discs to see if you can match the minimum number of moves. Then do the same for 4 and 5 discs.



9.4 The Binomial Theorem

There is a very important and useful formula that involves the natural numbers and the binomial $(a + x)$. This formula is called the **binomial theorem.**

The formula gives a short cut for finding values of products such as $(a + x)^2$, $(a + x)^3$, $(a + x)^4$, $(a + x)^5$, and so on.

You may already know the following products.
$(a + x)^2 = (a + x)(a + x) = a^2 + 2ax + x^2$
$(a + x)^3 = (a + x)(a + x)(a + x) = a^3 + 3a^2x + 3ax^2 + x^3$

The binomial theorem is stated as follows.

THEOREM

$$(a + x)^n = C(n,0)a^nx^0 + C(n,1)a^{n-1}x^1 + C(n,2)a^{n-2}x^2 + C(n,3)a^{n-3}x^3 + \ldots + C(n,r)a^{n-r}x^r + \ldots + C(n,n-1)a^1x^{n-1} + C(n,n)a^0x^n,$$

where $n \in \mathbb{N}$

Note 1 The value of $C(n,r)$ is $\dfrac{n!}{(n-r)!\,r!}$, where $n! = n(n-1)(n-2)\ldots(3)(2)(1)$.

2 The expansion of the product has $n + 1$ terms.

Example 1

Expand the product $(a + x)^4$.

Solution

Use the binomial theorem

$$(a + x)^n = C(n,0)a^nx^0 + C(n,1)a^{n-1}x^1 + C(n,2)a^{n-2}x^2 + C(n,3)a^{n-3}x^3 + \ldots + C(n,r)a^{n-r}x^r + \ldots + C(n,n-1)a^1x^{n-1} + C(n,n)a^0x^n.$$

Here $n = 4$.

Thus,
$$(a + x)^4 = C(4,0)a^4x^0 + C(4,1)a^{4-1}x^1 + C(4,2)a^{4-2}x^2 + C(4,3)a^{4-3}x^3 + C(4,4)a^{4-4}x^4$$

Now $C(4,0) = \dfrac{4!}{(4-0)!0!} = 1$ recall that $0! = 1$

$$C(4,1) = \frac{4!}{(4-1)!1!} = \frac{4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 1} = 4$$

$$C(4,2) = \frac{4!}{(4-2)!2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = 6$$

$$C(4,3) = \frac{4!}{(4-3)!3!} = 4$$

$$C(4,4) = \frac{4!}{(4-4)!4!} = 1$$

Therefore, $(a + x)^4 = a^4x^0 + 4a^3x^1 + 6a^2x^2 + 4a^1x^3 + 1a^0x^4$
$= a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$ ■

Pascal's Triangle

Observe the coefficients of the expansion of $(a + x)^n$ for $n = 1, 2, 3$, and 4.

value of n	expansion of $(a + x)^n$	Pascal's triangle
1	$1a^1 + 1x^1$	1 1
2	$1a^2 + 2ax + 1x^2$	1 2 1
3	$1a^3 + 3a^2x + 3ax^2 + 1x^3$	1 3 3 1
4	$1a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + 1x^4$	1 4 6 4 1

Note: In Pascal's triangle, the numbers on the left and right of each row are both 1. Each of the other numbers is the *sum* of the two numbers on each side of it in the line *above*.

Thus, the next line in the triangle, corresponding to $n = 5$, will be
1, 1 + 4, 4 + 6, 6 + 4, 4 + 1,1 or 1, 5, 10, 10, 5, 1

Each line of Pascal's triangle can also be written in terms of $C(n,r)$.

$$\begin{array}{c} C(1,0)\ \ C(1,1) \\ C(2,0)\ \ C(2,1)\ \ C(2,2) \\ C(3,0)\ \ C(3,1)\ \ C(3,2)\ \ C(3,3) \\ C(4,0)\ \ C(4,1)\ \ C(4,2)\ \ C(4,3)\ \ C(4,4) \end{array}$$

For $C(n,r)$'s the note above means that, for example, $C(4,2) = C(3,1) + C(3,2)$ and $C(3,1) = C(2,0) + C(2,1)$

In general, the following is true: $C(n + 1,r) = C(n,r - 1) + C(n,r)$

You will be given the opportunity to prove this in the exercises that follow.

Example 2 Expand $(3m - 2y)^4$.

Solution Since $n = 6$ the expansion you need is

$$(a + x)^6 = C(6,0)a^6x^0 + C(6,1)a^{6-1}x^1 + C(6,2)a^{6-2}x^2 + C(6,3)a^{6-3}x^3 + C(6,4)a^{6-4}x^4 + C(6,5)a^{6-5}x^5 + C(6,6)a^{6-6}x^6$$

where $a = 3m$ and $x = -2y$. Thus,

$$\begin{aligned}(3m + (-2y))^6 &= C(6,0)(3m)^6(-2y)^0 + C(6,1)(3m)^{6-1}(-2y)^1 + C(6,2)(3m)^{6-2}(-2y)^2 \\ &\quad + C(6,3)(3m)^{6-3}(-2y)^3 + C(6,4)(3m)^{6-4}(-2y)^4 \\ &\quad + C(6,5)(3m)^{6-5}(-2y)^5 + C(6,6)(3m)^{6-6}(-2y)^6 \\ &= 1(729)m^6 + 6(243)m^5(-2)y + 15(81)m^4(4)y^2 \\ &\quad + 20(27)m^3(-8)y^3 + 15(9)m^2(16)y^4 + 6(3)m^1(-32)y^5 + 1(1)(64)y^6 \\ &= 729m^6 - 2916m^5y + 4860m^4y^2 - 4320m^3y^3 \\ &\quad + 2160m^2y^4 - 576my^5 + 64y^6\end{aligned}$$

■

Check that the value of the $C(n,r)$'s, make up line $n = 6$ of Pascal's triangle.

Because the binomial theorem involves the natural numbers $\mathbb{N}$, you can prove the theorem using mathematical induction. The theorem will be stated using the sigma notation Σ.

In proving the theorem you will make use of the formula
$C(n + 1,r) = C(n,r - 1) + C(n,r)$
You will also use a property of the sigma notation about changing the limits of sigma, namely,

$$\sum_{r=t}^{m} a_r = \sum_{r=t+d}^{m+d} a_{r-d}$$

Example 3 Use mathematical induction to prove the binomial theorem

$$(a + x)^n = \sum_{r=0}^{n} C(n,r)a^{n-r}x^r$$

Solution *Step 1*: Let $n = 1$. $L.S. = (a + x)^1$, $R.S. = \sum_{r=0}^{1} C(n,r)a^{1-r}x^r$, that is,

$$R.S. = C(1,0)a^1x^0 + C(1,1)a^0x^1 = 1a + 1x = a + x = L.S.$$

Step 2: Assume the formula is true for $n = k$.

Assume $(a + x)^k = \sum_{r=0}^{k} C(k,r)a^{k-r}x^r$ (*)

Step 3: Prove the formula is true for $n = k + 1$, that is,

prove $(a + x)^{k+1} = \sum_{r=0}^{k+1} C(k + 1,r)a^{k+1-r}x^r$

But $(a + x)^{k+1}$

$= (a + x)(a + x)^k$ This is the L.S. of (*) multiplied by $(a + x)$.

$= (a + x)\sum_{r=0}^{k} C(k,r)a^{k-r}x^r$ using step 2

$= a\sum_{r=0}^{k} C(k,r)a^{k-r}x^r + x\sum_{r=0}^{k} C(k,r)a^{k-r}x^r$

$= \sum_{r=0}^{k} C(k,r)a^{k-r+1}x^r + \sum_{r=0}^{k} C(k,r)a^{k-r}x^{r+1}$

Now by changing the limits of the summation,

the second summation $\sum_{r=0}^{k} C(k,r)a^{k-r}x^{r+1}$ can be written $\sum_{r=1}^{k+1} C(k,r - 1)a^{k-r+1}x^r$

Thus $(a + x)^{k+1} = \sum_{r=0}^{k} C(k,r)a^{k-r+1}x^r + \sum_{r=1}^{k+1} C(k,r - 1)a^{k-r+1}x^r$

In order to combine these two summations you must write each summation so that each has the same limits. You can accomplish this by removing the first term from the first summation and the last term from the second summation.

$$(a + x)^{k+1} = C(k,0)a^{k+1}x^0 + \sum_{r=1}^{k} C(k,r)a^{k-r+1}x^r + \sum_{r=1}^{k} C(k,r - 1)a^{k-r+1}x^r + C(k,k)a^0x^{k+1}$$

$$= C(k,0)a^{k+1}x^0 + \sum_{r=1}^{k} [C(k,r) + C(k,r - 1)]a^{k-r+1}x^r + C(k,k)a^0x^{k+1}$$

This expression may be simplified using the following facts.

1. $C(k,0) = 1 = C(k + 1,0)$
2. $C(k,k) = 1 = C(k + 1,k + 1)$
3. $C(k,r) + C(k,r - 1) = C(k + 1,r)$

$$(a + x)^{k+1} = C(k + 1,0)a^{k+1}x^0 + \sum_{r=1}^{k} C(k + 1,r)a^{k-r+1}x^r + C(k + 1,k + 1)a^0x^{k+1}$$

This may be combined under one summation giving

$(a + x)^{k+1} = \sum_{r=0}^{k+1} C(k + 1,r)a^{k-r+1}x^r$ which is what needed to be proven. ■

The term $C(n,r)a^{n-r}x^r$ is called the **general term** in the expansion of $(a + x)^n$. You will find questions on the general term in the exercises.

Note that since $C(n,r) = \dfrac{n!}{(n-r)!r!} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$

then $C(n,0) = 1$, $C(n,1) = n$, $C(n,2) = \dfrac{n(n-1)}{2!}$, $C(n,3) = \dfrac{n(n-1)(n-2)}{3!}\ldots$

$C(n,n-1) = n$, $C(n,n) = 1$

Hence an alternative form for the binomial theorem is

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}a^{n-r}x^r + \ldots + nax^{n-1} + x^n$$

The Binomial Theorem for n not a Natural Number

The binomial theorem has been proven for n a natural number. A similar result is true when n is *not* a natural number. In this case, however, there are three important differences.

First, the alternative form of the expansion of $(a + x)^n$, with factorials, must be used.

Second, instead of a series with a finite number of terms, you will have an *infinite* series.

Thirdly, the expansion is true only for certain values of a and x. Indeed, the expansion is true only for values of a and x such that $-1 < \dfrac{x}{a} < 1$.

The result (which will not be proven) is the following, where $n \in \mathbb{R}$ but $n \notin \mathbb{N}$.

$$(a+x)^n =$$
$$a^n x^0 + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \ldots + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}a^{n-r}x^r + \ldots \text{(an infinite number of terms)}$$

Frequently this statement is written for $a = 1$ to give the following, where $n \in \mathbb{R}$ but $n \notin \mathbb{N}$ and $-1 < x < 1$.

$$(1+x)^n =$$
$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots \text{(an infinite number of terms)}$$

9.4 Exercises

1. a) Write the rows of Pascal's triangle for $n = 1$ to $n = 8$.
   b) Evaluate 4! and 6!
   c) Evaluate $C(7,3)$.
   d) Evaluate $C(5,0)$, $C(5,1)$, $C(5,2)$, $C(5,3)$, $C(5,4)$, $C(5,5)$.
   e) Check that your answer for part d) is the same as your answer for row $n = 5$ in part a).

2. Expand each of the following. Do not simplify the $C(n,r)$'s.
   a) $(a + x)^4$ c) $(a + x)^6$ e) $(a + x)^8$
   b) $(a + x)^5$ d) $(a + x)^7$ f) $(a + x)^9$

3. Rewrite each part of question 2 by substituting the values for the $C(n,r)$'s. You may find these either by using the formula $C(n,r) = \dfrac{n!}{(n-r)!r!}$ or by using the appropriate row in Pascal's triangle.

4. Expand each of the following and simplify.
   a) $(a + y)^4$ c) $(m + z)^3$ e) $(a + 1)^8$
   b) $(b - c)^4$ d) $(2 + x)^5$ f) $(3 - b)^4$

5. Expand each of the following and simplify.
   a) $(a + 2b)^4$ c) $(3 - 2m)^3$ e) $(2x + 3a)^5$
   b) $(3a + 4b)^4$ d) $(4a - 5)^3$ f) $(1 - m^2)^6$

6. Find the first four terms in the expansion of each of the following. Do not simplify.
   a) $(a + b)^{40}$ c) $(3 + x)^{23}$ e) $(2m - 3t)^{25}$
   b) $(m - k)^{39}$ d) $(4 + 2a)^{85}$ f) $(1 + b^2)^{36}$

7. Expand.
   a) $\left(x + \dfrac{1}{x}\right)^4$ b) $\left(x - \dfrac{2}{x^2}\right)^5$

8. a) Find the general term for $\left(x^2 + \dfrac{1}{x}\right)^6$
   b) Find the term containing x^9 in the expansion of the binomial in part a).
   c) Find the term containing x^0 in the expansion of the binomial in part a). This term is called the term independent of x.

9. Prove each of the following facts about the relationship among an element of one row in Pascal's triangle and the elements above it to the left and right.
   a) $C(4,2) = C(3,1) + C(3,2)$
   b) $C(8,5) = C(7,4) + C(7,5)$
   c) $C(n + 1,r) = C(n,r - 1) + C(n,r)$

10. Show each of the following is true by writing each sum explicitly. For example, $\sum_{r=1}^{3} a_r = a_1 + a_2 + a_3$.
   a) $\sum_{r=1}^{8} a_r = \sum_{r=5}^{12} a_{r-4}$ b) $\sum_{r=t}^{m} a_r = \sum_{r=t+d}^{m+d} a_{r-d}$

11. The first two terms in the expansion of $(3 + kx)^7$ are $2187 + 20\,412x$. Find the value of k.

12. By substituting $a = x = 1$ in the expansion of $(a + x)^n$ show that
$$C(n,0) + C(n,1) + C(n,2) + C(n,3) + \dots + C(n,n - 1) + C(n,n) = 2^n$$

13. Use the expansion of $(a + x)^n$ to show that
$$C(n,0) - C(n,1) + C(n,2) - C(n,3) + \dots + (-1)^n C(n,n) = 0.$$

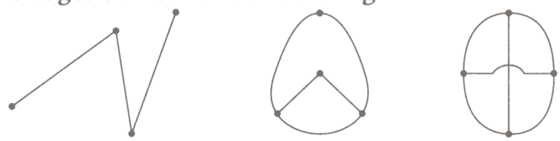
14. a) Find the first four terms in the expansion of $(1 + x)^{-2}$.
   b) Find the first four terms in the expansion of $(1 + x)^{\frac{1}{2}}$.

15. In your answers to question 14 a) and b), give x the value 0.02. Simplify these expressions to obtain approximate values for $(1.02)^{-2}$ and $(1.02)^{\frac{1}{2}}$. How do these values compare with the values of $(1.02)^{-2}$ and $(1.02)^{\frac{1}{2}}$ found using the $\boxed{y^x}$ key of your calculator?

16. a) Write a computer program for the expansion of $(a + x)^n$, $n \in \mathbb{N}$.
   b) Use your program to check your answers for questions 2, 4, 5, and 6.
   c) Adjust your program so that it will handle the first few terms in the expansion of $(a + x)^n$, $n \notin \mathbb{N}$.

Graph Theory

The graphs of graph theory are different from the graphs of lines, parabolas, circles, etc, with which you are familiar. In graph theory a *graph* is defined as a set of points called *vertices*, and a set of lines called *edges*, that connect pairs of vertices. The figure shows several graphs. Notice that the edges do not have to be straight lines.



The systematic study of graph theory began in the 18th century with the famous problems of the seven bridges of Koenigsberg, which you met in the introduction to this chapter.

Graph theory is used extensively to solve problems involved in the management of complex systems such as those in business and industry. The following is a simple example.

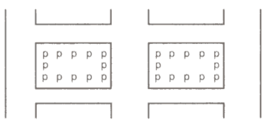


figure 1a

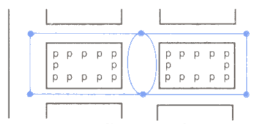


figure 1b

The map in figure 1a shows a section of a city where there are two blocks containing parking meters.

You are hired to find the most efficient route that a parking control officer should travel (on foot) to check the meters. You must consider two things.

1. The parking control officer must patrol all of the meters without retracing steps any more than is necessary.
2. The route should end at the same point it started, where the officer's car is parked.

To solve this problem you need to draw graphs. One such graph is indicated in figure 1b. Notice that each street intersection is a vertex and each sidewalk is an edge. Two possible routes covering this graph are indicated in figure 2a and 2b. It is clear that figure 2b is a better solution because it covers every edge (sidewalk) only once. A route that covers every edge only once is called an *Euler circuit*, provided that the route starts and finishes at the same vertex.

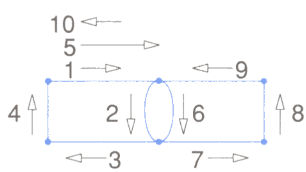


figure 2a

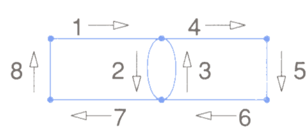


figure 2b

One of the first discoveries made in graph theory was that there are some graphs that do not have any Euler circuits. Two examples of such graphs are shown in figure 3, where it is impossible to start at a vertex and return to the same vertex unless you cover the same edge more than once.

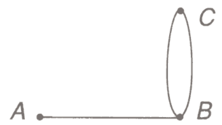


figure 3a

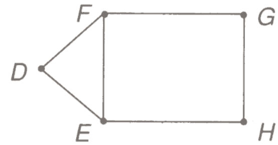


figure 3b

Euler was able to determine the conditions under which a graph had an Euler circuit. He used the concepts of valence and connectedness. The *valence* of a vertex in a graph is the number of edges meeting at that point. (Point *A* has valence 1, points *C*, *D*, *G*, and *H* have valence 2, while points *B*, *E* and *F* have valence 3.) A graph is *connected* if every pair of vertices is joined by at least one edge. The graph in figure 3a is not connected because points *A* and *C* are not joined by an edge.

Euler proved that a graph *G* has an Euler circuit if and only if the following two conditions are true.

1. *G* must be connected
2. Each vertex must have an even valence.

If you examine the graph for the parking control officer given above in figure 1b, you will see that both of these conditions are fulfilled.

It is interesting to note that the graph of the seven bridges of Koenigsberg does not have an Euler circuit. (See page 387.)

The following problems are among the many that can be solved using graph theory.

Computers, radios and TVs make use of printed circuits. These circuits are conductive paths on a sheet of nonconductive material. What conditions must hold for such a circuit to be able to be printed on a single nonconductive sheet?

A telephone company wishes to send long distance messages between cities at the least possible expense in transmission and in the construction of interconnecting telephone lines. What cities should be joined directly by telephone lines? What path should a telephone signal take to travel from city *A* to city *B*?

A salesperson must visit several cities always starting and ending at the same city. What route should be taken so that the cost of the trip will be a minimum?

What is the best way to prepare an airplane so that the airplane is on the ground for the least amount of time? Remember that passengers and baggage must be loaded and unloaded, the cabin must be cleaned, food must be brought on board and the airplane must be refueled.

Summary

- The *inductive property of* $\mathbb{N}$: let T be a subset of the natural numbers $\mathbb{N}$. Then T is the *entire* set $\mathbb{N}$, if and only if *both* of the following are true.
  a) 1 is a member of T.
  b) If k is a member of T, then $k + 1$ is also a member of T.
- *The principle of mathematical induction*: a statement involving the natural number n is true for every $n \in \mathbb{N}$ provided the following are true.
  a) The statement is true for $n = 1$.
  b) The truth of the statement for $n = k$ implies the statement is true for $n = k + 1$.
- The *three steps in a proof by mathematical induction.*
  Step 1: Show the statement is true for $n = 1$.
  Step 2: Assume that the statement is true for $n = k$.
  Step 3: Prove the statement is true for $n = k + 1$, *using the result of step 2.*

If the statement to be proved is not true for the first few natural numbers then step 1 must be done for the first number for which the statement is true.

- The principle of mathematical induction can only be used to prove a given formula is true. The principle does not help you to obtain such a formula. If a formula is not given you can try to guess a formula by examining results for $n = 1, 2, 3$, and 4. When you guess a formula you are making a *conjecture.*
- The *binomial theorem* for $n \in \mathbb{N}$:

$$(a + x)^n = C(n,0)a^n x^0 + C(n,1)a^{n-1}x^1 + C(n,2)a^{n-2}x^2 + C(n,3)a^{n-3}x^3 + \ldots + C(n,r)a^{n-r}x^r + \ldots + C(n,n-1)a^1x^{n-1} + C(n,n)a^0x^n$$

Note 1 The value of $C(n,r)$ is $\dfrac{n!}{(n-r)!r!} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$

2 The expansion of the product has $n + 1$ terms.

3 Using the sigma notation, the binomial theorem may be written

$$(a + x)^n = \sum_{r=0}^{n} C(n,r)a^{n-r}x^r$$

- *Pascal's triangle and the binomial theorem*

value of n	expansion of $(a + x)^n$	Pascal's triangle	
1	$1a^1 + 1x^1$	1 1	$C(1,0)$ $C(1,1)$
2	$1a^2 + 2ax + 1x^2$	1 2 1	$C(2,0)$ $C(2,1)$ $C(2,2)$
3	$1a^3 + 3a^2x + 3ax^2 + 1x^3$	1 3 3 1	$C(3,0)$ $C(3,1)$ $C(3,2)$ $C(3,3)$
4	$1a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + 1x^4$	1 4 6 4 1	$C(4,0)$ $C(4,1)$ $C(4,2)$ $C(4,3)$ $C(4,4)$

Note: In Pascal's triangle, the numbers on the left and right of each row are both 1. Each of the other numbers is the *sum* of the two numbers on each side of its in the line *above.*

- $C(n + 1,r) = C(n,r - 1) + C(n,r)$

Inventory

Complete each of the following statements.

1. Step 1 in the principle of mathematical induction usually shows that a statement is true for ______.

2. In step 2 in the principle of mathematical induction you assume that the statement is true for $n =$ ______, then in step 3 you ______ the statement is true for $n =$ ______.

3. Select the word in the bracket to make the statement true.
   a) If a formula is true for $n = 1$, $n = 2$, and $n = 3$, then the formula is (always, sometimes, never) true for all $n \in \mathbb{N}$.
   b) If you assume that a formula is true for $n = k$ and then are able to prove that it is true for $n = k + 1$, then the formula is (always, sometimes, never) true for all $n \in \mathbb{N}$.

4. You conjecture that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$.
   a) For $n = 1$, the *L.S.* equals ______, and the *R.S.* equals ______.
   b) For $n = k$, the *L.S.* equals ______, and the *R.S.* equals ______.
   c) For $n = k + 1$, the *L.S.* equals ______, and the *R.S.* equals ______.

5. You conjecture that
   $$(1 + 1)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{4}\right)\left(1 + \frac{1}{5}\right) \ldots \left(1 + \frac{1}{n}\right) = n + 1$$
   a) For $n = 1$, the *L.S.* equals ______, and the *R.S.* equals ______.
   b) For $n = k$, the *L.S.* equals ______, and the *R,S.* equals ______.
   c) For $n = k + 1$, the *L.S.* equals ______, and the *R.S.* equals ______.

6. You conjecture that $f(n) = \dfrac{n^3 + 3n^2 + 2n}{3}$ is a natural number for all $n \in \mathbb{N}$.
   a) For $n = 1$, the statement becomes ______.
   b) For $n = k$, the statement becomes ______.
   c) For $n = k + 1$, the statement becomes ______.

7. You conjecture that $2^n < n!$ for $n \geqslant 4$, $n \in \mathbb{N}$.
   a) For $n = 4$, the statement becomes ______.
   b) For $n = k$, the statement becomes ______.
   c) For $n = k + 1$, the statement becomes ______.

8. In the expansion of $(a + x)^5$, there are ______ terms. Unsimplified, these terms are ______.

9. Row $n = 8$ in Pascal's triangle is
   1 8 28 56 70 56 28 8 1
   Therefore, row $n = 9$ is ______.

Review Exercises

1. State the three steps in a proof using mathematical induction.

2. Prove the following statements using mathematical induction, where $n \in \mathbb{N}$.

   a) $4 + 11 + 18 + \ldots + (7n - 3) = \frac{n(7n + 1)}{2}$

   b) $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

   c) $1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$

   d) $1(2) + 2(3) + 3(4) + \ldots + n(n + 1)$
   $= \frac{n(n + 1)(n + 2)}{3}$

   e) $1(2) + 2(4) + 3(6) + \ldots + n(2n)$
   $= \frac{n(n + 1)(2n + 1)}{3}$

   f) $1(2)3 + 2(3)4 + 3(4)5 + \ldots$
   $+ n(n + 1)(n + 2) = \frac{n(n + 1)(n + 2)(n + 3)}{4}$

3. Conjecture and prove a formula for the sum of n terms of the series
   $1 + 7 + 19 + \ldots + (3n^2 - 3n + 1)$

4. Prove the following statements using mathematical induction, where $n \in \mathbb{N}$.

   a) $\frac{1}{1 \times 4} + \frac{1}{4 \times 7} + \frac{1}{7 \times 10} + \ldots$
   $+ \frac{1}{(3n - 2)(3n + 1)} = \frac{n}{3n + 1}$

   b) $\frac{1}{1 \times 5} + \frac{1}{5 \times 9} + \frac{1}{9 \times 13} + \ldots$
   $+ \frac{1}{(4n - 3)(4n + 1)} = \frac{n}{4n + 1}$

   c) $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \ldots$
   $+ \frac{1}{n(n + 1)} = \frac{n}{n + 1}$

   d) $\frac{1}{3^1} + \frac{1}{3^2} + \ldots + \frac{1}{3^n} = 1 - \frac{1}{2}\left(\frac{1}{3^n}\right)$

   e) $\sum_{s=1}^{n}(4s + 1) = \frac{n(4n + 6)}{2}$

5. Prove by mathematical induction

   a) $\sum_{s=1}^{n} s = \frac{n(n + 1)}{2}$

   b) $\sum_{s=1}^{n} s^2 = \frac{n(n + 1)(2n + 1)}{6}$

   c) $\sum_{s=1}^{n} s^3 = \left(\frac{n(n + 1)}{2}\right)^2$

6. Prove the following by mathematical induction
   $4 + 14 + 30 + 52 + \ldots + (3n^2 + n) = n(n + 1)^2$

7. Prove by mathematical induction
   $\frac{1}{3!} + \frac{5}{4!} + \frac{11}{5!} + \ldots + \frac{n^2 + n - 1}{(2n + 1)!}$
   $= \frac{1}{2} - \frac{n + 1}{(n + 2)!}$

8. Use mathematical induction to prove that each of the following is true.

   a) $\left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right)\left(1 - \frac{3}{4}\right) \ldots \left(1 - \frac{n}{n + 1}\right)$
   $= \frac{1}{(n + 1)!}$

   b) $\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{4}\right) \ldots \left(1 + \frac{1}{n + 1}\right)$
   $= \frac{n + 2}{2}$

9. Prove the following where $n \in \mathbb{N}$.

   a) $2n < 2^n$ for $n \geqslant 3$

   b) $(1.1)^n > 1 + \frac{1}{10^n}$ for $n \geqslant 2$

10. Where does mathematical induction *fail* when you try to use it to prove that $100n < n^2$ for all $n \in \mathbb{N}$?

11. Use mathematical induction to prove that $\frac{6n^5 + 15n^4 + 10n^3 - n}{30}$ is a natural number for all $n \in \mathbb{N}$.

12. Use mathematical induction to prove that $\frac{9^n - 4^n}{5}$ is a natural number for all $n \in \mathbb{N}$.

13. a) Show that if you falsely assume that $1 + 5 + 9 + \ldots + (4n - 3) = 2n^2 - n + 3$ is true for $n = k$, then the statement is also true for $n = k + 1$.

    b) Is the formula true for $n = 1$? for all $n \in \mathbb{N}$?

14. Suppose that n circles are drawn in a plane so that each circle intersects all of the others. No two circles are tangent. No three circles pass through the same point. Prove that the plane is divided into $n^2 - n + 2$ non-overlapping regions.

15. Expand each of the following and simplify.

a) $(a + x)^4$
b) $(3a + b)^5$
c) $(2 + x)^3$
d) $(2k - 5m)^6$

16. Show that the formula
$1! + 2! + 3! + \ldots + n! = 3^{n-1}$
is true for $n = 1$, $n = 2$, and $n = 3$.
Is the formula true for all $n \in \mathbb{N}$?

17. a) Turn back to the introduction to this chapter on page 384. Read again about the prince who had to open door after door. Use mathematical induction to prove that if the prince lived forever then he would be able to continue unlocking rooms.

b) Try to use mathematical induction to convince the cow with the ladder that she could climb the ladder to the moon, and beyond the moon.

18. Use mathematical induction to prove that

a) $(1)1! + (2)2! + (3)3! + (4)4! + \ldots + (n)n! = (n + 1)! - 1$

b) $(1)(2) + (2)(3) + (3)(4) + \ldots + n(n + 1) = \dfrac{n(n + 1)(n + 2)}{3}$

c) $\dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \ldots + \dfrac{1}{3^n} = \dfrac{1}{2}\left(1 - \dfrac{1}{3^n}\right)$

d) $\dfrac{1}{5} + \dfrac{1}{25} + \dfrac{1}{125} + \ldots + \dfrac{1}{5^n} = \dfrac{1}{4}\left(1 - \dfrac{1}{5^n}\right)$

19. Given n lines in a plane so that each line intersects all the other lines but no three lines are concurrent, show that the lines divide the plane into $\dfrac{n^2 + n + 2}{2}$ non-overlapping regions.

20. Use mathematical induction to prove that each of the following is a natural number for all $n \in \mathbb{N}$.

a) $\dfrac{6^n - 2^n}{4}$
b) $\dfrac{7^n - 2^n}{5}$
c) $\dfrac{8^n - 3^n}{5}$

21. Three consecutive terms in the expansion of $(1 + x)^n$ have coefficients 21, 35, and 35. Find the value for n.

22. Prove, by mathematical induction or otherwise, that

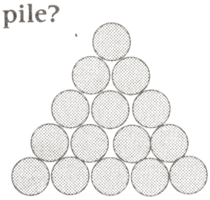
$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{r}x^r + \ldots + x^n$, where n is a positive integer and for $0 < r < n$, $\binom{n}{r} = \dfrac{n!}{r!(n - r)!}$

By using this result, or otherwise, and taking $\binom{n}{n} = 1$ find the values of

a) $\displaystyle\sum_{r=1}^{n}\binom{n}{r}$.
b) $\displaystyle\sum_{r=1}^{n}(-1)^r\binom{n}{r}$.
c) $\displaystyle\sum_{r=1}^{n}r\binom{n}{r}$

(80 H)

23. i) The diagram represents a pile of cylindrical logs; if there are n logs in the lowest row, how many logs are in the pile?



ii) Show that $\dfrac{1}{r(r + 1)} = \dfrac{1}{r} - \dfrac{1}{r + 1}$ for all positive r. Hence prove that

$$\sum_{r=1}^{n}\frac{1}{r(r + 1)} = \frac{n}{n + 1}$$

iii) Assuming that $\displaystyle\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n + 1)(2n + 1)$ show that

$\displaystyle\sum_{r=0}^{n}(r + 1)^2 = \frac{1}{6}(n + 1)(n + 2)(2n + 3)$ and determine the value of $\displaystyle\sum_{r=1}^{n-1}(r + 2)^2$

(82 SMS)